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Title	Looking beyond the lecture: A qualitative examination of pedagogical decision-making and student learning in upper-level mathematics courses
Authors(s)	Murphy, Ciara
Publication date	2026
Publication information	Murphy, Ciara. "Looking beyond the Lecture: A Qualitative Examination of Pedagogical Decision-Making and Student Learning in Upper-Level Mathematics Courses." University College Dublin. School of Mathematics and Statistics, 2026.
Publisher	University College Dublin. School of Mathematics and Statistics
Item record/more information	http://hdl.handle.net/10197/33384

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**Looking beyond the lecture:
A qualitative examination of pedagogical decision-making
and student learning in upper-level mathematics courses**

by

Ciara Murphy

This thesis is submitted to University College Dublin in fulfilment
of the requirements for the degree of:

Doctor of Philosophy

UCD School of Mathematics and Statistics

Head of School: Prof. Nial Friel

Supervisor: Prof. Maria Meehan

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June 2026

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Abstract

Research on the teaching and learning of university mathematics has primarily focused on what happens inside the classroom. This is in spite of acknowledgment from mathematicians and mathematics education researchers that most of student learning occurs independently outside of class. The primary aim of this research is to look beyond the lecture, to gain a more holistic insight into the nature of undergraduate mathematics teaching and learning. We focus on upper-level courses, which to date have been under researched and are subject to different institutional and disciplinary demands than introductory courses. We report on two qualitative, interview-based studies. In the first, four mathematicians participated in a series of three, one-on-one interviews about their instructional design of an upper-level, lecture-based course that they had recently taught. In contrast to previous work that has focused on the lecture or individual instructional components, we examine participants' pedagogical decision-making in relation to all instructional components of their courses, including the notes, lectures, homework, tutorial classes, and final examinations. We identify two tensions in participants' decision-making: the content and problem-solving tensions. We explore the factors that give rise to the tensions, how participants attempted to manage them through their course design, and the implications for future research aimed at augmenting and improving lecture-based instruction. An important factor of influence on mathematicians' instructional decision-making is their pedagogical content knowledge. Prior research on, and theoretical development of, pedagogical content knowledge has primarily occurred in the K-12 context. We combine and adapt two pedagogical content knowledge theoretical frameworks, developed in K-12 science education, for use in undergraduate mathematics. The applicability of the theoretical framework is demonstrated using one participant's interview data. Our second empirical study focused on student learning in upper-level, lecture-based courses. Ten recent graduates were interviewed about the collaboration they undertook on their mathematics

work outside of their lectures. We report on the commonalities in terms of who participants collaborated with, how their collaboration was structured, as well as the benefits attributed to collaboration and the factors that were perceived as constraining the practice. Additionally, we present a case study of two participants to demonstrate how collaborating students can be considered a community of practice and explore the impact that students' perceptions of the nature of research mathematics can have on their learning practices. We also explore participants' perceptions of the concept of undergraduate mathematics students being assessed on group work. Together, our findings provide further insight into the nature of teaching and learning in upper-level courses, with an emphasis on what happens outside the lecture. We conclude the thesis by outlining the implications of our work for future research and the practice of undergraduate mathematics instructors.

Statement of Original Authorship

I hereby certify that the submitted work is my own work, was completed while registered as a candidate for the degree stated on the Title Page, and I have not obtained a degree elsewhere on the basis of the research presented in this submitted work.

Acknowledgements

My deepest thanks go to my supervisor Prof. Maria Meehan. It has been a pleasure to collaborate with such a talented researcher and educator over the past number of years. Thank you for being so generous with your time and energy. Your support, advice, and our many conversations and emails have made completing this PhD an enjoyable and formative experience. I consider myself very fortunate to have begun my career under your guidance.

Although he played no formal role in my PhD, I would also like to thank Assoc. Prof. Ted Cox for being so giving with his time. I am very grateful for the wisdom, advice, and anecdotes you have shared with me over the past four years. I have learned a lot from watching the energy and commitment you bring to your teaching. To both Maria and Ted, I will miss seeing you each morning.

I would also like to express my thanks to the wider community of the School of Mathematics and Statistics in UCD. To Dr Róisín Neururer, thank you for the many coffees over the last number of months – I have looked forward to them each morning. To Dr Áine Byrne and Dr Michael Mackey, thank you for serving as members of my research studies panel and for your guidance throughout my PhD. I'd also like to thank the staff in the school office for all of their work that often happens behind the scenes. I am particularly grateful to my fellow PhD students in the school. Completing a PhD can at times be a lonely endeavour and I am very grateful to have been surrounded by a supportive community. I'd especially like to express my appreciation for Brian and Giulio, who I have been on this journey with from day one and have become great friends.

My thanks also go to the graduates and staff members of the School of Mathematics and Statistics who participated in both of my research studies. I am deeply appreciative that you were so giving with your time.

Completing this PhD marks the end of a period in my life that I will look back on warmly. I have been very fortunate to have several new experiences over the past four years: from learning tin whistle, to sea swimming, to hiking and mountain running. Thank you to all my friends who have made this time so enjoyable. The last four years would have looked very different without two people. To Joe, it can be quite a challenge to live, and work in the same building, with someone for four years. Despite our bickering, I think we've managed it well and you've become like a brother to me. To Pádraig, thank you for all of your support and patience as I finished this thesis. I am very grateful to have you in my life and for our many weekend adventures that have brought me much happiness.

And finally, I'd like to express my gratitude to my family: Suzanne, Liam, Cillian, and Milo. In particular, thank you to my parents whose support has come in many forms: culinary, financial, and emotional. Thank you for giving me the space to find my own way, and for always being there.

Publications

The material in Chapter 5 has been published in *The Journal of Mathematical Behavior*:

Murphy, C., & Meehan, M. (2024). Undergraduate students' collaboration on homework problems in advanced mathematics courses. *The Journal of Mathematical Behavior*, 76, 101200. <https://doi.org/10.1016/j.jmathb.2024.101200>

The material in Chapter 6 is a peer-reviewed conference paper included in the *Proceedings of the Tenth Conference on Research in Mathematics Education in Ireland (MEI 10)*:

Murphy, C., & Meehan, M. (2025). A case study of undergraduate mathematics students' collaboration. In S. Nic Mhuirí & M. Ryan (Eds.) *Proceedings of the Tenth Conference on Research in Mathematics Education in Ireland (MEI 10)* (pp. 156–161). Dublin City University. <https://doi.org/10.5281/zenodo.17398479>

The material in Chapter 7 is a peer-reviewed conference paper included in the *Proceedings of the Ninth Conference on Mathematics Education in Ireland (MEI 9)*:

Murphy, C., & Meehan, M. (2023). Undergraduate mathematics students' views on assessed group work. In A. Twohill & S. Quirke (Eds.) *Proceedings of the Ninth Conference on Research in Mathematics Education in Ireland (MEI 9)* (pp. 196–203). Dublin City University. <https://doi.org/10.5281/zenodo.10037327>

The material in Chapter 3 is under review for publication in *The Journal of Mathematical Behavior*.

The material in Chapter 4 is under review for publication in the *International Journal of Research in Undergraduate Mathematics Education*.

During my PhD, I also contributed to a peer-reviewed conference paper which does not appear as a chapter of this thesis:

Murphy, C., & Meehan, M. (2025). Mathematicians' goals for their upper-level, proof-based mathematics courses. In S. Cook, B. P. Katz & K. Melhuish (Eds.) *Proceedings of the 27th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1007–1013). SIGMAA on RUME.

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Chapter 1

Introduction

While mathematics has been taught and learnt for millennia, research in mathematics education has much more recent origins, with the first doctorates in the field awarded at the beginning of the 20th century (Kilpatrick, 2020). Many early researchers were psychologists interested in how children think and learn about mathematics and, in an attempt to emulate the natural sciences, they mainly conducted laboratory-based studies (Kilpatrick, 2020). However, it was later recognised that these kinds of tests fail to capture the context of how participants are taught (Kilpatrick, 2020). Consequently, research also began to consider the practices of mathematics teachers (Kilpatrick, 2020). This shift away from an exclusively cognitive focus is illustrative of the ‘social turn’ (Lerman, 2000) that occurred in mathematics education research in the 1990s (Inglis & Foster, 2018). During this period, researchers recognised that learning mathematics is not a purely individual, cognitive process. The social context in which learning occurs was considered equally important (Inglis & Foster, 2018).

Over the course of the 20th century the field of mathematics education research continued to grow and expand, with the publication of the first issues of *Educational Studies in Mathematics* and the *Journal for Research in Mathematics Education* in 1968 and 1970 respectively considered key milestones (Inglis & Foster, 2018). One aspect of the research field’s development was an expansion in terms of the educational levels considered. While early research focused on the K-12¹ level, from the 1990s onwards researchers also began attending to the teaching and learning of university mathematics (Winsløw et al., 2018). Mirroring the development of the wider discipline, initial research in undergraduate mathematics education (RUME) had a strong cognitive orientation. Reflecting on the development of RUME, Artigue (2021) remarked that early work principally investigated “students’ learning processes, thinking modes, conceptions and difficulties” (p. 7). There was a particular emphasis on students’ understanding of topics from introductory courses such as calculus and real analysis, likely motivated by high failure rates in these courses which act as a gatekeeper to many majors across the sciences (Artigue, 2021). As the field developed, research moved away from exclusively focusing on student cognition, and social and affective issues were also

¹A US-based term to describe the 13 years of pre-tertiary education from Kindergarten to Year 12 of school.

considered (Dreyfus et al., 2025). Additionally, there has been a shift from “investigations focusing on the student to a more balanced interest in both student and teacher” (Artigue, 2021, p. 12), with university teaching practices becoming a research topic in their own right.

As a subdiscipline of mathematics education research, the ultimate aim of RUME is to improve undergraduate mathematics instruction (Schoenfeld, 2000). Achieving this goal requires a comprehensive understanding of the nature of current teaching and learning practices (Schoenfeld, 2000). Herbst and Chazan (2023) outline two approaches to educational research: an instrumental approach and a fundamental approach. In an instrumental approach, teaching is considered a variable that can be manipulated to improve certain outcomes (e.g. student achievement). An example of this kind of research in a RUME context might be investigating the potential influence of the mode of delivery of homework (online or pen and paper) on student achievement (e.g., Hauk et al., 2015). In contrast, a fundamental approach views teaching as a “phenomenon in the social world that exists in response to societal and institutional conditions of existence” (Herbst & Chazan, 2023, p. 190) and seeks to understand its nature, rather than prescribing some desirable way of teaching. The research in this thesis adopts a fundamental approach. We are broadly interested in gaining insight into the nature of teaching and learning in undergraduate mathematics.

Surveying the existing literature on the teaching and learning of university mathematics, we see a focus on what happens inside the classroom. Mathematicians’ in-lecture practices have been investigated (Artemeva & Fox, 2011; Fukawa-Connelly et al., 2017; Paoletti et al., 2018; Woods & Weber, 2020) and their efficacy debated (Halmos et al., 1975; Leron & Dubinsky, 1995). Furthermore, research has examined students’ in-class learning practices (Fukawa-Connelly et al., 2017) and queried the level of understanding that they emerge from class with (Lew et al., 2016). We question whether this focus on in-lecture teaching and learning is appropriate in the context of undergraduate mathematics. Mathematicians and mathematics educators agree that, regardless of the format of in-class instruction, most of university students’ learning occurs out of class (Ellis et al., 2015; Rupnow et al., 2021; Wu, 1999). However, research has been slow to examine the teaching and learning that happens outside the classroom (Rupnow et al., 2021). The overarching goal of our research is to begin to address this gap by investigating the nature of undergraduate mathematics teaching and learning *beyond the lecture*. In particular, we focus on upper-level courses as they have been under researched (Florensa et al., 2025) and are likely subject to different disciplinary and institutional demands than introductory courses (Johnson et al., 2025; Ko et al., 2024).

In this thesis, we examine two particular aspects of teaching and learning in upper-level mathematics courses. Existing research on mathematicians’ instructional practices and pedagogical decision-making has largely focused on the lecture (e.g., Artemeva & Fox, 2011;

Fukawa-Connelly et al., 2017; Johnson et al., 2018; Paoletti et al., 2018; Woods & Weber, 2020). There have been a number of isolated studies on various other instructional components of undergraduate mathematics courses (e.g., Iannone & Simpson, 2015; Mesa, 2022; Rupnow et al., 2021). However, there has been limited research examining mathematicians' overarching pedagogical decision-making in relation to *all* instructional components of their courses (Pinto & Karsenty, 2018). The first strand of our research addresses this gap by examining mathematicians' pedagogical decision-making across upper-level, lecture-based courses. The second strand of our research is concerned with student learning in these types of courses. Existing research has noted that collaborating with peers can be a source of cognitive, social, and affective support (Brown et al., 2005; Johnson & Hanson, 2015; Rupnow & Feliciano-Semidei, 2025; Seymour & Hewitt, 1997; Solomon et al., 2010). Additionally, while students' engagement with homework has been identified as important (Ellis et al., 2015; Rupnow et al., 2021), we have limited insight into the nature of students' out-of-class learning (Dorko, 2021; Lew & Zazkis, 2019). We connect these two topics by examining how upper-level students collaborate on their mathematics work outside of their lectures. In summary, the two strands of our research address the following research questions:

- **RQ1:** How do mathematicians make pedagogical decisions about their instructional design of upper-level, lecture-based courses?
- **RQ2:** How do upper-level students collaborate on their mathematics work outside of lectures?

Two empirical studies were designed and conducted to address these research questions. Both studies were qualitative in nature as we aimed to gain an in-depth insight into mathematicians' and students' perspectives on teaching and learning, and prioritised conducting research with participants in their natural contexts (Creswell, 2013). Addressing our first research question, we designed and conducted what we refer to throughout the thesis as the 'lecturer study'. This was an interview-based study in which four mathematicians participated in a series of three, one-on-one interviews about an upper-level course that they had taught in the previous three years. The interviews were designed to examine participants' pedagogical decision-making across all instructional components of their courses including the notes, lectures, homework, tutorial classes, and final examination. Two papers based on this study are included in Chapters 3 and 4 of the thesis. In Chapter 3, we draw on Herbst and Chazan's (2011, 2023) theory of practical rationality to analyse the interview data for evidence of tensions in the form of competing individual and socio-technical decision-making factors. Two tensions are presented: the content and problem-solving tensions. The content tension

centres on the pressure participants perceived to cover certain content due to disciplinary obligations, which conflicts with a lack of time in the course and the need to accommodate diverse student backgrounds. The problem-solving tension is rooted in participants' goal of engaging their students in authentic problem-solving, which is constrained by institutional assessment norms and limited control over students' out-of-class work. By including a vignette of each participant, we illustrate the variety of ways in which they attempted to manage the tensions. The implications in terms of informing future research on augmenting lecture-based instruction are addressed.

An influence on mathematicians' instructional decision-making that has been widely discussed in the mathematics education research literature is their pedagogical content knowledge (Shulman, 1986). In Chapter 4, we present a theoretical paper in which we combine and adapt two pedagogical content knowledge theoretical frameworks developed in K-12 science education (Carlson et al., 2019; Magnusson et al., 1999) for use in undergraduate mathematics. Three characteristics of the proposed theoretical framework that make it well suited to analysing pedagogical content knowledge in this context are highlighted. We demonstrate how the theoretical framework can be operationalised using template analysis (Brooks et al., 2015; King, 2012) to code interview data and illustrate its applicability using one participant's interviews.

To address our second thesis-level research question, we designed and conducted what we refer to as the 'student study'. Ten recent graduates of degrees in the mathematical sciences participated in one-on-one, semi-structured interviews about the collaboration on their mathematics work that they undertook as students. The interviews were analysed for commonalities in participants' accounts using reflexive thematic analysis (Braun & Clarke, 2022). The findings of this analysis are presented in Chapter 5. We examine participants' descriptions of how they collaborated on homework problems and with whom. Additionally, we explore their perceptions of the affordances of collaborating on homework, as well as the factors they perceive as constraining their engagement in the practice. The findings provide initial insight into the nature of upper-level students' out-of-class learning.

In Chapter 6, we present a case study of two participants from the student study who worked together as undergraduate students and shared the ambition of becoming research mathematicians. Drawing on Wenger's (1998) theory of communities of practice, we illustrate how the three characteristics of a community of practice (mutual engagement, joint enterprise, and shared repertoire) manifested in the practices of the group. Additionally, we explore how the participants' perceptions of the nature of research mathematics influenced how they attempted to align their collaborative practices with their imagined futures as research mathematicians. Preparing future mathematicians has been identified as a goal in upper-level

courses (Woods & Weber, 2020) and this paper provides insight into the student perspective on the issue.

While the main focus of the student study was on the collaboration that students undertake informally outside of class, we also asked participants about their experiences and perceptions of assessed group work (students being asked to work together to submit a piece of coursework for assessment). In Chapter 7, we describe how participants questioned whether the purpose of implementing assessed group work is necessarily to learn more mathematics. Additionally, participants specified that if assessed group work were to be implemented in a course then particular attention should be paid to how students are assigned to groups and the mechanism by which they are assessed. The relationship between our thesis-level research questions, empirical studies, and thesis chapters is represented diagrammatically in Figure 1.1.

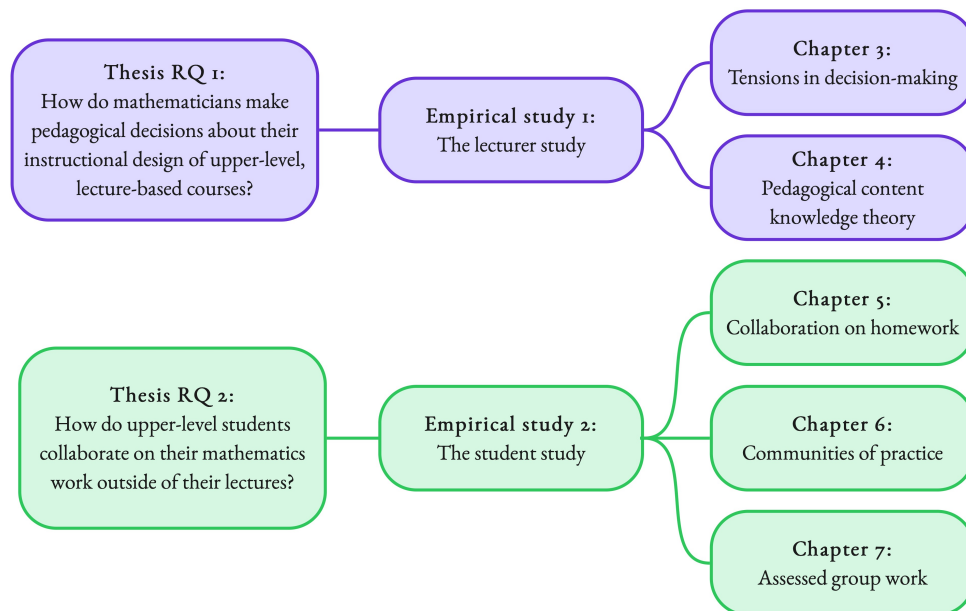


Figure 1.1: An overview of the relationship between the thesis-level research questions, two empirical studies, and thesis chapters.

Each of the five chapters based on our two empirical studies are papers that have either been published or prepared for publication. Therefore, in each chapter relevant background and methodological details are included and the implications of the results are discussed. The thesis contains three additional chapters. In the current chapter, we have provided an introduction to our research and an overview of the thesis structure. In Chapter 2, we outline our research methodology. While relevant methodological details are provided in

Chapters 3-7, in this chapter we take the opportunity to provide further details on the study designs, participants, and ethical considerations. We also describe the research paradigm, epistemological and ontological beliefs that guided our research. Additionally, we discuss quality criteria in qualitative research, with a focus on researcher reflexivity (Berger, 2015). Due to journal-imposed word count restrictions, researchers' engagement in reflexivity is not frequently addressed in the methodology sections of published articles (Trainor & Bundon, 2021). This was an issue we faced when preparing our research for publication. In Chapter 2, we take the opportunity to describe our engagement in reflexivity, drawing on examples from the data analysis process in the student study.

The thesis concludes with Chapter 8 in which we discuss the contributions, limitations, and implications of our research. Chapters 3-7 each include a discussion of the significance of our findings. However, in Chapter 8 we look across our work as a whole. We summarise our main findings and explain how they address our two thesis-level research questions. We also talk about the key recurring themes in the thesis. Additionally, we discuss the theoretical and methodological contributions of the thesis. This is followed by a reflection on the limitations of the studies. As previously mentioned, the research in this thesis adopts a fundamental approach (Herbst & Chazan, 2023), with the primary objective of providing further insight into the nature of teaching and learning in upper-level, lecture-based mathematics courses. However, we also consider our research to have implications in terms of guiding future study and informing the practice of undergraduate mathematics instructors. We conclude the thesis by discussing these implications of our work.

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Chapter 2

Methodology

2.1 Introduction

In this chapter, we provide an overview of the design of both the lecturer and student studies. Both studies were interview based. For each, we describe the participants and interview design, in addition to various ethical considerations. We also discuss our rationale for selecting the qualitative data analysis methods used in each of the studies. Both studies were conducted in University College Dublin, a large, PhD-granting university in Ireland. We provide additional details about the research context in Section 2.3. Additionally, we outline the research paradigm, and ontological and epistemological beliefs that shaped the research. We conclude the chapter by considering quality criteria in qualitative research, with a particular focus on reflexivity.

2.2 Research paradigm, ontological, and epistemological beliefs

Qualitative research is typically characterised by its focus on meaning, interpretation, and studying subjects in their natural contexts (Creswell, 2013). However, qualitative research should not be thought of as a single approach to research (Braun & Clarke, 2022b). It can be underpinned by several different, and potentially conflicting, research paradigms (Braun & Clarke, 2022b). A research paradigm is defined as “the basic belief system or world view that guides the investigator” (Guba & Lincoln, 1994, p. 105). Research paradigms consist of a set of interconnected ontological and epistemological beliefs. Ontology relates to the nature of reality and addresses questions such as ‘What exists?’ and ‘What is real?’ (Braun & Clarke, 2022b). Epistemology relates to the nature of knowledge and addresses questions such as ‘What can be known?’ and ‘How can we come to know?’ (Braun & Clarke, 2022b). Guba and Lincoln (1994) distinguish between four major research paradigms: positivism, postpositivism, critical theory, and constructivism.

The research in this thesis is underpinned by a postpositivist research paradigm (Guba & Lincoln, 1994). Postpositivism is a refinement of positivism which dominated scientific research for centuries (Braun & Clarke, 2022b). Positivism assumes the existence of a singular, objective reality and that researchers can observe this reality directly without influencing it. Postpositivism draws on critical realism, a form of ontology which assumes the existence of a singular reality but “recognises that human practices always shape how we experience and know this” (Braun & Clarke, 2022b, p. 169). Therefore, it is not possible to gain direct access to reality. This differs from constructivism which draws on relativism as its ontological position (Guba & Lincoln, 1994) and maintains that there is no singular reality and that reality is a product of human practices and interaction (Braun & Clarke, 2022b).

In terms of its epistemological stance, positivism assumes that it is possible for researchers to set aside their values and biases so that they play no role in the research process (Guba & Lincoln, 1994). In the postpositivist research paradigm, this is no longer assumed to be the case (Guba & Lincoln, 1994). Postpositivism acknowledges that researchers’ perspectives vary and can influence the knowledge-generation process (Braun & Clarke, 2022b). Constructivism can be considered to go one step further and its epistemological position is that the researcher and subject are intrinsically linked and that findings are ‘created’ as the investigation proceeds (Guba & Lincoln, 1994). Postpositivism is now the dominant research paradigm underpinning social science research (Braun & Clarke, 2022b). However, some researchers have criticised post-positivism for its attempt to sit in between positivism and constructivism without taking a definitive ontological or epistemological position (Braun & Clarke, 2022b).

By adopting postpositivism as my research paradigm, I assume that there is a singular reality but that this is never fully knowable. The knowledge that I generate is influenced by the past experiences and perceptions of participants, as well as my own. I acknowledge that it is not possible for me to set aside my biases and values completely. It is therefore important that I engage thoroughly in researcher reflexivity (Berger, 2015) to monitor the influence that my personal perspective is having on the research process (see Section 2.6.1 for a full discussion of the role of reflexivity in the research process). Additionally, I am aware that the findings that I generate provide only partial and incomplete insight into the research topics.

2.3 Research context

Both empirical studies reported on in this thesis were conducted in University College Dublin (UCD). UCD is Ireland’s largest university with around 41,000 students, 35% of whom are international students (University College Dublin, 2026). The School of Mathe-

matics and Statistics in UCD is the largest mathematics department in Ireland. The school employs roughly 65 academic faculty with research interests in pure and applied mathematics, statistics, and mathematics education. A range of undergraduate degrees are offered with possible specialisms including pure mathematics, applied and computational mathematics, theoretical physics, statistics, and financial mathematics. The department also offers masters' and doctoral programmes, with around 70 PhD students currently enrolled in the school.

UCD operates a modularised system. Full-time, undergraduate students study an average of 12 courses per year (in UCD courses are referred to as modules). Classes are held during the 12-week autumn and spring trimesters. Each of these teaching periods is followed by a revision week and a two-week examination period. Undergraduate mathematics modules are predominantly assessed through closed-book examination. This is typical of undergraduate mathematics degrees in the UK and Ireland (Iannone & Simpson, 2011; Iannone & Simpson, 2022; National Forum for the Enhancement of Teaching and Learning in Higher Education, 2016). It is standard practice for mathematics instructors in UCD to write their own course notes rather than assigning a course textbook. Instructors typically have autonomy in creating homework assignments (commonly referred to as problem sheets) for their courses. They may ask students to submit solutions to these problems for credit (usually worth between 10-30% of a student's overall grade) or to sit a midterm test consisting of similar problems. Homework problems may be discussed in tutorial classes (similar in nature to what may be referred to as recitation sessions in a US context). For first- and second-year courses these tutorial classes are typically facilitated by PhD students in the school, who may also assist with grading. For upper-level courses, the course instructor is responsible for facilitating tutorial classes and grading students' work.

2.4 Lecturer study methodology

2.4.1 Overview of study design

The aim of this study was to examine mathematicians' instructional design of upper-level, proof-based courses. The relevant extant research has predominantly focused on the lecture, neglecting other aspects of instruction (Rupnow et al., 2021). Additionally, very limited research has considered the relationship between various instructional components by considering teaching across a course (Pinto & Karsenty, 2018). To examine mathematicians' decision-making across an upper-level course, we decided to focus on a section of content consisting of at least one of each of the following: definition, theorem, example, and proof. An interview schedule was designed to investigate how the section of content was addressed

in mathematicians' design of their notes, lectures, homework, homework solutions, tutorials, and final examination. The interview design was informed by Schoenfeld's (2010, 2011) theory of resources, orientations, and goals and the idea that goals operate on multiple levels. We initially asked specific questions that were rooted in participants' pedagogical decisions before moving on to broader questions in an attempt to investigate the influence of beliefs and orientations on decision-making. The questions were divided between three interviews with the first focused on the notes and lectures, the second on homework and tutorials, and the final interview on assessment. Further information on the study design and the full interview schedule is provided in Chapter 3.

Four mathematicians participated in the series of one-on-one interviews in summer 2023. Each interview lasted an average of one hour and was audio-recorded and transcribed. The interviews were analysed using template analysis (Brooks et al., 2015; King, 2012) for evidence of tensions in participants' decision-making. The findings of this analysis are presented in Chapter 3. In Chapter 4, we present a theoretical paper in which two PCK frameworks (Carlson et al., 2019; Magnusson et al., 1999) are adapted and combined for use in undergraduate mathematics education. We utilise one participant's interviews from this study to illustrate the combined framework's applicability to the undergraduate mathematics context. In the following subsections, we provide further details on participants, ethical considerations, and data analysis methods associated with the study.

2.4.2 Participants and ethical considerations

The four participants in this study were research mathematicians employed in UCD's School of Mathematics and Statistics. We employed purposive sampling (Braun & Clarke, 2013), inviting tenured faculty with at least 15 years' teaching experience who had taught an upper-level, proof-based course in the last three years to participate. We emailed five faculty members inviting them to participate, providing an information sheet with further details about the study aims. Four agreed to participate. We specifically decided to interview experienced, tenured mathematicians as Mesa et al. (2020) suggested that staff on short-term contracts may be more likely to follow institutional demands and may have less time available to spend on teaching due to the need to work on research and future funding applications. We endeavoured to have a gender representation that reflects that of the department. Three males and one female participated in the study. Additionally, we attempted to have some variation in the research interests of participants. Two participants would be considered algebraists, and the other two mathematicians conduct research in analysis.

The study was approved by the UCD Office of Research Ethics (reference: LS-LR-23-104-Murphy-Meehan). The primary ethical consideration we identified when designing this study was that participants could be potentially identifiable in publications due to contextual information provided about the courses and topics they were interviewed about. This was highlighted to potential participants in the information sheet. The interviews were on a non-sensitive topic and we offered to provide a copy of the planned questions in advance of the interviews. One participant availed of this. Additionally, we sent a final draft of any manuscripts detailing findings of the study to participants prior to submission. Each participant signed a consent form prior to their first interview.

2.4.3 **Choice of data analysis method**

When selecting a data analysis method for this study, we wanted to be able to maintain a focus on individual participants as well as analysing the commonalities across their accounts. One approach that we considered was using interpretative phenomenological analysis (IPA) (Smith et al., 2022). Unlike qualitative methods such as reflexive thematic analysis which can be flexibly used in conjunction with a number of different research paradigms (Braun & Clarke, 2022b), IPA is a qualitative methodology and specifies “guiding theoretical principles, appropriate research questions and study designs, ideal methods of data collection, as well as analytic procedures” (Braun & Clarke, 2013, p. 180). Grounded in phenomenology, the core focus of IPA is on examining personal lived experience and how participants make sense of experiences (Smith, 2011). IPA has an idiographic focus, beginning with in-depth analysis of individual cases before searching for patterns across cases (Smith et al., 2022; Smith, 2011). For this reason, IPA studies typically have small sample sizes and data usually consists of one-on-one interviews (Smith et al., 2022). IPA acknowledges the interpretative work of the researcher and researcher reflexivity is encouraged (Willig, 2021). Smith (2011) talks about the double hermeneutic, a core concept in IPA, whereby the “researcher is trying to make sense of the participant trying to make sense of what is happening to them” (p. 10).

We were drawn to IPA primarily because of its dual analytic focus on individual perspectives as well as commonalities. Additionally, its methodological specifications in terms of small sample size and interview data fit well with our study design. We undertook the first steps of IPA analysis (writing exploratory notes and experiential statements) on one participant’s interviews. However, at this stage we identified two incompatibilities between IPA and our research aims. IPA emphasises bottom-up analysis and it is not advised that researchers employ theoretical frameworks in the analytic process (Smith et al., 2022). The design of the interview schedule for the study was informed by Schoenfeld’s (2010, 2011)

theory of resources, orientations, and goals and we had preliminarily planned to utilise the theoretical framework to guide our analysis. However, this would not be possible under IPA which emphasises staying as close as possible to participants' experiences and sense-making. Another issue that we identified was an incompatibility between IPA's emphasis on "examining people's perceptions of major experiences happening to them and which engage hot cognition" (Smith, 2019, p. 167) and our research focus. We were examining mathematicians' experiences of designing upper-level mathematics courses. Reviewing our data, we felt that our participants' experiences did not align with Smith's (2019) description of emotionally laden events that require wrestling to come to terms with. This is underpinned by the fact that IPA has frequently been used in health psychology to examine patients' experiences of coming to terms with a life-altering diagnosis (Smith & Osborn, 2015).

For these reasons, we decided that IPA was ultimately not the best fit for our study. We reviewed the qualitative methodology literature for data analysis methods that would facilitate an idiographic approach and the use of existing theory in the analysis process. One of the most commonly used qualitative data analysis methods is thematic analysis (TA), with Braun and Clarke (2019, 2022b) distinguishing between three different types: reliability TA, codebook TA, and reflexive TA. The three differ in terms of their philosophical assumptions and analytic procedures (Braun & Clarke, 2022b), with reliability TA most closely aligned with positivism and reflexive TA perhaps being considered the most deeply qualitative. Codebook approaches to TA are considered to lie between these two poles, utilising more structured coding methods associated with reliability TA, while being firmly embedded within a qualitative philosophy like reflexive TA (Braun & Clarke, 2022b).

One frequently used form of codebook TA is template analysis (Brooks et al., 2015; King, 2012). Template analysis emphasises the use of hierarchical coding and central to the method is the development of a coding template. The method sits between purely 'bottom-up' and 'top-down' methods of analysis. Existing theory and findings from relevant literature can be incorporated into the initial coding template as *a priori* codes. However, there is also scope for inductive, bottom-up coding of the data. Template analysis allows researchers to maintain an idiographic focus. King (2012) outlines how it is possible to present the findings of template analysis as a series of main themes followed by individual case studies to illustrate each theme. Alternatively, one could start by presenting individual case studies and then discuss the similarities and differences between these cases (King, 2012). Template analysis is most commonly used to analyse one-on-one interviews (Brooks et al., 2015; King, 2012). Template analysis is compatible with a range of epistemological positions and can be approached from a perspective where the researcher's role in the analytic process is

acknowledged and researcher reflexivity encouraged (Brooks et al., 2015). Consequently, practices such as calculating inter-rater reliability are not endorsed (Brooks et al., 2015).

We decided to use template analysis (Brooks et al., 2015; King, 2012) in this study as it allowed us to incorporate theoretical frameworks into our analysis and to maintain a focus on individual participants by presenting case studies. In Chapter 3, we provide further details on how template analysis was used to code the interview data. In Chapter 4, we adapt and combine two PCK theoretical frameworks (Carlson et al., 2019; Magnusson et al., 1999) for use in undergraduate mathematics education. Template analysis is used in an instrumental case study (Stake, 1995) of one participant's interviews to demonstrate the applicability of the proposed framework.

2.5 Student study methodology

2.5.1 Overview of study design

The general aim of this study was to gain insight into students' learning experiences in upper-level university mathematics courses. Specifically, we were interested in the collaboration they undertake informally outside of their lectures. We chose to adopt an interview-based design as we were interested in gaining an in-depth understanding of the nature of this form of collaboration, as well as students' perceptions of its advantages and the factors that constrain it. Ten recent graduates of undergraduate degrees in the mathematical sciences participated in the study. The graduates participated in semi-structured, one-on-one interviews in spring 2022. The interview schedule was designed to investigate the nature of participants' informal collaboration with peers by asking details about who they worked with, what material they worked on, as well as when and where they had collaborated. We also asked participants if they had ever been formally asked to work in a group during their degree (e.g. to complete a group project) and, if relevant, how they found this experience. Additionally, we were interested in participants' perceptions of the benefits of the informal collaboration they had undertaken. We also asked about the messages participants may have received from their instructors about whether collaboration was encouraged, or indeed permitted. Graduates were also asked about their impressions of the collaborative nature of research mathematics. Additional methodological details and the full interview schedule for the study are provided in Chapter 5.

Each interview lasted an average of around 30 minutes and was audio-recorded and transcribed. The ten interviews were analysed for commonalities using reflexive thematic analysis (Braun & Clarke, 2022b). We found that participants most commonly collaborated

on homework. In Chapter 5, we report on the themes we generated focusing on the nature of participants' collaboration on homework, as well as their perceptions of its benefits and factors that constrain the practice. While this analysis was focused on identifying commonalities in participants' accounts, we also noted variation in the graduates' collaborative practices that resonated with Wenger's (1998) theory of communities of practice. We explore this aspect of the data further in a case study of two participants presented in Chapter 6. Template analysis (Brooks et al., 2015; King, 2012) was used as the method of data analysis for this case study. In Chapter 7, we outline participants' views on assessed group work. Reflexive thematic analysis (Braun & Clarke, 2022b) was used to generate three themes centred on participants' views on being assessed as a group. In the subsequent subsections, we provide further details on the participants, ethical considerations, and data analysis methods associated with the study.

2.5.2 Participants and ethical considerations

The study was approved by the UCD Office of Research Ethics (reference: LS-E-22-40-Murphy-Meehan). At the time of interview, the ten participants were recent graduates of UCD's four-year Bachelor of Science undergraduate degree. A combination of convenience sampling followed by snowball sampling was used (Braun & Clarke, 2013), with recruited participants asked to recommend other graduates who may be interested in participating. We did not require participants to have experience of collaborating during their undergraduate degree, however all those who agreed to participate had this experience. The decision to interview graduates, rather than current students, was driven by ethical considerations. We were planning to ask participants about their collaboration on homework assignments which they may have submitted for credit. We felt that students who were still in the process of obtaining their degree may be reluctant to discuss their collaborative practices openly. Additionally, at the time of conducting the interviews I was an undergraduate mathematics student in UCD. We believed that it would not be ethical or methodologically sound for me to interview classmates with whom I may have personally collaborated. An additional ethical consideration was protecting the anonymity of participants. While we report the gender distribution of our sample (four females and six males), we do not specify the gender of individual participants. To this end, each participant was assigned a gender-neutral pseudonym. All participants were provided an information sheet about the study and signed a consent form prior to being interviewed.

2.5.3 Choice of data analysis methods

Braun and Clarke's (2006) paper on thematic analysis is "now one of the most cited academic papers of recent decades" (Braun & Clarke, 2019, p. 590). Therefore, it is unsurprising that thematic analysis is named as the method of qualitative data analysis in many published academic papers across a wide range of disciplines. However, thematic analysis "refers not to a singular approach, but rather to a cluster of sometimes conflicting approaches, divergent both in procedure and underlying philosophy, but which share an interest in capturing patterns in data" (Braun & Clarke, 2021, p. 333). One approach to thematic analysis is Braun and Clarke's (2019, 2022b) reflexive thematic analysis (RTA). RTA differs from other forms of thematic analysis in its conceptualisation of qualitative data analysis as inherently situated and subjective. Braun and Clarke emphasise that data analysis should be viewed as an active and creative process in which researcher subjectivity is capitalised on, rather than controlled for. The coding process is described as requiring a "continual bending back on oneself – questioning and querying the assumptions we are making in interpreting and coding the data" (Braun & Clarke, 2019, p. 594). Themes are conceptualised as the analytic outputs of the coding process. They are actively created, rather than found, by the researcher.

Our decision to utilise RTA in this study was based on several factors. The first factor was that RTA aligned well with our research aims. Beginning the study, we were broadly interested in examining the nature of students' out-of-class collaboration, their perceptions of the practice, as well as the factors that may shape or influence their participation in collaboration. RTA is well-suited to research with these types of aims (Braun & Clarke, 2022a). Additionally, RTA does not require research questions to be fixed from the outset of the study. When conducting RTA, it is very common for initial research questions to be quite broad and to subsequently "become more focused or expand or even shift in focus, as data collection and analysis progresses" (Braun & Clarke, 2022a, p. 12). Given the limited research on students' out-of-class practices, we were reluctant to narrow the scope of our analysis prematurely. Using RTA allowed us to broadly consider students' out-of-class collaboration before focusing in on collaboration on homework tasks after realising that this was the primary focus of students' collaborative efforts.

A second factor that influenced our decision to use RTA was that since there was very limited pre-existing research on the topic, we opted to conduct an inductive analysis. We aimed to gain a broad insight into students' experiences of collaborating without potentially narrowing our analytic focus by employing a theoretical framework. RTA was compatible with conducting this kind of inductive, less structured analysis. The final factor that we considered was that we were conducting the analysis from two quite disparate perspectives.

At the time of conducting the interviews, I was a final-year undergraduate mathematics student with personal experience of collaborating with my peers. Thus, I would be considered an insider researcher (Braun & Clarke, 2022b). On the other hand, my supervisor was coming to the data with the perspective of a professor of mathematics. RTA's emphasis on researcher reflexivity allowed for these two perspectives to be capitalised on as a "resource for knowledge production [. . .] rather than a must-be-contained threat to credibility" (Braun & Clarke, 2021, pp. 334-335). (More detail on how we engaged with reflexivity throughout the data analysis process is provided in section 2.6.1.)

In Chapter 6, we present a case study of two participants focused on analysing their collaborative practices through the lens of Wenger's (1998) theory of communities of practice (CoP). For this analysis, we chose to use template analysis (Brooks et al., 2015; King, 2012) as the more structured approach to coding facilitated the incorporation of the CoP framework into our analysis. The focus of our analysis in Chapter 5 is on the collaboration students undertake informally outside of their lectures. However, we also asked participants of their experiences and perceptions of assessed group work (when students are formally asked to work together and are assessed on this work). We analysed this aspect of our data using RTA (Braun & Clarke, 2022b) and report on the three themes generated in Chapter 7.

2.6 Quality criteria in qualitative research: A focus on reflexivity

Qualitative and quantitative research differ in terms of their underlying theory, approaches, and goals. Therefore, quantitative quality criteria such as reliability, validity, and generalisability either do not apply to, or must be interpreted differently when considered in relation to, qualitative research (Braun & Clarke, 2013). Given the range of different approaches and underpinning theories associated with qualitative research, it has been argued that "if a piece of research is to be evaluated, it needs to be evaluated on its own terms" (Finlay, 2006, p. 325). However, there have been attempts to develop quality criteria that are aligned with the nature of qualitative research. For instance, the concept of transferability (Finlay, 2006; Lincoln & Guba, 1985) has been proposed to encourage qualitative researchers to provide a detailed description of the context of their study so that readers can evaluate its applicability to their own setting. Additionally, concepts such as dependability (Finlay, 2006; Lincoln & Guba, 1985) and documentation (Finlay, 2006; Henwood & Pidgeon, 1992) promote the practice of maintaining audit trails documenting the research process. How-

ever, perhaps one of the most widely discussed criteria for quality in qualitative research is reflexivity (Berger, 2015; Finlay, 2002, 2006).

Berger (2015) defined reflexivity as follows:

It means turning of the researcher lens back onto oneself to recognize and take responsibility for one's own situatedness within the research and the effect that it may have on the setting and people being studied, questions being asked, data being collected and its interpretation. As such, the idea of reflexivity challenges the view of knowledge production as independent of the researcher producing it and of knowledge as objective. (p. 220)

Engaging in reflexivity requires acknowledging one's positionality and the impact this has on the research process (Berger, 2015). One aspect of a researcher's positionality that is particularly important is the extent to which they have personal experience of the topic being studied. Researchers can be broadly classified as either insider or outsider researchers (Braun & Clarke, 2022b; Hill & Dao, 2021) depending on the extent to which they have a pre-existing connection with the group or topic being researched. However, this is not a straightforward dichotomy, and it is possible to identify as both an insider and outsider researcher to varying degrees (e.g., Baz, 2024; Britton, 2020). Additionally, as a study progresses, researchers continue to have new experiences which may offer them new perspectives on their data (Berger, 2015; Braun & Clarke, 2022b; Trainor & Bundon, 2021).

When I conducted the interviews for the student study, I was a final-year undergraduate mathematics student with personal experience of collaborating on homework. This meant that I identified with the experiences of the participants and would be considered an insider researcher (Braun & Clarke, 2022b; Hill & Dao, 2021). My perspective shaped all aspects of the research process, including the choice of topic. Within the qualitative paradigm it is accepted that the topics that one chooses to study are an "expression of personal interests and values" (Wilkinson, 1988, p. 494). I was interested in studying collaboration based on my own experiences of its value to my learning and I was curious to see whether that would be replicated in the accounts of others. My status as an insider researcher also influenced the data generation process. It likely made it easier for me to recruit participants and to establish a rapport with them (Berger, 2015; Hill & Dao, 2021). Additionally, having completed the same undergraduate programme as the participants I had a good understanding of the context in which their collaboration occurred. My perspective on the interviews also influenced the data analysis process. In Section 2.6.1, I will elaborate on how I utilised strategies for engaging in reflexivity, suggested in the qualitative methodology literature, to monitor the impact of my positionality on the data analysis process.

The perspective I brought to the lecturer study was quite different. As an undergraduate student, I had been taught by each of the participants in the study. This was the primary perspective I brought to the research. I identified as an outsider researcher (Braun & Clarke, 2022b; Hill & Dao, 2021) and was curious to gain insight into the participants' perspectives on courses I had completed as a student. I found that my status as an outsider researcher made the data analysis process for this study challenging. In the student study, I felt that I understood the interviews easily and that the risk of me misinterpreting participants' meaning was relatively low. However, having no personal experience of lecturing, the interviews with the mathematicians felt much more unfamiliar. This was highlighted by the contrast between my supervisor's initial reactions to the data and my own. As a mathematics lecturer, my supervisor was able to relate and compare the participants' accounts with her own experience. For instance, in some of her initial notes on the data she commented "I immediately identified with the whole tract" and "exactly what I did the first time I taught [course]". In contrast, my initial notes often related back to my experience as a student in the courses: "I remember this question being quite fun" and "[their] enthusiasm for it comes across very strongly in [their] lectures". My positionality as an outsider had implications for my interpretation of the data. I had to remember to let participants' own words guide the analysis and to draw on my supervisor's insider perspective to minimise the risk of misinterpretation.

In the following section, I provide further details on my engagement in reflexivity during the data analysis process by drawing on examples from the student study.

2.6.1 Engaging in reflexivity in the data analysis process

In the qualitative methodology literature, several practices have been suggested to help researchers engage with reflexivity during the data analysis process. For instance, it has been proposed that maintaining a 'reflexive journal' (Braun & Clarke, 2022b), 'research diary' (Collins & McNulty, 2020), or 'audit trail' (Berger, 2015) can be a useful tool to monitor and review one's decision-making and reactions to both the data and ongoing analysis. It can be particularly useful to review these reflections after taking a break from the analytic process. Some authors advise that taking time away from analysing qualitative data can allow researchers to maintain "critical distance" (Collins & McNulty, 2020, p. 207) from the analysis and data. Berger (2015) advises that repeatedly reviewing preliminary analysis "offers an opportunity to view the same material through [a] 'new lens' and identify where one's own experience interfered with accurately understanding interviewee's report" (p. 230). Another suggested strategy for identifying ways in which a researcher's positionality may be interfering with the analytic process is collaboration (Berger, 2015; Braun & Clarke, 2022b;

Collins & McNulty, 2020). Involving a colleague in the analytic process can offer a fresh perspective on the data and help researchers to identify alternative meanings they may not have previously considered (Berger, 2015).

Although useful advice on engaging in reflexivity is available in the qualitative methodology literature, there are limited examples of these strategies being implemented in practice. Oftentimes, the methodologies appearing in published research articles consist of short, polished summaries of the method employed and fail to “reflect the complexity of the authors’ experience” (Collins & McNulty, 2020, pp. 203-204). Where reflexivity is discussed, it is frequently limited to a short summary of the research team’s relation to the topic or population being studied (Trainor & Bundon, 2021). The practice of including these types of positionality statements is becoming more common in published research on undergraduate mathematics education (e.g., Dobie & MacArthur, 2022; Shultz et al., 2024; Tremaine et al., 2022). While this is undoubtedly a positive step, it is still relatively uncommon for the practical realities of reflexivity to be addressed in published research. This absence is likely the result of journal-imposed word count restrictions (Trainor & Bundon, 2021). However, accounts in which researchers fully expose their thought processes, decision making, and active role in the analytic process are important in terms of helping novice qualitative researchers develop their understanding of how reflexivity can be practically engaged with.

In what follows, I provide three examples of how I utilised three strategies from the qualitative methodology literature for engaging in reflexivity. These include collaborating on data analysis, taking breaks from the analytic process, and maintaining an audit trail. The three examples are drawn from the data analysis for the student study (Chapter 5 in particular). As previously mentioned, I was an undergraduate mathematics student at the time of conducting the interviews for this study. As such, I identified as an insider researcher (Braun & Clarke, 2022b; Hill & Dao, 2021). Therefore, in engaging in reflexivity I was particularly mindful of the possibility that I would focus on similarities between participants’ experiences and my own which may leave other aspects of the data underexamined (Berger, 2015) or that I might impose my personal experiences and biases on the participants’ accounts (Hill & Dao, 2021).

2.6.1.1 Collaboration

Before beginning the data analysis for the study, both my supervisor (referred to below as MM) and I (referred to below as CM) wrote and exchanged individual researcher reflections (Braun & Clarke, 2022b) on our thoughts on, and experiences of, the research topic: students collaborating on undergraduate mathematics. At the time, I was a recent mathematics

graduate who had had very positive experiences of working with peers. In my researcher reflection, I acknowledge that I felt very familiar with the research topic and was partially motivated to pursue research on it because of my own experience as a student:

I think it is safe to say that the topic of collaboration between mathematics students is one to which I was never going to come to ‘cold’. Indeed my initial motivation for wanting to explore the topic was born out of my own experience.
(CM, July 2022)

I identified strongly as an insider researcher and reflected that both the “context behind the participants’ experiences” and the interviews felt “incredibly familiar” (CM, July 2022). As a result of my personal experience of collaborating as an undergraduate student, I naively thought that there was little that I would consider ‘new’ or ‘surprising’ in the participants’ accounts. While I now understand that I was “taking for granted similarities and overlooking certain aspects of participants’ experience” (Berger, 2015, p. 224), at the time I was left wondering what kind of interesting story I could tell about the data.

MM’s perspective and initial reflections on the data were different to mine. Her primary perspective on the data was that of a mathematics lecturer. While I felt that the nature of students’ collaboration was somewhat self-evident, as an instructor who was more removed from the undergraduate experience, MM found the participants’ accounts insightful:

I was . . . coming to the data with a real sense of curiosity as a maths lecturer. I was seeing the other side of the story – when we ask students to work as a group, what do they think about it? And when they work on maths together – how do they navigate that? Do they learn? I was getting an insider view as to how students collaborated (or not) during undergraduate study and I was delighted.
(MM, July 2022)

Reading MM’s initial reflections on the data, I was aware that we had differing perspectives on the topic. I felt that my position was that of an insider researcher while MM’s perspective was more closely aligned with that of an outsider. I had read that collaborating on qualitative data analysis with someone who offers an alternative perspective can be a strategy for enhancing one’s engagement with reflexivity (Berger, 2015; Braun & Clarke, 2022b; Collins & McNulty, 2020). However, beginning the analytic process, I was unsure as to how we could capitalise on our seemingly disparate viewpoints and what the practical implications for the analysis would be.

After completing two rounds of coding of the data, I began the process of theme development. At the time, I was conscious that a common issue with qualitative research utilising reflexive thematic analysis (RTA) is a confusion between themes and topic summaries

(Braun & Clarke, 2021, 2022b). Themes should draw together potentially disparate data around a “central concept or idea” (Braun & Clarke, 2021, p. 324). In contrast, topic summaries are surface-level synopses of what participants said and may be drawn directly from responses to interview questions. As a beginning qualitative researcher, I felt compelled to ensure that I was doing RTA the ‘correct’ way. I debated whether it would be possible or ‘correct’ to develop a viable theme that focused on an aspect of the nature of the participants’ collaboration, for instance the people that they collaborated with. I was concerned that these potential themes were perhaps too simple and whether we may be in danger of developing topic summaries as opposed to true themes. As I questioned whether the nature of students’ collaboration warranted its own theme, MM suggested I might be overcomplicating the analysis:

I think plain speaking might work best here. Say it how it is. They work mainly on problem sheets, sometimes all stuck, sometimes one stuck and others know, how the process works (alone and together), and their implicit rules of collaboration. (MM, August 2023)

Braun and Clarke (2022b) advise that the phases of analysis are rarely linear and that researchers should be prepared to move back and forth between stages. Only after writing the first draft of the findings did I realise that in attempting to not provide a pure summary of students’ collaboration, I had inadvertently overcomplicated the themes to the extent that they failed to cohere around a single unifying idea: “I think some of the issues with themes 1 and 2 might be to do with the fact that they are both attempting to do two things” (CM, December 2023).

At this point, I realised that my familiarity with the research topic was interfering with the theme development process. Braun and Clarke (2022b) advise that collaborating on data analysis in RTA is not about researchers striving for consensus or complete agreement. They advise that the value of working with another researcher on the analysis lies in being exposed to alternative interpretations of the data and helping you to “reflect on whether your particular standpoints or experiences are (problematically) limiting and constraining how you engage with the data” (Braun & Clarke, 2022b, p. 274). Gaining insight into MM’s perspective as a lecturer on the data helped me to identify what was interesting about undergraduate students’ collaboration and to realise that there was a worthwhile story to be told about the participants’ accounts, as long as I did not let my familiarity with the research topic constrain the analytic process.

2.6.1.2 Taking breaks from the analysis

Beginning the data analysis process, I was aware that one challenge my status as an insider researcher could pose was separating my own personal experiences from those of the participants. My own (positive) experiences of collaborating as an undergraduate student formed part of my motivations for undertaking the study. In my researcher reflection I acknowledge:

I suppose I was curious as to whether I would see aspects of my own personal experience replicated in the accounts of others. I think reading the interviews through this sort of lens does have its advantages but equally poses potential limitations of which it is probably important to be aware. (CM, July 2022)

Discussing her experience as an insider researcher, Berger (2015) reflected that she had to make “a constant deliberate effort to . . . curb the drive to compare experiences” (p. 224). I experienced a similar tendency to compare participants’ experiences of collaboration with my own. However, I was aware that this could lead to me dismissing certain aspects of participants’ accounts. I was conscious to remind myself that it was the participants’ story we were trying to tell, not my own.

After engaging in an initial process of analytic note-taking on the data, I took a break from the analysis. Revisiting the notes I had made on one of the interviews a couple of months later, I wrote the following reflection:

Looking back at this interview, and the way in which I interpreted it in July, one thing that stood out to me is the way in which I seem to be trying to establish whether [participant’s] experiences of collaboration were similar or different to mine . . . I think I kind of cast [participant] as maybe not collaborating in the manner in which I was familiar and perhaps took a slightly negative view of the way in which I seemed to see [participant] as consulting other people or fishing for information. I think I missed that these relationships were *transactional*, not in a one-way kind of sense, but that there was an element of give and take. (CM, March 2023)

In the qualitative methodology literature, it is advised that taking breaks from the analysis can allow researchers the space to identify where aspects of their own experience may be interfering with the analysis (Berger, 2015; Collins & McNulty, 2020). By taking a few months away from the analysis before re-reading my initial reflections on the interviews, I realised that I was dismissing aspects of the participants’ experiences of collaboration that I felt were not aligned with my own. In doing so, I was imposing my own conception of undergraduate students’ collaboration onto participants’ accounts and, in effect, telling my story as opposed to theirs.

Prior to beginning the analysis, I had read that as an insider researcher I should be cautious about not imposing my own experience on the participants and remember that “while we may have shared an experience, it was different for each, and I want to learn theirs” (Berger, 2015, p. 223). However, it was only through taking breaks and revisiting the data that I developed a full awareness of the extent to which my own experience was shaping my interpretation of the data. As I discussed in my researcher reflection, this awareness was important in terms of capitalising on the familiarity being an insider researcher offers, while ensuring that this familiarity did not compromise the depth and integrity of the analysis:

My position as an insider researcher means the interviews feel very familiar. This gives me a sense of closeness to the participants’ experiences as long as I am careful not to overlook the potential heterogeneity of student experience when choosing to examine others’ experiences in relation to my own. (CM, July 2022)

2.6.1.3 Maintaining an audit trail

Prior to beginning the data analysis for this study, I had read that “the positions that inform our research are multiple, complex, and evolving” (Braun & Clarke, 2022b, p. 15). When I conducted the interviews for the study, I was an undergraduate mathematics student, and this was the primary lens through which I viewed the data. Having had personal experience of collaborating on mathematics homework with my friends, I felt a strong affinity with the experiences of the participants. The data felt very familiar and I identified strongly as an insider researcher.

The data analysis process for the study continued over a period of two years during which time I had started a PhD in mathematics education and had begun tutoring a first-year mathematics course. Having completed my mathematical studies, I became more removed from the undergraduate student experience and developed an identity as a tutor of university mathematics. Examining the audit trail I maintained during the analytic process, it is evident that my new perspective as a tutor was influencing my interpretation of the data. Through my work as a tutor, I witnessed the beginning of the formation of friendships and collaborations between the first-year students. This drew my analytic attention towards the initial process of students finding peers with whom they are compatible with as collaborators and friends. As I noted in my audit trail, I became curious to see whether I would find reference to this process in participants’ accounts:

The other thing was looking at the first years and thinking that really the long-term/time element allows for students to ‘suss out’ those with common factors (like similar academic level, motivation, interest etc.), but also more generally

to find people that they enjoy spending time with, who then go on to become friends. (CM, October 2023)

While it is one thing to read that a “researcher’s position may be fluid rather than static” (Berger, 2015, p. 231), I was interested to see evidence in my audit trail that my experience as a tutor had influenced the aspects of the data I was being drawn to. Having initially viewed the data from the perspective of an undergraduate mathematics student, over the course of the analytic process I had developed alternative perspectives that were less tied to my role as an insider researcher. This realisation reinforced the fact that “insiderness is dynamic” (Collins & McNulty, 2020, p. 204). I understood more clearly that reflexivity requires ongoing engagement and that “researchers must continually ask themselves where they are at any moment in relation to what they study” (Berger, 2015, p. 231). I found that maintaining an audit trail throughout the analytic process was a practical way in which I could monitor how my perspectives as a former undergraduate mathematics student and beginning university tutor were informing the analysis.

2.7 Conclusion

In this chapter, we have provided an overview of the research designs of the lecturer and student studies. Further details are provided in Chapters 3 and 5 respectively. We explained that the research in this thesis is guided by a postpositivist research paradigm (Guba & Lincoln, 1994). While this research paradigm can be criticised for failing to take a definitive ontological and epistemological stance (Braun & Clarke, 2022b), there are two main positions of the paradigm that we consider important to our research. The first is the existence of a singular reality independent of human practice, while the second is that this reality is never fully knowable and our research is intrinsically influenced by the researchers’ beliefs and past experiences. This belief is reflected in our emphasis on the importance of researcher reflexivity. In this chapter, we illustrated how reflexivity can be engaged in during the qualitative data analysis process. In Chapter 8, we discuss the importance of the availability of such accounts of the doing of reflexivity in the context of research in undergraduate mathematics education.

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2. Details of the Paper
Title of Paper: 'I think the best I can aim for is to be happy enough': The tensions that mathematicians experience when designing upper-level, lecture-based courses
Current Status of the Paper: ☐ Drafted ☐ Submitted ☒ Under review ☐ Under revision ☐ Accepted for publication ☐ Published
Name of Journal (if known): The Journal of Mathematical Behavior
Digital Object Identifier (if published):
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Chapter 3

‘I think the best I can aim for is to be happy enough’: The tensions that mathematicians experience when designing upper-level, lecture-based courses

Abstract

Recent research has provided greater insight into the factors that influence mathematicians’ decision to lecture. Consequently, there has been increased focus on augmenting lecture-based instruction, rather than replacing it. Pursuing this goal requires an understanding of mathematicians’ perspectives on, and concerns about, lecture-based instruction. However, the focus of extant research has been on the lecture itself, overlooking that there is more to lecture-based instruction than what happens in class. In this paper, we examine four mathematicians’ pedagogical decision-making across an upper-level, lecture-based course they have recently taught. We focus on a section of content from each of the four courses and examine how it is addressed in participants’ design of their lectures, notes, homework, tutorial classes, and final examinations. Drawing on Herbst and Chazan’s (2011, 2023) theory of practical rationality, we examine interview data for evidence of tensions in the form of competing individual and socio-technical decision-making factors. Two tensions are presented: the content and problem-solving tensions. The content tension centres on the pressure participants perceive to cover certain content due to disciplinary obligations, which conflicts with a lack of time in the course and the need to accommodate diverse student backgrounds. The problem-solving tension is rooted in the participants’ goal of engaging their students in authentic problem-solving, which is constrained by institutional assessment norms and limited control over students’ out-of-class work. We present vignettes of each

participant to illustrate the ways in which they attempt to manage the tensions through their course design, before concluding by discussing the implications of our findings.

Keywords: lecture, homework, examinations, mathematicians, lecturers, decision-making

3.1 Introduction

Despite significant efforts to develop research-informed, student-centred curricula for undergraduate mathematics topics (e.g., Larsen et al., 2013; Leron & Dubinsky, 1995), most courses are still taught through lecture (Johnson et al., 2018). The limited impact of research on student-centred instruction contributed to a call for mathematics education researchers to investigate the reasons why mathematicians believe in the efficacy of lecture-based instruction (Fukawa-Connelly et al., 2016). Research addressing this call has identified several factors of influence on mathematicians' pedagogical decisions including their professional obligations (Herbst & Chazan, 2023; Johnson et al., 2025; Mesa et al., 2020), content coverage concerns (Johnson et al., 2018; Johnson et al., 2013; Woods & Weber, 2020), and pedagogical goals such as preparing future mathematicians (Woods & Weber, 2020) or preparing students to learn outside of class (Mitchell et al., 2026). The increased awareness of the complexity of mathematicians' pedagogical decision-making has led mathematics education researchers to conclude that instructional innovations that are not designed with mathematicians' concerns in mind are likely to continue to have limited impact (Dawkins & Weber, 2023; Johnson et al., 2025). Consequently, there has been increased focus on augmenting and improving lecture-based instruction, rather than seeking to replace it (e.g., Alcock, 2018; Dawkins & Weber, 2023; Johnson et al., 2025). Pursuing such research requires an in-depth understanding of lecture-based instruction, and mathematicians' perspectives on, and concerns about, their instructional practices. To date, research has predominantly focused on the lecture (e.g., Artemeva & Fox, 2011; Fukawa-Connelly et al., 2017; Paoletti et al., 2018; Woods & Weber, 2020), neglecting the fact that there is more to lecture-based instruction than what happens in class.

The goal of this study is to examine mathematicians' pedagogical decision-making *across* a lecture-based course, accounting for the interconnected nature of instructional components (Pinto & Karsenty, 2018). Four mathematicians participated in a series of three, one-on-one interviews about an upper-level course they had recently taught. The interviews focused on a specific section of content from each course and examined how it was addressed in the design of the lectures, course notes, homework, tutorials, and final examinations. Drawing on the theory of practical rationality (Chazan et al., 2016; Herbst & Chazan, 2011, 2023), we examine participants' accounts for evidence of tensions in the form of competing individual

and socio-technical decision-making factors. In this paper, we present two tensions: the content tension and problem-solving tension. We include a vignette of each participant illustrating how they attempted to manage the identified tensions in their course design. In summary, our study addresses the following two research questions:

- **RQ1:** What tensions do mathematicians experience when designing upper-level, lecture-based courses?
- **RQ2:** How do they attempt to manage these tensions through their course design?

We propose that our study contributes to the research literature in three main ways. First, most research on the teaching of university mathematics has been conducted on introductory courses such as calculus and linear algebra (Florensa et al., 2025). By focusing our study on upper-level courses, we provide insight into an under-researched instructional context. Second, research on mathematicians' pedagogical practices has tended to examine instructional components in isolation from one another, failing to account for the relationship between components and the overarching influence of goals and beliefs (Pinto & Karsenty, 2018). Our study adopts a holistic approach, providing insight into how mathematicians think about their courses as a whole. Third, by investigating the tensions that mathematicians experience when designing lecture-based courses, our findings have the potential to inform future work on augmenting and improving lecture-based instruction.

3.2 Literature review

University mathematics instruction can be broadly classified as either lecture-based or student-centred depending on the extent to which students are responsible for generating mathematical ideas in the classroom (Melhuish et al., 2022). Significant research efforts have been dedicated to the development and dissemination of research-informed, student-centred curricula but the impact of this work has been modest (Fukawa-Connelly et al., 2016; Johnson et al., 2018; Melhuish et al., 2022). Recent research has highlighted the complexity of mathematicians' pedagogical decision-making and the many competing factors that shape their instructional decisions (Johnson et al., 2018; Johnson et al., 2025; Mesa et al., 2020; Woods & Weber, 2020). One implication of this research is that instructional innovations that are compatible with lecture-based instruction and account for mathematicians' concerns may have greater impact on the teaching of undergraduate mathematics (e.g., Alcock, 2018; Dawkins & Weber, 2023; Johnson et al., 2025).

Compared to the extensive work on developing student-centred instruction, research on augmenting lecture-based instruction is still in its early stages, with Melhuish et al. (2022)

identifying it as a direction for future research. Pursuing this line of research requires a deep understanding of mathematicians' concerns about their current pedagogical practices. However, a major weakness of the current literature base is that researchers have largely treated lecture-based instruction as being synonymous with the lecture. This neglects the fact that there is more to university teaching than what occurs in the classroom (Rupnow et al., 2021). Pinto and Karsenty (2018) elaborate on this point, noting the tendency for mathematicians' teaching practices to be analysed "in isolation from, rather than as part of, a series of interconnected instructional activities that constitute a course" (p. 132). They criticise this approach stating that it fails to account for the complex relationships between various elements of instruction and neglects to consider the broader course context which is influenced by an instructor's beliefs and expectations. While very few studies have examined mathematicians' pedagogical decision-making across a course (cf. Pinto & Karsenty, 2018; Weber, 2004), there have been isolated studies on various aspects of lecture-based courses. We structure the remainder of this review by providing an overview of the findings of this research.

3.2.1 Research on lectures and notes

Lectures are the predominant form of instruction in undergraduate mathematics, with over 80% of Johnson et al.'s (2018) sample self-identifying as lecturers. Observational studies have provided insight into what mathematicians do in lectures, describing a pedagogical genre referred to as 'chalk talk' where instructors write formal mathematics on the board, add informal comments orally, and ask questions with typically limited wait times (Artemeva & Fox, 2011; Fukawa-Connelly et al., 2017; Paoletti et al., 2018). Many mathematics educators and some mathematicians have expressed dissatisfaction with lecture-based instruction. Some have criticised the fact that lectures operate on a transmission-based model, offering limited participatory opportunities for students and failing to incite meaningful mathematical learning (Halmos et al., 1975; Leron & Dubinsky, 1995; Paoletti et al., 2018). Mathematics education research has also raised questions about the understanding that students emerge from their lectures with and whether they interpret their instructors accurately in class (Krupnik et al., 2018; Lew et al., 2016). Based on these and other criticisms, significant resources have been invested in developing research-informed, student-centred curricula for a range of mathematical topics (e.g., Larsen et al., 2013; Leron & Dubinsky, 1995). However, the impact of such efforts has been modest (Melhuish et al., 2022).

The limited impact of research on student-centred instruction led to the call for mathematics education researchers to "better explore the reasons that mathematicians appear to

believe strongly in their current practice” (Fukawa-Connelly et al., 2016, p. 281). Johnson et al. (2018) conducted a survey-based study to investigate the factors that influence mathematicians’ decision to lecture. They found that the decision to lecture was primarily linked to two beliefs: that lecture is the best way to teach and that it is the only instructional method that allows for adequate content coverage. In relation to content coverage concerns, Johnson et al. (2018) reported that the constraint did not appear to originate from departments. Instead, they suggested that it may be internal and linked to beliefs about what content should be covered in a course. This aligns with prior research by Johnson et al. (2013) who noted that one participant’s content coverage concerns were linked to his belief that students should develop a “certain level of sophistication” (p. 749) in group theory which could be achieved by covering at a minimum the first isomorphism theorem.

Interview-based studies have provided further insight into the reasons why mathematicians believe in the efficacy of lecturing. Woods and Weber (2020) interviewed eight participants from a research-intensive university in the United States about why they engage in common in-lecture practices. Drawing on Schoenfeld’s (2011) theory of resources, orientations, and goals they found that lecturing aligned well with participants’ goals of content coverage, modelling authentic mathematical thinking, and preparing future mathematicians. The benefits of lectures have also been framed in terms of preparing students for studying outside of class (Mitchell et al., 2026; Pritchard, 2010; Weber & Fukawa-Connelly, 2023; Wu, 1999). Weber and Fukawa-Connelly (2023) interviewed 13 participants in a two-week workshop on a branch of set theory about the benefits they perceive of lecture attendance on their mathematical research and learning. A key finding was that participants did not expect to immediately understand the content presented. Instead, they viewed lectures as useful in terms of facilitating future study of material by providing a big picture of the topic, highlighting interesting results, and illustrating how to think about ideas. Mitchell et al. (2026) built on this work by interviewing seven mathematicians about the benefits of lecture attendance that they perceive for their students. They found that the participants sought to offer students similar benefits of lecture attendance as those identified by Weber and Fukawa-Connelly (2023). The authors conclude that learning from lectures may constitute an important disciplinary practice and this may be one reason why the adoption of non-lecture, student-centred instruction has been limited.

The research literature points to other reasons for why lecture-based instruction remains prevalent in university mathematics. Mathematicians typically receive little to no formal pedagogical training (Winsløw et al., 2021). Research has highlighted that successfully enacting student-centred instruction requires pedagogical content knowledge beyond that which a mathematician might develop through experience lecturing (Johnson & Larsen,

2012; Speer & Wagner, 2009; Wagner et al., 2007). Additionally, research examining the experiences of mathematicians implementing student-centred curricula has documented the tensions that they can experience (Johnson et al., 2013, 2025; Mesa et al., 2020). For instance, mathematicians can experience a tension between their obligation to cover a certain amount of content and the additional time that is required in class to allow for student activity (Johnson et al., 2013, 2025; Mesa et al., 2020). While it is possible for mathematicians to become more confident and competent implementing student-centred instruction over the course of multiple iterations (Andrews-Larson et al., 2019), doing so requires considerable time and effort. Dawkins and Weber (2023) propose that, when deciding how to teach, mathematicians tend to *satisfice* (Simon, 1972), selecting the first suitable option, rather than spending considerable time and energy finding the ‘optimal’ approach.

While considerable research has investigated mathematicians’ practices in, and perspectives on, lectures, significantly less research has focused on notes in university mathematics. Mesa (2022) investigated mathematicians’ design of lecture notes and the resources they draw on in the design process. This research indicates that there may be less uniformity in instructors’ design of lecture notes than in their approach to lecturing. For instance, some participants in the study did not share notes with students and used the writing of notes mainly as a way of preparing for lectures. Others annotated notes live in class or released notes in advance or after lectures. While relatively little is known about mathematicians’ design of, or perspectives on, notes, mathematics education researchers have proposed the use of a particular style of notes referred to as guided notes (Iannone & Miller, 2019). Guided notes consist of pre-prepared notes with gaps that students fill in during class. Their use has been found to alter students’ note-taking behaviour, with students more inclined to record their instructors’ oral comments, although the implications of this remains unclear (Iannone & Miller, 2019).

3.2.2 Research on homework

As students transition from the K-12 to university level, they are expected to complete significantly more learning independently outside the classroom (Ellis et al., 2015; Pepin, 2014). Consequently, homework constitutes an important learning resource in terms of guiding out-of-class learning in undergraduate mathematics courses (Ellis et al., 2015; Rupnow et al., 2021). Most of the extant research on homework in university mathematics courses has focused on the student experience or examining the relationship between performance on homework and student success (Rupnow et al., 2021). Very few studies have investigated the design of homework as an instructional practice (cf. Rupnow et al., 2021), although research

has indicated the many decisions instructors must make when implementing homework in their courses. For instance, Ellis et al. (2015) identified three salient features of university mathematics homework: the content included, how it is structured, and the associated feedback system.

In relation to the content of homework, Rupnow et al. (2021) conducted one of the only investigations of how mathematicians choose homework problems in upper-level courses. Task-based interviews were conducted with six mathematicians from two research-intensive universities in the United States. Participants were asked to create a homework assignment on Groups and Group Isomorphisms by selecting problems from an Abstract Algebra textbook. A major finding of their study was that participants perceived students' engagement with homework problems as essential to their learning and that participants' selection of problems was guided by their broader pedagogical goals for student learning. The authors noted that one limitation of their study was that it was not conducted on participants' personal instructional practices and that the unnaturalistic setting may have influenced their results.

When deciding how to structure homework, instructors must consider issues such as whether to assign online or written homework, and whether homework should be completed individually or in groups (Ellis et al., 2015). An additional issue that has been identified in the general education literature (Harland et al., 2015), and also specifically in relation to mathematics (Halcrow & Dunnigan, 2012) is how students are incentivised to work on homework tasks. Instructors must decide if homework is to be voluntary, linked to a benefit (such as additional marks in the examination), or obligatory (Neuhaus-Eckhardt & Siller, 2026). If a mathematician decides that students' homework should be graded then they must also consider the associated feedback system. Moore (2016) conducted an interview-based study with four participant mathematicians to investigate how they grade student proofs. When grading, participants focused on evaluating students' understanding of the proof as much as the validity of the proof itself. This contributed to variation in the marks assigned by individuals as participants perceived students' level of understanding differently. Moore noted that participants had not taught the courses from which the proofs used in the study were taken. This meant that participants were forced to make assumptions about the course context which may have influenced the results.

3.2.3 Research on examinations

Iannone and Simpson's (2011, 2022) examination of the 'assessment diet' in undergraduate mathematics degrees in the UK revealed that closed-book examinations are the predominant assessment method. In the general higher education literature, instructors' as-

assessment choices have been linked to their beliefs about teaching and learning (Samuelowicz & Bain, 2002). This has also been found to be the case in mathematics (Iannone & Simpson, 2015). Investigating mathematicians' views on examinations, Iannone and Simpson found that a primary factor that influenced lecturers' decisions about assessment were their beliefs about the relative importance of various forms of knowledge. Participants reported valuing mathematical understanding, creativity, and imagination. They felt that these attributes could be assessed in a closed-book examination by asking students to tackle unseen problems or apply familiar procedures in new situations. However, participants felt constrained in asking these sorts of questions by the need to accommodate for weaker students who might fail if the examination was too difficult. Some lecturers felt that other assessment methods such as projects or oral examinations had the potential to assess the forms of mathematical knowledge that they value but felt constrained in implementing these alternative methods due to a lack of resources.

3.3 Theoretical framework

In this paper, we draw on Herbst et al.'s (2024) conceptualisation of tensions in undergraduate mathematics instruction as two or more competing priorities which make legitimate appeals to the instructor and of which they may or may not be aware. To specify the nature of these competing priorities, we draw on the theory of practical rationality (Chazan et al., 2016; Herbst & Chazan, 2011, 2023) which proposes that the decisions a teacher makes can be explained in terms of a combination of individual and socio-technical factors. Herbst and Chazan (2023) conceptualise individual decision-making factors as consisting of stable and malleable individual characteristics. Examples of stable individual characteristics include one's identity and life experience. Malleable individual characteristics include beliefs, knowledge, and teaching experience. Socio-technical decision-making factors are subdivided into norms and professional obligations (Herbst & Chazan, 2023). Norms represent what teachers are typically expected to do in instructional situations. Professional obligations capture teachers' accountability to various stakeholders including students, the discipline, the institution, and society. The dependency of teacher decision-making on individual and socio-technical factors is illustrated in Figure 3.1.

Herbst and Chazan (2023) acknowledge the connection between their conceptualisation of individual decision-making factors and Schoenfeld's (2010, 2011) theory of resources, orientations, and goals (ROGs), which has been frequently used in research on undergraduate mathematics education (e.g., Hannah et al., 2011; Paterson et al., 2011; Pinto, 2013; Woods & Weber, 2020). In this study, we utilise practical rationality (Chazan et al., 2016;

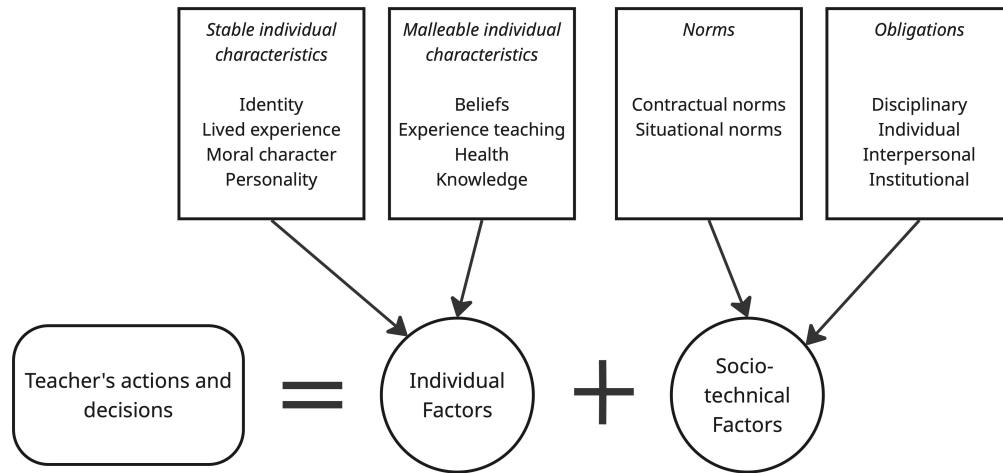


Figure 3.1: Practical rationality's conceptualisation of teacher decision-making as dependent on a combination of individual and socio-technical factors (Herbst & Chazan, 2023).

Herbst & Chazan, 2011, 2023) as our overarching theoretical framework and connect it to Schoenfeld's theory of ROGs to specify the nature of individual factors. We elaborate on our conceptualisation of individual and socio-technical decision-making factors in the below subsections.

3.3.1 Individual decision-making factors

We draw on Schoenfeld's (2010, 2011) theory of ROGs to specify the nature of individual decision-making factors (Herbst & Chazan, 2023). Schoenfeld proposed that the decisions a teacher makes can be understood as a function of their ROGs. Goals are described as "the things that people consciously or unconsciously set out to achieve" (Schoenfeld, 2011, p. 459) and are conceptualised as operating on multiple levels. An instructor may have overarching goals for their course, as well as associated subgoals. Schoenfeld (2010) explains that teachers must attend to multiple and oftentimes conflicting goals. A teacher's decision-making regarding goal prioritisation is influenced by their orientations. Orientations is used as an "inclusive term to encompass beliefs, dispositions, values, tastes and preferences" (Schoenfeld, 2011, p. 15). Most relevant to the study of mathematicians' pedagogical decision-making is the role of beliefs about mathematics, teaching, and students (Melhuish et al., 2022; Rupnow, 2023; Speer, 2008). Schoenfeld (2010, 2011) also identifies the availability of resources as an influence on teachers' decision-making. In particular, knowledge-related, social, and material resources are emphasised.

3.3.2 Socio-technical decision-making factors

While Schoenfeld's (2010, 2011) theory of ROGs has been widely used, some researchers have critiqued the framework's failure to "take up the broader social, long-term and institutional dimensions of teachers' thinking" (Petropoulou et al., 2020, p. 352). Herbst and Chazan (2023) propose that their theory of practical rationality complements person-centred approaches to analysing teachers' decision-making by also accounting for the social and institutional context in which they are teaching. The central thesis of the theory is that a teacher's decisions and actions should not be reduced to the "voluntary expression of individual teachers" (Herbst & Chazan, 2023, p. 193). They conceptualise socio-technical factors as aspects of a teacher's context that either constrain or enable their actions. Herbst and Chazan list two types of socio-technical factors. The first is norms which they define as "normal or unmarked behavior that is tacitly expected in a setting" (Herbst & Chazan, 2011, p. 411). For example, lecturing (Johnson et al., 2018) or assessing students through closed-book examinations (Iannone & Simpson, 2011, 2022) might be considered norms in undergraduate mathematics instruction.

The second category of socio-technical factors is professional obligations. Herbst and Chazan (2011, 2023) name four different stakeholders to whom teachers are obligated: students, the discipline, the institution, and society. The obligation to students consists of recognising and responding to each individual's cognitive and affective needs. The obligation to the discipline is the expectation that a teacher will represent the knowledge and practices of mathematics faithfully and accurately. The institutional obligation refers to the expectation that an instructor will conform to departmental and university-wide policies and expectations. Attending to the obligation to society (also referred to as the interpersonal obligation) involves representing and cultivating socially desirable values and managing the social dynamics within the classroom.

Obligations can be formal rules, or they may be informal and relate to "commitments that a mathematician feels they should satisfy to be a respectable member of their disciplinary culture" (Johnson et al., 2025, p. 148). For example, Johnson et al. (2025) found that mathematicians can feel obligated to provide a rigorous answer to all questions posed in a course even if there is no formal rule requiring them to do so. Even if an obligation is informal, it can still constrain the actions of an instructor, regardless of whether it is an actual constraint or shared by others (Asada, 2026). Obligations can be perceived differently or recognised to different extents by individual instructors (Asada, 2026; Herbst & Chazan, 2023; Ko et al., 2024; Mesa et al., 2020). For instance, Mesa et al. (2020) suggested that staff on short-term contracts may give greater consideration to institutional obligations

than tenured faculty. Instructional contexts also influence the way in which obligations are experienced. For instance, Ko et al. (2024) compared how university instructors and high school teachers recognise professional obligations and found that university instructors recognise the institutional obligation more than high school teachers but the individual and interpersonal obligations significantly less. Herbst and Chazan (2023) suggest that there is likely to be similarities and differences in the “practical rationality of teaching different courses” (p. 196). Johnson et al. (2025) agree with this hypothesis suggesting that professional obligations are not only discipline specific but also depend on the context of the course being taught. For instance, participants in their study were “comfortable situating the first linear algebra course in R^n but thought references to R^n should be avoided in the second linear algebra course” (p. 159). As another example of the course-dependent nature of obligations, Ko et al. (2024) proposed that institutional obligations may be more relevant in introductory/lower-division courses where practices such as prescribed textbooks and common final examinations are more relevant.

3.3.3 Individual and socio-technical decision-making factors in tension

Obligations may compete or conflict with one another (Chazan et al., 2016). One obligation could conceivably justify a decision that could be viewed as ill-advised on account of another obligation (Ko et al., 2024). For example, an instructor could experience a conflict between their institutional obligation to cover the course curriculum and their obligation to students to pause in class and respond to questions. Similarly, individual factors such as goals may be competing, with teachers forced to make in-the-moment decisions about which goal to prioritise (Schoenfeld, 2010). Additionally, research has demonstrated that individual and socio-technical factors can also conflict with one another in the decision-making process. For example, Johnson et al. (2025) illustrated how mathematicians may *believe* in the potential value of active learning but may choose not to enact it as it conflicts with one or more *obligations* which increase the ‘cost’ associated with implementing it. Similarly, while investigating perceived barriers to implementing equitable instructional practices, Asada (2026) noted that even if an instructor *believes* that an instructional practice is likely to achieve one of their pedagogical *goals* they “may still feel unable to enact the practice if it conflicts with their professional obligations to their students, institution, discipline, or society” (p. 4).

In this study, we are interested in the tensions experienced by mathematicians teaching upper-level, lecture-based courses. Following Herbst et al. (2024), we conceptualise tensions as competing priorities or considerations and utilise the theory of practical rationality (Chazan

et al., 2016; Herbst & Chazan, 2011, 2023) to specify the nature of these priorities. We examine our data for evidence of competing individual and socio-technical decision-making factors and investigate how the resulting tension is managed by participants. In Section 3.4, we explain how we operationalised our theoretical framework to code the data.

3.4 Methodology

3.4.1 University and department context

University College Dublin (UCD) is a large, PhD-granting university in Ireland. Approximately 41,000 students are enrolled in UCD, which is a campus-based university located south of Dublin city centre. The School of Mathematics and Statistics in UCD is the largest in Ireland, offering undergraduate, masters', and doctoral programs. There are approximately 65 full-time faculty employed in the school with research interests in pure and applied mathematics, statistics, and mathematics education. Additionally, there are approximately 70 doctoral students enrolled annually in the school.

3.4.2 Participants

Four research mathematicians from UCD School of Mathematics and Statistics, participated in this study. We employed purposive sampling (Braun & Clarke, 2013), inviting tenured faculty, each with at least 15 years' teaching experience, who had taught an upper-level, proof-based mathematics course in the last three years to participate. We endeavoured to recruit participants with a variety of research interests and institutions of undergraduate study and to have a gender representation that reflects that of the discipline of mathematics in the university. Throughout the paper, we will refer to the four participants using the gender-neutral¹ pseudonyms Dr Boyne, Dr Dargle, Dr Foyle, and Dr Liffey.

3.4.3 Course contexts

Each of the four participants in this study was interviewed about an upper-level, proof-based mathematics course they had taught in the previous three years (courses in UCD are referred to as modules). These courses are taught to third- and fourth-year undergraduate students who are typically majoring in either pure or applied mathematics. The average enrolment in these courses is usually between 10 and 40 students. UCD operates a modularised

¹Throughout the paper, we will use the pronoun 'he' when referring to the four participants.

system, and full-time undergraduate students usually study 12 courses over the Autumn and Spring trimesters. Each trimester consists of 12 weeks of classes, followed by a three-week revision and examination period. The in-class instructional component of these upper-level courses typically consists of lectures and tutorials² in which homework problems are discussed. In upper-level courses, both the lectures and tutorials are generally delivered by the course instructor. It is not common practice for mathematics instructors in UCD to select a course textbook and most instructors design their own course notes and homework assignments. Mathematics courses in UCD are predominantly assessed through a closed-book final examination (a closed-book final examination accounts for at least 70% of a student's grade in each of the four courses). Research conducted in a UK context has indicated that this is typical of undergraduate mathematics degrees (Iannone & Simpson, 2011, 2022).

Details about the content covered in each course participants were interviewed about is provided in Table 3.1.

Table 3.1: Summary of the contexts of courses that participants were interviewed about

Participant	Course name	Content summary
Dr Boyne	Coding Theory	Introductory course on algebraic coding theory. Topics covered include coding theoretic bounds; error correction; code existence, construction, equivalence, and optimality. Classic algebraic codes are presented.
Dr Dargle	Measure Theory and Integration	Topics covered include sigma algebras, measurable and integrable functions, and the Lebesgue integral. Some applications to functional analysis and potential theory are covered.
Dr Foyle	Galois Theory	Course covers necessary field theory content to build up to Galois correspondence theory. Latter half of course is focused on applications of the Galois correspondence, including to some classic problems.
Dr Liffey	Topology	Introductory course on Topology. Topics covered include topological spaces, bases, closed sets and accumulation points, continuous function, product and quotient spaces, compactness, and connectedness.

²In upper-level mathematics courses in UCD, tutorials are conducted with the whole class group. They are similar in nature to what may be referred to as recitation sessions or classes in a US context.

3.4.4 Study design and data generation

The goal of this study was to examine mathematicians' instructional decision-making across a course, rather than restricting our focus to one specific instructional component e.g. the lecture. To add specificity to the interview questions, we asked each participant to select a portion of content from their course consisting of at least one of each of the following: definition, example, theorem, and proof. We asked that the section of content they picked be reflective of the main focus of the course (rather than introductory/background material), be a topic that they may ask an examination question on, as well as one that might feature in the homework assignments. The section of content chosen by each participant is listed in Table 3.2.

Table 3.2: The section of content chosen by each participant

Participant	Course name	Section of content chosen
Dr Boyne	Coding Theory	Bose–Chaudhuri–Hocquenghem (BCH) codes
Dr Dargle	Measure Theory and Integration	Existence and uniqueness of the completion of measure spaces
Dr Foyle	Galois Theory	Galois correspondence theorem and the theorem of the primitive element
Dr Liffey	Topology	Topological compactness

The study was designed to follow this section of content throughout the course and to examine how it was addressed in the participant's design of the notes, lectures, homework, tutorial, homework solutions, and final examination. We asked each participant to provide the lecture notes, homework assignments, and past examination questions relevant to their chosen section of content. These were then used in the design of the interview questions. The focus of the first interview was on the mathematicians' course notes and lectures. In relation to the course notes, we asked participants why they had decided to include each piece of content in the relevant section of notes and why they had stated it in the way that they had. We also asked how they would present the content to students in lectures. The first interview concluded with a series of more general questions on notes and lectures. Some of these questions were based on research on mathematics lectures (e.g., Fukawa-Connelly et al., 2017; Lew et al., 2016; Paoletti et al., 2018).

The second interview examined the mathematicians' instructional design of homework assignments. Based on the interview protocol used by Rupnow et al. (2021), for each problem related to the relevant content topic we asked why the mathematician had decided to include the problem and what they hoped students might learn from engaging with it. We also asked participants more general questions about how they structure homework in their courses. For

instance, participants were asked whether they ask students to submit solutions to homework problems for assessment, whether they release solutions to homework problems, and whether they organise tutorial classes and what format these take.

The final interview focused on participants' design of the assessment component of their courses. We asked participants why they decided to (not) have students take midterm tests. All four courses are predominantly assessed through closed-book examination. For relevant examination questions, we asked the mathematician why they had decided to include the problem, what they were hoping to examine, and whether students had seen a similar problem in the lectures or homework. Participants were asked more general questions about their views on closed-book examinations and responded to a summary of the findings of Iannone and Simpson (2015) on mathematicians' views on examinations. The interview concluded with questions about participants' perceptions of the relationship between different aspects of instruction and their overarching reflections on teaching undergraduate mathematics. An overview of the study design is provided in Figure 3.2.

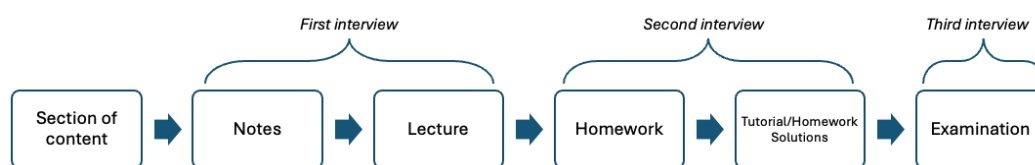


Figure 3.2: An overview of the study design.

Ethical approval for the study was granted by the Human Research Ethics Committee of UCD. The four mathematicians participated in the series of three, one-on-one interviews with the first author in summer 2023. The interviews each lasted on average one hour and were audio-recorded and transcribed.

3.4.5 Data analysis

We chose template analysis (Brooks et al., 2015; King, 2012), a form of thematic analysis, as the method of data analysis for this study. Template analysis emphasises the use of hierarchical coding and central to the method is the development of a coding template. The method sits between purely ‘bottom-up’ and ‘top-down’ forms of analysis. Existing theory and findings from relevant literature can be incorporated into the initial coding template as *a priori* codes. However, there is also scope for inductive, bottom-up coding of the data.

The first stage in conducting template analysis is constructing an initial coding template. Our theoretical framework informed this process. In this study, we were interested in identifying tensions caused by competing individual and socio-technical factors (Herbst &

Chazan, 2023). Therefore, our initial coding template was divided into two categories: codes which related to individual factors and those related to socio-technical factors. We included the four types of professional obligations identified in practical rationality as *a priori* codes under the socio-technical factors category. Utilising Schoenfeld's (Schoenfeld, 2010, 2011) theory we included resources, orientations, and goals as *a priori* codes under the individual factors category. Reflecting the fact that goals operate on multiple levels (Schoenfeld, 2010, 2011), we included subcodes to capture participants' goals that might apply at the level of the programme, course, or a particular instructional component e.g. the lecture. A list of all *a priori* codes included in our initial coding template is shown in Table 3.3.

Table 3.3: List of *a priori* codes included in the initial coding template

Individual factors
Resources
Orientations
Goals
Programme-level goals
Course-level goals
Instructional component-level goals
Goals for notes
Goals for lecture
Goals for homework
Goals for tutorial
Goals for solutions
Goals for in-semester assessment (e.g. midterms)
Goals for examination
Socio-technical factors
Obligations to the discipline of mathematics
Obligations to students as individuals
Obligations to the institution
Obligations to society

After constructing the initial coding template, we began the coding process using the qualitative data analysis software NVivo 12. In some instances this involved deductively applying *a priori* codes (such as the four types of professional obligations) to the data. Template analysis also allows for inductive coding and as new codes were created we added them to our coding template. For example, under the *a priori* code 'orientations' we added subcodes to capture the influence of beliefs about teaching, students, and mathematics (Rupnow, 2023; Speer, 2008) on participants' decision-making.

We coded the interviews participant by participant. Once we had finished the coding of a participant's three interviews, we examined our analysis for evidence of tensions in the form

of competing individual and socio-technical factors. If an apparent tension was identified we made note of it accompanied by a short description and the relevant quotes from the interview data. We repeated this process so that we had a list of apparent tensions for each of the four participants. At this stage, we examined the lists of tensions identified for each participant for commonalities and focused on identifying tensions that were experienced in some form by at least three of the four participants (a similar approach was taken by Mesa et al. (2020) in identifying the tensions reported in their study). In this paper, we outline two identified tensions which we refer to as the content tension and problem-solving tension.

3.5 Findings

In this paper, we present two tensions experienced by our participants when designing their upper-level, lecture-based courses: the content tension and the problem-solving tension. For each tension, we describe the conflicting factors that give rise to the tension. We also present a vignette of each participant, illustrating the variety of ways in which they attempted to manage the two tensions.

3.5.1 The content tension

This tension centres on participants perceiving a pressure to cover certain content in their courses, largely due to disciplinary obligations, but simultaneously feeling constrained in achieving their content-related goals due to various factors such as lack of time in the course and the need to accommodate for students' background knowledge. The disciplinary pressures and contextual constraints that give rise to the content tension are illustrated in Figure 3.3.

The fact that mathematicians perceive a pressure to cover a certain amount of content in their courses has been widely documented (e.g., Johnson et al., 2018; Johnson et al., 2013, 2025; Woods & Weber, 2020). This pressure has been framed in terms of an institutional obligation to adequately prepare students for subsequent courses or, in the case of coordinated courses, to conform to the pacing of other sections (Johnson et al., 2013, 2025; Woods & Weber, 2020). However, the upper-level nature of the courses in our study, which were all taught to a single cohort of students, meant that these considerations were less relevant. Instead, we found that participants' content-related goals for their courses largely stemmed from their disciplinary obligations.

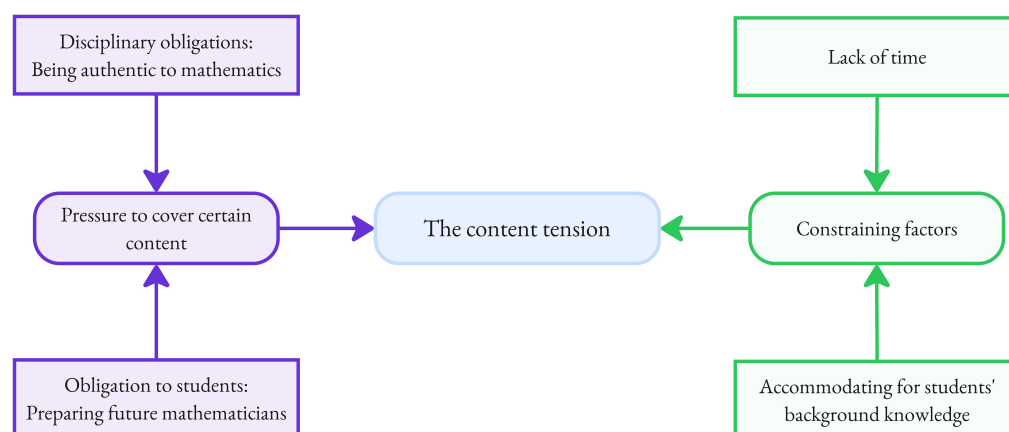


Figure 3.3: An illustration of the relationship between the conflicting factors that give rise to the content tension.

3.5.1.1 The influence of disciplinary obligations: Being authentic to mathematics

Each participant was teaching an upper-level course on a branch of mathematics related to their research area. A primary concern participants held when deciding on their course curriculum was fulfilling an apparent obligation to represent the relevant branch of mathematics accurately. To satisfy this obligation, participants considered it essential that they present particular content. For instance, Dr Foyle considered it imperative that he reached the fundamental theorem of Galois theory in his course: “I have to get to the Galois correspondence”. Participants’ content-related goals appeared to be informed by disciplinary norms for what content should be included in a course on particular branches of mathematics. For example, Dr Boyne demonstrated his adherence to such norms, describing his algebraic coding theory course curriculum as “standard”:

Dr Boyne: I wanted to give a standard course in algebraic coding theory that I think anyone meeting coding theory for the first time should do. That was my principle. So there were basic things, like, and the key word is *algebraic* coding theory, so that would mean cyclic codes, BCH codes, Reed Solomon codes, so they’re all really important, as far as I am concerned, in that topic.

Participants’ obligation to represent their discipline accurately also manifested in their approach to proof in their courses. Exemplifying this, Dr Liffey reflected on the core principle in mathematics that truth must be backed up by proof. Consequently, he believes it essential that proofs are presented in undergraduate courses to represent the disciplinary values accurately:

Dr Liffey: I mean mathematical proof are, I think they form the central component of critical thinking in the context of a maths degree. You know, in modern mathematics, statements are not accepted as true unless they are backed up by watertight mathematical proof . . . So I think you have to include mathematical proofs in a mathematics degree.

While participants' decision to include certain content in their courses seemed to stem from adherence to disciplinary norms, we also saw evidence of instructors tailoring their course curriculum dependent on their personal research experience. For example, Dr Foyle talked about how getting students to work in characteristic³ p was, to him, "fundamentally important". However, he clarified that this is not standard across the discipline, mentioning that there "are some books on Galois theory that only work in characteristic zero". When asked about his decision to cover characteristic p , Dr Foyle explained that it was primarily motivated by its importance to his own research:

Dr Foyle: It's important to my research. I think it's also important to research in number theory and algebraic geometry. But there are mathematicians who only work in characteristic zero. So they probably wouldn't consider it important . . . And it is interesting, you can actually tell if you look, I've looked at other people's notes online and you can tell when there's somebody who's area is analysis and teaches Galois theory. They have far less stuff on characteristic p and they have a lot more arguments that use the complex numbers.

3.5.1.2 The influence of obligations to students: Preparing future mathematicians

The pressure participants perceived to cover certain content in their courses was principally related to their disciplinary obligations. However, we also saw evidence of the influence of obligations to students. When asked whether he would consider alternative assessment methods in his Topology course, Dr Liffey explained that he felt that this may require reducing the content covered in his course. He believed that this could leave students inadequately prepared to progress to further studies in mathematics and could make it more difficult for them to be admitted to prestigious universities:

Dr Liffey: Other methods of assessment might require the actual module content to become unreasonably sparse. So, and this really matters actually . . . because, you know, you want to be able to give the high-flying students the best chances they can to go on to be at the best universities they can be at. But on the other hand, you can't sort of gear the course simply for that.

³The characteristic of a field is the smallest positive integer n such that adding the multiplicative identity to itself n times gives the additive identity. If no such positive integer exists the characteristic is defined to 0. The characteristic of a field must either be 0 or a prime number p .

While Dr Liffey acknowledged that it would be unfair to cater the course solely towards the needs of high-achieving students, his obligation to this cohort does appear to have shaped his content-related goals.

3.5.1.3 Factors in tension with participants' content-related goals: Lack of time in the course and a need to account for students' background knowledge

As outlined, participants had well-developed content goals, largely related to their disciplinary obligations. However, we identified a tension between these content goals and other factors such as a lack of time in the course and a need to account for students' background knowledge. Dr Liffey explained that he believes the modularised system gives students limited time to reflect on the content that is being presented to them. Consequently, there is a limit to the amount of content that can be covered in a course without overwhelming students:

Dr Liffey: I think there's also the potential to overload the students, especially in a 12-week module where students have very little time to revise before exams, so they don't, in this sort of modular system, students do not get the chance to think deeply about what they've seen.

Dr Boyne expressed similar sentiments. He felt that time constraints force instructors to make decisions about the amount of content that can realistically be covered in the timetabled number of classes:

Dr Boyne: You have time constraints. So you've constraints on what you can deliver and how much time you have to prepare things. And then you have to pick a theme. So you're kind of maybe choosing a theme that you want to achieve. And then that's going to be delivered over a maximum of 24 lectures. So you're picking some stuff I guess, focusing on that and . . . putting in factors of like what do they know? What can I tell them based on what they know?

Here, Dr Boyne touches on the need to consider students' background knowledge when making decisions about course content. This was another factor that we identified as being in tension with participants' content-related goals. For instance, a key priority for Dr Foyle was emphasising the algebraic nature of his Galois theory course. However, Dr Foyle remarked that this was complicated by his students' lack of background in algebra and their tendency to think in terms of analysis:

Dr Foyle: When I taught it there wasn't a whole lot of algebra before this module. So it could have almost been a first algebra module for some people. Which, by the way, was another challenge. I found some people would think very much in terms of analysis, they will always think in terms of complex numbers.

Dr Dargle also experienced a tension between his content goals and his students' background knowledge. Dr Dargle expressed an orientation towards making his course on measure theory as interesting as possible for students. In accordance with this orientation, he decided to cover the Banach Tarski⁴ paradox as he believed "it was the most interesting thing you could do in the measure theory course". However, achieving this content-related goal was complicated by the fact that there were students from a variety of majors including pure, applied, and financial mathematics in the course. Dr Dargle explained that this "does present some problems because, you know, the financial maths students wouldn't have done the kind of group theory you need for say the Banach Tarski paradox". Consequently, a compromise is required and Dr Dargle makes the proof of the Banach Tarski paradox non-examinable explaining: "I say 'relax, it won't be on the exam'. So you have to be aware that the different groups have different backgrounds".

3.5.2 Managing the content tension

We now present two vignettes to illustrate the ways in which participants managed the content tension. In the first, we examine how Dr Foyle compromises on the content he includes in his Galois theory course and uses course notes to optimise limited class time. In the second, we discuss how Dr Liffey observes his obligation to convey the importance of proof to students while not overwhelming them with excessive content.

3.5.2.1 Dr Foyle

We identified Dr Foyle attempting to manage the content tension in two main ways. The first is compromising on the content he includes in his course. For example, Dr Foyle spoke about how when he initially taught Galois theory he followed Emil Artin's (1942) approach. He explained that Artin "took offense" (Zassenhaus, 1964, p. 2) at the central role the theorem of the primitive element for finite separable extensions played in the proof of the main correspondence theorem: "one of the reasons he [Artin] didn't like it is that the proof has to be totally different for finite and for infinite fields. They're completely different proofs and he didn't like that. Nobody, nobody likes that". Emil Artin (1942) reformulated Galois theory, removing the dependence on the theorem of the primitive element. While Dr Foyle initially followed this approach, more recently he has made the decision to prove the theorem of the primitive element earlier in the course and use it to prove the main correspondence theorem (following the approach outlined by Michael Artin (1991)). Dr Foyle explained that

⁴The Banach-Tarski paradox states that the unit sphere can be decomposed into a finite number of disjoint pieces (as few as five) which can be simply translated and rotated to form two unit spheres of the same size.

this was a “pragmatic” decision as it makes some proofs shorter, allowing him to move more quickly through the course:

Dr Foyle: It makes the proofs shorter and gave me more time to do other things. I felt we were getting a bit bogged down in long technical proofs, which you could completely skip a proof, but I don’t think that’s a good idea to be reflective of Galois theory.

Dr Foyle reflected that while he sees the mathematical merits in Artin’s (1942) approach, ultimately one has to come to a viable compromise that accounts for the limited time available in the course:

Dr Foyle: And I used to kind of, okay in theory I agree with him, but in reality, we have to . . . be pragmatic, teach our module. Try and get in important things and at the same time not get too carried away with long and detailed proofs.

The second way we identified Dr Foyle managing the content tension was through his use of notes which allow him to optimise the limited time available in class. By providing students with complete typed notes for the course, Dr Foyle relieves himself of the obligation to cover all details with students in class. Instead, he can focus on the important parts and has more time to do examples and provide informal explanations:

Dr Foyle: I felt that it’s better for me to give out the notes because I’ll have all the details in the notes. If I give them everything, I don’t have to do every single detail in the lecture. I can say here’s an important bit, here’s an important bit, you can read the rest yourself. So it allows me more time in the lecture to do examples, to do informal things, or have discussions, and let the students read the notes themselves.

Adopting this approach reflects Dr Foyle’s beliefs about the value of lectures in students’ learning: “I feel like the lecture is for highlighting the important bits and for doing examples”. It also allows him to satisfy his obligation to cover a certain amount of content within limited class time by delegating some material for independent student reading: “the odd time I might even skip a proof. I might say to the class ‘this is straightforward, I’ll leave it to you to read’”.

3.5.2.2 Dr Liffey

One way in which participants’ disciplinary obligations manifested was in their approach to proof in their courses. Dr Liffey reflected that it is important that students understand that nothing can be claimed as true in mathematics unless backed up by proof. While this is a key

disciplinary value, he also acknowledged that proving every statement presented in the course could lead to students being overburdened with little time to properly digest the content:

Dr Liffey: It is tempting to prove every statement that you write down and that's what I experienced as a student. But I think you can ask, and should ask, whether that's actually really beneficial to students to prove absolutely everything. I mean, you do need to get across the fact that nothing can be claimed as being true unless you have a proof to back it up. But I think there's also the potential to overload students, especially in a 12-week module.

Dr Liffey talked about attending a professional development course in which he was prompted to think about his approach to proof in his teaching. As a result, he decided to deviate from the style of teaching he experienced as a student and to present fewer proofs in his lectures:

Dr Liffey: I consulted some maths education literature while I was doing the [professional development course]. It's had a marginal effect . . . we had to do a project, I focused on proofs in mathematics and I did, I suppose I did adjust my approach slightly in the sense that, nowadays, I put in fewer proofs into the lecture than I did before. Because when I was a student, basically everything, everything that was stated was proved. And nowadays I don't necessarily think that's the best approach. I think that most things that are stated should be proved. But it's not necessary to prove everything.

Presenting fewer proofs in class allows Dr Liffey to move more quickly through the course content and allows his students more time to think and reflect on their learning. However, to satisfy his disciplinary obligations, Dr Liffey believes there ought to be a proof available to students for every statement in the course. As a compromise, he has decided to include less important proofs in the appendices of his course notes. In this way, he feels he includes a more manageable amount of content in his course while still observing the key idea that mathematical truth requires proof:

Dr Liffey: These days, I relegate proofs, some proofs, to the appendices. So in that way, you know, I'm still kind of observing the central idea, the message that truth in mathematics needs to be backed up with a proof, and the students who are minded to do so can read them if they want, but you know, by putting them into the appendices and making them non-examinable means that, you know, every student has the space then to focus on getting the fundamentals right.

Dr Liffey explicitly acknowledges that teaching requires compromise, reflecting: "I think it's hard to be perfectly happy. I think I'm happy enough. I think the best I can aim for is to be happy enough".

3.5.3 The problem-solving tension

This tension centres on participants' desire to engage their students in problem-solving⁵ practices that are authentic to the discipline but being constrained in achieving this goal by factors such as a lack of control over what happens outside the classroom and the obligation to adhere to institutional assessment norms. The conflicting factors that give rise to the tension are illustrated in Fig. 3.4.

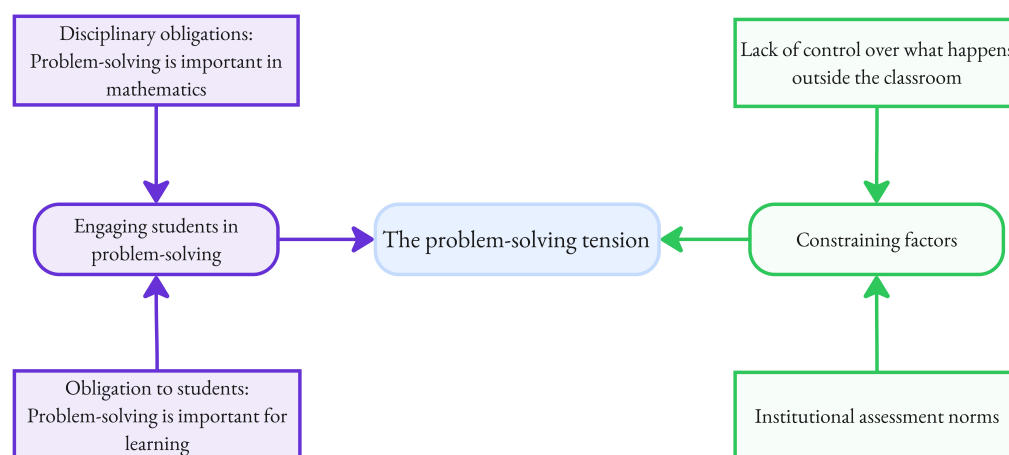


Figure 3.4: An illustration of the relationship between the conflicting factors that give rise to the problem-solving tension.

3.5.3.1 The influence of disciplinary obligations: Problem-solving is important in mathematics

Participants' goal of engaging their students in problem-solving appeared linked to their disciplinary obligations. As we were considering upper-level courses, being authentic to the discipline and preparing students to progress to further studies in mathematics were important goals. Problem-solving has been identified as a key element of the work of mathematicians (Burton, 2004). Indeed, Halmos (1980) remarked “the mathematician’s main reason for existence is to solve problems . . . therefore, what mathematics really consists of is problems and solutions” (p. 519). Consequently, participants were motivated to engage students in problem-solving practices that are representative of the discipline and to enable them to

⁵We acknowledge that the term *problem-solving* has several different meanings and conceptualisations within mathematics education research depending on the focus (see Neururer (2024) for an overview). In this paper, we use the term to describe the practice of lecturers setting appropriate homework problems and students working to solve these problems and emphasise the distinction between our use of the term and other bodies of research.

develop associated skills. For instance, Dr Dargle emphasised the importance of learning to struggle and persevere on problems:

Dr Dargle: If you didn't do that [struggle on problems], I mean, you couldn't become a mathematician if, if you knew how to solve all your undergraduate problems right? The whole point of it is that you hit these problems that you haven't met before and if all you do is spend, like a minute thinking about it and saying you know 'I can't do it' then, you know, you're not going to be a mathematician. It's about looking at them, seeing which ones interest you, spending as much time as your interest dictates and hopefully some of the problems you crack them and some you won't.

3.5.3.2 The influence of obligations to students: Problem-solving is important for learning

Participants' goal of engaging students in problem-solving also stemmed from their obligations to students and their belief that working on problems is beneficial to student learning. The participants viewed the lecture as only the beginning of students' learning, with the expectation that most of the learning would occur outside the classroom. For instance, Dr Dargle reflected: "the amount of time you get to spend with students is fairly small in comparison to the amount of time it takes a student to learn something like measure theory". Similarly, Dr Boyne explained:

Dr Boyne: Most of the work has to be done essentially, in terms of time, it has to be done autonomously . . . so you get the main ideas across in the lecture and then the students should go and puzzle over those themselves.

Participants believed that students primarily learned outside of class by working on homework problems. When asked about optimal student engagement in his course, Dr Boyne replied:

Dr Boyne: I think the most important thing is to do the problems. If you really want to get an A+, but more than just getting an A+, if you want to actually get an understanding of the module, you should do the problems.

Dr Foyle also emphasised the importance of attempting problems, suggesting that mathematical learning stems from the effort students put into solving them:

Dr Foyle: I feel strongly that students should try problems. That's how you learn in maths. I think everybody knows that's how you learn maths is by trying questions. Even if you don't get them, you've got to try them.

Reflecting on his own experience as a student, Dr Liffey felt that solving difficult problems helped him to consolidate his understanding of course material. Consequently, he views problem-solving as an essential activity for students in his own course:

Dr Liffey: That's where I think that they do most of their active learning, Yeah, that's where they have to really engage with, they have to dig into the material I mean that's, again it's coming down to personal circumstances, I found that it was only after answering questions, and in particular hard questions, that I really felt that I was on top of what was happening.

Thus, participants were motivated to engage their students in problem-solving both because it is an important disciplinary practice, as well as an important learning activity for students.

Participants' goal of engaging their students in authentic problem-solving conflicted with two factors. The first is that instructors have limited control over what students do outside of class. The second factor is that mathematicians have an institutional obligation to assess their students in a particular way. We examine how these two factors conflicted with mathematicians' goals for authentic problem-solving below.

3.5.3.3 Factor in tension with participants' problem-solving goals: Limited control over what happens outside the classroom

While some lecturers report engaging their students in problem-solving in class (Johnson et al., 2018), it is reasonable to assume that most of undergraduate students' problem-solving occurs out of class while working on homework. This was the case in our participants' courses. Participants had beliefs about how students should work on their homework problems to maximise their learning. However, they also acknowledged that they have limited control over what happens outside the classroom. For instance, some participants worried that students do not attempt problems when homework is voluntary (Neuhaus-Eckhardt & Siller, 2026):

Dr Boyne: I think generally students aren't going to do work unless it's being graded. So if you say, I've done that many times, 'Can you hand in your homeworks?' and if you're not grading them they just won't hand them in.

However, participants also recognised that they were teaching upper-level courses and had to assume that their students would have the maturity to manage their own learning. For example, Dr Dargle explained that he does not believe it is appropriate to over-assess students in advanced courses:

Dr Dargle: I don't think it's particularly appropriate to assess higher-level maths modules all the time, You know if you're working to . . . what are you trying to do? You're just trying to force people to work all the time. I mean they're big boys and girls, they have to be able to take the responsibility for that.

Dr Foyle explained that when he teaches first- and second-year students he makes homework obligatory (Neuhaus-Eckhardt & Siller, 2026) and has it count towards students' grades. However, he believes that this is not appropriate in upper-level courses for two reasons. Firstly, he feels that upper-level student should be given more flexibility and "time to do things in their own time". Secondly, he believes that third- and fourth-year students should understand the importance of working on homework problems to their learning: "you would hope that by stage three or four the students, they know what's going on and what they should be doing". However, Dr Foyle did express some uncertainty about this decision and questioned whether students may end up spending more time on courses with more frequent continuous assessment components: "I mean maybe I suffer because of this, because I find that if other modules have a deadline, students will spend time on that module and won't spend time on my module".

Participants also spoke about how they have limited control over the manner in which students work on homework problems. For example, participants were generally unopposed to students working together on homework problems but expressed concern that it could lead to students simply copying solutions from peers. This was perceived as detrimental to students' learning: "you can watch other people do maths all day long, you don't learn" (Dr Foyle). However, as Dr Liffey acknowledges it is impossible to control what happens outside the classroom and participants were forced to accept that some students may go about their work on problems in a suboptimal manner:

Dr Liffey: I don't mind [students working together on homework] because I can't control what happens outside the classroom. So I write the questions with the expectation that, you know, there is going to be some collaboration and that can be positive, I mean provided, you know, you get a group of students who, who work together you know, and everybody makes some sort of contribution. In practice, you know, that ideal state of affairs often isn't there.

Another point that participants had strong views on was students' use of homework solutions. All participants believed that students' learning from engagement in problem-solving is contingent on them struggling on problems prior to viewing the solution. For example, Dr Boyne explained:

Dr Boyne: The point is that they should think about them [the problems] without seeing the solution. The whole point isn't really whether you get it right or

wrong, I know that students think that it is. The point is really, you know, trying to, when you're trying to answer the question that's when you learn stuff. And if you're seeing the solution ahead of time, you're just not making that step.

Dr Foyle felt similarly, and when asked what a student would miss out on if they did not attempt homework problems before seeing the solutions, responded: "Oh, you miss out on learning. . . . You don't learn, I think, if you don't try problems". For this reason, participants expressed misgivings about providing homework solutions to students. Dr Liffey believed that providing full solutions to students could hinder students' active engagement with problems:

Dr Liffey: I dislike giving full solutions. So I stop at giving hints or partial solutions. I don't like giving full solutions to everything because I feel like it's analogous to agreeing to, you know, run some sort of five-kilometre race, and then just getting in the car and driving to the finish line. You know, the student hasn't engaged actively with the material.

Dr Dargle also expressed doubts about providing solutions to students. He felt that simply reading and memorising solutions is not sufficient for students' to develop problem-solving skills:

Dr Dargle: If you give everything there, for the solution of a problem, an exercise, it's true, like the student learns perfectly how to do that problem but they don't learn how to do problems. In other words, if the full solution is there in front of them, and all they do is read that full solution, it's non-transferable skills. The skills that are important in your programme are, that we want, the transferable skills means being able to solve problems. And that's not like learning off a finite list. It's attempting them and seeing these connections.

While the mathematicians expressed concern that the availability of solutions might detract from the quality of students' engagement in problem-solving, all participants provided students with some form of solutions (either partial or full) to homework problems in their course. Other considerations factored into this decision. Participants recognised that students may feel frustrated by their perceived lack of progress when stuck on problems without access to solutions:

Dr Dargle: A student would have to put in a certain amount of time on their own trying stuff. Working through the problem sheets and realising that they can't do something. I think there's, yeah, that's a difficult one. I mean students generally want to see some return on the time they put into something and will see an inability to solve a problem as a lack of return on time spent and it isn't. They are the hard yards that bear fruit later on.

They were also aware that students are studying multiple courses simultaneously and have limited time available to work on problems for each:

Dr Boyne: At the undergraduate level you should be sensitive to the fact that not everyone is going to be hugely invested in it [the course] and they can only put in so much time into this, and they only have so many hours in the week to cover all their subjects.

Given the many demands on students' limited time, participants believed it would be unfair not to offer any support to students when they are stuck on problems. For this reason, Dr Liffey has come to the compromise of providing partial homework solutions, consisting of hints and tips for how to solve problems:

Dr Liffey: The other extreme would be to not give any solutions and I think that's not good either because students do have limited time to address things. And I think, to give no solutions at all would just simply be frustrating. So I try to find some sort of middle way. And I'm not, you know, I guess I'm happy enough. I'm not perfectly happy with it, but it is what it is.

Another option to support students struggling on problems might be to help them in class. Participants in this study held tutorial classes in their courses in which solutions to homework problems were discussed. However, they reported that typically students don't tend to ask many questions and are often reluctant to discuss their mistakes and misunderstandings. For example, Dr Liffey explained that while he does want more student interaction in his tutorials it can be hard to achieve this: "if I get few questions in the tutorial, then the tutorial runs the risk of turning more into a lecture, which is not ideal because I do want there to be greater interaction in the tutorial". Similarly, Dr Foyle remarked that while he would be glad to talk to students about problems, he finds that he is approached relatively infrequently: "I'm more than happy to talk to people about it [homework and test solutions], but I find students generally don't". Dr Dargle also reflected that students are often reluctant to approach instructors when stuck on problems or to discuss mistakes made on homework submissions:

Dr Dargle: Maybe students don't like to have mistakes pointed out so much. So you have to be aware of that. Like a submission that's marked, it's like the confessional box. We don't really talk about it outside. Unless a student comes and says 'oh I didn't really understand this', but that's actually very rare and that surprises me.

3.5.3.4 Factor in tension with participants' problem-solving goals: Obligation to adhere to institutional assessment norms

Participants had the goal of engaging their students in authentic problem-solving which they characterised as attempting challenging problems and persevering through periods of stuckness. However, we identified a tension between this goal and participants' obligation to adhere to institutional assessment norms. These assessment norms were to predominantly assess their students through closed-book examinations and for the examinations to be two hours in length⁶.

Participants believed that it would be possible to assess students' engagement in problem-solving in an examination setting by asking them to solve unseen problems (i.e. problems that had not appeared in the homework). Dr Liffey reflected: "I think we would like to see students answering [in examinations] more *unseen questions and constructing their own arguments* [emphasis added], because that would demonstrate the kind of mathematical creativity that is prized among mathematicians". Similarly, speaking about the final examination in his coding theory course, Dr Boyne commented: "I'd like to see a bit more of a contribution from the student, because *problems that you've already seen, they're not really problems* [emphasis added]". We inferred that Dr Dargle and Dr Foyle also held this belief from the fact that they did not want students to be able to rely on memorisation to answer the problems posed in their examinations. For example, Dr Foyle stated that he tries to "design an exam which isn't based on memory". Additionally, Dr Dargle feels that "there's a bit of responsibility, not just to provide, you know, snapshots of images that the students can commit to memory [and reproduce in an examination]".

While participants believed it was possible to assess students' engagement in problem-solving by asking them to solve unseen questions, they felt constrained in the extent to which they could include these types of questions on examinations that is two hours in length. Dr Boyne reflected that the two-hour examination length is inadequate to expect students to solve unseen problems:

Dr Boyne: If you've never thought about something in a particular way, you're not going to do it in 20 minutes . . . I don't know that an exam is the right time

⁶In UCD, final examinations are centrally administered. The university takes responsibility for issuing a central examination timetable, photocopying examination papers, and for proctoring the examinations in a large examination centre. Up to 2,000 students can be accommodated at a single sitting. Although it is possible to hold one-hour examinations, the norm is for final examinations to be two hours in duration. Generally, three-hour examinations are discouraged unless necessary for professional accreditation. An instructor can choose an alternative assessment and does not need to avail of service provided centrally, but then responsibility for the organisation lies with the instructor. It is the norm for mathematics courses in UCD (and indeed courses in many UCD departments) to have centrally organised two-hour closed-book examinations.

to, you should be giving people more time in the exam if you're really interested in that [having students solve unseen problems] and we don't, we give them two hours.

Similarly, Dr Liffey stated that he would include more unseen and challenging problems in examinations “only if the exams are longer. Like I said, three hours, I think that would give the students room to, time to make those creative arguments. Two hours is not enough”. Participants associated problem-solving with attributes like creativity, imagination, and an ability to make new connections. Dr Foyle questioned the extent to which it is possible to assess whether a student has these attributes within the current examination system:

Dr Foyle: I feel that, you know, testing to see whether a student has some imagination, it's hard to do that in a two-hour exam. Because students might have plenty of imagination but just might not make that connection in those two hours.

Another factor that constrained participants' inclusion of unseen questions on their examinations was their obligation to accommodate for varied student abilities. Dr Boyne explained that he includes only a small number of unseen questions on his examination as: “if all the test is like that, then everybody is going to get zero”. Dr Liffey expressed a similar sentiment, explaining that posing challenging problems in an examination setting risks negatively impacting many students' grades:

Dr Liffey: Generally speaking, if you find yourself writing a test or an exam and you find yourself writing questions that you find interesting then this is a bad sign. Because if the lecturer finds the questions interesting then that means that the students, most of the students, are going to get killed. So you have to write boring questions . . . That's just a dilemma, but you have to stick to the quite vanilla, uninteresting questions, uninteresting to you, because otherwise you run the risk of massacring students.

As Dr Liffey acknowledges, he experiences tension around this issue. While he feels obligated not to pose overly challenging problems in examinations, he also does not “want the more able students to be bored”. However, Dr Liffey is aware that future mathematicians represent only a minority of his course cohort: “the reality is that, I mean, the vast majority of students who major in mathematics here, they do not go on to become mathematicians or do graduate level mathematics courses”. As he is obligated to all students in his course, his priority must be to write an examination that accommodates for a range of student abilities.

3.5.4 Managing the problem-solving tension

We present two vignettes to illustrate the ways in which participants managed the problem-solving tension. In the first, we examine how Dr Dargle encourages students to attempt their homework problems, and accommodates for his varied student cohort. Additionally, we describe how Dr Dargle attempts to support students in problem-solving by providing partial solutions and outline his approach to asking unseen questions in examinations. In the second vignette, we examine how Dr Boyne arrived at the compromise of providing full homework solutions after the relevant tutorial class has taken place. We also describe his approach to balancing difficulty in examinations.

3.5.4.1 Dr Dargle

We identified Dr Dargle attempting to manage the problem-solving tension in a number of ways. In order to encourage students to engage with the homework problems in his measure theory course he assigns a grade to students' homework submissions. However, he explains that the main purpose of grading students' work is to encourage their engagement:

Dr Dargle: I put a mark out of five on each submission, but to be honest the marks start at three. So it's basically three, four, or five. And three means 'more work' and four is 'fine' and five is 'grand, good'. They don't have to be perfect. *It's about keeping up with some of the ideas and practicing [emphasis added].*

Dr Dargle allows students to choose two problems out of a problem sheet consisting of around ten problems to submit solutions to. In doing so, he fulfils his obligations to all students in his course by allowing more able students to demonstrate their ability to solve more difficult problems while not penalising weaker students in the course:

Dr Dargle: So some students, you know, okay, they're just trying to get through it . . . they just want to pass, they want to get their credit and their marks out of five for the problem sheet. Others maybe want to show off a little bit, 'I can do this' and that's okay. So there's room there for students to show off a little bit that they can do harder problems, they get exactly the same credit at the end of the day. But you know, they will get a "Well done".

Like the other participants, Dr Dargle believes that struggling on problems is an essential part of problem-solving. While he recognises that students would likely prefer to be given full solutions to homework problems, he refrains from doing so out of an obligation to ensure that students are actively engaging with the problems and not just memorising solutions to reproduce in an examination setting:

Dr Dargle: The main reason is that putting out full solutions, I don't find it benefits students. I mean, I know it's what a student would like, maybe at the end that, okay I can get the solution, I can learn off the solution so if it comes up [on the final examination], I've learnt it. But there's a bit of responsibility, not just to provide, you know, snapshots of images that the students can commit to memory but that they have process these solutions.

As Dr Dargle explains, he believes that his decision to provide only partial solutions is ultimately beneficial to students' mathematical learning. However, he is also aware that it can be frustrating for students to be stuck on problems in time-limited conditions. For this reason, he provides students with partial solutions consisting of hints and tips: "you want to provide enough that the student can get through it on their own and feel that yeah, I know the solution now. But they also have to put some effort into it".

We discussed that participants felt constrained in assessing their students' engagement in problem-solving by the two-hour length of examinations and their obligation to accommodate for a range of student abilities. One strategy Dr Dargle employs to manage this tension is including a number of exercises and problems in his course notes. Dr Dargle provides an overview of how students might approach solving these in class, with the expectation that students will work on the solutions in their own time. Dr Dargle then picks some of these exercises to be included on the examination: "if they've done the exercise after class time, they get their reward here on the exam paper because it came up". In doing so, Dr Dargle encourages his students to engage in problem-solving outside the constraints of a time-limited examination. By including some problems that have appeared in the homework and notes on the examinations, Dr Dargle is enabling students of a range of abilities to succeed. However, given the upper-level nature of the course, preparing and identifying future mathematicians was also an important goal. Dr Dargle explains that he generally includes a challenging, unseen problem on his examinations which he assigns a low number of marks to. He reflects that doing so allows him to identify and challenge the really talented mathematics students in the class without necessarily impacting the grades of other students:

Dr Dargle: Most years, I would have something on the paper, some part of a question, where very few students get it. Maybe nobody gets it or maybe one student gets it. And there'd be a couple of marks going for it. It's unlikely to make a difference in the grade, but I would say that's a maths student, that's someone who could progress to do more mathematics.

3.5.4.2 Dr Boyne

Dr Boyne does not have students submit solutions to their homework problems, however he organises a midterm test consisting of similar problems to encourage student engagement. He notes that this decision was primarily influenced by the time commitment that would be involved in grading students' homework each week: "I think the resources for grading, it's a logistics thing really". Consistent with the other participants, Dr Boyne believes that struggling on problems is very important that providing solutions can detract from this process. However, aware that being stuck can be a stressful experience for students, Dr Boyne has arrived at the compromise of providing full solutions to homework after the relevant tutorial has taken place:

Dr Boyne: I've had many journeys on this one. But I give the solutions. I try not to give them before the tutorial. So what I like is to try and do a lag on them. So it's after the tutorial has happened that they would usually see the solutions.

Dr Boyne believes that there is a correct way in which solutions should be used, but ultimately has accepted that he cannot control how students approach their learning:

Dr Boyne: I think solutions are very useful, but you have to use them in the right way. So at a certain point, you just have to go okay, like they're adults and I can give them this advice and at the end, hopefully they'll take the advice which is like to try and do it themselves. And then so really good students will just want to try and do their own solutions anyway. But like, like, you know, people have exams that they are worried about and maybe your subject isn't their dream subject, it's just another thing they have to do.

Given the upper-level nature of the course, identifying and preparing future mathematicians are relevant goals. For this reason, Dr Boyne asks a couple of unseen, challenging problems on his examinations to assess his students' problem-solving abilities. However, he reflects that he is obligated to all students in the course and that the cohort of potential future mathematicians cannot be his primary consideration when writing the examination:

Dr Boyne: I would always have some questions that everyone should be able to answer if they've done the work. And that should be at least 50% and usually would be more. And then . . . *I'm interested in knowing who is got the kind of, the sort of mind maybe that could handle a more interesting question. But I don't want that to be the dominant thing in the exam* [emphasis added]. The exam is really about how well do they know the course?

The compromise that Dr Boyne has adopted in relation to the issue of unseen questions is to ask students to solve slight modifications of problems that have been posed on the homework:

“What I do is ask problems that they may not have seen before, but they wouldn’t be a million miles away from something that they have seen before, and I find that difference is enough to distinguish people”.

3.6 Discussion

In this paper, we examined four participant mathematicians’ instructional design of upper-level, lecture-based courses in a research-intensive university in Ireland. Drawing on the theory of practical rationality (Herbst & Chazan, 2023), we analysed interview data for evidence of tensions in the form of competing individual and socio-technical decision-making factors. Two tensions were presented: the content and problem-solving tensions. The first centred on the pressure participants perceived to cover certain content due to disciplinary obligations which was constrained by factors such as a lack of time and the need to accommodate for students’ background knowledge. The second focused on participants’ goal of engaging students in problem-solving, driven by their obligations to students and the discipline, which conflicted with institutional assessment norms and a lack of control over students’ out-of-class work. In four vignettes, we illustrated how the participants attempted to manage these two tensions through their course design. We propose that our study builds on the existing literature in three main ways. First, we examine pedagogical decision-making in upper-level courses, which have been under-researched (Florensa et al., 2025). Second, we adopt a holistic whole-course approach to examining lecture-based instruction, in contrast to existing research which has tended to examine instructional components in isolation from one another (Pinto & Karsenty, 2018). Third, by examining the tensions mathematicians experience when enacting lecture-based instruction we provide a foundation for future research on augmenting lecture-based instruction, which is grounded in mathematicians’ concerns. We examine each of these three contributions in further detail below.

The first way in which our study builds on the research literature is by examining mathematicians’ pedagogical decision-making in upper-level courses, which to date have been under-researched (Florensa et al., 2025). Prior research has suggested that practical rationality and instructor decision-making should be considered at a course-specific level (Herbst & Chazan, 2023; Johnson et al., 2025; Ko et al., 2024). In agreement with this research, our findings indicate that the level of a course is a factor of influence in mathematicians’ decision-making. We highlight two ways in which this manifested in our data. The transition from school to university mathematics has been conceptualised in terms of a transformation or shift in the didactic contract (Brousseau, 1997), with undergraduate students expected to complete significant learning outside of class (Ellis et al., 2015; Pepin, 2014). Recognising

the importance of this independent work and taking responsibility for one's own learning is a key challenge for first-year university students (Pepin, 2014). However, it has been posited that over the course of their studies, students can develop "to the point of knowing what it means to do mathematics" (Larsen, 2017, p. 245), appreciating the necessity of out-of-class study. Our findings suggest that mathematicians expect this kind of maturity from their upper-level students and that this is reflected in their course design. For example, while Dr Foyle makes homework obligatory (Neuhaus-Eckhardt & Siller, 2026) in the first- and second-year courses that he teaches, he believes that upper-level students should understand the importance of working on homework problems without the need for any form of external motivation. Similarly, Dr Dargle believes that it is inappropriate to over-assess upper-level courses as the students should be able to take responsibility for their own learning.

A second way the upper-level course context influenced participants' decision-making was through the prioritisation of disciplinary obligations. It has been proposed that the nature of professional obligations varies between different instructional contexts (Johnson et al., 2025; Ko et al., 2024). Our findings support this proposition. A key concern for participants when selecting the content to cover in their courses was representing the relevant branch of mathematics accurately. Participants felt compelled to include particular content to achieve this goal, which in some cases was shaped by their personal research experience. For example, Dr Foyle believed it important to cover characteristic p in his Galois theory course on the basis of his own research experience, but acknowledged that this is not standard across the discipline. We propose that mathematicians' content-related goals may be less strongly linked to their disciplinary obligations in introductory courses such as calculus and linear algebra. In these types of courses, the priority may be preparing students for future courses, and institutional obligations may play a larger role. The influence of disciplinary obligations was also evident in participants' goal of preparing future mathematicians (see also Woods and Weber (2020)). As students in these courses were in their penultimate or final year, preparing them to potentially go on to postgraduate mathematical study was a priority for some participants. This goal informed Dr Liffey's selection of content in his course as he sought to ensure that students had covered sufficient content to facilitate their progression to postgraduate study. It also influenced Dr Dargle's emphasis on problem-solving as an important mathematical skill in his course, stating that students could not go on to become mathematicians without learning to struggle and persevere on problems. Preparing future mathematicians also shaped participants' approach to assessment. For instance, both Dr Dargle and Dr Boyne explained that they generally include a challenging, unseen problem for a low number of marks on their examinations, with the goal of identifying students who could potentially progress to further mathematical study.

While adhering to their disciplinary obligations appeared to be a key influence on our participants' instructional decision-making, we do not claim that this necessarily generalises to all mathematicians who teach upper-level courses. A direction for future research could be to use a confirmatory study like that of Mejía-Ramos and Weber (2014) to investigate the extent to which this finding generalises. Should it be found to generalise more broadly, the implications for undergraduate mathematics education must be considered. Such a finding could suggest that upper-level courses hold particular disciplinary significance for mathematicians that is linked to their role in training the next generation of mathematicians. As Woods and Weber (2020) suggest, this might mean that mathematicians will be reluctant to depart from lecture-based instruction in these courses unless convinced of the efficacy of alternatives in terms of training future mathematicians.

The second way in which our study contributes to the literature is by looking across a lecture-based course and considering mathematicians' decision-making regarding instructional components beyond the lecture. A prominent focus of extant research on mathematicians' pedagogical decision-making has been on understanding the decision of whether to enact lecture-based or student-centred instruction (e.g., Fukawa-Connelly et al., 2016; Johnson et al., 2018; Woods & Weber, 2020). Concerns about lecturing include that students can fail to understand the main points that their instructor is attempting to convey (Lew et al., 2016) and that students can misinterpret their lecturer (Krupnik et al., 2018). However, such criticisms overlook the fact that lecture-based instruction is not synonymous with lectures. Our participants viewed the lecture as playing an important role in terms of introducing students to content and, as Dr Foyle remarked, "highlighting the important bits and . . . doing examples". However, they did not expect their students to emerge from lectures with a complete understanding of the material presented and emphasised that developing such understanding requires significant out-of-class study. This finding aligns with recent research that found that mathematicians do not expect students to develop a complete understanding of material in class and that a significant goal of in-class instruction is preparing students to study independently (Mitchell et al., 2026). Therefore, like other researchers (Rupnow et al., 2021), we advocate that future research aimed at evaluating the effectiveness of lecturing should adopt a broader conception of the success/failure of lectures than the degree of understanding that a student emerges with. It is important to consider that when lecturing, mathematicians are also attending to other goals such as preparing students to learn out of class by providing an overview of content, inspiring students, highlighting what is important, and modelling how to think about mathematics (Mitchell et al., 2026; Pritchard, 2010; Weber & Fukawa-Connelly, 2023; Wu, 1999).

Additionally, we found that participants did not view lectures in isolation, but considered the relationship between lectures and other instructional components. For example, we presented vignettes of Dr Foyle and Dr Liffey describing how they used their course notes in conjunction with their lectures to manage the content tension and satisfy their disciplinary obligations. More broadly, we found that participants' goals were reflected in their design of multiple instructional components and that they considered the integration between various aspects of the course. For instance, engaging students in problem-solving was viewed as essential to their learning and also important in terms of adhering to disciplinary obligations. This goal was reflected in participants' design of homework, homework solutions, as well as the final examinations of their courses.

Our findings suggest that mathematicians do not think about individual instructional components in isolation. They consider the integration of various instructional components and their course goals, beliefs, and professional obligations shape their design of the overall course. However, this approach to course design has not been reflected in existing research which has traditionally examined mathematicians' design of, and perspectives on, individual instructional components (e.g., Iannone & Simpson, 2015; Mesa, 2022; Rupnow et al., 2021; Woods & Weber, 2020). Adopting such an approach fails to capture the interconnected nature of course components as well as the influence of an instructor's goals and beliefs on overall course design (Pinto & Karsenty, 2018). As Fortune and Keene (2021) phrased it, "instruction is a complex act; thus, research must be done at a scale that captures that complexity" (p. 135). The obvious limitation associated with such research is the potential for an unmanageable quantity of data to be generated. We found that focusing on a particular section of content for each course in our study had three benefits. First, it ensured that the data generated remained at a scale that was manageable for analysis. Second, by basing our study on participants' personal pedagogical practices we avoided the influence that conducting research in an unnaturalistic setting may have had on our results (Moore, 2016; Rupnow et al., 2021). Third, we were interested in the influence of individual decision-making factors such as beliefs, which we were aware are often unconsciously held and not easily articulated (Leatham, 2006). By grounding our questions in particular decisions participants had made, we were able to infer the influence of decision-making factors from participants' actions and provided rationale. A direction for future research could be to adapt our study design in order to scale up and out (Melhuish, 2018) and examine mathematicians' overall course design with a larger group of participants and in various instructional contexts such as other countries or course levels.

The final way in which our study contributes to the research literature is by examining the tensions mathematicians experience when enacting lecture-based instruction. To date, a

significant focus of research on lecture-based instruction has been on examining why mathematicians engage in this form of teaching (e.g., Johnson et al., 2018) and how it aligns with their pedagogical goals and beliefs (e.g., Woods & Weber, 2020). In comparison, a central concern of research on student-centred instruction has been on examining the limitations of mathematicians' pedagogical content knowledge that can prevent them from implementing non-lecture instruction effectively (e.g., Johnson & Larsen, 2012; Speer & Wagner, 2009; Wagner et al., 2007), as well as documenting the tensions they can experience when enacting student-centred instructional methods (e.g., Johnson et al., 2013; Mesa et al., 2020). While we have an understanding of the difficulties mathematicians can experience when implementing student-centred instruction, the literature has tended to portray an unproblematic relationship between mathematicians and lecture-based instruction. Our study challenges this portrayal by presenting two tensions our participants experienced when designing upper-level, lecture-based courses. We do not contend that these two tensions necessarily generalise to a broader population of mathematicians. However, our study does suggest that not all mathematicians are entirely satisfied with their current teaching methods.

As suggested by Dawkins and Weber (2023), our participants tended to satisfice (Simon, 1972), arriving at a compromise in response to the tensions they experienced without necessarily knowing if a better alternative exists. We presented a vignette of each participant illustrating the variety of creative ways in which they attempted to manage the content and problem-solving tensions. However, participants also expressed doubt and uncertainty about the approaches they adopted. For instance, Dr Foyle decided not to make homework obligatory (Neuhaus-Eckhardt & Siller, 2026) in his Galois theory course, but questioned whether this might lead to students spending less time on his course than others. Additionally, Dr Liffey explicitly acknowledged that he has compromised on issues in his teaching, reflecting: "I think the best I can aim for is to be happy enough".

In recent years, there has been acknowledgement of the limited impact of efforts to replace lecture-based instruction with student-centred curricula (Melhuish et al., 2022). The limited success of these efforts has been partially attributed to a lack of consideration of mathematicians' pedagogical goals and orientations, as well as their professional obligations (e.g., Dawkins & Weber, 2023; Johnson et al., 2025; Mitchell et al., 2026; Woods & Weber, 2020). Consequently, there has been increased focus on examining ways in which lecture-based instruction can be augmented (e.g., Alcock, 2018; Dawkins & Weber, 2023; Johnson et al., 2025). We propose that one productive way of pursuing such research could be to use the tensions that mathematicians experience when enacting lecture-based instruction as the starting point. One approach could be to connect mathematicians with relevant research that might help in informing their approach to managing tensions in their teaching. For example,

participants in our study admitted to having limited insight into students' out-of-class learning and might find research on students' approaches to, and perceptions of, completing homework informative (e.g., Lew & Zazkis, 2019; Murphy & Meehan, 2024; Neuhaus-Eckhardt & Siller, 2026). Additionally, some participants discussed their use of course notes to complement their in-class teaching and may be interested in mathematics educators' research on partial notes (e.g., Iannone & Miller, 2019). Other issues raised by participants might prove fruitful for collaborative research between mathematicians and mathematics education researchers. For instance, participants questioned the correct approach to providing students with homework solutions, an issue that has become more complex as the use of artificial intelligence among university students becomes increasingly prevalent (Stöhr et al., 2024). Such an approach to collaborative research would have the advantage of being grounded in mathematicians' concerns, mindful of the various factors that influence their pedagogical decision-making.

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Appendix to Chapter 3

3.A Interview schedule

Note: The below are context-free interview questions for a generic course. The questions that were asked to participants in the interviews were customised to their specific course context, using teaching artefacts such as notes, homework problems, and past examination questions.

Interview 1

Section 1 – Introduction and overall context of the course

- Where did you complete your undergraduate degree?
 - What kind of teaching did you experience over the course of your undergraduate degree?
- How long have you been teaching?
- What is your area of research?
- Do you think your research experience has an influence over your teaching of the course?
- When did you first teach the course and how many times have you taught it since?
- In the course, what would you say your overarching goals for student learning are?
 - Prompts: content, skills?
- Would you be able to give me a brief overview of the structure of the course?
 - Prompts: lectures, notes, homework, tutorial classes, assessment (contextual information).

Section 2 – Learning outcomes associated with [section of content]

- Why would you consider [section of content] to be an important result in this course?

- What understanding of this result would you like students to develop over the course?
- Are there any specific mathematical techniques associated with the result you would like students to learn? Or, alternatively, is there anything you would like students to be able to use the result to do?

Section 3 – The notes

For each definition, theorem, example, and proof related to [section of content]:

- Why did you decide to include this piece of content in the course?
- Did you make any decisions about how it was stated in the notes?

More general questions on notes:

- What resources did you use when you first prepared your notes?
 - When you first taught the course did you prepare your notes in advance or as the course progressed?
 - * If the latter did what was happening in lectures change the direction of what you taught?
 - Do you make any changes to your notes year-on-year?
- How do you decide what to include in your notes/what content to cover in the course?
- Have you spent a lot of time getting your notes to where they are now?
- Do you see the notes as primarily a resource for yourself or the students?
- Do you publish your notes for students in advance of the lecture? Why?
 - Do you think there would be any benefit in them reading the notes ahead of the lecture?
- Do you expect students to take their own notes in lectures?
 - What would you recommend students record in these notes?
 - * Do you offer them any specific advice regarding this?

Section 4 – The lecture

- What do you do to prepare for giving a lecture?

- Do the notes play a role in your preparation for lectures?
- Do you make decisions about what you will or will not present/write out?
- Do you make decisions about what you will say/the main points you will make?
- Do you make decisions about at which points you might ask a question or engage the class?
- Do you think about or plan the use of whiteboard(s)/tablet(s) in your lectures?
- Do you have a lesson plan/an image of the lecture in your head when you go to class?
 - Do you ever find the lecture deviating from your plan? Why?
- Do you think your lectures have deviated between course iterations?
- After a lecture, what would you recommend a student do? Go back over the lecture?
 - What does this look like and how long should it take?
- Overall, what do you see as the value of lectures to students' learning of mathematics? What do you see as the added benefit for the student who attends the lecture, rather than just reading the notes alone?

Interview 2

Section 5 – Homework, homework solutions, and the tutorial

- For each problem relevant to the [section of content] (Rupnow et al., [2021](#)):
 - Why did you decide to include this question?
 - What did you hope students would learn by attempting this problem?
 - How do you think students might engage with this problem?
- Out of these questions which would you expect students to find most difficult? Why?
- In general, when creating a problem sheet do you think about balancing the level of difficulty across the problems?
- Have you used the same problem sheets every time you have taught the course or do you make changes year-on-year?

- What resources did you use when creating the problem sheets?
- Do you ask students to submit solutions to homework problems for grading?
 - Why/why not?
- What kind of format are your tutorial classes and what do you see as their purpose?
- Do you expect students to have attempted all the problems before the tutorial class?
 - How long do you expect students to spend working on homework problems?
- Do you give solutions to your homework problems?
 - Why/Why not?
- If a student doesn't attempt the problems and waits to see the solution, what do you think they miss out on?

Interview 3

Section 6 – Test(s)/Midterm(s)

- Do you give a midterm test(s) in the course?
 - Why/why not?
- What do you see as the primary purpose of such a test?
- How do you decide on the timing of the test?
- How do you gauge the appropriate level of difficulty and how long it will take students to complete problems?
- What kind of feedback do you give on a test like this?
- If a student were to get an A grade in a midterm/test, what do you think that demonstrates?

Section 7 – The final examination

- For each final examination question related to [section of content]:
 - Why did you decide to include this question?

- What was it that you were hoping to examine?
 - Is it unseen or have students seen a similar-style question as part of their course-work?
- Out of these questions/parts which would you expect students to find most difficult? Why?
- How do you go about balancing the difficulty of an exam? Do you find this to be a difficult task?
- In general, how happy would you say you are about the way in which you currently assess students?
 - In an ideal world is there anything you would like to change?
- If someone were to get an A grade on this exam, what do you think it demonstrates?

In 2015, two mathematics education researchers called Paola Iannone and Adrian Simpson published findings from a survey- and interview-based study they conducted with mathematicians in the UK, examining their views on closed-book examinations (Iannone & Simpson, 2015). They found that while the interviewed mathematicians felt that closed-book examinations had the “potential to assess some of the dimensions they value [such as] understanding, imagination, creating new connections” and to generally distinguish between “stronger and weaker mathematicians” (p. 75), there was a general consensus that the current reality is that examinations are failing to live up to this potential.

For instance one mathematician commented:

So there are at least two styles of written exam which I can identify - those that are competency testing and those which are looking deeper and trying to see whether the student has some imagination, some ability to connect things that nobody’s shown them were connected before. [...] A standard written exam, [...] to be honest is nowadays more or less totally a competence type exam. (Iannone & Simpson, 2015, p. 77)

As a result some felt “that written examinations no longer differentiate between students on the basis of the kinds of mathematical ability which many of the lecturers value. Instead, they are seen to reward effort, diligence and memory and this impacts on how students prepare for exams and how they approach learning material” (p. 77). One mathematician elaborated on this saying:

So with closed book exams and the sorts of questions we've asked, I think if you have a good memory you're half way there and literally half way of the marks. What else? In fact, on the whole, in order to get a good mark even on an exam, if you have been conscientious and you've done all the coursework and you've revised and you've done questions from past papers and so on, you will do well. . . I think anyone who has actually just put the work in will do well in exams. They don't actually need to be that talented. (Iannone & Simpson, 2015, p. 77)

- Do you have any comments on these findings? To what extent do you agree or disagree?

Section 8 – Conclusion

- Over the past three interviews, we have discussed your preparation of the notes, lectures, problem sheets, tests and final exam for the course. Would you be able to sum up how you would view optimal student engagement across these different components?
- Would you be able to sum up what you hope a student would have learned/how they might have developed mathematically after completing the course?
- What do you enjoy most about teaching advanced, proof-based undergraduate mathematics courses like this one?
- What do you view as the biggest challenges in the preparation and delivery of a course like this one?
- Over the past couple of interviews, you have discussed your decisions in relation to the preparation of the course resources. What would you say are the greatest influences on your decision making process in relation to your teaching?
 - Experience as a teacher/lecturer?
 - Experience as a student?
 - Discussions with colleagues?
 - Professional development courses?
 - Workshops/conferences?
 - Mathematics education literature?
 - Other materials?

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Examination Candidate UCD Student Number: 18322691
Research Degree for which thesis is being submitted: Doctor of Philosophy
Title of Research thesis: Looking beyond the lecture: A qualitative examination of pedagogical decision-making and student learning in upper-level courses
2. Details of the Paper
Title of Paper: Towards a PCK theoretical framework for undergraduate mathematics education: Drawing on insights from K-12 science education
Current Status of the Paper: ☐ Drafted ☐ Submitted ☐ Under review ☒ Under revision ☐ Accepted for publication ☐ Published
Name of Journal (if known): International Journal for Research in Undergraduate Mathematics Education
Digital Object Identifier (if published):
AUTHOR CONTRIBUTIONS
Part A: Contribution of Examination Candidate <i>A full description the Examination Candidate's intellectual and practical contribution to the research presented in the paper. It would be helpful for the Examination Committee if an estimated percentage of the candidate's contribution to this paper could be provided below. Each section of the paper should be listed separately i.e. Introduction, Methods, Figure 1, Table 2 etc. The examination candidate should either be the sole author/primary author or a major contributor to the paper. Please also review the UCD Authorship Policy when completing this section.</i>
Percentage contribution of the candidate to the paper : 75%
Description of Examination Candidate's contribution for each section of paper: (Max 500 words): I developed the research aims, incorporating feedback from my supervisor. I reviewed relevant background literature. With my supervisor, I collaboratively wrote the interview schedule. I conducted the interviews for the study was responsible for their transcription. I was primarily responsible for the data analysis, which developed through discussions with my supervisor. I wrote all sections of the manuscript, incorporating feedback from my supervisor. I also produced the figures and tables for the manuscript.

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Contribution of Co- Author No.1	25% contribution <i>Brief outline of contribution: (Max 200 words)</i> As my supervisor, Maria provided essential advice and guidance in conceptualising and developing the methodology for the study. I was responsible for conducting the interviews, however Maria provided valuable feedback on my interviewing technique. Maria took on the role of experienced supervisor and collaborator during the data analysis process. She was familiar with the data and we discussed the analysis as it progressed. Maria reviewed and provided feedback on all sections of the manuscript.
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Chapter 4

Towards a PCK theoretical framework for undergraduate mathematics education: Drawing on insights from K-12 science education

Abstract

Over the four decades since its inception, pedagogical content knowledge (PCK) has become a major research focus in K-12 mathematics and science education. In contrast, relatively little research has been conducted on PCK in university mathematics. Most of the extant research has examined the PCK required to enact student-centred instruction and has utilised theoretical frameworks developed in the K-12 context. Researchers have cautioned against this practice, but there has yet to be significant focus on adapting PCK theoretical frameworks for use in undergraduate mathematics. This paper addresses this gap by adapting and combining Carlson et al.'s (2019) and Magnusson et al.'s (1999) PCK frameworks, developed in K-12 science education, for use in university mathematics. By drawing on insights from science education, the proposed framework has three characteristics that make it well suited to examining PCK in this context. First, it accounts for teacher beliefs and orientations which have been identified as important influences on mathematicians' pedagogical decision-making. Second, it incorporates PCK components such as knowledge of assessment and knowledge of curriculum, which have been understudied in a mathematical context but are relevant to the work of undergraduate instructors. Third, it distinguishes between PCK developed individually through teaching practice and PCK derived from research. This distinction is important given that mathematicians typically receive limited pedagogical training. We demonstrate how the proposed framework can be operationalised to analyse interview data by applying it to a mathematician teaching an upper-level, lecture-based course.

Keywords: pedagogical content knowledge, mathematical knowledge for teaching, science education, lecture-based courses, teaching knowledge

4.1 Introduction

In recent years, there has been increasing acknowledgement that mathematicians can develop teaching knowledge and skills through experience and reflection, along with calls for more genuine collaboration to capitalise on this knowledge (e.g., Dawkins & Weber, 2023; Johnson et al., 2025; Stewart, 2023). One form of teaching knowledge that mathematicians may develop is pedagogical content knowledge (PCK), which was conceptualised by Shulman (1986) as the knowledge required to teach particular content effectively. Over the forty years since its inception, PCK has become a prominent focus of research in K-12 mathematics and science teacher education, with considerable attention given to understanding its nature and development (Chan & Hume, 2019; Depaepe et al., 2013). In contrast, a recent review of teaching knowledge in undergraduate STEM instruction highlighted that research on PCK in this context is scarce (Andrews et al., 2022). In undergraduate mathematics specifically, PCK studies have largely focused on the knowledge required to enact student-centred or inquiry-oriented forms of instruction (Melhuish et al., 2022), with comparatively little attention paid to lecture-based courses (cf. Delgado-Rebolledo & Zakaryan, 2020). This gap is especially pronounced in upper-level courses, which remain broadly under-researched (Florensa et al., 2025).

The initial goal of our study was to analyse the characteristics of the PCK of a mathematician teaching an upper-level, lecture-based course. To conceptualise PCK, we selected Ball et al.'s (2008) mathematical knowledge for teaching (MKT) framework, which, while developed in the elementary school context, represented the most prominent and widely used model in mathematics education (Depaepe et al., 2013). Researchers have previously cautioned against automatically extrapolating PCK frameworks such as the MKT model from the K-12 to university context (Andrews et al., 2022; Speer et al., 2015). Their concerns were confirmed during our preliminary data analysis which identified limitations of using the MKT framework to analyse PCK in the context of upper-level, lecture-based mathematics courses. These limitations included the framework's largely cognitive focus which fails to account for affective factors such as teachers' beliefs and a focus on certain components of PCK that, we contend, fails to reflect the diversity of teaching tasks associated with the post-secondary setting. To date there has been little focus on refining and adapting PCK theoretical frameworks for use in undergraduate mathematics and this gap forms part of the motivations for this paper.

We begin to address this gap by adapting and combining two PCK theoretical frameworks developed in the K-12 science education context (Carlson et al., 2019; Magnusson et al., 1999) for use in undergraduate mathematics education. Our decision to consider theoretical

frameworks from science education is due to PCK research traditionally being siloed by discipline (Andrews et al., 2022). This means that mathematics education researchers have been slow to engage with the significant theoretical development of PCK that has occurred in science education (Andrews et al., 2022; Fukaya et al., 2025). Our proposed framework, drawing on Carlson et al. (2019) and Magnusson et al. (1999), has three characteristics that make it suited to analysing PCK in the context of undergraduate mathematics courses. First, unlike the MKT model (Depaepe et al., 2013; Johns & Burks, 2023), it accounts for the role of teacher beliefs and orientations, which have been shown to be particularly important factors in mathematicians' pedagogical decision-making (e.g., Melhuish et al., 2022; Pinto, 2019; Rupnow, 2023; Weber, 2004; Woods & Weber, 2020). Second, drawing on Magnusson et al.'s (1999) framework expands our conceptualisation of PCK to include knowledge components, such as knowledge of assessment and knowledge of curriculum, that have been understudied in a mathematical context (Andrews et al., 2022), but are very relevant to the work of university mathematics instructors. Third, leveraging Carlson et al.'s (2019) model enables us to distinguish between enacted PCK which is individual and developed largely through teaching practice, and collective PCK which encompasses research-derived canonical knowledge. We consider this distinction to be important in the undergraduate context where instructors typically don't receive formal pedagogical training (Sarkar et al., 2024; Winsløw et al., 2021) and are more likely to develop their PCK through their own teaching experiences (Lawrie et al., 2019). These theoretical aims are summarised in the following research question:

- **RQ:** How can pedagogical content knowledge (PCK) theoretical frameworks, developed in a K-12 science education context, be adapted and combined for use in the context of undergraduate mathematics courses?

As mentioned, our decision to consider PCK theoretical frameworks developed in another disciplinary context was prompted by preliminary data analysis using Ball et al.'s (2008) MKT framework. These data comprised of a series of three interviews with a mathematician about his teaching of an upper-level, lecture-based mathematics course at a research-intensive university in Ireland. We draw on these data as an instrumental case study (Stake, 1995), illustrating the application of our proposed theoretical framework rather than serving as the main object of inquiry. We structure the paper by first providing an overview of the theoretical developments of PCK in the K-12 science and mathematics education literature, as well as outlining previous research on PCK in university mathematics. We then provide a summary of the study context. This is followed by a description of how Carlson et al.'s (2019) and Magnusson et al.'s (1999) frameworks were adapted and combined based on both our

preliminary data analysis and review of the literature. We demonstrate how the framework can be operationalised using template analysis (Brooks et al., 2015; King, 2012) and applied to interview data. We conclude with a discussion of the implications of our work in relation to the broader literature base.

4.2 Background literature

In 1986, Shulman introduced the concept of pedagogical content knowledge (PCK) to the education research literature, conceptualising it as knowledge of teaching and learning that is specific to a particular topic or subject. His initial description of PCK consisted of two main components: the first concerned “the ways of representing and formulating the subject that make it comprehensible to others” (Shulman, 1986, p. 9), while the second included “an understanding of what makes the learning of specific topics easy or difficult” (p. 9). Shulman’s conceptualisation of PCK occurred in response to the artificial distinction he perceived between content and pedagogy in the preparation and professional development of K-12 teachers (Depaepe et al., 2013). Accordingly, much of the subsequent empirical research on, and theoretical development of, PCK has occurred in the K-12 context (Sarkar et al., 2024), with a particular focus on mathematics and science teachers (Fukaya et al., 2025).

In this review, we outline some of the major theoretical developments of PCK in the K-12 context. As this work has largely been siloed by discipline (Andrews et al., 2022), we begin by reviewing PCK theory in the context of mathematics, followed by science. We then provide an overview of the contextual differences between teaching at the K-12 versus university level, as well as the limited literature on PCK in undergraduate mathematics.

4.2.1 PCK in K-12 mathematics education

Teachers’ PCK has become an increasingly prominent topic of research in K-12 mathematics education with a recent systematic review identifying 237 studies on the subject published in the last ten years (Grigaliūnienė et al., 2025). One of the most frequently used theoretical frameworks to study PCK in this context is Ball et al.’s (2008) mathematical knowledge for teaching (MKT) framework (Depaepe et al., 2013). Ball et al.’s framework builds on the work of Shulman (1986) by adding specificity to the definition of PCK and connecting the concept to teachers’ content knowledge. The MKT framework is practice-based, having been developed by studying the forms of knowledge employed by US-based elementary teachers in the course of their work.

The MKT framework consists of six knowledge components, three of which are considered forms of PCK while the other three relate to subject matter knowledge. Knowledge of content and students (KCS) is a teacher's ability to anticipate students' reactions to content, including what they are likely to struggle with and how they might approach tasks. Knowledge of content and teaching (KCT) includes knowing appropriate ways to represent and sequence content and what examples and activities best support student learning. Knowledge of content and curriculum (KCC) is a teacher's ability to situate content within the context of a particular curriculum. The three subject matter knowledge components of the MKT framework are common content knowledge, specialised content knowledge, and horizon content knowledge (see Ball et al. (2008) for a description of these components).

The two most frequently investigated components of PCK in mathematics education are KCS and KCT (Depaepe et al., 2013; Grigaliūnienė et al., 2025). Although KCC is included in the MKT framework, it has been studied less often (Andrews et al., 2022). While the MKT model has been widely used, it has been criticised for adopting a largely cognitive perspective on teachers' knowledge, failing to account for affective factors such as teachers' beliefs (Depaepe et al., 2013; Johns & Burks, 2023). Andrews et al. (2022) note that researchers' choice of PCK theoretical framework appears to be largely dictated by disciplinary norms and called on researchers to consider how PCK has developed theoretically in other disciplines. In the next section we provide an overview of PCK in K-12 science education as much of the theoretical development of PCK has occurred in this context (Fukaya et al., 2025).

4.2.2 PCK in K-12 science education

Since Shulman's (1986) conceptualisation of PCK, several science education researchers have proposed theoretical developments of the concept (e.g., Grossman, 1990; Magnusson et al., 1999; Park & Oliver, 2008). The most widely used PCK framework in science education is that of Magnusson et al. (1999) (Chan & Hume, 2019). The framework consists of five components: orientations towards teaching (OTT), knowledge of curriculum (KC), knowledge of instructional strategies and representations (KISR), knowledge of students' understanding (KSU), and knowledge of assessment (KA). Three of these components correspond to PCK components included in the MKT framework: KC aligns with KCC, KISR with KCT, and KSU with KCS¹ (Andrews et al., 2022). The two components without a counterpart in the MKT model are KA and OTT. KA refers to a teacher's knowledge of what content and skills are important to assess, as well as appropriate assessment methods. Magnusson et al.

¹For the remainder of the paper we will use Magnusson et al.'s (1999) terminology when referring to PCK components included in both the Ball et al. (2008) and Magnusson et al. (1999) frameworks.

conceptualise OTT as consisting of a teacher's beliefs about teaching science at a particular grade level which shape and influence the other PCK components. Aside from OTT, each component included in Magnusson et al.'s framework consists of two subcomponents (definitions of these subcomponents are included in Appendix 4.A). The relationship among the components and subcomponents of Magnusson et al.'s model is illustrated in Figure 4.2.1.

Since the publication of Magnusson et al.'s (1999) framework, science education researchers have raised theoretical concerns about PCK. Specifically, researchers have questioned the nature of the relationship between beliefs and PCK (Carlson et al., 2019; Friedrichsen et al., 2011). Additionally, they have debated whether PCK constitutes collective, research-informed knowledge or whether it is personal knowledge derived through teaching experience (Carlson et al., 2019; Settlage, 2013). To address these concerns a meeting of science education researchers, referred to as the PCK summit (Berry et al., 2015), was organised. Their work contributed to the development of the consensus model of PCK (Gess-Newsome, 2015), which advanced the theoretical development of PCK by defining it as personal knowledge and distinguishing it from canonical knowledge which is "generated by research or best practice" (p. 33). Additionally, it clarified that teacher orientations and beliefs should not be considered a component of PCK but instead conceptualised as an amplifier or filter for how instructors apply knowledge to practice.

The theoretical refinement of PCK continued at a second PCK summit (Hume et al., 2019) which culminated in the development of the refined consensus model (Carlson et al., 2019). A key contribution of this framework was the conceptualisation of PCK as consisting of three distinct realms: collective PCK (cPCK), personal PCK (pPCK), and enacted PCK (ePCK). Enacted PCK is defined as the knowledge and skills utilised by an instructor in a particular instructional context. It applies both to the knowledge drawn on when interacting with students in class, as well as when planning and reflecting on instruction. Personal PCK is conceptualised as a "reservoir of knowledge and skills that [a] teacher can draw upon during the practice of teaching" (p. 85). Personal PCK is considered a superset of enacted PCK and when an instructor utilises an element of their personal PCK in practice it becomes enacted. Deciding on what knowledge to enact in an instructional context requires teachers to engage in pedagogical reasoning. Carlson et al. explain that a teacher's beliefs influence the construction of personal (and therefore enacted) PCK, meaning it is unique to individual instructors. Collective PCK is public knowledge possessed by more than one person and includes research-informed canonical knowledge. Carlson et al. also allow for the possibility of collective PCK to be developed locally and shared among the members of a department or school. Collective PCK exists at varying grain sizes from the broad discipline level to the more fine-grained topic and concept levels.

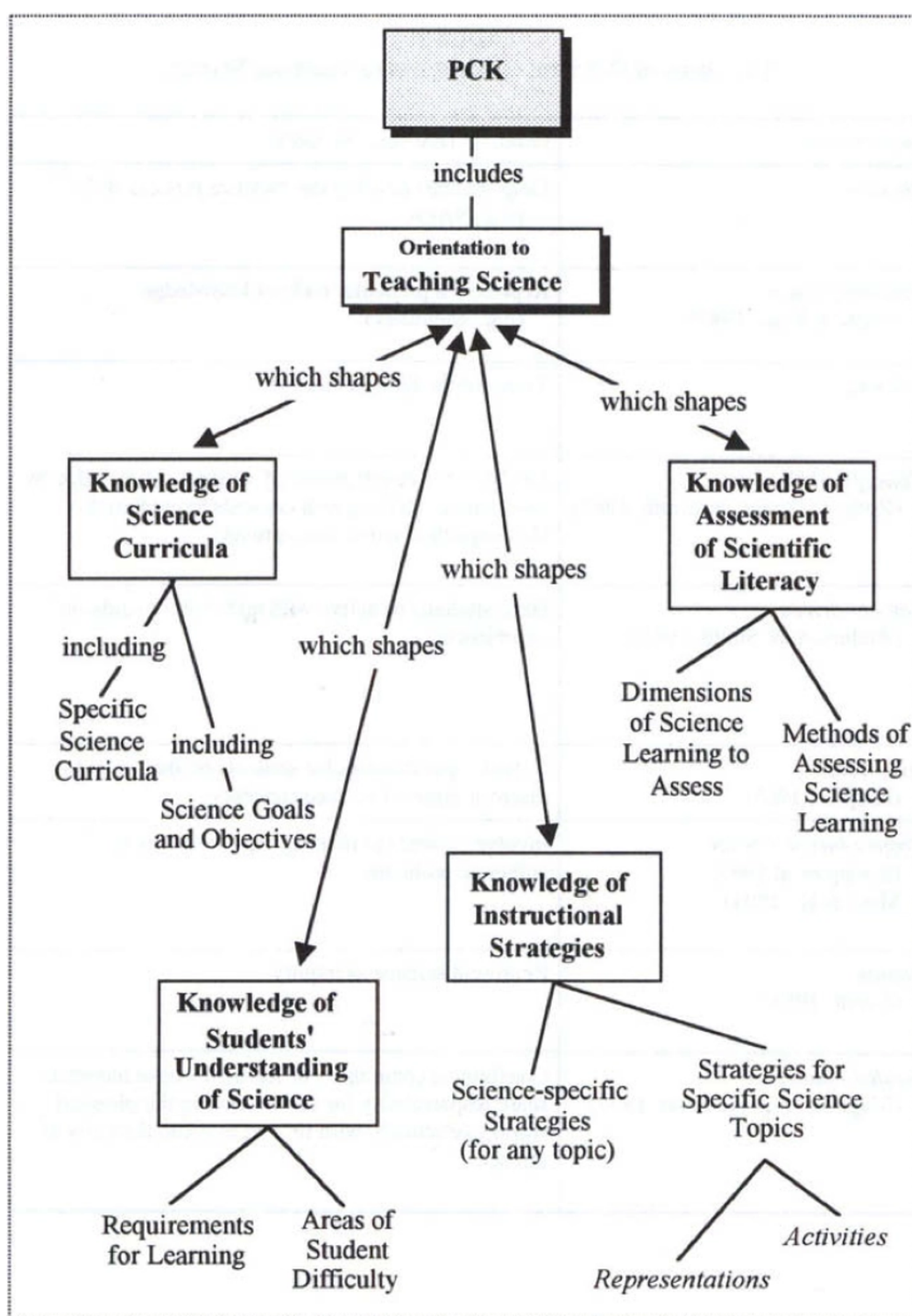


Figure 4.2.1: Magnusson et al.'s (1999) PCK framework (Magnusson et al., 1999, p. 99).

The refined consensus model also emphasises the influence of the learning context which is conceptualised as an amplifier and filter that separates individual PCK (enacted and

personal) from what others know and do. Additionally, Carlson et al. (2019) note that PCK can interact with other professional knowledge bases including generic pedagogical knowledge, knowledge of students, curricular knowledge, assessment knowledge, and content knowledge. These forms of educational knowledge are generic and not connected to a specific discipline. Carlson et al. state that two-way knowledge exchanges between the various components of their model are possible and that any exchange is influenced by a teacher's beliefs and attitudes. The refined consensus model is represented diagrammatically in Figure 4.2.2.

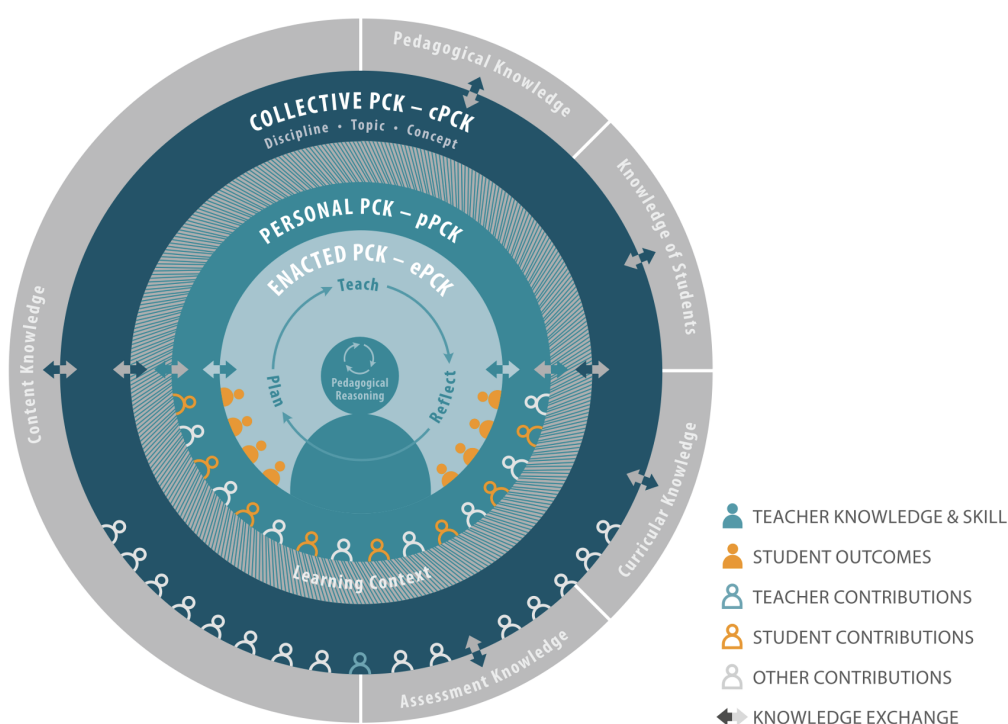


Figure 4.2.2: The refined consensus model (Carlson et al., 2019, p. 83).

4.2.3 Differences between mathematics instruction in the K-12 and university contexts

In contrast to the rather extensive literature on PCK in the K-12 context (e.g., Chan & Hume, 2019; Depaepe et al., 2013), relatively little research has been conducted on PCK in the general higher education context (Sarkar et al., 2024), nor specifically in relation to university mathematics (Andrews et al., 2022; Melhuish et al., 2022). Researchers have cautioned against automatically extrapolating research findings and theoretical constructs from the K-12 literature to the university context (Andrews et al., 2022; Speer et al., 2015) as there are many contextual differences that must be considered. For instance, the nature of teaching

at the K-12 and university level differs. In undergraduate mathematics, in-class instruction is classified as either lecture-based or student-centred depending on the extent to which students are responsible for generating disciplinary ideas (Melhuish et al., 2022). Lectures are the most common form of instruction (Johnson et al., 2018), with mathematics education researchers expecting this to remain the case for some time to come (Lew et al., 2022). At the undergraduate level, most of student learning occurs outside of the classroom (Rupnow et al., 2021; Wu, 1999). This necessitates a shift in the role of the instructor, with greater emphasis placed on preparing students for out-of-class learning (Ellis et al., 2015; Rupnow et al., 2021). University instructors also typically teach larger class groups which leaves less time for in-class interaction with students (Sarkar et al., 2024). Furthermore, they generally have more flexibility in choosing course content when compared to K-12 teachers who may have to follow a set syllabus (Lawrie et al., 2019). An additional contextual difference relates to assessment. The predominant form of assessment in university mathematics in the UK and Ireland is closed-book examination (Iannone & Simpson, 2011, 2022; National Forum for the Enhancement of Teaching and Learning in Higher Education, 2016). Typically, course instructors in these countries have autonomy over designing their own examination questions, which contrasts with the K-12 setting where teachers prepare students for examinations that are set by national examination boards.

Another difference between teaching at the K-12 and university level relates to the nature of teacher preparation. University teachers typically have very well-developed content knowledge but receive little to no pedagogical training or mentoring (Andrews et al., 2022; Sarkar et al., 2024; Winsløw et al., 2021). In the absence of formal pedagogical training, research has indicated that the primary influence on mathematicians' pedagogical decision-making is their beliefs about mathematics, teaching, and student learning (Melhuish et al., 2022; Pinto, 2019; Rupnow, 2023; Speer, 2008; Woods & Weber, 2020). This influence is illustrated by mathematicians' decision to lecture despite the availability of research-informed, student-centred curricula (e.g., Larsen et al., 2013; Leron & Dubinsky, 1995). Research investigating the modest impact of student-centred curricula has found that "mathematicians teach via lecture because they believe lecture is the best way to teach" (Melhuish et al., 2022, p. 6). Reasons that mathematicians believe in the efficacy of lecture-based instruction include that it allows them to model the doing of mathematics (Pritchard, 2010; Woods & Weber, 2020), that well-delivered lectures can motivate and inspire students (Pritchard, 2010; Rodd, 2003), that lecturing is necessary to cover all of the course material (Fukawa-Connelly et al., 2016; Woods & Weber, 2020; Wu, 1999) and that lecture-based instruction is successful in training future mathematicians (Woods & Weber, 2020).

The implication of this research is that it is now recognised that research on undergraduate mathematics instruction cannot proceed without due consideration of mathematicians' beliefs and goals (e.g., Dawkins & Weber, 2023; Johnson et al., 2025; Stewart, 2023). This is reflected in our proposed theoretical framework which accounts for the influence of orientations and beliefs on mathematicians' PCK. We conclude this review by outlining previous research on PCK in university mathematics education.

4.2.4 Prior research on PCK in university mathematics

Very limited research has been conducted on PCK in undergraduate mathematics with Andrews et al.'s (2022) review identifying only five relevant papers (Delgado-Rebolledo & Zakaryan, 2020; Johnson & Larsen, 2012; Speer et al., 2015; Speer & Wagner, 2009; Wagner et al., 2007). Speer et al. (2015) presented a theoretical paper in which they proposed that components of Ball et al.'s (2008) MKT model need to be considered differently in elementary, secondary, and university settings. Delgado-Rebolledo and Zakaryan (2020) utilised the mathematics teacher's specialised knowledge model (Carrillo-Yañez et al., 2018), an evolution of the MKT framework, to illustrate the connection between one lecturer's mathematics knowledge and PCK.

The remaining three papers all focus on the PCK required to enact forms of student-centred instruction. Wagner et al. (2007) presented a case study of a mathematician, Rossa, using an inquiry-oriented curriculum for teaching differential equations for the first time. The authors presented four challenges that Rossa faced when enacting the curriculum and proposed that these challenges can be viewed in terms of a lack of PCK in this context. For instance, Rossa struggled when coordinating classroom discussions as he lacked the ability to anticipate and evaluate the ideas presented by students. Speer and Wagner (2009) built on this work, presenting another case study of the PCK needed to enact the same inquiry-oriented differential equations curriculum. Their focus was on analysing the knowledge required to provide analytic scaffolding during whole-class discussions. The study drew on Shulman's (1986) conceptualisation of PCK as well as Ball et al.'s (2008) MKT framework. The authors argued that despite having strong content knowledge, the success of their participant's efforts to provide analytic scaffolding was constrained by his lack of knowledge of the curriculum and the typical ways in which students think about particular tasks or content. Johnson and Larsen (2012) elaborated on the work of Speer and Wagner (2009), presenting a case study of a mathematician enacting an inquiry-oriented abstract algebra curriculum for the first time. The authors utilised the MKT framework to analyse the knowledge needed to listen to and comprehend students describing their mathematical struggles. Three cases are presented

to illustrate how the participant was primarily constrained in understanding her students' thinking by her lack of KCS (Ball et al., 2008).

We note that most of the reviewed literature on PCK in student-centred instruction has utilised Ball et al.'s (2008) MKT model. We propose that as the MKT framework was developed in the interactive elementary school context, its components may map on well to student-centred instruction where anticipating, interpreting, and responding to student thinking in class are important aspects of teaching (Johnson & Larsen, 2012; Speer & Wagner, 2009). While lecture-based instruction can incorporate student-focused activity (Johnson et al., 2018), there is generally less in-class interaction with students. We suggest that the MKT framework, with its emphasis on in-class interaction with students, may fail to capture the range of PCK associated with enacting lecture-based instruction.

4.3 Study context

In this paper, we present a PCK theoretical framework adapted for use in undergraduate mathematics education, drawing on the work of Carlson et al. (2019) and Magnusson et al. (1999). The impetus for developing this theoretical framework was preliminary analysis, using Ball et al.'s (2008) MKT framework, of a series of three, one-on-one interviews with a mathematician about his teaching of an upper-level, lecture-based course. We use the same data as an instrumental case study (Stake, 1995) to illustrate the application of the proposed theoretical framework. We take this opportunity to introduce the study participant, instructional context, and interview design.

4.3.1 Participant and instructional context

Dr Doyle², the participant in this study, is a tenured mathematician employed at a large, PhD-granting university in Ireland. Dr Doyle completed his PhD in analysis and has not completed any formal pedagogical training. His educational background is typical of those employed as mathematicians in research-intensive universities in Ireland. We employed purposive sampling (Braun & Clarke, 2013), inviting Dr Doyle to participate on the basis of having 20 years' teaching experience and being well-regarded in the department for his teaching excellence.

Dr Doyle was interviewed about an upper-level, proof-based course on measure theory that he teaches to a cohort of around 40 third- and fourth-year students. Topics including the Lebesgue measure, sigma algebras, measurable functions, and integrable functions are

²To protect the identity of our participant we refer to them using the pseudonym Dr Doyle.

covered. The course is taught in a lecture-based format, which is typical of undergraduate mathematics instruction (Johnson et al., 2018). Students receive typed notes in advance of lectures with gaps for select proofs and examples which Dr Doyle covers with students in class (a format similar to guided notes (Iannone & Miller, 2019)). There are four homework assignments in the course, each worth 5% of a student's final grade. Students submit solutions to their choice of two questions from a set of around 10 problems, working individually or as a group. After Dr Doyle has finished grading a particular homework assignment, he organises a tutorial class (similar in nature to a recitation session) in which the problems are discussed. Partial solutions, containing hints and tips for solving the problems, are then provided to students. A two-hour, closed-book examination accounts for the remaining 80% of a student's grade.

At the time of interview, Dr Doyle had taught this course five times which allowed us to gain insight into the full "teaching cycle" (Andrews et al., 2022, p. 8) of planning, enactment, and reflection. We focused on an upper-level, proof-based course as these courses typically have relatively small enrolments and a single instructor which eliminates potential issues associated with coordination across instructors (Johnson et al., 2018). Additionally, in upper-level courses instructors are typically less constrained in terms of their content selection (Lawrie et al., 2019).

4.3.2 Data generation

Ethical approval for this study was provided by the Human Research Ethics Committee of University College Dublin. In the summer of 2023, Dr Doyle participated in a series of three, one-on-one, semi-structured interviews with the first author about his teaching of the measure theory course. The interviews lasted an average of an hour and were audio-recorded and transcribed. The interviews were designed to examine the characteristics of Dr Doyle's PCK in the context of his own teaching practice (Andrews et al., 2022). Carlson et al. (2019) note that PCK consists not only of the knowledge utilised by teachers when interacting directly with their students but also to the knowledge that they draw on when "planning instruction and reflecting on instruction" (p. 84). Consequently, they explain that PCK is evident in a teacher's choice of instructional strategies and in their rationale for pedagogical decisions.

Our study design reflects Carlson et al.'s (2019) conceptualisation of PCK. In each interview, Dr Doyle was asked to explain his rationale for instructional decisions made in the context of his own teaching practice. We asked Dr Doyle to select a portion of content consisting of at least one of each of the following: definition, example, theorem, and proof. The section of content that he chose was the completion of measure spaces. We requested

that Dr Doyle provide various teaching artefacts from his course relevant to the completion of measure spaces including notes, homework problems, and past examination questions. These were used in the design of the interview questions. The focus of the first interview was on Dr Doyle's design of his course notes and lectures. The second interview examined Dr Doyle's instructional design of the homework assignments in his course, while the final interview focused on assessment.

4.4 Development of the theoretical framework

The initial goal of this study was to examine the characteristics of the PCK of a mathematician teaching a lecture-based course, given the lack of research on PCK in this context (Melhuish et al., 2022). Due to the prevalence of its use in mathematics education research (Depaepe et al., 2013), we initially considered using Ball et al.'s (2008) MKT model to conceptualise PCK. However, having used the framework to conduct some preliminary analysis of Dr Doyle's interviews, we found that there were aspects of our data that were not being captured by the model. For example, affective issues such as the role of Dr Doyle's beliefs were not being addressed which is a common critique of the MKT model (Depaepe et al., 2013; Johns & Burks, 2023). Aware of research that has highlighted the important role of teacher beliefs and orientations in undergraduate mathematics instruction (e.g., Melhuish et al., 2022; Pinto, 2019; Rupnow, 2023; Speer, 2008; Weber, 2004; Woods & Weber, 2020), we felt that this was a major limitation. Additionally, while KCC is included as a component of the MKT framework, most studies employing the model focus on KCS and KCT (Depaepe et al., 2013). This leaves important aspects of PCK in the university context unexamined. For example, assessment has been identified as a nuanced and complex issue in both the general higher education literature (e.g., Harland et al., 2015), as well as specifically in relation to mathematics (Iannone & Simpson, 2015), but is not addressed by the MKT framework.

Aware that PCK theoretical frameworks have traditionally been siloed by discipline (Andrews et al., 2022), we decided to examine how PCK has been conceptualised in the science education literature given that significant theoretical development of PCK has occurred in this context (Chan & Hume, 2019; Fukaya et al., 2025). Having reviewed this literature (e.g., Carlson et al., 2019; Chan & Hume, 2019; Gess-Newsome, 2015; Magnusson et al., 1999), we decided to utilise Magnusson et al.'s (1999) framework which is the most widely used PCK model in K-12 science education (Chan & Hume, 2019). Adopting this framework introduced new components of PCK (KA and OTT) into our conceptualisation of PCK which were very relevant to our data and the undergraduate context. Additionally, the framework's

inclusion of PCK subcomponents allowed for a more detailed and nuanced analysis of each component.

However, having conducted some preliminary analysis with Magnusson et al.'s (1999) model, we identified additional limitations of our conceptualisation of PCK. The first related to the need for clarification on the role of beliefs and orientations. Magnusson et al. included orientations as an overarching component of PCK which influences the other four knowledge components. However, we were also aware that contemporary knowledge theory typically separates knowledge and beliefs (Andrews et al., 2022). The second limitation related to the need for clarification on the distinction between PCK that is developed through personal reflection on teaching practice, and that which constitutes research-informed pedagogical practice. While the former is likely of greater relevance to the undergraduate context where instructors typically receive little to no pedagogical training (Andrews et al., 2022; Sarkar et al., 2024; Winsløw et al., 2021) and thus limited exposure to educational research, Magnusson et al.'s conceptualisation of PCK offered no means of addressing this distinction.

To address these limitations, we reviewed the work of science education researchers (Berry et al., 2015; Hume et al., 2019) who have continued to focus on the theoretical development of PCK since the publication of Magnusson et al.'s (1999) framework. Based on this literature, we decided to combine Carlson et al.'s (2019) refined consensus model with Magnusson et al.'s (1999) four PCK components (KA, KC, KISR, and KSU) which we use to specify the components of enacted and personal PCK. Adopting the refined consensus model (Carlson et al., 2019) allows us to distinguish between collective PCK, which encompasses research-informed pedagogical practices, and personal and enacted PCK, which consist of individual knowledge derived through teaching experience. Given that mathematicians typically receive little to no pedagogical training (Andrews et al., 2022; Sarkar et al., 2024; Winsløw et al., 2021), they likely receive less exposure to collective PCK than K-12 teachers.

We build upon Carlson et al.'s (2019) refined consensus model by connecting it to Magnusson et al.'s (1999) PCK components: KSU, KISR, KA, and KC. These categories allow us to specify the characteristics of enacted and personal PCK. Adopting contemporary knowledge theory, we diverge from Magnusson et al.'s initial conceptualisation of PCK by not including OTT as a component of PCK (Andrews et al., 2022; Carlson et al., 2019; Gess-Newsome, 2015). Instead, we consider beliefs and orientations as acting as a lens or amplifier, influencing the personal PCK an instructor constructs from either interacting with collective PCK (e.g. through pedagogical training) or reflecting on experience (Carlson et al., 2019; Gess-Newsome, 2015).

4.5 Operationalisation of the framework

We now explain how the proposed PCK theoretical framework can be operationalised and applied to interview data, using Dr Doyle as an instrumental case study (Stake, 1995). We utilise template analysis (Brooks et al., 2015; King, 2012), a form of thematic analysis, as the method allows for deductive coding using pre-existing theory without limiting the potential for inductive coding. Template analysis emphasises hierarchical coding and central to the method is the development of a coding template which may incorporate existing theory in the form of *a priori* codes (Brooks et al., 2015).

To code for the characteristics of enacted PCK, we included four components from Magnusson et al.'s (1999) framework (KISR, KSU, KA, KC) as well as their associated subcomponents. As previously discussed, we conceptualise beliefs and orientations as acting as a filter or lens which influences the application and construction of PCK. We use Magnusson et al.'s term 'orientations towards teaching' to operationalise this influence of beliefs in our coding. We also include four subcodes to capture how beliefs and orientations affect the application and construction of the four components of enacted PCK. Personal PCK and collective PCK were also included as *a priori* codes in our coding template. The same subcodes as listed under enacted PCK apply for personal PCK. In line with Carlson et al.'s (2019) model, we included discipline, topic, and concept as subcodes under collective PCK to allow for the grain size to be specified. A list of codes included in the coding template is shown in Table 4.5.1.

In line with Carlson et al.'s (2019) conceptualisation of PCK, we applied the coding template to our data by coding instances where Dr Doyle explicitly referred to his knowledge and also inferred its existence from his actions and rationale provided for pedagogical decisions. To provide further insight into our coding process, we have included the complete final coding template in Appendix 4.A. For each code included in Table 4.5.1, we provide a definition that explains how the code was operationalised in coding the data, as well as an example of how the code applies to the data and an associated quote. In most cases, we drew strongly on Magnusson et al.'s (1999) definition of the various subcomponents of PCK and made relatively minor adjustments to account for the mathematical context of our study. Where our operationalisation of a code differs more significantly from Magnusson et al.'s original definition this is indicated in the coding template. This mainly applies to the subcomponents of KC where the influence of differences between the K-12 and undergraduate contexts was most evident.

Table 4.5.1: A list of the codes included in the final coding template.

Enacted PCK
KISR: Knowledge of instructional strategies and representations
Subject-specific KISR
Topic-specific KISR
<i>Representations</i>
<i>Activities</i>
KSU: Knowledge of student understanding
Knowledge of requirements for learning
Knowledge of areas of student difficulty
KA: Knowledge of assessment
Knowledge of dimensions to assess
Knowledge of methods of assessment
KC: Knowledge of curriculum
Knowledge of goals and objectives
Knowledge of specific curricular programme
Orientations towards teaching
OTT: Orientations towards teaching
OTT-KISR
OTT-KSU
OTT-KA
OTT-KC
Personal PCK
<i>Same subcodes as listed under enacted PCK.</i>
Collective PCK
Discipline
Topic
Concept

4.6 Application of the theoretical framework

In this section, we apply the framework to interview data, using Dr Doyle as an instrumental case study (Stake, 1995) to demonstrate the insights the framework can provide into the characteristics of the PCK of a mathematician teaching an upper-level, lecture-based mathematics course. We structure this section by first discussing collective PCK, followed by personal and enacted PCK.

4.6.1 Collective PCK

In our analysis, we found little evidence of Dr Doyle having engaged with collective PCK in the form of research-informed canonical knowledge. Indeed, Dr Doyle remarked:

“I’d have to say professional development courses are low down on my list [of influences on his teaching]. Maths ed research literature, I’d have to hold my hands up and say I’m not very well informed”. Consequently, Dr Doyle reflected that his instructional decisions were not typically informed by education research or theory. Instead, they appeared to be largely influenced by his beliefs, or in some cases were driven by practical necessity: “I do wonder when you’re doing these interviews whether you might overanalyse whether there was a great pedagogical theory behind this and, initially, no there wasn’t. It was just pure practicality”.

While research-informed pedagogical knowledge does not appear to have been a primary influence on Dr Doyle’s teaching, we did see some evidence of locally developed collective PCK shared between Dr Doyle and his colleagues. For instance, he and his colleagues would each discuss the content they were covering in their courses contributing to a shared knowledge of curriculum in the department. Dr Doyle also appeared to have been influenced by collective knowledge of how to structure a course on measure theory which he was exposed to through textbooks: “most measure theory textbooks kind of follow the same lines, start with the real line and abstract from there, you know, doing the abstract stuff and then focusing on examples”.

4.6.2 Personal and enacted PCK

In contrast to collective PCK, personal and enacted PCK are individual and largely developed through teaching practice. Given that the study was designed to examine Dr Doyle’s PCK in the context of teaching a specific course, the majority of the PCK we identified in the interviews was enacted. Dr Doyle did compare his approach to assessment in the measure theory course to that which he adopts in a first-year course he also teaches: “it’s obviously more involved than maybe the quiz sheets you’d give to the first-year class, but the concept is essentially the same”. We take this as an indication that Dr Doyle’s PCK extends beyond that which he uses in the context of the measure theory course. However, given the nature of the study design, we cannot comment further on the nature of his personal PCK. We now examine the characteristics of Dr Doyle’s enacted PCK under Magnusson et al.’s (1999) four components. The influence of his orientations and beliefs on the construction and application of the various PCK components is also discussed.

4.6.2.1 Knowledge of curriculum (KC)

In the K-12 context, KC refers to a teacher’s understanding of externally created curricula. National or state-level curricular policies are examples of sources of this form of knowledge (Magnusson et al., 1999). A key contextual difference between teaching at the K-12 and

university level is that university instructors have greater autonomy in determining course content, particularly in upper-level courses (Lawrie et al., 2019). In the context of Dr Doyle's teaching, there was no externally mandated syllabus for the measure theory course. Instead, through an iterative process of reflection Dr Doyle made decisions regarding which content was most important for his students to be exposed to. We conceptualise the knowledge that Dr Doyle drew on in this curricular design process as his KC.

While mathematicians typically have well-developed content knowledge (Winsløw et al., 2021), Dr Doyle explained that this does not necessarily translate into an understanding of which content should be included in a course curriculum:

Dr Doyle: The first time you teach something, you don't realise what's important. It takes quite a lot of consideration and reflection on what you're doing and also familiarity with what comes later on to realise, okay, that's not so important, I don't need to prove that, we won't be using that again, or just there's nothing crucial in that.

As Dr Doyle described, knowing which content to include in his measure theory course evolved through a process of reflection over multiple course iterations, reflecting the plan, teach, reflect cycle of the refined consensus model (Carlson et al., 2019). To exemplify this refinement process, we examine Dr Doyle's decision-making regarding which proofs to cover in class with his students. Dr Doyle noted that when he initially planned the course curriculum, his starting point was the notes he had from his time learning measure theory as a student: "I suppose the first thing was to look back at notes I had from my own learning. They were very good, very detailed". However, reflecting on his experience of learning measure theory, Dr Doyle felt that there were an excessive number of proofs presented which hindered his enjoyment of the subject:

Dr Doyle: One of the things I remember when I learnt measure theory was that . . . everything was proved. Right, so the open sets are Lebesgue measurable, and the half open sets are Lebesgue measurable and there were all these proofs and they all seemed exactly the same . . . there were lots of repeated arguments, lots of kind of tricks that were used over and over again, and the extra proofs didn't illuminate anything . . . You're not getting a feel for the subject if you just do too much detail.

This experience contributed to Dr Doyle's belief that proving every statement in class with students can be pedagogically unproductive³. Consequently, when discussing his rationale

³Typically, Dr Doyle includes the proofs that he doesn't cover in class in a set of typed notes that gets released to students. That way if students are interested in the details of any proof it is accessible to them.

for choosing to cover a particular proof, he demonstrates his understanding of the purpose the proof serves in his curriculum and the benefits of demonstrating it to students:

Dr Doyle: I think the proof [of the completion of measure spaces] illuminates a lot of points that have been introduced up to then and it would be remiss not to go over the proof. Some proofs you can leave out, they don't really add anything. But I think that this is one that is important for students to see and be aware of. It's really, the statement is fundamental, the proof is very accessible and I think . . . when you kind of give students the fact that here's a fundamental thing and here's a proof that you can understand, there's an accomplishment to that.

Dr Doyle's knowledge of which content to include in his measure theory course has also been informed by his perceptions of measure theory as a subdiscipline of mathematics and the purpose of the course within students' overall programmes of study. Dr Doyle described measure theory as foundational to other areas of analysis:

Dr Doyle: Measure theory is like a set of rules. People don't do research in measure theory or not really. It's not like group theory or functional analysis where you're investigating new spaces. It's like the bricks and mortar of analysis.

Thus, he is aware that the primary goal of the measure theory course must be preparing students for further courses in analysis:

Dr Doyle: Measure theory is not the most interesting subject. It's real kind of nuts and bolts. So, you know, I'm happy if students come out and they've learnt the concepts and they're well-prepared to go into [courses like] probability theory afterwards.

While the knowledge that the measure theory course plays a primarily preparatory role in students' programmes informs much of Dr Doyle's content selection, he also expresses an orientation towards making the course as interesting as possible for students. This is reflected in his decision to cover the Banach-Tarski paradox despite it not typically being included in undergraduate courses on measure theory:

Dr Doyle: There was other stuff that I thought was missing from most measure theory courses. The Banach-Tarski paradox is one. I don't know if you know it but it's like this really cool fact and it's generally not covered in an undergraduate course because there isn't time to cover it and . . . because students haven't done free groups and the geometry also . . . but I wanted to include it. It was the most interesting thing you could do in a course on measure theory.

Thus, Dr Doyle demonstrates his KC in his understanding of which content to include in the measure theory course and his overarching goals for the course. This has been influenced

by several factors including his reflections on his experience as a student, his perceptions of measure theory, and his orientation towards making the course as interesting as possible for students.

4.6.2.2 Knowledge of instructional strategies and representations (KISR)

As discussed above, the primary way Dr Doyle's KC is enacted in the context of the measure theory course is through the design of the course curriculum. In addition to deciding on the content to be included in the measure theory course, Dr Doyle must also consider how this content is presented to students. This is where his KISR becomes relevant. While in some instructional contexts it may be standard practice to adopt a course textbook, in the measure theory course Dr Doyle prepares his own notes for students (this is common practice for undergraduate mathematics courses taught in University College Dublin and in Ireland more generally). By examining Dr Doyle's decision-making regarding the design of his notes and lectures, we gain insight into the nature of his KISR which is strongly influenced by his beliefs about student learning (Magnusson et al., 1999).

Dr Doyle's KISR varies in terms of scope, with some instructional strategies and representations enacted at the course level while others are specific to particular content. An example of an instructional strategy that Dr Doyle implements at the course level is his use of typed notes with gaps in them which are released to students in advance of lectures (akin to what have been described in the literature as guided notes (e.g., Iannone & Miller, 2019)). Dr Doyle justifies his decision to prepare notes in this style by suggesting that they assist him in optimising the limited time in lectures with students:

Dr Doyle: The amount of time you get to spend with students is fairly small in comparison to the amount of time it takes a student to learn something like measure theory. So, you don't have time to kind of 'hmm let's sit down and think about a problem for a couple of hours'. You don't have that luxury, so you have to kind of pick and choose, so those bits [the gaps] are the bits that I choose to spend time on.

An example of the type of content that Dr Doyle prioritises covering in class with students is proofs. This decision is based on his belief in the value of students seeing theorems proven live in class:

Dr Doyle: If it's printed on the slide, here's line one of the proof, and here's line two of the proof, and here's line three of the proof, and here's the Halmos tombstone at the end, we've finished the proof, we haven't done anything. . . . I feel that it goes better where we're writing it out. The pace is good. The idea that I'm actually trying to figure it out as I'm doing it.

Dr Doyle thinks carefully about how to present proofs in class to best support student learning. Drawing on his own experience as a student of measure theory, he believes that presenting every detail of a proof has the potential to obscure its main ideas:

Dr Doyle: When I took measure theory as a student everything was proven in detail and I actually found that looking back . . . I didn't think it was terribly interesting because you weren't really discovering anything and you didn't get a kind of bird's-eye view of what was happening in the proof.

Consequently, rather than covering complete proofs in his lectures, he presents them as a series of key 'steps'. For example, as shown in Figure 4.6.1, he divides the proof of the completion of measures spaces into five main steps.

Theorem 2.18 *Let $(X, \mathfrak{X}, \mu)$ be a measure space. Then there is a smallest complete measure space $(X, \overline{\mathfrak{X}}, \overline{\mu})$ such that $\mathfrak{X} \subset \overline{\mathfrak{X}}$ and $\mu(E) = \overline{\mu}(E)$ for any $E \in \mathfrak{X}$.*

Proof.

Step 1. Define $\overline{\mathfrak{X}} = \{A \subset X : \exists E, F \in \mathfrak{X} \text{ such that } E \subset A \subset F \text{ and } \mu(E \setminus F) = 0\}$.
Verify that $\overline{\mathfrak{X}}$ is a σ -algebra.

Step 2. Define $\overline{\mu}$ on $\overline{\mathfrak{X}}$ by $\overline{\mu}(A) = \mu(E)$ where A and E are as in the definition of $\overline{\mathfrak{X}}$.
Verify that $\overline{\mu}$ is well-defined.

Step 3. Show that $\overline{\mu}$ is a measure and agrees with μ on $\mathfrak{X}$.

Step 4. Show that $(X, \overline{\mathfrak{X}}, \overline{\mu})$ is complete.

Step 5. $(X, \overline{\mathfrak{X}}, \overline{\mu})$ is the smallest measure space with these properties.

□

Figure 4.6.1: Dr Doyle's presentation of the proof of the completion of measure spaces.

In his lectures, Dr Doyle handwrites partial details of each step of the proof and might prove one or two steps in detail, leaving the others as an exercise for students to complete independently. While Dr Doyle does not believe it would be reasonable to expect a student to be able to independently prove a theorem such as the completion of measure spaces, he feels that presenting proofs in this manner makes students more active participants in the proof process:

Dr Doyle: I like this idea of, okay, here are the steps of the proof: how many of those steps could you do? And you're not coming up with the proof yourself, that would be a very difficult thing to do, but you know, if I say here's [five] points:

‘Do any of them make sense to you? Do you think you could prove those as an exercise?’.

Dr Doyle’s approach to presenting proofs and use of ‘gappy’ notes are examples of his KISR that are enacted across the measure theory course. However, he is also deliberate about how he presents specific content to support student learning. For instance, consider his statement of the definition of a complete measure space shown in Figure 4.6.2.

Definition 2.17 A measure space $(X, \mathfrak{X}, \mu)$ is called *complete* if we have the following: any subset F of a measurable set E of measure 0 is itself measurable. That is, $E \in \mathfrak{X}, \mu(E) = 0, F \subset E$ implies $F \in \mathfrak{X}$.

Figure 4.6.2: Dr Doyle’s statement of the definition of a complete measure space.

When asked about his decision to state the definition in both words and mathematical notation, Dr Doyle remarked: “I think the mathematical notation removes all kinds of ambiguity and it might be the version that you can call on”. However, while stating the definition using mathematical notation may be more in line with disciplinary norms, Dr Doyle also provides the definition in words as he considers it more accessible to students:

Dr Doyle: You could present it purely from the mathematical notation point of view. But maybe that’s more difficult to internalise . . . if you’re trying to internalise what the notion of completeness is, saying subsets of measure zero sets are measurable is a little bit more friendly.

By including both forms of the definition in his notes, Dr Doyle demonstrates his knowledge of how the content in his course can be represented to best support student learning.

4.6.2.3 Knowledge of student understanding (KSU)

Having taught the course five times, Dr Doyle’s KSU in the context of measure theory appears well developed. He remarked that his knowledge of common student errors and misconceptions has expanded over multiple course iterations: “once you’ve taught it three or four times, the errors that students make are repeated from year to year. So I know now what errors students will make on their submission, on their problems”. This knowledge informs Dr Doyle’s selection of homework problems designed to address areas of student difficulty (a practice that was also identified by Rupnow et al. (2021) in an abstract algebra context). For example, Dr Doyle discussed his inclusion of a problem that requires students to work with preimages of sets, anticipating that they may find this confusing:

Dr Doyle: Students are uncomfortable with the notation and the concept of a preimage of a set under a function. They see f^{-1} and they, I think, think of functions like inverse sine function, inverse tan function, those types of inverse functions . . . Preimages they're less comfortable with, so the exercise is there mainly to develop that notion.

Dr Doyle also demonstrates an understanding of the cognitive demands associated with tasks and problems posed in the course. For instance, he recognised that the thought processes required of students to solve problems are not necessarily accurately represented by their final solutions: “so it's only a line to write down but there's thought involved in it. And then the other direction has a lot more to write down, but in some sense, it's probably less abstract”.

While Dr Doyle demonstrates and utilises KSU in the measure theory course, he also acknowledges the limitations of this component of his PCK. He recognises that the lecture-based nature of his course limits opportunities for interaction with students describing his classes as “not really discursive”. The only sort of insight Dr Doyle gains into student understanding in class comes from attempting to interpret students' non-verbal cues: “blank looks, okay we need to go through this in more detail”. Therefore, Dr Doyle is reliant on grading students' homework and examinations to develop his understanding of what aspects of the course students are struggling with. However, he admits that there are limits to the insight you can gain into student thinking from grading their work:

Dr Doyle: A student will have ‘this set is measurable because it's a countable collection’ and I say ‘no, it's not a countable collection, it's a countable union of intervals’ . . . Are you spotting the difference between the union of sets and a collection of sets? And so that might be a slip of the pen, or it could be the student is just operating on the wrong level . . . [its] very hard to spot on paper if a student is getting these different levels.

Therefore, while Dr Doyle demonstrates an understanding of common student errors and the cognitive demands of tasks in the measure theory course, his insight into student thinking appears limited by the lecture-based nature of the course.

4.6.2.4 Knowledge of assessment (KA)

In Ireland and the UK, university mathematics courses are predominantly assessed through closed-book examinations (Iannone & Simpson, 2011, 2022; National Forum for the Enhancement of Teaching and Learning in Higher Education, 2016). While closed-book examinations represent the ‘standard’ form of assessment in university mathematics, instructors do have autonomy in terms of writing their own examination questions. The

measure theory course adheres to this assessment norm with a final examination accounting for 80% of students' grades. Dr Doyle demonstrates his KA by determining what content to assess students on and how to structure his examination questions.

Measure theory is a proof-based mathematics course and therefore Dr Doyle considers it important to assess students' knowledge of the proofs in the course. The standard way of doing this might be to ask students to state and prove various theorems in the examination. However, during the COVID-19 pandemic, examinations were open book and these types of questions were no longer viable: "the exams were open book, there's no point asking a student to prove something and yet the coursework still involved going through a lot of proofs. So how to check that a student had looked at these proofs?". As a result, Dr Doyle was prompted to expand his knowledge of methods of assessing students' understanding of proofs. A method that he trialled was to provide students with an outline of a proof and ask them to justify why certain steps are valid. For example, an examination question that Dr Doyle has previously asked students on the proof of the completion of measure spaces is shown in Figure 4.6.3.

Let $(X, \mathfrak{X}, \mu)$ be any measure space. The following argument outlines the construction of its *completion* $(X, \overline{\mathfrak{X}}, \overline{\mu})$.

- (1) Let $\overline{\mathfrak{X}} = \{A \subset X : \text{there exist } E, F \in \mathfrak{X} \text{ with } E \subset A \subset F \text{ and } \mu(F \setminus E) = 0\}$.
 - (2) Define $\overline{\mu} : \overline{\mathfrak{X}} \rightarrow [0, \infty]$ by $\overline{\mu}(A) = \mu(E)$ where E is as given in (1).
- (i) Show that if $A \in \overline{\mathfrak{X}}$ then $A' \in \overline{\mathfrak{X}}$.
 - (ii) Let $A \in \overline{\mathfrak{X}}$. Given that the set E guaranteed in (1) may not be uniquely determined, explain why, in (2), $\overline{\mu}$ is well-defined.
 - (iii) Show that $(X, \overline{\mathfrak{X}}, \overline{\mu})$ is a complete measure space.

Figure 4.6.3: A previous examination question on the proof of the completion of measure spaces.

Reflecting on the success of this approach, Dr Doyle explained that he believed it encouraged students to focus on understanding, rather than memorising, proofs when revising. Consequently, he has continued to use this method of assessing proofs, even though examinations have since returned to being closed book:

Dr Doyle: I was surprised the first time to notice that students still have difficulty ... it's actually much more telling now. So, you know, explain this line in the proof, why does it follow from the previous lines, it's much more telling. So, students aren't answering those things by just having learnt them off. They

actually, in their revision, have to understand them, how do you get from one line to the next.

We also note the connection between this method of assessment and Dr Doyle's approach to presenting proofs as a series of steps which we discussed in relation to his KISR. When presenting a proof in class, Dr Doyle will cover some steps in detail leaving the rest as exercises for students to complete after class. Dr Doyle explained that he considers the steps he leaves as exercises to be "viable" examination questions and noted that "if they've done the exercise after class time, they get their reward here on the exam paper". This reflects Dr Doyle's belief that students should not be able to complete an examination based solely on memorisation. Students are introduced to the content in class with the expectation that they will complete additional study outside the classroom: "the idea is that students will have got pointers, but they will have to have done work on their own".

4.7 Discussion

Existing research on PCK in undergraduate mathematics has typically utilised theoretical frameworks developed in the K-12 context (Andrews et al., 2022). While researchers have cautioned against automatically extrapolating theoretical constructs from the K-12 to university context (Speer et al., 2015), there has been little focus on developing and adapting theoretical frameworks for use in university mathematics. In this paper, we adapt and combine Carlson et al.'s (2019) and Magnusson et al.'s (1999) frameworks, developed in the K-12 science education context, for use in undergraduate mathematics. Our justifications for proposing the presented framework, grounded in literature on undergraduate mathematics instruction, are threefold. Firstly, the proposed framework accounts for the role of teacher beliefs and orientations, which have been shown to be important factors of influence on mathematicians' pedagogical decision-making (e.g., Melhuish et al., 2022; Pinto, 2019; Rupnow, 2023; Weber, 2004; Woods & Weber, 2020). Secondly, existing research on PCK in university mathematics has largely focused on the knowledge required when interpreting and responding to student thinking in class (e.g., Johnson & Larsen, 2012; Speer & Wagner, 2009; Wagner et al., 2007). The proposed framework includes PCK components that are relevant to the work of instructors beyond the classroom. Finally, the proposed framework distinguishes between enacted/personal PCK which is developed by individuals through practice and collective PCK which is public and includes research-informed canonical knowledge. We contend that this distinction is important in the undergraduate context, where instructors typically receive

little pedagogical training or exposure to research-informed practices (Andrews et al., 2022; Sarkar et al., 2024; Winsløw et al., 2021).

For the reasons outlined, we believe that the proposed framework advances the conceptualisation of PCK in university mathematics. However, by no means do we wish to suggest that it represents a “perfect model” (Carlson et al., 2019, p. 91). Indeed, a major limitation of this paper is that we have illustrated the application of the framework to one specific case of a mathematician teaching an upper-level, lecture-based course in Ireland. We are conscious that PCK is highly contextual and factors such as the level of the course, institutional context, teacher experience, and national assessment norms are likely influential. For this reason, we call for future research to apply the proposed theoretical framework in other instructional contexts. We have demonstrated one means of operationalising the framework and applying it to interview data using template analysis (Brooks et al., 2015; King, 2012), with other methods also possible. It is only through testing the framework with larger and more varied samples that we can gain insight into its explanatory power and potential limitations.

While the proposed framework may have relevance across a range of undergraduate mathematics instructional contexts, its development was prompted by encountering limitations of applying the MKT framework (Ball et al., 2008) in the lecture-based context. Therefore, we propose that it may be particularly well-suited to examining PCK in this setting. Existing research on PCK in student-centred instruction has largely employed the MKT framework (e.g., Johnson & Larsen, 2012; Speer & Wagner, 2009), suggesting that researchers have found the theory useful in this context. We propose that as the MKT framework was developed in the interactive, student-focused elementary school context, its components may map onto student-centred instruction more naturally than lecture-based courses. For instance, research on PCK in student-centred undergraduate mathematics instruction emphasises the importance of KCS (Ball et al., 2008), particularly the ability to anticipate, interpret, and respond to student thinking in class (Johnson & Larsen, 2012; Speer & Wagner, 2009; Wagner et al., 2007). Lecture-based instruction can incorporate elements of student-focused activity (Johnson et al., 2018), but as Dr Doyle’s case illustrates, there is typically less in-class interaction with students. This means that KCS may be a less central form of PCK in this setting. However, Dr Doyle exemplifies other forms of PCK that lecturers may utilise outside of class time such as when planning the course curriculum, thinking about how to optimise limited class time, or designing appropriate methods of assessment. By including additional PCK components that capture these dimensions, the proposed framework provides a broader depiction of PCK that we suggest is perhaps more suited to the lecture-based environment.

As well as including additional PCK components, the proposed framework, drawing on Carlson et al. (2019), distinguishes between three realms of PCK: enacted, personal, and

collective. We propose that this distinction offers a productive reframing of discussions about the relationship between research and mathematicians' teaching practices. For example, research on PCK in student-centred instruction has tended to adopt a deficit perspective, highlighting the gaps in mathematicians' PCK that prevent them from implementing research-informed curricula effectively (Johnson & Larsen, 2012; Speer & Wagner, 2009; Wagner et al., 2007). Drawing on Carlson et al.'s three realms of PCK, we argue that these studies have documented the development of collective PCK on how these curricula should be enacted and that this collective PCK has not yet become part of mathematicians' personal PCK. Reframing the discussion in this manner shifts the focus from knowledge deficit on the part of mathematicians towards a consideration of ways in which the translation between collective and personal PCK can be facilitated.

Mathematics education researchers have attempted to encourage mathematicians to engage with mathematics education research through various dissemination efforts. However, the impact of this traditional model of professional development has been modest as indicated by the low uptake of research-informed, student-centred curricula (Melhuish et al., 2022). Indeed, Dr Doyle illustrates this point by acknowledging his limited engagement with the mathematics education research literature and professional development opportunities. Despite his lack of engagement with this body of knowledge, Dr Doyle discusses using some pedagogical practices that have been studied in the research literature. For instance, he uses a style of notes similar to guided notes (Iannone & Miller, 2019). Additionally, his approach to asking students to explain various steps in proofs in examinations is reminiscent of self-explanation training (Hodds et al., 2014). Interestingly, Dr Doyle was unaware of this research and adopted the mentioned pedagogical practices as a result of iterative cycles of planning, teaching, and reflecting on the outcomes of instructional efforts (Carlson et al., 2019). This process was heavily influenced by his beliefs about mathematics and learning, underpinning research that has highlighted the important role they play in pedagogical decision-making (e.g., Melhuish et al., 2022; Pinto, 2019; Rupnow, 2023; Weber, 2004; Woods & Weber, 2020).

Given the limited impact of the traditional model of professional development (Melhuish et al., 2022), we wonder whether it may be productive to consider alternative methods. We suggest one potential approach. Rather than beginning with developing collective PCK and trying to persuade mathematicians of its relevance, mathematics education researchers might instead begin by working collaboratively with mathematicians to identify and articulate their enacted and personal PCK. In doing so, researchers may find that mathematicians, like Dr Doyle, are already drawing on research-evidenced practices either knowingly or unknowingly. We propose that identifying such alignments could be a more productive basis

for collaboration, building on mathematicians' current teaching knowledge. It may also offer a more accessible entry way into education research for mathematicians, meeting them where they are rather than asking them to adopt a body of knowledge that sits outside their disciplinary training.

Our study presents an example of how such work could be pursued. Grounding our interview questions in Dr Doyle's current teaching practice and focusing on a specific piece of content proved effective in identifying the characteristics of his enacted PCK. We suggest that the proposed theoretical framework could be a useful tool for illustrating to mathematicians the connection between their current enacted PCK and relevant collective PCK that may be available in the research literature. Reflecting on over a decade of collaborative research with mathematicians, Stewart (2023) highlighted that the mathematicians she worked with often welcomed, and gravitated towards, theories. This suggests that mathematicians may not be unwilling to engage with research but that the challenge may lie in how that engagement is initiated and framed. Our suggestion of starting with mathematicians' current enacted and personal PCK and illustrating the potential connections to the research literature is one way towards the kind of sustainable and genuine collaboration that has been called for in the research literature (e.g., Dawkins & Weber, 2023; Johnson et al., 2025; Stewart, 2023).

4.8 Conclusion

In this paper, we presented a PCK theoretical framework for use in undergraduate mathematics, developed by adapting and combining Carlson et al.'s (2019) and Magnusson et al.'s (1999) frameworks, developed in K-12 science education. By responding to Andrews et al.'s (2022) call for researchers to reach across disciplinary boundaries, the proposed framework has three characteristics that make it better suited to analysing PCK in undergraduate mathematics when compared to the frequently used MKT model (Ball et al., 2008). These include accounting for the influence of orientations and beliefs, including additional knowledge components, and distinguishing between collective, research-informed PCK and that which is developed individually through experience. We also presented a method of using template analysis (Brooks et al., 2015; King, 2012) to operationalise the framework and demonstrated the kinds of insights it can provide into the PCK of a mathematician teaching an upper-level, lecture-based course in Ireland. We call for researchers to build on our work by applying the framework in a range of instructional contexts to develop further insight into its explanatory power and limitations. We posit that the framework may be particularly well-suited to analysing PCK in lecture-based contexts due to its broader conceptualisation which captures instructors' thinking beyond the classroom. Finally, we believe that the pro-

posed framework, particularly the distinction between enacted and collective PCK may offer a means of productively reframing how mathematicians are encouraged to engage with the research literature.

We note that one avenue for future work would be to further investigate the influence of subject matter on the theoretical conceptualisation of PCK. We were interested in studying PCK in the context of upper-level, lecture-based mathematics courses and consulted the science education PCK literature to assist with our theoretical conceptualisation. Both mathematics and science are considered ‘hard’ subjects under Biglan’s (1973) classification. However, in this paper we did not explore whether this similarity is important in terms of conceptualising PCK. We also note that Magnusson et al.’s (1999) framework, which was developed in a science education context, has previously been used to characterise PCK in the context of teaching advanced social science research methods (Nind, 2020). This suggests some level of applicability of PCK theoretical frameworks across the hard-soft classification boundary (Biglan, 1973). PCK has remained an active research topic among science educators since its conceptualisation as evidenced by the organisation of two PCK summits (Berry et al., 2015; Hume et al., 2019). We wonder whether the significant work on the theoretical development of PCK that has occurred in science education is perhaps a more important factor than the similarity of mathematics and science when considering the development of theoretical frameworks for use in undergraduate mathematics education. We call for subsequent research to more explicitly examine the influence of subject matter on the theoretical conceptualisation of PCK.

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Appendix to Chapter 4

4.A Coding template

Table 4.A.1: Coding template

Code	Operationalisation	Example of how code applies to the data	Illustrative quote
Enacted PCK	Enacted PCK refers to the PCK that is utilised in a particular context. Building on Carlson et al.’s (2019) refined consensus model of PCK, we are introducing four PCK components from Magnusson et al.’s (1999) framework to add specificity to the definition of enacted PCK. Below we define how we are operationalising each of the four components along with their subcomponents. Where our operationalisation differs significantly from Magnusson et al.’s definition we clarify the nature of these differences.		

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Code	Operationalisation	Example of how code applies to the data	Illustrative quote
KISR	Knowledge of instructional strategies and representations. This component consists of two categories which differ in terms of scope: <i>subject-specific KISR</i> relates to overall approaches to enacting strategies for mathematics instruction, whereas <i>topic-specific KISR</i> relates to instructional strategies that are specific to particular content (e.g. the completion of measure spaces).		

Continued on next page

Code	Operationalisation	Example of how code applies to the data	Illustrative quote
Subject-specific KISR	Knowledge of instructional strategies and representations related to mathematics as a discipline. Magnusson et al. (1999) note that a teacher's subject-specific KISR is closely related to their beliefs about and orientations towards teaching and learning mathematics. We operationalise this code as knowledge that is drawn on when enacting instructional strategies at the course level (rather than those that relate to specific topics). We acknowledge the influence of beliefs on this PCK component and, therefore, overlap with the code OTT-KISR is expected.	Throughout his course, Dr Doyle presents proofs as a series of main "steps". In lectures, he presents these "steps" to students and might prove one or two steps in detail, leaving the remaining steps as exercises for students to complete after class. The decision to present proofs in this manner is based on his belief that presenting all the details of a proof obscures the main idea of the proof for students (OTT-KISR).	"I like this idea of, okay, here are the steps of the proof: 'how many of those could you do?' And you're not coming up with the proof yourself, that would be a very difficult thing to do, but you know, if I say 'here's four points: do any of them maybe sense to you?' Do you think you could prove those as an exercise?"

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Code	Operationalisation	Example of how code applies to the data	Illustrative quote
Topic-specific KISR	Knowledge of instructional strategies and representations that are useful for helping students to comprehend specific content and topics. There are two categories of this form of knowledge: <i>representations</i> and <i>activities</i> . Topic-specific KISR is coded under either one of these two subcodes depending on whether it relates to a way of representing specific topics or activities that can help students learn a specific topic.		
<i>Representations</i>	We operationalise this code as knowledge of how to represent topics to students to assist their learning. Examples of this form of PCK could include deciding on how best to state definitions/theorems, knowledge of the most useful examples and non-examples of concepts, the use of analogies/informal explanations, making comparisons to other topics to aid understanding, the choice of a particular proof approach etc.	When Dr Doyle was proving the theorem of the completion of a measure space, he could have chosen between either a ‘non-constructive’ or ‘constructive’ approach to the proof. He decided to present the constructive proof to students as it gives them better insight into how the completion is constructed.	“I think it is natural actually to do the constructive approach here because it shows you really how you get the sets which are in the completion, and it kind of illuminates a little bit what are the missing ones? Where do the missing ones come from?”

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Code	Operationalisation	Example of how code applies to the data	Illustrative quote
<i>Activities</i>	We operationalise this code as knowledge of suitable activities (e.g. problems) that can help students learn specific concepts.	Dr Doyle assigns a homework problem that requires students to find the sigma algebra generated by a collection of subsets of a finite set. He explains that while most of the time going forward students will be dealing with sigma algebras on $\mathbb{R}$, this exercise allows them to gain familiarity with, and utilise, the definition of sigma algebras.	“The sigma algebras that you generally are going to be using as you move forward, things like Lebesgue measurable sets or Borel sets, they’re very difficult to describe. There’s too many of them, you can’t list them. So you know, you start off here with just a small finite set and you say, okay here’s a couple of sets that are in the sigma algebra, what else has to be there? So the student gets to play, experiment, like, okay, what happens if I take the union of these two? What happens if I intersect these three?”

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Code	Operationalisation	Example of how code applies to the data	Illustrative quote
KSU	Knowledge of Student Understanding. Knowledge about students which helps teachers assist students in developing specific mathematical understanding. Examples of this form of knowledge might include anticipating how students might approach a task/problem, what they may find confusing about a concept, what misconceptions/prior conceptions they may have about a topic, an ability to diagnose the nature of a student's misunderstanding. KSU consists of two categories of knowledge: <i>Knowledge of requirements for learning</i> and <i>Knowledge of areas of student difficulty</i>.		

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Code	Operationalisation	Example of how code applies to the data	Illustrative quote
Knowledge of requirements for learning	Magnusson et al. (1999) describe this form of knowledge as an understanding of the prerequisite knowledge required by students to learn specific concepts as well as an appreciation of the variety of different approaches students can take to learning a topic. We broaden Magnusson et al.'s definition by operationalising this code as knowledge of the cognitive demands placed on students as they internalise new concepts, solve problems, or construct proofs.	Dr Doyle discusses a homework problem that requires students to show that the completion of the Borel sigma algebra gives the collection of Lebesgue measurable sets. In discussing the solution to this problem, he addresses the cognitive demands of the task by highlighting the prior knowledge and logical steps required to successfully complete it.	“One direction is a one-line thing if you see why. But I mean the student still has to see why and if they ... haven't grasped the definition of what the Borel sets are and what the Lebesgue measurable sets are, they can't even see that direction ... there is a bit of a process ... it's only a line to write down but there's thought involved”

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Code	Operationalisation	Example of how code applies to the data	Illustrative quote
Knowledge of areas of student difficulty	We operationalise the code as knowledge of what topics/concepts/problems/tasks students may find difficult and the reasons why they might struggle with them. Potential reasons for student difficulty might include the abstract nature of the concept/topic, student misconceptions, students' lack of problem-solving skills, or a lack of familiarity with relevant problem-solving approaches.	When discussing the solution to a homework problem that required students to find the sigma algebra generated by the collection of all intervals of the form $(-\infty, n)$, Dr Doyle explained that students can struggle with expressing the solution in set notation even if they have an understanding of what should be in the sigma algebra.	"There's a problem for the student ... when you figure out that, okay, I can get intervals of the form seven to eight, of seven to nine where seven is included but the right-hand side is not. How do I now describe that sigma algebra? ... The set notation to describe all those things involves a few layers of thought. It's very difficult to write even when you know in your head what you're looking for"
KA	Knowledge of assessment. This knowledge component consists of two categories: <i>knowledge of dimensions to assess</i> and <i>knowledge of methods of assessment</i> .		

Continued on next page

Code	Operationalisation	Example of how code applies to the data	Illustrative quote
Knowledge of dimensions to assess	We operationalise this code as knowledge of what aspects of a student's learning are important to assess within the context of a particular subject/topic.	When discussing a past examination in his course that asked students to provide an example of a sigma algebra and measure on $[0, 1]$ for which the measure space is (i) complete and (ii) not complete, Dr Doyle talked about the importance of asking students to provide examples. He explained that a student's ability to provide an example is a good indicator of their understanding of a concept.	"Examples are important because, again, it really shows if the student has understood the definition. Now you can, because examples will be provided in the notes, you can learn off examples. But you still have to remember what example goes where. And yeah, if you can give an example, then you've kind of internalised the concept to some extent"
Knowledge of methods of assessment	We operationalise this code as knowledge of the potential methods that can be used to assess an aspect of students' learning as well as the advantages and disadvantages associated with various methods of assessment.	Dr Doyle asks students to submit four homework assignments over the course of the semester and marks each assignment out of five. However, the lowest potential mark he returns is 3/5 as he views the purpose of homework to be to encourage engagement with the course and does not want to penalise students for making mistakes prior to the final examination.	"When I mark the homeworks ... I return an integer mark out of five, but I only ever return three, four, or five. So the mark is, it's just an indicator ... like it's always three so that there's no real penalty for being wrong at that stage. You can make it up later"

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Code	Operationalisation	Example of how code applies to the data	Illustrative quote
KC	Knowledge of curriculum. Consists of two categories: <i>Knowledge of goals and objectives</i> and <i>knowledge of specific curricular programme</i> .		

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Code	Operationalisation	Example of how code applies to the data	Illustrative quote
Knowledge of goals and objectives	Magnusson et al. (1999) conceptualise this form of knowledge as understanding the goals and objectives for a subject and how these goals are reflected in the topics covered during the year. They identify curricular documents as potential sources of this form of knowledge. We adapt knowledge of goals and objectives to the undergraduate context by operationalising the code as relating to an instructor's goals and objectives for the course they are teaching (which they may have more flexibility in determining compared to the K-12 context). This includes understanding what the main learning outcomes of the course are for students, what skills students should develop, what content should be covered, and why that content is important for students to be exposed to.	There are students from various majors, including pure mathematics, applied mathematics, and financial mathematics in Dr Doyle's course. Dr Doyle understands why the measure theory course is important for students from each of these degrees and what the most important learning outcomes are for each.	"It's taught to these different cohorts ... for different reasons. So for pure maths students, they need to know about measures so they can go on to do certain Banach spaces, L^p spaces. Financial maths students are doing it because they need to know about integration and limits of integration and the kind of integration you do in first year isn't good enough for the kind of discontinuous processes you need in financial markets later on ... Applied maths students are looking at it ... because they would meet L^p spaces and Hilbert spaces in certain areas like maths physics, quantum stuff"

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Code	Operationalisation	Example of how code applies to the data	Illustrative quote
Knowledge of specific curricular programme	Magnusson et al. (1999) conceptualise this form of knowledge as understanding the various specific curricula and materials that might exist in the context of K-12 science education. We adapt knowledge of specific curricular programme to the undergraduate mathematics context by operationalising the code as knowledge of how a course/topic sits in the context of a student's overall programme of study. This includes both vertical and horizontal curricular knowledge i.e. an instructor's knowledge of how a topic covered in their course relates to those that students cover in other courses.	Dr Doyle includes remarks in the margins of his notes on how topics that are covered in the course relate to those which students have or will see in prior or subsequent courses. For example, he notes that in the context of probability theory a measurable function on a measure space would be called a random variable.	"I tend to add margin notes from year to year, just trying to make the link to this and that. Okay so, a measurable function on a measure space would be what probabilists call a random variable, that kind of stuff"

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Code	Operationalisation	Example of how code applies to the data	Illustrative quote
Orientations towards teaching (OTT)	Adopting modern PCK theory (e.g., Carlson et al., 2019; Gess-Newsome et al., 2015), we conceptualise orientations and beliefs as distinct from PCK (thus differing from Magnusson et al.'s original framework). We view a teacher's beliefs and orientations as acting as a lens or amplifier which influences both their application of PCK to their teaching practice (Gess-Newsome et al., 2015) as well as their construction of (personal) PCK from reflection on personal experience (Carlson et al., 2019). To code for the influence of beliefs and orientations we borrow the term 'orientations towards teaching' from Magnusson et al. and introduce four subcodes, each examining the interaction of beliefs and interactions with components of PCK.		

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Code	Operationalisation	Example of how code applies to the data	Illustrative quote
OTT-KISR	We operationalise this code as relating to the influence of orientations and beliefs on the application and construction of KISR.	Based on his own learning experience as a student, Dr Doyle believes that re-writing someone else's notes can be a great way of learning and consolidating understanding. This belief is reflected in his decision not to publish complete typed notes in measure theory as he feels there is value in students re-writing his handwritten lecture notes.	"The whole process of tidying up someone else's meanderings is a brilliant way to learn. And so I deliberately leave those bits, not messy as such but they're there for the students, and the students, I think, would benefit greatly from going through those and tidying them themselves. Either rewriting and transcribing, so that it's more legible. Or even typing them up themselves. Like, actually latexing a proof is a very good way to learn."
OTT-KSU	We operationalise this code as relating to the influence of orientations and beliefs on the application and construction of KSU.	N/A – Not found to be relevant to the data.	

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Code	Operationalisation	Example of how code applies to the data	Illustrative quote
OTT-KA	We operationalise this code as relating to the influence of orientations and beliefs on the application and construction of KA.	Dr Doyle's beliefs about the purpose of assessment influence his choice of assessment methods in the measure theory course. He states that he does not organise midterm tests in the course as he does not believe in over-testing and feels that the purpose of in-semester assessment should be to provide feedback to students on their learning rather than forcing them to study.	"For something of this level, I don't like midterms. I don't like over testing . . . I don't think it's particularly appropriate to assess higher level maths modules all the time . . . What are you trying to do? If you're trying to just force people to work all the time, I mean they're big boys and girls they have to be able to take responsibility for that. And also, it gives the impression that the purpose of the whole thing is assessment, which I don't really subscribe to. If the purpose is to inform students how they're doing and whether they have things more or less correct, the home-works work well for that."

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Code	Operationalisation	Example of how code applies to the data	Illustrative quote
OTT-KC	We operationalise this code as relating to the influence of orientations and beliefs on the application and construction of KC.	Dr Doyle has an orientation towards making the mathematics in the course as interesting as possible. One example of how this influenced his curricular design is his decision to include the Banach Tarski paradox in the course despite not all students having the necessary background knowledge as he believed it was one of the most interesting things you can do in a course on measure theory.	“The Banach Tarski paradox is one. I don’t know if you know it, but it’s like, this really cool fact. And it’s generally not covered in an undergraduate course, because there isn’t time to cover it and because it brings in stuff, you know, can you talk about free groups if you’re doing measure theory? Because the students haven’t done free groups. And the geometry also, quite a bit of geometry is required so. But I wanted to include that. It was the most interesting thing you could do in the measure theory course.”

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Code	Operationalisation	Example of how code applies to the data	Illustrative quote
Personal PCK	Personal PCK consists of the PCK available to an instructor to draw on in an instructional context. It is unique to individual instructors. Personal PCK is a superset of enacted PCK. When instructor uses their personal PCK in context it becomes enacted. The same subcodes as listed under enacted PCK apply here. We omit them in this coding template as, due to our study design, the majority of the PCK we identified was enacted.	An example of an indication that Dr Doyle's personal PCK extended beyond that which he uses in the context of the measure theory course (i.e. his enacted PCK) was his reference to the quiz sheets he uses to assess students in a first-year course he teaches. This suggests that his knowledge of assessment may differ depending on course level.	"It's obviously more involved than maybe the quiz sheets you'd give to the first-year class but the concept is essentially the same in that you have to practice the material and here is your opportunity to find out where you stand."

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Code	Operationalisation	Example of how code applies to the data	Illustrative quote
Collective PCK	Collective PCK is PCK that is public and shared by more than one person. It includes research-informed canonical knowledge but there is also the possibility that collective PCK can be developed locally and shared by members of a department or school. Collective PCK has three subcodes which specify the grain size from the broad discipline level to the more specific topic and concept level.	We found limited examples of the influence of collective PCK in Dr Doyle's interviews. He did discuss collective knowledge of how to structure a course in measure theory which he was exposed to through textbooks and also alluded to a collective knowledge of curriculum within the department.	"I think in measure theory most textbooks assume, kind of follow the same lines build it up from, start with the real line and abstract from there rather than, you know, doing the abstract stuff first and then focusing on examples."
Discipline	Collective PCK at the discipline level i.e. mathematics in general.		
Topic	Collective PCK at the topic level e.g. measure theory, calculus, linear algebra.		
Concept	Collective PCK at the concept level e.g. the completion of measure spaces, the first fundamental theorem of calculus, linear independence.		

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2. Details of the Paper
Title of Paper: Undergraduate students' collaboration on homework problems in advanced mathematics courses
Current Status of the Paper: ☐ Drafted ☐ Submitted. ☐ Under review ☐ Under revision ☐ Accepted for publication ☒ Published
Name of Journal (if known): The Journal of Mathematical Behavior
Digital Object Identifier (if published): https://doi.org/10.1016/j.jmathb.2024.101200
AUTHOR CONTRIBUTIONS
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Description of Examination Candidate's contribution for each section of paper: (Max 500 words): I developed the research aims, incorporating feedback from my supervisor. I reviewed relevant background literature. With my supervisor, I collaboratively wrote the interview schedule. I conducted the interviews for the study was responsible for their transcription. I was primarily responsible for the data analysis, which developed through discussions with my supervisor. I wrote all sections of the manuscript, incorporating feedback from my supervisor. I also created all figures and tables in the manuscript.

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Co-Author 1

Name of Author No.1	Maria Meehan -
Contribution of Co- Author No.1	25% contribution <i>Brief outline of contribution: (Max 200 words)</i> As my supervisor, Maria provided essential advice and guidance in conceptualising and developing the methodology for the study. I was responsible for conducting the interviews, however Maria provided valuable feedback on my interviewing technique. Maria took on the role of experienced supervisor and collaborator during the data analysis process. She was familiar with the data and we discussed the analysis as it progressed. Maria reviewed and provided feedback on all sections of the manuscript.

Signature of Supervisor

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Chapter 5

Undergraduate students' collaboration on homework problems in advanced mathematics courses

Abstract

While mathematicians and mathematics education researchers have acknowledged the importance of undergraduate mathematics students' learning outside of class time, little is known about what students actually do. The aim of this study is to examine one aspect of students' out-of-class learning: their collaboration with peers on homework problems. Ten interviews with recent graduates of mathematics degrees were conducted and analysed using reflexive thematic analysis. We examine participants' descriptions of how they collaborated on homework problems and with whom. Additionally, we explore their perceptions of the affordances of collaborating on homework, as well as the factors they perceive as constraining their engagement in the practice. Our study is an initial step towards developing a more complete understanding of undergraduate mathematics students' engagement with homework problems and out-of-class learning practices more generally. We discuss the implications of our findings in terms of guiding future research.

Keywords: undergraduate students, collaboration, homework, out-of-class learning, advanced mathematics

5.1 Introduction

Undergraduate mathematics instruction consists of in-class instruction, as well as the provision of opportunities for students to engage with content outside of class. Research on undergraduate mathematics education has primarily focused on in-class instruction (Rupnow et al., 2021). This is despite the fact that both mathematicians and mathematics education researchers have acknowledged that learning advanced mathematics requires considerable work on the part of students and that, regardless of the format of in-class instruction, the vast majority of this work will take place outside of class time (Lew et al., 2022; Rupnow et al., 2021; Wu, 1999). As yet, we know very little about what undergraduate mathematics students do beyond the classroom.

There are indications that engaging with peers on mathematics may be an important component of time spent by undergraduate students outside class (Brown et al., 2005; Iannone & Nardi, 2005; Seymour & Hewitt, 1997; Solomon et al., 2010). Research on collaborative learning at the undergraduate level more generally has found student-student collaboration to be beneficial to students (e.g., D. W. Johnson et al., 1998; Springer et al., 1999). However, the focus of this literature has been on the implementation of group-based activities during class. Students' collaborative engagement with mathematics more specifically has also been emphasised as a key tenet of some forms of student-centred learning (Laursen & Rasmussen, 2019; Melhuish et al., 2022), but once again the emphasis is on students working together during class time.

The initial goal of our study was to examine undergraduate mathematics students' collaboration *beyond the classroom*, especially in upper-level mathematics courses. To this end, ten interviews with recent mathematics graduates were conducted and analysed using reflexive thematic analysis (Braun & Clarke, 2022b). It became clear that the primary focus of participants' collaboration with peers was on working together on their assigned homework problems. This reflects the supposition that working on homework problems is the most common way by which undergraduate students engage with mathematics outside of class time (Dorko, 2019; Rupnow et al., 2021). Therefore, in this paper we focus on examining the collaboration undergraduate mathematics students engage in on their homework problems. Specifically, we examine participants' descriptions of how they collaborated on homework problems and with whom. Additionally, we explore their perceptions of the affordances of collaborating on homework, as well as the factors they perceive as constraining their engagement in the practice.

We structure this paper by first outlining background literature on in-class instruction and out-of-class learning in undergraduate mathematics courses, with a particular focus on

research that has been conducted on collaboration and homework in the context of undergraduate mathematics. This is followed, in Section 5.3, by a description of the context of the study and data collection and analysis methods. In Section 5.4, we present four themes addressing the nature and perceived affordances of participants' collaboration on homework problems, as well as the factors they perceived as constraining their engagement in the practice. Our findings contribute to the literature by providing initial insight into undergraduate mathematics students' out-of-class learning practices, which to date have been under-researched. While there are certain contextual features of our study that are important to consider in relation to our findings, we believe there is scope for our findings to generalise beyond our local context and to inform future research on homework and students' out-of-class learning more generally. This is discussed further in Section 5.5 where we outline three potentially productive ways in which future research could build upon our exploratory study.

5.2 Literature review

Students' work in undergraduate mathematics courses consists of engaging with the in-class instructional element of their courses in addition to their independent study out of class. Like Rupnow et al. (2021), we adopt the perspective that "mathematics instruction is not limited to the time spent in class" (p. 2) but extends to the provision of resources to guide students' out-of-class engagement with mathematics. To date, research has predominantly focused on the in-class aspect of instruction. Over the last decade, we have learned that the majority of undergraduate mathematics courses are taught through lectures (E. Johnson et al., 2018), what mathematicians' instructional practices in lectures are (Artemeva & Fox, 2011; Fukawa-Connelly et al., 2017), and why mathematicians choose to engage in such instructional practices (Woods & Weber, 2020). While we now have a good understanding of what happens in the undergraduate mathematics classroom (Woods & Weber, 2020), mathematicians and mathematics educators agree that most of a student's learning happens *out of class* (Lew et al., 2022; Rupnow et al., 2021; Wu, 1999).

The focus of this paper is on one aspect of students' out-of-class learning: their collaboration with peers on homework problems. Despite acknowledgement of the importance of undergraduate mathematics students' independent work, the topic remains under-researched. We therefore structure this review by first providing an overview of research on in-class instruction in undergraduate mathematics courses. Given our study's emphasis on collaboration, we focus on research that has sought to examine students' engagement in collaborative activities during class time. Following this, we highlight relevant studies that have provided insight into students' out-of-class learning practices. It has been posited that working

on homework problems is the most common way that undergraduate students engage with mathematics outside of class (Dorko, 2019; Rupnow et al., 2021). Our focus on students' collaboration on homework problems reflects this, and we provide an overview of research on homework in undergraduate mathematics courses.

5.2.1 Research on in-class instruction in undergraduate mathematics courses

In a recent review, Melhuish et al. (2022) distinguished between two forms of instruction in undergraduate mathematics courses: lecture-based and student-centred (or inquiry-based) pedagogies. The latter is differentiated from lecture-based instruction by the degree to which students are responsible for generating mathematical ideas. The majority of mathematics courses are taught through lecture (E. Johnson et al., 2018). Research examining the nature of mathematics lectures (e.g., Artemeva & Fox, 2011; Fukawa-Connelly et al., 2017), indicates that student-student collaboration does not form a large component of lecture-based instruction. However, some faculty who self-identify as teaching through lecture do report occasionally engaging students in small-group discussion or problem-solving during class time (E. Johnson et al., 2018). Indeed, some authors have advocated for the further incorporation of these and other similar in-class student activities as part of an effort to improve mathematics lectures (e.g., Alcock, 2018).

While it accounts for the minority of undergraduate mathematics teaching, students who do experience a form of student-centred instruction are much more likely to participate in collaborative activities as a component of their in-class learning. Melhuish et al. (2022) noted that “the commonality of student engagement and *collaboration* [emphasis added] can be found in different iterations of inquiry” (p. 12). Additionally, Laursen and Rasmussen (2019) identified “students *collaboratively* [emphasis added] process[ing] mathematical ideas” (p. 138) as one of the four key pillars they viewed as underpinning various forms of inquiry-based instruction. However, while collaboration is often positioned as a key component of student-centred instruction, “studies rarely address how that group work is enacted beyond description of the task” (Melhuish et al., 2024). Operationalising collaborative activities in the classroom is a major focus of the wider collaborative learning literature. Researchers have pursued the issue, and aimed to optimise collaboration, from a variety of different perspectives (see Slavin (1996) for a detailed review). We note that research on collaborative learning has found it to be generally beneficial to students academically and in other, more affective, ways (e.g., D. W. Johnson et al., 1998; Springer et al., 1999).

Given the emphasis on collaboration in the student-centred literature and its aim of engaging students in practices that are authentic to the discipline of mathematics (Laursen & Rasmussen, 2019), we conclude this section of the review by considering the collaborative nature of research mathematics. While often depicted in popular media as mystical, esoteric and inaccessible to all but the lone, usually male, genius” (Mendick et al., 2008, p. 25), there are indications that mathematics is becoming an increasingly collaborative academic discipline. Grossman (2005) reviewed patterns in mathematics research publications, published between 1940 and 1999. He found that the proportion of single-authored papers published has fallen from 91% in the 1940s to 54% in the 1990s. Given the relatively dated nature of these results, we reviewed the authorship of papers published in 2023 in the top twelve journals under “Mathematics” as ranked by Scimago JR. Single-authored papers accounted for 18% of the papers published in these journals in 2023, suggesting the continuation of this collaborative trend. (More detailed results of this review are available in Appendix 5.A.)

Mathematicians in Burton’s (1998) interview-based study also noted a cultural shift from an emphasis on individual work towards a more collaborative norm. Out of Burton’s sample of 70 mathematicians, only four reported working exclusively alone. Participants discussed benefits of collaborating to the quality of the research output, but also cited advantages less directly related to their mathematical work. For instance, a collaborative style of working was described as “less isolated” (p. 129), with participants enjoying sharing excitement about their work with colleagues.

5.2.2 Research on students’ out-of-class learning in undergraduate mathematics courses

There has been a growing call for future studies to look beyond the undergraduate mathematics classroom acknowledging that it “is not, and cannot be, the central venue for mathematical learning” (Rupnow et al., 2021, p. 1). It has been posited that the most common way that undergraduate students engage with mathematics outside of class time is by working on homework problems (Dorko, 2019; Rupnow et al., 2021). However, research on homework in the context of undergraduate mathematics courses is relatively sparse. Rupnow et al. (2021) investigated how mathematicians assign homework problems in an Abstract Algebra course. The primary reason their participants reported assigning homework problems was that they believed that engaging with these problems supports student learning. Five of their six participants positioned homework as essential to students’ learning of advanced mathematics, with one mathematician describing it as “the most important thing” (p. 5).

The authors called for future research to build on their work and examine whether students understand and/or share their instructors' goals in relation to homework.

We know relatively little about students' perspectives on, and approaches to, homework, particularly in upper-level courses. Ellis et al. (2015) investigated the nature of homework in Calculus I courses in five US-universities. They found that student participants in focus groups appreciated being assigned homework as it encouraged them to continually engage with the course content and offered a means of obtaining marks. Lew and Zazkis (2019) claimed that their study was among the first to provide insight into students' authentic at-home engagement with mathematics problems. They highlighted three students' varying approaches to a prove-or-disprove task and suggested that "some students spend a significant amount of time, producing several pages of work, when engaging in at-home proving" (pp. 13–14). The authors suggest that their data may be useful in illustrating to students what productive engagement with homework tasks might look like.

Dorko (2021) also commented upon variation in students' approaches to homework tasks in her examination of students' use of a 'see similar example' feature when completing online homework in a Calculus course. Of note was the apparent alignment between participants' use of the feature and their learning goals. Dorko noted that students in the study completed their homework individually and suggested that "future research could explore students' homework learning when they are working in pairs or small groups" (p. 24). While no research has specifically examined students' collaboration on their homework, there are indications that engaging with peers forms an important component of the time undergraduate mathematics students spend out of class. Participants in Seymour and Hewitt (1997) study, which examined the reasons for student attrition among science, engineering, and mathematics undergraduate students in the US, discussed the importance of making "effective use of peer study groups" (p. 172). Cognitive and affective benefits were attributed to membership of peer study groups which included "a firmer grasp of concepts" (p. 172) and "emotional support through difficulties" (p. 172).

The Student Experiences of University Mathematics project (Brown et al., 2005) was a longitudinal study that examined students' experiences of studying undergraduate mathematics over the course of their degree in two UK universities. The study highlighted that "building positive attitudes [towards their studies] related to students being able to share their ideas/problems with other students" (p. 6). The availability of "social space for mathematics students" (p. 6) was seen as key in fostering a sense of mathematical community among the undergraduate students. The lack of such a resource in one university was identified as contributing to a "sense of isolation" (p. 6), with students reporting finding it difficult to connect with peers.

Solomon et al. (2010) reported that student participants in focus groups also spoke of the importance of informal collaboration outside of class time. They valued the explanation of concepts and alternative views on problems peers could provide. The provision of social learning spaces was identified as key in facilitating these kinds of interactions between students. The same point was raised by the mathematicians interviewed in Iannone and Nardi's (2005) study. One participant elaborated "where our students learn best is in these interactive environments and things like these that they sit around and talk and talk all day. It appears to me that they learn more there than in many of our lectures" (p. 197). The undergraduate students' collaboration was characterised as necessarily unmediated by the participants. The mathematicians felt that while students could be supported and encouraged in their collaboration, any major form of external intervention would only serve to detract from the quality of the students' interactions.

While the outlined research has indicated that students' interactions with their peers may form an important component of the time they spend out of class, to our knowledge no study has specifically examined the nature of undergraduate mathematics students' collaborative work outside of class.

5.3 Methodology

The study's qualitative research design adopted a phenomenological approach (Willig, 2021). Ten graduates of mathematics degrees were interviewed about their experiences of collaborating, and reflexive thematic analysis (Braun & Clarke, 2022b) was used to analyse the data.

5.3.1 Participants

At the time of the interviews taking place in spring 2022, the ten participants were recent graduates (graduating in 2019, 2020, and 2021) of University College Dublin's (UCD) four-year Bachelor of Science undergraduate degree. After studying mathematics (including applied mathematics and statistics) and potentially other science subjects in their first two years, all participants in this study chose to major in either (pure) mathematics or applied mathematics, or pursue a joint major containing at least one of these two. The latter two years of these degrees consist mainly of core courses in mathematics and applied mathematics respectively, with optional courses from across the mathematical sciences. When providing examples from specific courses, participants mostly referenced the upper-level courses they took during these latter two years of their degree.

Four participants were female, while six were male. At time of data collection, all participants had completed, or were in the process of pursuing, postgraduate studies in mathematics and other disciplines. We believe this indicates that our participants were highly motivated undergraduate students and that this is an important contextual point to consider when interpreting our findings. We return to this point in Section 5.5.

Initially, graduates known to the authors were invited to participate and from there a form of snowball sampling was used. This meant that some participants graduated as members of the same year cohort and potentially worked together as undergraduate students¹. Table 5.3.1 summarises each participant's assigned, gender-neutral pseudonym, major(s), area of postgraduate study (mathematics or other), and year of graduation (2019, 2020, 2021) have been randomly designated as either year A, B, or C).

Table 5.3.1: A summary of the participants' major(s), area of postgraduate study, and year of graduation.

Pseudonym	Major(s) (Mathematics and/or Applied Mathematics)	Area of postgraduate study (Mathematics/ Other)	Year of graduation (A, B, C)
Quinn	Mathematics and applied mathematics	Mathematics	B
Nolan	Applied mathematics	Mathematics	B
Shea	Mathematics	Mathematics	A
Walsh	Applied mathematics	Other	A
Dunne	Mathematics and applied mathematics	Other	A
Sullivan	Applied mathematics	Mathematics	C
Reilly	Mathematics and applied mathematics	Other	A
Lynch	Applied mathematics	Mathematics	C
Brennan	Applied mathematics	Mathematics	A
Doherty	Mathematics	Mathematics	A

¹During the analysis it became apparent that both Dunne and Reilly, as well as Sullivan and Lynch, collaborated on their homework together as undergraduate students.

5.3.2 University context

UCD is large, research-intensive, PhD-granting university located in Ireland. Nearly 41,000 students are enrolled annually in UCD. Of this student body, roughly 3,000 students live on-campus with the remainder commuting each day to their classes. The School of Mathematics and Statistics in UCD is the largest mathematics department in Ireland. The school offers undergraduate, masters', and doctoral programmes and has approximately 75 doctoral students enrolled in the school.

UCD operates a modularised system. Over the course of two 15-week trimesters, full-time undergraduate students typically study 12 courses. (In UCD, courses are referred to as modules.) As is typical of undergraduate mathematics instruction (Melhuish et al., 2022), mathematics courses in UCD are taught predominantly through lecture. Instructors typically provide students with homework problems (commonly referred to as problem sheets). Instructors generally have autonomy in creating homework assignments for their courses. Students may have to submit solutions to homework problems for credit (usually worth no more than 20–30% of their overall grade). Alternatively, in some courses students might take midterm tests which are generally based on homework problems. Homework problems may form the basis of recitation classes (referred to as tutorials). Undergraduate mathematics students in UCD are predominantly assessed through closed-book examination. Research conducted in a UK context has indicated that this is typical of the general 'assessment diet' of undergraduate mathematics degrees (Iannone & Simpson, 2011, 2022).

5.3.3 Data generation

Ethical approval for this study was granted by the Human Research Ethics Committee of UCD. Participants took part in semi-structured interviews with the first author in spring 2022. The interviews lasted an average of 30 minutes and were audio-recorded and transcribed. Participants were asked whether they collaborated over the course of their undergraduate degree with no definition of collaboration in this context given. Participants were then asked to describe their experiences of collaborating with peers on their studies. They were asked about the other students they worked with, the tasks on which they collaborated, and how the collaboration was structured around these tasks. Their perceptions of the value of collaboration to their mathematical learning experience were explored. Participants were asked whether they ever worked on a credit-bearing group assignment or project and whether they would consider assessed group work to be appropriate in the context of undergraduate mathematics. We also sought participants' views on whether they considered mathematics to be a collaborative or individualistic discipline and the messages they may have received

from their instructors regarding whether collaboration was something to be encouraged or not. (The full interview protocol for this study is included in Appendix 5.B.)

We note that given its interview-based nature, the data generated for this study consists of self-report data. However, given that all participants were graduates, we felt that they were well-positioned to be open about their collaborative practices during their degree. In addition, we specified that participants be *recent* graduates to ensure that their recollections were as faithful to their experiences as possible.

5.3.4 Data analysis

We chose reflexive thematic analysis (RTA) (Braun & Clarke, 2022b) as our method of data analysis. RTA differs from other forms of thematic analysis in its focus on the inherently situated and subjective nature of qualitative data analysis and the active role of the research team in the analytic process (Braun & Clarke, 2022a). At the time of data collection, the first author was a final-year undergraduate mathematics student and, having had personal experience of collaborating with peers, would be considered an insider researcher (Braun & Clarke, 2022b). The second author is a professor of mathematics with research interests in mathematics education. The method's emphasis on researcher reflexivity allowed for these two disparate perspectives to be capitalised on as a "resource for knowledge production [] rather than a must-be-contained threat to credibility" (Braun & Clarke, 2021, pp. 334-335).

The analytic process followed the six phases outlined by Braun and Clarke (2022b). Both authors familiarised themselves with the data corpus through repeated reading of the interview transcripts and re-listening to the audio-recordings. Two rounds of inductive, semantic coding were completed by the first author, first using Microsoft Excel and then NVivo 12. Having a single coder is considered good practice in RTA (Braun & Clarke, 2022b). The second author reviewed the second round of coding and made comments with the goal of "enhanc[ing] reflexivity and interpretive depth, rather than [achieving] consensus between coders" (Braun & Clarke, 2022a, p. 9). The first author then began the process of theme development, clustering codes around different potential central ideas.

Braun and Clarke (2022b) emphasise that the phases of analysis are rarely linear, and researchers should be prepared to move back and forth between phases. We found that as we moved between initial theme development, reviewing the themes in relation to the data, and writing the findings, we iteratively refined and adjusted the candidate themes. This process resulted in an initial set of six candidate themes. It has also been noted that "research questions in reflexive TA . . . commonly evolve throughout the course of the research. The initial research question(s) can be quite open and constitute a 'starting point'

that might become more focused or expand or even shift in focus, as data collection and analysis progresses” (Braun & Clarke, 2021, p. 12). As discussed in Section 1, we choose to focus this paper on participants’ collaboration on their homework problems. Specifically, we decided to address the following research questions:

- **RQ1:** Who did participants report collaborating on their homework with?
- **RQ2:** How did participants report collaborating with others on their homework problems?
- **RQ3:** What did participants perceive as the affordances of, and factors that constrain, collaborating on homework?

Having finalised our research questions, we returned to the analysis and made some adjustments to our candidate themes. Four themes were finalised, one addressing each of the first two research questions, and two addressing the third. These four themes are summarised in Table 5.4.1.

Before describing the themes, we note that for all four themes, each participant’s interview contained instances of codes that ultimately contributed to the theme. In this sense, every participant can be viewed as contributing to each theme. However, as is typical of RTA (Braun & Clarke, 2022b), some themes pull together disparate, and in some cases contradictory, opinions or experiences of participants. To provide context and indicate the prevalence of a particular opinion or experience, we will follow Terry (2010) by using some quantifying language in our discussion of our themes. In Table 5.3.2, we provide a list of terms that we adhere to when describing the prevalence of a particular perspective or opinion expressed by the participants, along with the associated number of participants who were coded as having expressed the opinion.

Table 5.3.2: Description of prevalence and associated number of participants.

Description of prevalence	Number of participants
All	10
Most/Almost all	8 or 9
Many	6 or 7
Some	3, 4, or 5
A minority/A few	1 or 2

5.4 Findings

5.4.1 Overview

In this section we present four themes, summarised in Table 5.4.1, addressing undergraduate mathematics students' collaboration on their homework problems. In Section 5.4.2, we describe who the participants reported collaborating on their homework with (Theme 1). In Section 5.4.3, we present a three-phase workflow of how the participants reported collaborating on their homework problems (Theme 2). The focus of Section 5.4.4 is on the perceived affordances of collaborating on homework (Theme 3). This is followed, in Section 5.4.5, by a discussion of the factors participants perceived as constraining their collaboration on homework (Theme 4). While in this section we describe each of our four themes individually, we acknowledge that there are points of connection between the themes. In Section 5.5, we provide a holistic summary of our main findings, illustrating some of these connections, before discussing the significance of our results in the context of existing and future research.

5.4.2 Theme 1 – Collaboration on homework happens with specific peers

The focus of this theme is on the fact that all participants reported having a specific set of peers (henceforth referred to as their collaborators) with whom they routinely collaborated on their homework. In our analysis, we identified two main commonalities in the participants' descriptions of their collaborators. All participants identified their collaborators as friends, and many noted that they and their collaborators were alike in terms of academic motivations and ability. Participants viewed these two factors as important reasons for why their collaboration on homework worked well with these specific peers.

Before we examine commonalities in participants' descriptions of their collaborators further, we note that participants reported engaging in two different modes of collaboration. Eight of the participants reported working with collaborators on homework *as a group* (of typically between four and six members). However, two (Quinn and Shea) reported predominantly working with their collaborators on a *one-on-one* basis. We provide a vignette of both modes of collaboration to illustrate their differences.

5.4.2.1 Collaborating as a group: a vignette of Dunne and Reilly

During the analysis process, it became apparent that Dunne and Reilly had collaborated with one another during their undergraduate degree. They described collaborating on their homework assignments *as a group* of four students: “there was a quartet of us who all

Table 5.4.1: Summary of the four themes.

Theme name	Theme summary
Theme 1 – <i>Collaboration on homework happens with specific peers</i>	All participants reported collaborating on homework with a specific set of peers: • Participants collaborated either on a <i>one-on-one</i> basis or <i>as a group</i> • Collaborators were also friends • Collaborators tended to be alike in terms of academic motivations and ability
Theme 2 – <i>Collaboration happens when stuck on a problem</i>	We present a three-phase workflow for how the participants reported collaborating on homework problems: • <i>Individual problem-solving phase</i> in which a student attempts a homework problem by themselves and either solves it or gets stuck • <i>Collaborative engagement phase</i> in which the student and their collaborator(s) come together and engage in either <i>collaborative problem solving</i> or <i>collaborative helping</i> • <i>Individual reflection phase</i> in which the student considers their collaborative work and, if the problem is solved, writes out a solution to ensure understanding
Theme 3 – <i>The affordances of collaborating on homework</i>	All participants perceived affordances of collaborating on homework which included: • Benefits to mathematical learning • Companionship and support when stuck on problems • Deepened friendships with collaborators
Theme 4 – <i>Factors that constrain collaboration on homework</i>	Factors that constrained participants' collaboration on homework included: • Unclear messages from instructors on whether collaboration on homework is permitted • The individual nature of assessment • Doubts regarding the collaborative nature of mathematics

worked together on the maths” (Dunne). The group members met while taking the same core mathematics courses in their first year and ultimately majored in different combinations of mathematics and applied mathematics. However, they continued to collaborate on their shared assignments and found the various academic specialties within the group to be of benefit to their collaborative work:

Reilly: There were a few others whose degrees were slightly different [than my major], but that kind of almost helped us like if one person was stuck on a statistics module, there was a statistics person who could help out more and if there was maths, there was a maths person.

In terms of their compatibility as collaborators, Reilly described the group as having “similar work ethic” and “always working at the same times”. Dunne had similar thoughts:

Dunne: Oh well motivation is the key thing. We were there, we were all committed. We were all putting in all of the hours, like all of the hours. And so like, we were all the same sort of, we want to do this, we need to do this, it’s awful sometimes and we will like scream and cry about it, but we’re eventually going to do this and we’re not going to give up.

The group’s shared determination to succeed was reflected in the name of the group chat that they used to text one another:

Dunne: We have a WhatsApp chat that’s called ‘Ahhh’ from our undergrad days, which just encompasses most of our frustration about what we were texting about a lot of the time.

While texting allowed the group to continue working together in the evenings or on weekends, their collaboration primarily occurred on campus. Both Dunne and Reilly talked about their group’s well-established routine of meeting to work together in a particular social learning space on campus: “we had some assignments that took hours and we would be sitting there in [building name]” (Reilly). The group would meet “for breakfast” to discuss the progress they had made on their current homework assignments and “throughout the day, other than our morning meetings, like you could genuinely find one of us there” (Dunne).

5.4.2.2 Collaborating one-on-one: a vignette of Quinn

Like Dunne and Reilly, Quinn also reported collaborating on mathematics homework with “particular people”. Quinn had “one or two” preferred collaborators, however their collaboration occurred on a *one-on-one* basis rather than as a group: Quinn: I’d usually probably go to them one-on-one rather than like, it wasn’t like we have a group chat where

we all discussed everything or anything, it was more kind of spontaneous and I'd reach out to them, or they'd reach out to me.

Quinn's collaboration on homework appears to have been more sporadic than that of Dunne and Reilly, who reported meeting up as a group to discuss their assignments almost every day. This may be partly due to Quinn's description of themselves as being "bad for going in [to campus] sometimes" and tending to work from home, rather than attending their lectures and tutorials. However, Quinn also found collaborating on homework assignments on a one-on-one basis to be more effective than working on problems as a group:

Quinn: I found that I wasn't great at working on something together. I always found I was better at like giving it a go and then discussing and then giving it a go and then discussing. . . . If I sat down with two or three people, I'd end up dossing and chatting basically, I couldn't help myself. I needed to like go off by myself, kind of in the bunker, and sit down and spend a couple of hours at it. And then if I wasn't getting anywhere, or if I got some or whatever, then go off and discuss it with someone and then maybe go back again is kind of how I tended to work.

Quinn described the people they particular enjoyed collaborating with as those who were similarly enthusiastic about the content: "it was always more fun for me to collaborate with people who weren't mopey about it I guess, and who were like, interested and would get excited about a problem". They subsequently described one collaborator in more detail:

Quinn: One guy in particular that I worked with quite a bit who, he was kind of better at like the algebra-type modules and I was more, I was better at the analysis-type modules and that kind of complemented quite well.

Quinn talked about how they enjoyed capitalising on one another's complementary strengths by reciprocally helping one another with their algebra and analysis homework: "whenever there was an algebra question, I'd go to him. If there was an analysis question, he'd come to me and it, like, it was kind of, it was nice".

5.4.2.3 Collaborators are friends

Whether they collaborated as a group or on a one-on-one basis, all participants identified the people they worked with as friends. Most participants explicitly referenced the fact that getting to know their collaborators preceded their work together on homework problems. Nolan recalled that while they did not initially collaborate with peers on homework, they may have chatted about their workload more generally:

Nolan: I had a couple of friends before Christmas in first year, which I'm now aware that a lot of people didn't. We were all commuting, or some form of commuting, so we were all hanging around all day. We just saw the same faces over and over again, so we had a chat. And we may not have done the assignment together, but we definitely would have kept each other on top of what things were due. And then by second semester, there's a bit more collaboration.

Some participants indicated that, towards the beginning of their degree, they were reluctant to be open and honest about their progress on homework assignments with students who they did not yet trust:

Dunne: In first year, it's sort of like 'Oh is that what you got?', 'Oh okay. Yeah, no, I worked on that.' And like you're not all admitting to each other that you spent 10 hours on one question and like this is the like fourth version of your answer.

More generally, the extent of participants' collaboration on homework assignments built over the course of their degree as it took time for them to "get to know people better" (Nolan) and collaboration only really commenced once "you've settled in and made your friends" (Reilly). For Brennan, the importance of this initial period of getting to know one's peers was highlighted during their master's degree, which was conducted online due to the coronavirus pandemic. Reflecting on the lack of collaboration they experienced during this degree, Brennan said:

Brennan: I would have said the bigger thing was . . . it [collaboration] was always just with the people who I was friends with anyway, and then the fact that the masters was all online you never got to know anyone. Like, the only person I ever had any level of collaboration with was the person who came from the same course in UCD and did the same masters.

Brennan's experience helps highlight the fact that for most of our participants, their collective work on homework problems was secondary to, and developed as a consequence of, their relationship as friends:

Brennan: There would have been probably three of four modules that I did without anyone else I knew doing them. In those ones, there wouldn't have been really any collaboration at all. So it was definitely driven by like, knowing people in the courses already, as opposed to getting to know people just for that.

5.4.2.4 Collaborators have similar academic motivations and abilities

All interviewed participants appeared to have been highly motivated and interested in their studies and many identified their collaborators as being like them in this regard.

For instance, Doherty described themselves and their collaborators as “the typical kind of academic-type people. We were just very much interested in our studies and like trying to do the best that we could”. Participants and their collaborators appeared to share the goal of understanding, rather than just obtaining, the solutions to their homework problems. Some, therefore, resented when other students, with whom they did not regularly collaborate, asked to effectively copy their solution to a problem. For instance, Lynch recalled:

Lynch: It was a bit of a sticking point that I didn't like sending on assignments [solutions] unless like somebody like, I'd worked on it with them . . . there was definitely a few times people asked ‘Oh could you send on the solutions?’ and I'm like ‘I spent 15 hours getting this, I'm not just sending it to you.’

This contrasted with Lynch's experience of working with their regular group of collaborators, all of whom shared the same goal of understanding their homework assignments:

Lynch: Nobody wanted to get the answers off. It was, it was like a bit competitive, but in a good way that you'd always be trying to figure it out, like the cool idea to figure out the problem. Nobody was just kind of like I'm just here to get answers. It was all very much trying to figure it out.

In addition to being alike in terms of their academic motivations, most participants described their collaborators as being similar to them in terms of their prior academic achievement. They perceived this as being important to the success of their collaboration. Some participants talked about issues that arose when they attempted to work with peers who may have been further ahead on a particular assignment or tended to work at a faster pace. For instance, Dunne mentioned that when they and their group of collaborators were unable to solve a particular problem, they might approach someone else in the course who they perceived as being potentially “further ahead”. Dunne recalled that this was “sometimes frustrating because they would be like ‘Obviously the answer is this.’ Could we remove the word obviously and go back a few steps please?”

Some participants specifically described themselves and their collaborators as high-achieving students. These participants reported that while they were happy to help other students in their class, they found that collaborating on homework problems was only really viable with those they perceived as being at a similar academic level:

Lynch: It's a hard one because . . . it's easy to help someone when they're not on the same level, but it's hard to have like a proper back and forth with somebody not on the same level as you. So like, you're always happy to help someone, but when you're stuck on something, like there's a natural group you kind of tend towards I suppose.

Generally, participants appeared to perceive collaboration on homework as most effective when working with peers of comparable academic ability. For instance, Reilly reflected on the gradual process of finding other students with whom one can successfully collaborate on homework:

Reilly: I can remember more so in the early years, when you're getting to grips with things, if you're in a group and the other people, either like they were far ahead of you in terms of their knowledge, you kind of felt like you were floating along after them and you didn't understand. Or the other way, like if you thought you were ahead of them, like you were dragging them along. But I think that kind of filters itself and you find the group of people.

5.4.3 Theme 2 – Collaboration happens when stuck on a problem

The focus of this theme is on participants' descriptions of how they collaborated on their mathematics homework, with an emphasis on the state of being stuck as a catalyst. In our analysis, we focused on identifying commonalities in the participants' accounts of their collaborative process. To illustrate these commonalities, we present a three-phase workflow that reflects the 'typical' course that participants' collaboration on homework problems took. The first phase is an initial *individual problem-solving* phase in which a student attempts a homework problem independently. If the student gets stuck, a phase of *collaborative engagement* on the problem commences. The mechanics of this phase depends on whether (at least) one of the student's collaborators has managed to solve the problem individually or if all collaborators are stuck. If the student and their collaborator(s) remain stuck after a period of collective work on the problem, they might seek help from an instructor or other students with whom they do not regularly collaborate. In the case that the student's work with their collaborators results in a potential solution to the problem, the student then checks and writes out the solution in full independently. We refer to this final stage as the *individual reflection* phase. The three-phase workflow is represented diagrammatically in Fig. 5.4.1.

Before examining each of these phases in further detail, we wish to emphasise that a student's approach to completing their mathematics homework is likely highly dependent on a range of variables, including the relative difficulty of the course content, the student's workload, the time they have available, and their individual goals and motivations. We do not wish to suggest that this three-phase workflow captures the nature of every mathematics student's (or indeed each of our participant's) collaboration on homework problems. Our goal in presenting the workflow is to illustrate the commonalities in our participants' accounts, with a particular focus on how their collaboration centred around the state of being stuck.

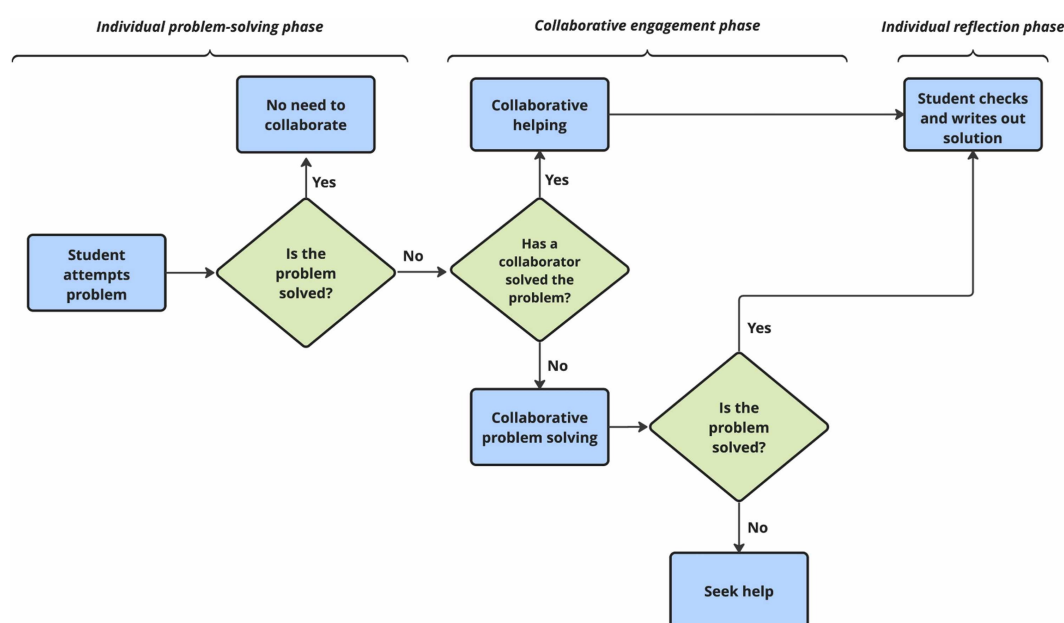


Figure 5.4.1: Three-phase workflow representing the ‘typical’ course that participants’ collaboration on homework problems took..

5.4.3.1 Individual problem-solving phase

Almost all of the participants explicitly stated that they would typically not collaborate on a homework assignment before attempting it individually. They talked about this initial period of independent work allowing them to assess the relative difficulty of problems and to determine which they might need to collaborate on:

Nolan: Maybe get an assignment on a Thursday, you go home at the weekend, make a bash at it. You might text the other person and then you’d agree this is an assignment that needs to be done as a group and you meet up whenever you had a few hours.

Participants considered collaboration unnecessary if they judged that they would be able to complete a homework assignment independently:

Quinn: If it was a module that everybody kind of agreed was relatively fine, it was much more a case of people kind of just saying ‘Oh did you do the whatever assignment?’ And like ‘Oh yeah, I did it’ or ‘Oh yeah I am going to look at it tonight, should be fine’ and ‘Oh yeah it’s pretty fine, you’ll be grand.’ And we wouldn’t really collaborate on that stuff.

This meant that more difficult material and problems tended to induce more extensive collaboration as “you’re going to have more points where you get stuck and more opportunity for [it] to happen” (Lynch).

Some discussed an intrinsic motivation to try to solve their homework problems by themselves before approaching their collaborators: “I always wanted to try it [a homework assignment], you know, try get something myself first” (Brennan). Attempting a problem individually was also deemed important to the success of subsequent collaboration on it. For instance, Sullivan felt it was “important that everyone puts in a couple hours before so that they’ve had, they’ve had time to formulate their own thoughts”.

5.4.3.2 Collaborative engagement phase

All participants identified the homework problems they collaborated on as the ones on which they or a collaborator had gotten stuck. The nature of participants’ collaboration on a problem depended on whether one of their collaborators had managed to solve the problem they were stuck on, or if they were all stuck. We refer to the case where a student solves a problem and assists a collaborator who is stuck on it as *collaborative helping*. While this type of exchange appears momentarily asymmetric, over time collaborators would alternate between being the one giving and receiving help. Quinn described it as “a bit of a kind of quid pro quo thing where someone couldn’t do something, but they helped you in the past and you’d be like ‘here, this is how I did it’”. In some cases, collaborators would help one another in accordance with their individual strengths:

Doherty: Different people would shine in different modules. So if we ever kind of had a stats module that would kind of be my favorite kind of ones, whereas the others didn’t really love it . . . But the kind of, I don’t know, like Number Theory or Galois Theory, that was just not for me. So I would just be like, ‘I’m so lost, please help.’

Some participants stressed that when they received help from a collaborator on a homework problem, it did not extend to them copying the entire solution. Sullivan stressed that within their group of collaborators:

Sullivan: What was important to us is that everybody got the spirit of what was going on, that you kind of understood the steps that you might take if you wanted to do a problem and then you could go and do it yourself.

Accordingly, Sullivan and their collaborators restricted the help they gave one another on homework problems to discussing high-level ideas as: “it didn’t feel good to be copying what others are doing. And so, if you didn’t really get something and you felt like someone just told you what to do, that never felt good”.

The case of all collaborators being stuck on a problem resulted in what we refer to as *collaborative problem solving*. Many participants recalled that they and their collaborators

used to meet up at a particular location on campus and it was here that they would come together to work on the problem on which they were all stuck. This involved brainstorming ideas for how to approach the problem: “there would generally be a point where we were all stuck, so then there’d be like a massive brainstorming session of like try this, try this, try this” (Dunne). Participants would “bounce back and forth ideas” (Shea) with collaborators who could help them assess the viability of potential approaches:

Lynch: Just having somebody to bounce ideas off and just yeah, like have an idea and spend maybe an hour going through the motions of oh this could be right and realising after all the time it’s wrong, you have somebody being like that’s wrong let’s move on.

Some talked of building on one another’s thinking, with exposure to alternative views on a problem sparking inspiration:

Quinn: Primarily what would happen is everyone kind of would have given it a bit of a go and then . . . everyone would exchange what things they tried to get to like a level. And then someone would figure out something else, and then we get back to a level and eventually someone [would figure the final solution out]. So it would feel a bit like someone cracked it, but really it was a collaborative thing.

Participants also spoke of the value of collaborating on a problem in terms of an increased generation of ideas and assisting with the identification of errors. For instance, Sullivan recalled working with their group of collaborators on one particularly difficult problem in their final year:

Sullivan: There’s just so much algebra going on. And there’s so many little tricks that each of us individually had gotten a bit muddled along the way and lost bits. When we sat and we did it on a board together it was a case of one at a time someone would point out that this was something that someone had done wrong, or ‘oh I found this trick online, could we try applying that?’ And eventually we got through the whole problem, you know, after a few hours work.

In this quote, Sullivan references collaborators sharing hints and tricks they found online with one another. We note that some participants mentioned consulting online resources, such as mathematics help forums, and sharing anything of potential relevance they found with collaborators during this collaborative problem-solving phase.

If a group or pair of collaborators were still stuck on a homework problem after working on it together, they would either seek further help, take a break from working, or, if an assignment deadline was approaching, submit whatever work they had so far. Some participants talked about seeking help from students with whom they did not regularly collaborate.

For instance, Lynch mentioned that they might “reach out” to people in their course who preferred to mainly work “on their own, rather than in groups”. Many participants mentioned that they may have approached instructors for a hint on tricky problems. Some participants felt more justified in approaching their instructor for help at this stage, feeling that they and their collaborators’ collective lack of success solving a particular problem demonstrated its difficulty:

Sullivan: It would usually be going to a lecturer next, because by the time, if that’s the case then we’ve probably all had a go on our own, all looked online to see if we could find anything useful and couldn’t do it together.

5.4.3.3 Individual reflection phase

Many participants reported that if their collaborative engagement on a problem resulted in a potential solution, each collaborator would engage in a period of individual reflection on the potential solution. This involved working through any outstanding details and writing out the solution in full: “you try to then write it in a way that you understood what happened” (Nolan). This reflected these participants’ goal of understanding, as opposed to just obtaining, a solution to their homework problems: “I wanted to be able to understand [a solution] and reproduce it independently once I knew what it was” (Shea).

5.4.4 Theme 3 – The affordances of collaborating on homework

In this theme, we focus on participants’ perceptions of the affordances of collaborating on homework. All participants felt that collaborating on their homework benefitted their mathematical learning and understanding of material. Additionally, participants felt that feelings of hopelessness and isolation associated with being stuck alone on a problem were lessened by the companionship of their peers. Many also spoke of social benefits which included deepened friendships with collaborators. Given that research on collaborative learning has “to date foregrounded the cognitive dimension” (Baker et al., 2013, p. 19), we note that the affordances participants perceived of collaborating on homework had cognitive, social, and affective elements. However, we are also conscious of recent research that has illustrated the inherent complexity of student-student collaborations and the interdependent nature of the “cognitive, social, and affective mechanisms” (Kontorovich, 2023, p. 1537) that may be at work. For this reason, we feel that to classify the perceived affordances of collaborating on homework as *strictly* cognitive, social, or affective would misrepresent the complex and situated nature of collaboration.

5.4.4.1 Benefits to mathematical learning

Given that the participants were not externally incentivised to collaborate on their homework, it is perhaps unsurprising that all participants reported that they felt doing so benefitted their learning. Sullivan and Lynch both talked about appreciating being exposed to their collaborators' skill sets and problem-solving approaches. Lynch, an applied mathematics graduate, valued the physical insight into problems provided by a collaborator who studied physics: "one of our friends [studied theoretical physics] and so when it comes to physics questions like, you kind of listened to him talk about like, his intuition". Sullivan discussed learning to internalise and adopt some of the problem-solving approaches exhibited by his peers:

Sullivan: I feel like ways that they would approach problems are so, used to be so foreign to me, and now some of what they do still is, but I can appreciate it and it comes to mind sometimes when I start a problem.

More generally, many participants talked about how they appreciated the insight into collaborators' alternative conceptualisations of material they gained when working on problems together:

Nolan: It's also seeing different people's way of understanding the work. Like the lecturer teaches it in one way, someone else hears it and interprets it differently ... I think the more you hear the same thing in different ways the better you understand it.

Doherty specifically valued the informal nature of peers' explanations. These informal explanations allowed them to return to formal statements in course notes with increased comprehension:

Doherty: I found like, a lot of times where like, I would just be like 'Oh that makes like complete sense.' I just didn't get it from the way it was written in the notes. Especially if it's something that's like written very, like a formal theorem or whatever I'd be like 'What's he actually doing?' And then if someone just explains it in like very informal language, it kind of makes more sense and then I can understand the more formal version of it better.

Some participants also viewed explaining concepts to collaborators as beneficial in terms of consolidating and solidifying one's own understanding:

Quinn: I'm a big believer in like having to verbalise maths, because you can think you get it and it's kind of scrambled up in your head. But when you have to put words on it you, it really makes, you have to straighten it up in your head and figure out what's actually going on.

5.4.4.2 Companionship when stuck on problems

Mathematicians have identified getting stuck on problems as a natural and important part of learning and doing mathematics (Burton, 2004; Rupnow & Fukawa-Connelly, 2024). All participants spoke of getting stuck on problems when completing homework assignments. Given that the participants were constrained in terms of the time they had available to work on these assignments, experiencing periods of being stuck on problems was depicted as potentially stressful. Many participants spoke of the value of having collaborators in terms of helping to alleviate this stress. Even when their collaborators could not help them solve the problem, they derived comfort from the knowledge that they were not the only one struggling. This was highlighted for Brennan during his master's degree which they completed with minimal collaboration due to it being conducted online:

Brennan: It felt a lot worse getting stuck on something when you're on your own and you're the only one getting stuck on it as opposed to like, when you're working together, you're all getting stuck on something or you know you're putting in ideas for the different things that people are getting stuck on. It definitely felt a lot more helpless when you get stuck on something and you're like, there is no one to go to and say 'Are you getting stuck on this as well?'

As Brennan touches on, a key comfort for participants was knowing that they had collaborators they could go to for help. This extended to situations where participants got stuck on a problem while working individually:

Nolan: When I got stuck in a rut at home, doing my assignments I was just like, I'm going to go in on Monday, have a conversation and suddenly you do about twice as much work in the next hour that you did all weekend.

Seeing that others were also struggling helped participants to normalise that getting stuck on problems is a natural part of learning mathematics:

Quinn: I think the big this is, well, it [collaboration] puts into perspective the fact that like a degree is hard and if you're struggling with something . . . there's definitely somebody else who's struggling. And that's just great to like, feel like you've got someone else. You're not the only person struggling.

5.4.4.3 Deepened friendships with collaborators

In Section 5.4.2.3, we discussed most participants' characterisation of friendship as an important basis for collaboration on homework. Some participants also spoke about how they felt their friendships with collaborators deepened as a result of their work together. The time

they spent together working on homework assignments had a social quality. Lynch spoke warmly of the of the blended nature of working and socialising with friends: “we have like a good group of friends that everyone was like working hard and then also, like, had fun”. They recalled occasions “hanging out” with collaborators where they could “end up talking about the problem sets and the maths” and “might spend the evening working together”. Some participants explicitly recognised strengthened friendships as a key benefit of collaborating on homework. For instance, Walsh reflected:

Walsh: Massive social benefit there as well, you know, that you actually get to know the people who are on your course and kind of suffer with them as it were. . . . That was a very big part of, you know, your total social interactions really you know in college is sort of group work and working together over different stuff.

5.4.5 Theme 4 – Factors that constrain collaboration on homework

While all participants attributed benefits to collaborating on homework assignments, the focus of this theme is on the factors that some participants perceived as constraining their collaboration. All participants reported that their instructors generally failed to explicitly address whether collaboration on homework assignments was permitted. Some participants’ general interpretation was that they could collaborate on their homework provided they did not copy and were happy to work within these parameters. For others, their instructors’ failure to explicitly address the issue left them doubting the legitimacy of collaborating on homework. A few specifically expressed concern about collaborating on homework assignments that they would then submit individually for credit. Others reported that the individual nature of final examinations served to motivate them to not become over-reliant on collaboration. Additionally, most participants perceived mathematics as an individualistic discipline or reported having no impression as to its collaborative nature. For some this perception caused them to question whether collaborating on homework problems was authentic to the discipline of mathematics. We examine each of these factors further in the subsections below.

5.4.5.1 Unclear messages from instructors on whether collaboration on homework is permitted

All participants reported that the messages they received from their instructors on whether collaboration on homework assignments was permitted or not were generally vague and unclear. Many did recall individual instructors who encouraged them to work together in tutorials or allowed them to submit homework assignments as a group. However, it was

felt that in most cases the issue was not addressed explicitly, and it was “very much left up to students” (Lynch) to assess the legitimacy of collaborating on homework assignments.

Many participants suspected that their instructors were aware of the extent to which students were collaborating on homework, despite rarely addressing the topic explicitly: “there was always, I think, the underlying assumption that we were all working on our assignments together” (Dunne). Brennan, therefore, felt that the onus was on instructors to intervene if they did not want students to be collaborating on homework: Brennan: It was rarely mentioned, but like I think it would be a stretch to say, like, they didn’t know fairly well that it was happening and that, you know, if there was someone [an instructor] who really wanted that [collaboration on homework] to not be a thing, I think it sort of feels like it would be on them to say it because they know it’s happening.

Shea similarly noted that their instructors never expressed concern about students working together. They were among a number of participants whose general interpretation was that they could collaborate on their homework assignments, provided they were not copying:

Shea: I remember I sent a lot of emails to lecturers being like ‘hey, I am working and talking with X, Y, and Z’ and they never responded being like ‘well, you shouldn’t do that, you should be working on your own.’ The implicit thing I got was that collaboration was encouraged but that copying wholesale was not.

These participants felt that by ensuring their collaboration on homework problems never crossed into the territory of copying, they were working within the parameters set by their instructors.

However, for others the fact that their instructors must have been aware of their collaboration on homework and yet did not address it explicitly left them doubting the legitimacy of the practice. For instance, Quinn recalled feeling that working together on homework assignments “wasn’t necessarily what the lecturers wanted us to do” and that:

Quinn: We shouldn’t be doing it, yeah. That it was our dirty little secret that we did this [collaborate] and they didn’t know. Or maybe it was a case of they knew but they didn’t let on that they knew, and no one said anything. It shouldn’t really be happening; we definitely got a bit of an impression of that.

5.4.5.2 The individual nature of assessment

The participants in this study were predominantly assessed through closed-book examinations. In some courses, they submitted homework assignments for credit. Given that their instructors were generally unclear on whether collaboration on homework was permitted, some participants experienced a sense of guilt about working together on these assignments.

Sullivan reported feeling slightly constrained in their collaboration on homework assignments in contrast to the sense of freedom they experienced working with peers on past examination problems:

Sullivan: I think that those were the clear moments when we really worked well together, . . . because for assignments there was always a tiny bit of guilt, maybe about maybe we're copying each other, stuff like this. But when it came to studying for finals, and just doing past problems, that was when we really got to work together and really go through problems together.

Other participants felt compelled to ensure that they balanced working collaboratively and individually on homework problems since they would ultimately be assessed individually. For instance, Reilly reflected:

Reilly: I think it's important to do some individually and not to lean too much. Because at the end of the day, when you sit in your final exam it's going to be by yourself. You need to make sure that you know the stuff without them. There is a balance, I think both are important.

5.4.5.3 Doubts regarding the collaborative nature of mathematics

Most of the participants perceived mathematics as an individualistic discipline or reported having no strong impression as to its collaborative nature. In contrast to the general opinion of our participants, Nolan and Brennan both felt that research mathematics was collaborative, basing their opinions on seeing their instructors interact with one another. Lynch distinguished between pure and applied mathematics, perceiving the former as being individualistic and the latter collaborative. Sullivan said that while at the beginning of their degree they “thought it was the man on his own in a room writing on a piece of paper”, after completing undergraduate research projects and beginning postgraduate studies they have “realised that each person can only do so much on their own and each person can only approach a problem in so many ways on their own and really good work only gets done by groups”. However, the stereotype of the lone genius appeared to persist for Sullivan who remarked that: “there are still some very brilliant people who put papers out on their own and do very, very well”. Lynch similarly positioned individual mathematical achievements as potentially superior to collaborative forms of research: “you need to be much better than I am to work on your own the whole way through a research career”.

For some participants, their perception of mathematics as an individualistic discipline gave them cause to consider the legitimacy of their collaboration. Doherty distinguished between their collaborative approach to learning undergraduate mathematics and the practices

of research mathematicians: “I think as like, a mathematician in like the real world, I see it as extremely individual. But as mathematics in university, I would kind of see it as you collaborate with your friends to get by”. Quinn also expressed doubts about the collaboration they engaged in during their undergraduate degree, questioning whether they had behaved in a manner authentic to the discipline of mathematics: “I definitely kind of would have come out of my degree thinking that maths was a solo venture and maybe we were kind of, we weren’t strictly being kosher by working together”.

5.5 Discussion

Despite acknowledgement from both mathematics educators and mathematicians that most of undergraduate mathematics students’ learning takes place out of class (Lew et al., 2022; Rupnow et al., 2021; Wu, 1999), the predominant focus of research to date has been on what happens in the classroom. In this paper, we report on an exploratory study examining one aspect of undergraduate mathematics students’ out-of-class learning: their collaboration on homework problems. Specifically, the goal of our study was to examine who students collaborate on their homework with (RQ1), how this collaboration works (RQ2), and what students perceive to be the affordances and constraints of collaborating on homework (RQ3). Ten graduates from mathematics degrees from a large, research-intensive, Irish university were interviewed and RTA (Braun & Clarke, 2022b) was used to analyse the data.

In Section 5.4, we presented four themes addressing our three research questions. While each theme has a unique focus, centring on a particular aspect of the interview data, we also wish to acknowledge the interconnected nature of our results. Previous research has noted the “complexity inherent to student-student collaborations” (Kontorovich, 2023, p. 1537). It is therefore not surprising that we were unable to entirely separate out the issues of who our participants collaborated with, how they collaborated on homework, and the associated affordances and constraints they perceived. We now take the opportunity to provide a holistic summary of our main findings, illustrating the points of connection between our four themes.

Our participants reported that collaboration on a homework problem was precipitated by getting stuck on it after attempting it by themselves. They recalled working on problems that they were stuck on with specific peers, who were also their friends and similar to them in terms of academic level and motivation. If their collaborators were also stuck on the problem they would collectively brainstorm and discuss various ideas for solving the problem. The fact that participants and their collaborators were at a similar academic level and trusted one another meant they were comfortable and able to understand and build upon one another’s ideas. In some cases, participants recalled instances where one of their

collaborators helped them with a problem that they were stuck on. However, over time collaborators would alternate between being the one giving and receiving help on a particular problem ensuring that there was both give and take in the collaborative relationship. Our participants emphasised the distinction between their collaboration on homework problems and copying. The importance of maintaining this distinction was reinforced for some by vague messages from their instructors on whether collaboration on homework was permitted and doubts about the collaborative nature of research mathematics. Despite these factors that constrained some participants' engagement in collaboration, there were a variety of benefits that encouraged participants to collaborate on homework. Discussing problems was perceived as beneficial to mathematical learning, and working on homework with friends was depicted as social and fun. Even when collaboration did not result in a solution to the problem they were stuck on, participants derived comfort from the knowledge that they were not the only one struggling. When collaboration did result in a potential solution, participants stated that they would then individually check and write out this solution, reflecting the fact that they would ultimately be assessed individually.

In the preceding paragraph, we have presented a summary of the main commonalities identified in our participants' accounts of collaborating on homework, illustrating the interconnected nature of the four themes presented in Section 5.4. We wish to emphasise that, given the interview-based nature of this study, our findings do not support the conclusion that all undergraduate students collaborate on their homework in the manner depicted by our participants. Indeed, this type of statistical-probabilistic generalisability is not a meaningful goal of qualitative research in general (B. Smith, 2018). There are a number of contextual factors that we feel are important to consider in relation to our findings. For instance, the participants in this study were highly motivated and high-achieving, undergraduate students as evidenced by their progression to postgraduate study. Additionally, participants mainly spoke about their collaboration on homework in upper-level courses. We posit that there may be differences between students' collaboration at the beginning, as opposed to the latter stages, of their degree. To date, most of the research on undergraduate mathematics education has been conducted in the US (Kinnear et al., 2024). Our study was conducted in an Irish context. It is therefore important to consider the potential differences between the setting of our study and the 'typical' US context.

Acknowledging the particular context in which this research was undertaken is key to accurately interpreting our results. However, we do believe that there is scope for our findings on undergraduate students' collaboration on homework to generalise beyond our local context in two ways. Firstly, our results generalise through transferability (B. Smith, 2018) which refers to "qualitative research that is richly contextualised in a way that allows the reader to

make a judgement about whether, and to what extent, they can safely transfer the analysis to their own context or setting” (Braun & Clarke, 2022b, p. 143). By highlighting potentially important aspects of the context of our study, we enable readers to evaluate the potential applicability of our results to other contexts and cohorts of students. Secondly, we believe our findings are significant in terms of the scope they offer for analytic generalisability (B. Smith, 2018). In qualitative research that generalises in this manner “it is the concepts or theories that are generalizable, not the specific context or population” (p. 141). The findings of our exploratory study indicate potentially productive directions for further research on the important topic of students’ out-of-class learning in undergraduate mathematics courses. We conclude this discussion section by outlining three ways in which future research could build upon our findings.

Previous research has indicated that mathematicians regard students’ engagement with homework problems as a key component of their out-of-class learning (Rupnow et al., 2021). However, to date, how undergraduate students engage with mathematical tasks outside of the classroom and clinical interview settings has not been widely studied (Lew & Zazkis, 2019). Indeed, Rupnow et al. (2021) cited exploring how students actually engage with the problems their instructors assign them as an important direction for future research. In this study, we have provided an existence proof that, at least for some students, collaborating with peers is a key aspect of their engagement with homework. We have also provided initial insight into the nature of this collaboration by presenting a three-phase workflow, represented diagrammatically in Fig. 5.4.1, that captured the commonalities in our participants’ descriptions of how they collaborated on their homework problems. However, we acknowledge that this workflow does not capture the nature of all students’ collaboration on homework problems. Indeed, in Section 5.4.2.2 we presented a vignette of Quinn who, in contrary to the three-phase workflow outlined, reported collaborating in a ‘stop-start’ manner, whereby they would iteratively discuss a problem with peers and work individually. Therefore, our first suggested direction for future research is for further examination of the variety of ways in which students collaborate on their homework, building upon the workflow we presented in Fig. 5.4.1. Pursuing this line of research might involve the “scaling up (increasing numbers) [and] scaling out (using samples with differing characteristics)” (Melhuish, 2018, p. 9) of our study. We posit that the latter, scaling out, may be of greater interest in this case. While we have hypothesised about a number of contextual factors that may have influenced our participants’ collaborative practices, further research, with different cohorts of students, is required to evaluate whether these are indeed factors of influence.

One finding from our study that would be particularly interesting to investigate the generalisability of is whether it is common for students to work on their homework with

specific peers. All participants in our study reported collaborating on their homework with particular peers, with some describing their membership of seemingly well-defined groups of collaborators of stable membership. One such example is that of Dunne and Reilly, described in Section 5.4.2.1. Examining Wenger's (1998) "indicators that a community of practice has formed" (p. 126), we posit that some of these groups of collaborators could be considered communities of practice. Cox (2005) noted that in Wenger's (1998) treatment of communities of practice his focus is on "the impact on individual identity [emphasis added]" (p. 527). This emphasis on identity is reflective of how the theoretical framework has previously been utilised in mathematics education research (e.g., Solomon, 2007). Our focus in this paper was on identifying commonalities in participants' descriptions of their collaboration on homework problems. This was reflected in our choice of RTA as the data analysis method (Braun & Clarke, 2022b), which enables researchers to spotlight patterns of meaning across participants' accounts. However, our second suggested research direction involves investigating the prevalence of communities of collaborating students and using Wenger's (1998) theory in conjunction with an idiographic data analysis method (e.g., J. A. Smith et al., 2022) to examine the impact of membership of such communities on students' identity formation.

Generally, participants in this study identified the companionship and support of their collaborators as a key benefit of collaborating on homework. This is notable as, despite the importance mathematicians attribute to students' work on homework tasks (Rupnow et al., 2021), "little is known about the pedagogical actions that will encourage constructive engagement on these homework activities" (Melhuish et al., 2022, p. 17). Our participants depicted collaboration as something that enhanced their engagement with homework in a variety of ways. The affordances they identified align with the benefits of students' out-of-class collaboration that have been identified in other qualitative research (Brown et al., 2005; Seymour & Hewitt, 1997; Solomon et al., 2010). This suggests that some benefits of students working with peers may generalise beyond local contexts and this could be further investigated by conducting a qualitative meta-synthesis (B. Smith, 2018). While our findings suggest that collaborating on problems can be a productive way of students engaging with homework, they also point to factors that can both facilitate and constrain students' collaboration on homework. Therefore, our final suggestion for future research relates to investigating ways in which undergraduate mathematics students can be supported in productively collaborating on their homework.

In particular, our findings point to two factors that appear to influence how, and to what extent, students engage in collaboration on their homework. Firstly, some of our participants questioned the legitimacy of their collaboration in light of the fact that their instructors did not

explicitly address whether collaboration on homework was encouraged, or indeed permitted. This raises questions about instructors' views on students' working together on homework. Are instructors aware of the extent and nature of their students' collaboration on homework problems? How do they view the distinction between collaboration and copying? Do they feel they communicate their views on the issue clearly to students? We believe that clarifying instructors' views on these issues and investigating ways in which they can be supported in conveying their expectations around collaboration to students is a productive avenue for future research.

Secondly, some of our participants' perceptions of mathematics as an individualistic discipline caused them to doubt their collaboration on homework. While we can only comment on the perspectives of our ten participants, we note that most perceived mathematics as an individualistic discipline or reported having no strong impression as to its collaborative nature. Participants appeared unaware of the important role collaboration plays in research mathematics, as indicated by published research (Burton, 1998; Grossman, 2005) and our own review of the authorship of recently published mathematics journal articles (see Appendix 5.A). Given that some feel that undergraduate mathematics courses should prepare future mathematicians (Woods & Weber, 2020) and be reflective of the nature of research mathematics (Laursen & Rasmussen, 2019), we feel that this apparent discrepancy warrants further investigation. A confirmatory survey like that of Mejía-Ramos and Weber (2014) could be used to examine to what extent the perspectives of our participants generalise to the wider population of undergraduate mathematics students.

5.6 Conclusion

In this paper, we examined one aspect of undergraduate mathematics students' out-of-class learning: their collaboration on homework problems. Despite mathematicians and mathematics educators' emphasis of the importance of students' out-of-class learning and engagement with homework, the topic has been under-researched in the mathematics education literature. Our exploratory study on the issue involved using RTA to analyse interviews with ten graduates of mathematics degrees from an Irish university. Our findings demonstrate the ways in which collaboration can form an important and productive part of engaging with homework for some students. We believe that our results are particularly significant in terms of the direction they can provide to future research on students' out-of-class learning. We have hypothesised about a number of contextual features of our study which may have influenced our participants' engagement in collaboration, and advocate for research to investigate students' collaboration on homework in other contexts and cohorts of students. Addition-

ally, our findings indicate that collaborating on homework, and in particular the support and friendship of collaborators, can form a hugely positive aspect of students' undergraduate mathematics learning experience. Therefore, we believe that an avenue for future research with potentially important pedagogical implications is to investigate ways in which students can be supported in effectively collaborating on homework.

We note that the data collection for this study was conducted in spring 2022, prior to the launch of Open AI's ChatGPT in November 2022 (OpenAI, 2022). For this reason, the influence of AI tools was not raised in the interviews. However, in the subsequent years AI tools have become increasingly prevalent in higher education (Stöhr et al., 2024) and more mathematically proficient (Hubert et al., 2026). While not relevant to the present study, we think it interesting to consider the potential implications of the availability of AI tools on students' collaboration. Getting stuck on problems was a key catalyst in terms of prompting participants to collaborate on homework. We wonder whether, as the mathematical proficiency of AI tools increases (Hubert et al., 2026), students may be more likely to consult AI tools when stuck rather than their peers? However, we also note that the participants in our study stressed the importance of consolidating their collaborative efforts individually and appeared to appreciate the learning opportunities associated with homework. This suggests that some students may approach the use of AI tools in a mature and considered manner. We call for future research to build on our study and to investigate the influences of AI tools on students' collaboration and engagement with homework more broadly.

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Appendix to Chapter 5

5.A Investigation of the authorship of mathematics journal articles

Table 5.A.1: Number and percentage of single- and multiple-authored publications in 2023 in the top twelve journals under “Mathematics” as ranked by Scimago JR

Journal Name	Single Author	Two Authors	Three Authors	Four Authors	Five Authors	More than Five	Total	% Single Author	% Two Authors	% Three or more Authors
Acta Numerica	2	2	1	2	0	0	7	28.6	28.6	42.8
Annals of Mathematics	7	13	3	5	2	1	31	22.6	41.9	35.5
Publications de l’Institut des Hautes Etudes Scientifiques	2	2	3	0	0	1	8	25.0	25.0	50.0
Journal of the American Mathematical Society	4	12	4	2	0	0	22	18.2	54.5	27.3
Acta Mathematica	2	4	3	0	1	0	10	20.0	40.0	40.0
Inventiones Mathematicae	19	25	19	8	1	0	72	26.4	34.7	38.9
Communications on Pure and Applied Mathematics	6	34	25	4	1	0	70	8.6	48.6	42.8
Duke Mathematical Journal	15	22	17	7	1	0	62	21.4	31.4	47.2
Archive for Rational Mechanics and Analysis	12	44	38	13	0	0	107	11.2	41.1	47.7
Stata Journal	16	12	9	4	5	2	48	33.3	25.0	41.7
Biometrika	1	19	29	7	5	0	61	1.6	31.1	67.3
Journal of the European Mathematical Society	23	46	35	2	2	0	108	21.3	42.6	36.1

5.B Interview schedule

Section 1 – Background

- What undergraduate degree did you do at university?
 - Why did you choose that degree?
 - When did you graduate?
 - What did you do after you graduated?

Section 2 – What is the nature of informal collaboration in undergraduate mathematics programs?

- Did you collaborate with your peers during your mathematics degree?
 - Can you tell me a little bit about that?
 - Did you collaborate from the start of the degree? Why?/Why not?
 - At what stage in the program did you feel comfortable collaborating with peers? Why?
 - At what stage in the program did you feel you benefitted most from collaboration? Why?
- Can you tell me a little about the people you collaborated with?
 - What brought you together?
 - Was there a consistent group of you or did it vary depending on the module?
 - Did you mainly collaborate with people one-on-one or was there a group of you who collaborated together?
- In what ways would you say that those you collaborated with were similar to you?
 - For example, mathematical knowledge and skills, motivation, aims?
- In what ways would you say that those you collaborated with had skills/knowledge that complemented yours?
 - For example, mathematical ability, motivation, aims?
- What type of things did you collaborate with other people on? How did it work?
 - Problem sheets?

- Understanding lecture notes?
 - Revising for exams?
- Do you have an example from a specific module? Any more examples?
- Did you ever experience an instance where you and your peers were stuck on a problem, but collectively managed to solve it?
- What would you do if the group was collectively unable to solve a particular problem/understand a particular concept?
- Where did your collaboration take place?
 - Was it informal or more formally organised?
- Before the pandemic, did you collaborate with peers online? In what ways?
- In your opinion, are there any aspects of peer collaboration that you think cannot be replicated successfully online?
 - What was your experience during the pandemic?
- In your opinion, are there any aspects of peer collaboration that are better in an online environment?
 - What was your experience during the pandemic?
- Did you ever have to work on a credit-bearing group assignment or group project?
 - Can you give me an example? How did you find the experience? What percentage of the grade did it account for?
 - Do you have another example? How did you find the experience? What percentage of the grade did it account for?
- Were you given allocated groups or were you allowed to pick your own groups?
- Do you think this (allocated groups/not) had an impact on your experience?
- Were credit-bearing group assignments or group projects a common occurrence in your modules?
 - What type of modules was it most common in, for example, ACM, MATH, STAT etc.?

- Do you think group assignments or projects are appropriate assessments in mathematics modules?
 - What about pure mathematics specifically? Applied mathematics? Statistics?
- In your opinion, does the nature of collaboration on a credit-bearing assignment or group project differ from the nature of the collaboration you engage in more informally with peers?
 - What do you feel you learned from participating in credit-bearing assignments or group projects?

Section 3 – What are students’ perceptions of the value of informal collaboration to their mathematics learning?

- Do you feel you have benefitted from informal collaboration over the course of your undergraduate studies?
 - In what ways?
 - Did your mathematical learning benefit?
 - Aside from learning, what are the other benefits of collaborating with peers in your opinion?
- How do you view the balance between working on your own versus working with others on mathematics?
- Do you feel there are any disadvantages to collaborating?
 - Can you give me an example of when collaboration did not work?

Section 4 – Messages about informal collaboration in undergraduate mathematics

Now I would like to explore your perceptions of the messages that lecturers gave, either explicitly or implicitly, about whether collaboration was encouraged or not in their modules. We have already talked about modules where collaborating on a credit-bearing group assignment or group project was part of the assessment of the module, so obviously in this instance, lecturers explicitly wanted you to collaborate. Now let me turn to when you had to submit credit-bearing individual work as part of the module assessment:

- Were lecturers generally clear about whether collaboration was permitted or not?

- Did some make it very explicit that collaboration was not permitted?
 - Did some leave it vague about whether collaboration was permitted or not?
 - In general, what percentage of the final grade were these assignments worth?
- Finally, leaving credit-bearing individual and group assignments aside, what messages did you receive, implicitly or explicitly, from lecturers about whether collaborating with peers was to be encouraged or not?
 - Do you think they expected you and your peers to be discussing the module and collaborating informally together on it?
 - Or do you think they assumed that everyone worked by themselves?
- At the end of your undergraduate degree, did you have a sense of whether mathematicians collaborated on research or not?
 - What informed your sense of this?

Section 4 – Closing questions

- To sum it up what do you feel were instances of your most successful collaborations during your undergraduate degree?
 - Did the type of task/subject matter/level of module play a role, in your opinion?

Acknowledgement of Authorship Form

This form must be completed for each paper to be included and placed in front of that paper in the thesis. The purpose of this is to ensure that shared authorship is appropriately explained in a thesis. Please review the [Guidelines for Theses from a Collection of Papers](#)¹ and [UCD Authorship Policy](#) when completing this form. A word version of this document is available on the [Graduate Studies Document Repository](#).

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2. Details of the Paper
Title of Paper: A case study of undergraduate students' collaboration
Current Status of the Paper: ☐ Drafted ☐ Submitted. ☐ Under review ☐ Under revision ☐ Accepted for publication ☒ Published
Name of Journal (if known): Proceedings of the Tenth Conference on Research in Mathematics Education in Ireland (MEI10)
Digital Object Identifier (if published): https://doi.org/10.5281/zenodo.17398479
AUTHOR CONTRIBUTIONS
Part A: Contribution of Examination Candidate <i>A full description the Examination Candidate's intellectual and practical contribution to the research presented in the paper. It would be helpful for the Examination Committee if an estimated percentage of the candidate's contribution to this paper could be provided below. Each section of the paper should be listed separately i.e. Introduction, Methods, Figure 1, Table 2 etc. The examination candidate should either be the sole author/primary author or a major contributor to the paper. Please also review the UCD Authorship Policy when completing this section.</i>
Percentage contribution of the candidate to the paper : 75%
Description of Examination Candidate's contribution for each section of paper: (Max 500 words): I developed the research aims, incorporating feedback from my supervisor. I reviewed relevant background literature. With my supervisor, I collaboratively wrote the interview schedule. I conducted the interviews for the study was responsible for their transcription. I was primarily responsible for the data analysis, which developed through discussions with my supervisor. I wrote all sections of the manuscript, incorporating feedback from my supervisor.

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Part B: Contribution(s) of Co-Authors to this paper

Please outline briefly the intellectual and practical contribution of any co-authors to the paper.

Co-Author 1

Name of Author No.1	Maria Meehan -
Contribution of Co- Author No.1	25% contribution <i>Brief outline of contribution: (Max 200 words)</i> As my supervisor, Maria provided essential advice and guidance in conceptualising and developing the methodology for the study. I was responsible for conducting the interviews, however Maria provided valuable feedback on my interviewing technique. Maria took on the role of experienced supervisor and collaborator during the data analysis process. She was familiar with the data and we discussed the analysis as it progressed. Maria reviewed and provided feedback on all sections of the manuscript.

Signature of Supervisor

Name of Principal Supervisor	Maria Meehan
<i>I confirm that the information provided in this form is an accurate reflection of the intellectual and practical contribution of the examination candidate to this paper. I can confirm that the student is a major contributor to this paper.</i>	

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Chapter 6

A case study of undergraduate mathematics students' collaboration

Abstract

We report on a case study from a research project examining undergraduate mathematics students' collaboration on homework. We examine the collaborative practices of two participants from the larger study who worked together on homework in a group that we propose is a community of practice. These two participants are chosen due to their aspirations of pursuing a career in research. While preparing future mathematicians has previously been identified as a goal in university mathematics, we examine the issue from the student perspective. Using communities of practice as our theoretical frame, we analyse how the two participants attempted to align their collaborative practices as undergraduates with their imagined futures as research mathematicians.

Keywords: research, students, communities of practice, collaboration

6.1 Introduction

In this paper, we report on a case study from a research project in which we examined undergraduate mathematics students' collaboration on homework (Murphy & Meehan, 2024). As part of the larger study, ten recent graduates from a large, research-intensive university in Ireland were interviewed about the collaboration they undertook on homework problems. Reflexive thematic analysis (Braun & Clarke, 2022) was used to identify commonalities in participants' accounts. We found that participants collaborated on homework with specific peers who were also their friends and tended to be like them in terms of academic ability and interests. Participants reported working on problems alone initially and collaborating at the point at which they got stuck. All participants felt they benefitted from collaborating. We also identified constraints on participants' collaboration which included doubts regarding the collaborative nature of research mathematics.

Our initial analysis of the interview data was focused on identifying commonalities in participants' accounts. However, we also noted idiosyncrasies in participants' collaborative practices and felt that the indigenous way in which participants negotiated their response to constraints resonated with Wenger's (1998) theory of communities of practice (CoP). In this paper, we take the opportunity to explore this aspect of our data further by presenting a case study of two participants, Sullivan and Lynch¹, who collaborated as undergraduates in a group which we propose can be considered a CoP. We substantiate our claim that this group is a CoP by illustrating the manifestation of three characteristics of a CoP in their practice of collaborating on homework. We chose to focus on these two participants as both spoke of their ambitions to pursue a career in research mathematics. Prior research has identified preparing future mathematicians as a goal of undergraduate mathematics education (Woods & Weber, 2020). In this paper, we examine the issue from the student perspective. Specifically, we utilise the theory of CoPs to examine how Sullivan and Lynch attempted to *align* their practice of collaborating as undergraduates with their *imagined* futures as mathematicians.

6.2 The collaborative nature of research mathematics

In popular media, mathematics has traditionally been portrayed as a solitary activity (Barba, 2018). Contrary to this stereotype, there are indications that mathematics is becoming increasingly collaborative. Grossman (2005) reviewed patterns in mathematics research publications and found that the proportion of single-authored publications fell from 91% in the 1940s to 54% in the 1990s. Murphy and Meehan (2024) conducted a review of the authorship

¹Sullivan and Lynch are gender-neutral pseudonyms.

of articles published in 2023 in the top twelve “Mathematics” journals as ranked by Scimago JR. Single-authored papers accounted for 18% of this sample, suggesting the continuation of the collaborative trend. Burton’s (2004) interview-based study with 70 mathematicians from the UK and Ireland provides insight into the nature of research collaboration. Burton found that all but four participants reported working either co-operatively across disciplines or collaboratively with other mathematicians. Participants discussed cognitive and affective benefits of collaborating. They also talked about the role of competition within mathematical culture and reported that while the discipline has become increasingly collaborative, competition is pervasive and often co-exists with collaboration.

6.3 Theoretical framework

Stated succinctly: communities of practice (CoPs) are “groups of people who share a concern or a passion for something they do and learn how to do it better by interacting regularly” (Wenger-Trayner & Wenger-Trayner, 2015, p. 1). Wenger (1998) describes three characteristics of a CoP as the *mutual engagement* of participants negotiating a *joint enterprise* over time and developing a *shared repertoire* in the process. *Mutual engagement* involves establishing ways of working. Examples of the *shared repertoire* of a CoP include the discourse used, routines created, and the creation of artifacts. Simplistically, the *joint enterprise* of a CoP can be thought of as its members’ understanding of what the enterprise is about, its aims and ideals. Wenger (1998) cautions that joint enterprise goes beyond a “statement of purpose” and is negotiated by the community as they determine “what is important and why it is important, what to do and not to do” (p. 81). Wenger notes that CoPs are not “self-contained entities” (p. 79). Their practice can be influenced by external constraints to which they respond in an indigenous manner.

While *engaging* in practice is one way that an individual can identify as belonging to a CoP, Wenger (1998) presents two other modes of identification: *imagination* and *alignment*. Identifying with a CoP through *imagination* is described as “creating images of the world and seeing connections through time and space by extrapolating from our own experience” (p. 173). *Imagination* can enable us to see our engagement in practice in a different light and may be affected by stereotypes we hold. Identifying with a CoP through alignment involves “coordinating our energy and activities in order to fit within broader structures” (p. 174). In Farnsworth et al. (2016) Wenger discusses the example of a girl who *imagined* herself as a marine biologist. In terms of *alignment*, she had to consider what practices she needed to engage in so that she could ultimately *engage* in her dream job.

Using the CoP theoretical framework, we articulate our research questions as follows:

- **RQ1:** How are the three characteristics of a CoP manifested in the practices of a group of undergraduate mathematics students collaborating on homework?
- **RQ2:** How do members of this CoP *align* their practice of collaborating on homework with their *imagined* future as research mathematicians?

6.4 Methodology

At time of interview, Sullivan and Lynch were recent graduates of a mathematics degree from a large, research-intensive Irish university. During the interviews, it became apparent that Sullivan and Lynch had collaborated on homework within the same group of peers. This is likely a result of the snowball sampling technique used during participant recruitment, with participants asked to recommend peers who might also consider participating. At the time of interview, both Sullivan and Lynch were pursuing graduate study in the mathematical sciences with the goal of conducting research and obtaining a PhD.

The first author conducted one-on-one, semi-structured interviews with participants which lasted an average of 30 minutes and were recorded and transcribed. Participants were asked to describe their experiences of collaborating with peers – who they collaborated with, what they collaborated on, how the collaboration process worked, and their perceptions of its benefits to their learning. We also asked the graduates whether they believed that research in mathematics is a collaborative or individualistic endeavour. (See Murphy and Meehan (2024) for the full interview schedule.)

Template analysis (Brooks et al., 2015) was chosen as the method of data analysis for this study. Initial coding was conducted by the first author before being reviewed and refined collaboratively by both authors. To address RQ1, *mutual engagement*, *joint enterprise*, and *shared repertoire* (Wenger, 1998) were used as a priori codes to examine how these characteristics of a CoP manifested in the practices of the participants. To address RQ2, we drew on data analysis undertaken as part of the larger study using reflexive thematic analysis (Braun & Clarke, 2022). Specifically, we examined instances of codes related to participants' perceptions of the collaborative nature of research mathematics in Sullivan and Lynch's interviews and reframed our analysis using Wenger's concepts of *imagination* and *alignment*.

6.5 Findings

To address RQ1, we introduce Sullivan, Lynch and their collaborators and substantiate our claim that they can be considered a CoP by illustrating the *mutual engagement*, *joint enterprise*, and *shared repertoire* of the group. In their final year, Sullivan and Lynch collaborated extensively with a group of five or six students. Both participants stated that the main factor that united the group was their mathematical interest and research ambitions:

Lynch: I think the group in fourth year, like, very much what kind of, like, made us friends was the fact we are, like, very into the maths and physics. That was the main thing that drove the group.

Sullivan: We were people with kind of similar ambitions, similar interests. All wanted to do research. From there, it kind of naturally progressed to talking about the work and then doing the work together.

The group's *mutual engagement* consisted of their collective work on homework problems. Sullivan explained that through working together the group became friends and enjoyed the social aspect of collaborating on homework:

Sullivan: In kind of academic level, we all typically got similar grades which is definitely something that we thought was important because you all want to contribute an equal amount. And I think we're all quite social relative to the typical people who study maths and so we quite enjoyed the aspect of having like a social work together.

As Sullivan and Lynch's community of collaborating peers engaged in the practice of working together on homework, they developed a *shared repertoire* which consisted of a routine and shared discourse. Both participants explained how they routinely met in a room on campus which contained multiple blackboards. Lynch exemplified how they would work on homework problems individually and ask one another questions when stuck:

Lynch: The way we'd work a lot in final year was we'd, it was in one of those rooms with like three sets of double blackboards and we'd literally just stand there, and we'd all work away on the problem sheet on our own on each blackboard. And you'd kind of go like 'Oh what are you doing for this thing here?'. And you're like 'I've tried this thing' and you'd try it yourself. Very much at the high level, not like high level, but yeah, you're not going like 'Here's my solutions, look at this'. It's very much, 'Alright, I'm a bit stuck, what are you thinking?'.

As Lynch illustrated, one characteristic of the group's practice was that they restricted their collaboration on homework to high-level discussion. This was important to the members, and

we consider it part of their *joint enterprise*. For the remainder of this paper, we focus on one aspect of the CoP's negotiation of their *joint enterprise*. As previously stated, members of the CoP all shared the ambition of pursuing postgraduate mathematical studies and careers in research. We propose that, as undergraduates, they identified with the community of research mathematicians through *imagination*. We address RQ2 by examining how they *aligned* their practice of collaborating on homework with their perceptions of appropriate collaboration in the context of research mathematics.

When asked about their impressions of the collaborative nature of research mathematics, Sullivan reflected that at the beginning of their undergraduate studies they “thought it was the man on his own in a room writing on a piece of paper”. However, after completing an undergraduate research project with a collaborative research group they came to realise: “each person can only do so much on their own and each person can only approach a problem in so many ways on their own and really good work only gets done by groups”. Similarly, Lynch reflected that the research group that they completed a research project with would “all chat about, like, recent research in the area, and like, it definitely seemed quite collaborative”. While both Sullivan and Lynch recognised the potential for mathematical research to be collaborative, they emphasised individual intellectual ability as key to their *imagined* futures as researchers: “if you want to go into research you do need to be able to work independently” (Lynch). Additionally, both positioned individual academic achievement as intellectually superior to work completed by groups. The stereotype of the lone genius appeared to persist for Sullivan: “there are still some very brilliant people who put papers out on their own and do very, very well”, while Lynch reflected: “you need to be much better than I am to work on your own the whole way through a research career”.

We now examine how Sullivan and Lynch attempted to *align* their practice of collaborating on homework with their perceptions of collaboration in a research context. While both appreciated the merits of collaboration, they emphasised the importance of maintaining one's independent ability: “I think collaborating is great and I think it's something that doesn't get done enough but you can't do away with individual work” (Sullivan). This emphasis on individual work manifested in the practice of the CoP, with each member initially attempting homework problems independently and only collaborating on problems that multiple members “were really struggling with” (Sullivan). As previously mentioned, the CoP's collaboration on these difficult problems was restricted to high-level discussion. Sullivan explained: “it'd be sketching ideas on the board. Usually, we would try to avoid doing out explicit things because that didn't always feel in the spirit of it”. Lynch elaborated that when the group was stuck on a difficult problem, each member wanted to be the one to

figure the solution out. The emphasis on individual achievement contributed to a competitive atmosphere, which Lynch reported that the group enjoyed:

Lynch: Nobody wanted to get the answers off. It was, it was like a bit competitive, but in a good way, that you'd always be trying to figure it out, like, the cool idea to figure out the problem.

However, the normalised competition in the CoP and the emphasis on individual academic ability appears to have contributed to Sullivan and Lynch, at times, doubting their own mathematical capabilities. Sullivan reflected on how they did not enjoy feeling reliant on others when stuck on a problem: "It didn't feel good to be copying what others are doing and so, if you didn't really get something and you felt like someone just told you what to do, that never felt good". Lynch expressed similar feelings:

Lynch: It can stress you out sometimes, it you're in this group of people that are doing really well and you're just not getting it for a time, or you feel a bit stupid and you're like, 'Oh my God, I'm really behind', like, I can't understand something.

Lynch questioned their ability to work independently, which they perceived important to their *imagined* future as a researcher: "I was a little bit worried that I was kind of using it too much as a crutch and if you want to go into research you do need to be able to work independently".

6.6 Discussion

In this paper, we illustrated the existence of CoPs of undergraduate mathematics students collaborating on homework. We highlight two main implications. Firstly, our study emphasises the importance of accurately representing the nature of research to undergraduate students. Both participants identified their experience of completing undergraduate research projects as key in informing their perceptions of the collaborative nature of research. We call for further research on undergraduate students' perceptions of mathematical research and how these perceptions can be accurately informed. Secondly, we wish to highlight the competition that appeared to be a normal part of the CoP's collaborative practice. Research would suggest that the co-existence of competition and collaboration is prevalent in research mathematics (Burton, 2004). However, we note the way in which it caused Lynch and Sullivan to doubt their independent mathematical capabilities. Given that we are reporting on a case study situated in a particular context, we call for further research to examine the co-existence of collaboration and competition in research mathematics and its influence on the perceptions and practices of undergraduate students.

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1. Examination Candidate Details
Examination Candidate Name: Ciara Murphy
Examination Candidate UCD Student Number: 18322691
Research Degree for which thesis is being submitted: Doctor of Philosophy
Title of Research thesis: Looking beyond the lecture: A qualitative examination of pedagogical decision-making and student learning in upper-level courses
2. Details of the Paper
Title of Paper: Undergraduate mathematics students' views on assessed group work
Current Status of the Paper: ☐ Drafted ☐ Submitted. ☐ Under review ☐ Under revision ☐ Accepted for publication ☒ Published
Name of Journal (if known): Proceedings of the Ninth Conference on Research in Mathematics Education in Ireland (MEI9)
Digital Object Identifier (if published): https://doi.org/10.5281/zenodo.10037327
AUTHOR CONTRIBUTIONS
Part A: Contribution of Examination Candidate <i>A full description the Examination Candidate's intellectual and practical contribution to the research presented in the paper. It would be helpful for the Examination Committee if an estimated percentage of the candidate's contribution to this paper could be provided below. Each section of the paper should be listed separately i.e. Introduction, Methods, Figure 1, Table 2 etc. The examination candidate should either be the sole author/primary author or a major contributor to the paper. Please also review the UCD Authorship Policy when completing this section.</i>
Percentage contribution of the candidate to the paper : 75%
Description of Examination Candidate's contribution for each section of paper: (Max 500 words): I developed the research aims, incorporating feedback from my supervisor. I reviewed relevant background literature. With my supervisor, I collaboratively wrote the interview schedule. I conducted the interviews for the study was responsible for their transcription. I was primarily responsible for the data analysis, which developed through discussions with my supervisor. I wrote all sections of the manuscript, incorporating feedback from my supervisor.

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Submission Date for Thesis 26/06/2026

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Chapter 7

Undergraduate mathematics students' views on assessed group work

Abstract

Previous research has examined the implementation of collaborative learning and the conditions under which it may be optimally effective. One means of implementing collaborative learning in the university setting is through assessed group work, whereby students are asked to work together to submit a piece of coursework for assessment. In the context of undergraduate mathematics, assessed group work is a relatively novel means of assessment, and previous research has documented undergraduate mathematics students' preference for the retention of the status quo of assessment through closed-book examinations. This paper builds on previous research, related to both collaborative learning and assessment, by examining undergraduate mathematics students' views on assessed group work. To this end, semi-structured, one-on-one, interviews were conducted with ten recent graduates from undergraduate degrees in the mathematical sciences. These interviews were analysed using reflexive thematic analysis. Findings indicate an uncertainty among students as to the purpose of implementing assessed group work in undergraduate mathematics degrees. Additionally, participants demonstrated concern regarding mechanisms of assessment and allocation of groups in any proposed implementation of assessed group work. We discuss potential implications for practice, highlighting the importance of considering students' views when evaluating and implementing novel assessment methods.

Keywords: assessed group work, collaborative learning, assessment, students

7.1 Introduction

While there is no well-established definition of collaborative learning, we will follow the broad conceptualisation of Johnson et al. (2014) and define it to be a set of methods in which students “work together to maximize their own and each other’s learning” (p. 87). Much of the research that has been conducted on collaborative learning to date has focused on its implementation and evaluating the conditions under which it might be optimally effective (e.g., Johnson et al., 2007; Springer et al., 1999). One means of implementing collaborative learning in a university context is through assessed group work, whereby students are asked to work together to submit a piece of coursework for assessment.

Undergraduate mathematics degrees in the UK and Ireland, the setting of this study, are predominantly assessed through closed-book examination (Iannone & Simpson, 2011, 2022; National Forum for the Enhancement of Teaching and Learning in Higher Education, 2016). This means that assessed group work is a rather novel concept for undergraduate mathematics students. Previous research, examining undergraduate mathematics students’ perceptions of oral performance assessment (Iannone & Simpson, 2015b), has advocated for the representation of students’ views when evaluating and implementing non-standard assessment methods. We hope to contribute to research on undergraduate mathematics students’ assessment preferences, and collaborative learning in the context of mathematics, by addressing our research question:

- **RQ:** What are undergraduate mathematics students’ views on assessed group work?

7.2 Literature review

Slavin (1996) remarked that it could be considered a point of irony that researchers who have examined aspects of collaborative learning have “often operated in isolation from one another, almost on parallel tracks” (p. 44). There is no singular, well-established definition of collaborative learning and different researchers have focused on various aspects in their analyses of the issue (Slavin, 1996). However, we will follow the broad conceptualisation of Johnson et al. (2014), defining collaborative learning as encompassing a set of methods in which students “work together to maximize their own and each other’s learning” (p. 87). We additionally distinguish assessed group work as instructor-mandated group work where students work together to submit a piece of coursework for assessment.

Previous research on collaborative learning has generally focused on the implementation of collaborative instructional methods by teachers, evaluating their effects and the conditions

under which they may be optimally effective (e.g., Johnson et al., 2007; Springer et al., 1999). Slavin (1996) outlines four broad types of theoretical perspectives from which researchers can come to the issue of collaborative learning: motivational, social cohesion, developmental, and cognitive elaboration. Each perspective places emphasis on different aspects of collaboration between students, and this is reflected in researchers' recommendations for the implementation of collaborative learning. The focus of motivational perspectives is primarily on the "reward or goal structures under which students operate" (Slavin, 1996, p. 44), and places emphasis on all group members feeling a sense of individual accountability to the group. Social cohesion perspectives are concerned with the importance of positive interpersonal relationships between peers and contend that when students feel a connection with one another, learning and peer-teaching occur as a natural consequence. The emphasis of developmental perspectives is on the learning that results from the resolution of cognitive conflicts and promotes opportunities for students to share and hear opposing viewpoints. Adopters of a cognitive elaboration perspective believe that an effective means for students to learn is through "explaining the material to someone else" (Slavin, 1996, p. 50) and emphasise peer-teaching as a result.

Much of the research referenced above relates to a general educational context. Research on students' assessment preferences has demonstrated the importance of not automatically accepting findings from the general education literature as valid in a mathematical context (Iannone & Simpson, 2015a). Closed-book examinations are the primary means of assessment in undergraduate mathematics degrees in the UK and Ireland (Iannone & Simpson, 2011, 2022; National Forum for the Enhancement of Teaching and Learning in Higher Education, 2016). Despite a general tendency, documented in the general education literature, for students to prefer less traditional forms of assessment, undergraduate mathematics students appear to favour the retention of the status quo of being assessed, primarily, through closed-book examinations (Iannone & Simpson, 2015a). Students were found to prefer, and deem most fair, assessment methods they perceived as being best able to distinguish between *individual* student's mathematical proficiencies. To this end, it was found that "any assessment that can be completed in group . . . may be deemed to be unfair" (Iannone & Simpson, 2015a, p. 1058). Additionally, some mathematics students have exhibited concern regarding the possibility of other students' performance impacting their grade in the context of assessed group work (Iannone & Simpson, 2017).

While relatively little is known about undergraduate mathematics students' engagement in, and perceptions of, collaboration in the context of mathematics, mathematicians' views on the issue have been documented to a greater extent. The collaborative nature of mathematics emerged as a prominent theme in the work of Burton (1998), who conducted an

interview-based study with 70 mathematicians from the UK and Ireland. In contrast to the often-purported image of the solitary mathematician, only four of the 70 interviewed mathematicians reported working on mathematics entirely alone and many spoke of a cultural shift towards a collaborative norm.

7.3 Methodology

As the aim of this study is to understand undergraduate mathematics students' views on, and experiences of, collaborating with their peers, a descriptive phenomenological design was chosen. One-on-one, semi-structured interviews were conducted in spring 2022 with ten recent graduates (graduating in 2019, 2020 and 2021) from degrees in the mathematical sciences from a large Irish university. Initially, graduates known to the authors were invited to participate. From there a form of snowball sampling was used, with recruited participants asked to recommend other graduates who may be interested in participating. We sought participants' views on both assessed group work and informal, voluntary collaboration with peers undertaken outside of their lectures and tutorials. The interviews lasted an average of 30 minutes and were audio-recorded and transcribed.

We undertook reflexive thematic analysis (Braun & Clarke, 2022) based on descriptive phenomenology in a manner that was guided by Sundler et al. (2019). Sundler et al. (2019) advise that rather than “attempting to set aside one’s experiences and assumptions” (p. 735), researchers should recognise and reflect on “personal beliefs, theories or other assumptions that can restrict the researcher’s openness” (p. 735). At the time of data collection, the first author was a final-year undergraduate mathematics student and, having had personal experience of working with peers on group assessments, would be considered an insider researcher (Braun & Clarke, 2022). The second author is a mathematics lecturer and mathematics education researcher. The method’s emphasis on researcher reflexivity allowed for these two disparate perspectives to be capitalised on and used as a “resource for knowledge production [...] rather than a must-be-contained threat to credibility” (Braun & Clarke, 2021, pp. 334-335).

Braun and Clarke’s (2022) six-stage process was followed. The second author took on the role of an “experienced qualitative researcher [and] supervisor” (Braun & Clarke, 2022, p. 273), both participating in and supporting the first author in the analysis process. Both authors familiarised themselves with the data corpus through repeated reading of the interview transcripts and re-listening to the audio-recordings. Throughout this process, notes were made on any initial ideas, and both authors wrote reflections on their position in relation to the data and topic. For the purposes of this paper, our research question is concerned with

participants' perceptions of assessed group work. Therefore, the data set which was coded was "all instances in the corpus where that topic is referred" (Braun & Clarke, 2006, p. 79). This data set was coded by the first author using the qualitative data analysis software NVivo 12. An inductive approach was taken, and no pre-existing theoretical framework was used. However, the first author was aware that her coding was likely influenced by prior reading of literature on assessment and collaboration and the issues highlighted therein. A mixture of semantic and more latent codes was used. For instance, while the code 'Individual marks don't really work' captures an explicit, overt meaning in the data, the code 'Different students have different goals' centred on the more implicit narrative that students should be free to work towards their own individual goals.

The first author then reviewed her codes and associated data extracts and wrote a short analytic summary of each. She then began to sort the code labels into candidate themes, which were explored in relation to one another and the entire data set, using visual thematic mappings. Through this process she decided that a candidate theme, focused on whether there was something different about group assessment in mathematics as opposed to other subjects, was under-developed as it was not addressed fully enough in the data. The remaining three themes were then named and defined, and a final visual thematic mapping produced.

7.4 Findings

Existing research has identified closed-book examinations as the prevalent assessment method in undergraduate mathematics degrees in the UK and Ireland (Iannone & Simpson, 2011, 2022; National Forum for the Enhancement of Teaching and Learning in Higher Education, 2016). This is reflected in the fact that the mathematics graduates interviewed as part of this study reported limited experience of group assessment. Their experiences of assessed group work were largely confined to a small number of group projects, mainly statistics and programming related, completed in various modules, and the option, in a few instances, to submit homework as a group as opposed to individually. In order to address our research question, participants were asked for their views on assessed group work based on their own limited experience and were also asked to respond to the hypothetical proposal of assessed group work being made a larger component of the 'assessment diet' in undergraduate mathematics. This paper reports three themes in relation to our research question: 'Is the purpose, necessarily, to learn more mathematics?', 'Not when my grade depends on it', and 'Navigating freedom and fairness in relation to group composition'.

A scepticism among participants as to whether group work has direct benefits to students' mathematical learning became the focus of the theme 'Is the purpose, necessarily, to learn

more mathematics?’. This was particularly evident in discussions regarding the prospect of ‘divvying up’ work among group members in mathematics. For instance, some participants mentioned that when asked to complete a problem sheet as a group, perhaps the most obvious approach would be to divide up the relevant questions among group members. However, some participants did not deem this suitable in the context of mathematics. They felt it important that they understand all questions on a problem sheet, rather than just a subset: “If four of us do one question each, then I don’t feel like I know how to do the other three questions” (Brennan). Other participants spoke of issues arising when each of the constituent group member’s questions were checked prior to submission:

Shea: It was mainly me checking their work to be honest. So I checked [another student’s] work and then unfortunately there were a bunch of errors and issues with it that we then kind of needed to fix quickly.

Shea’s account implicitly raises the issue of heterogeneous student abilities in group assessments. Some participants deemed the purpose, and perhaps the value, of group work as allowing for the “delegat[ion of] some of the teaching” (Quinn) and effectively to:

Shea: Get some stronger students and weaker students to work together and maybe the stronger students can learn by having to teach other people and the weaker students can learn from both the lecturer and the other students.

There was no concrete consensus as to whether assessed group work definitively benefits all students’ learning and a number of participants hypothesised about whether the decision to implement assessed group work might stem from a place of pragmatism, in relation to grading, on the part of lecturers. Some participants did speak to the value they saw in group work in providing an opportunity for students to learn to work with others: “I’m not sure you necessarily learn more maths by submitting as a group, but I think you probably learn more skills, life skills as it were” (Walsh).

The value they saw in peer-teaching and in learning life skills meant that some graduates were not entirely opposed to the concept of assessed group work. However, many approached the prospect with a degree of apprehension. Their concerns, in relation to the assessment of group work, became the focus of the theme ‘Not when my grade depends on it’. A recurring sentiment was that individual student’s ambitions, in relation to assessment, ought to be respected. However, as Lynch illustrates, tension can arise when individual student’s goals are in conflict with one another: “some people [...] are happy enough to get a certain grade and that’s completely fine but it’s not fair throwing them in with someone who wants to do better”. The dilemma of how to reconcile heterogeneous student goals and ambitions

appeared to lie at the heart of some students' issues with assessed group work: "There's two outcomes, either it's one or two people in the group working really hard that like, get everybody an A+ or it just brings everybody closer to an average" (Lynch).

Also relevant to the above theme were participants' discussions of how inequity in student workload can lead to conflict and feelings of resentment. It was felt by some participants that, *if* assessed group work were to be implemented, there should be some means of ensuring, or incentivising, the accountability of individual members to the group. Some suggestions included implementing a system of individual marks, or individual interviews between students and the lecturer in order to gauge understanding. As Quinn phrased it, it was felt there ought to be "some reward for actually trying". However, some participants also expressed doubts as to the feasibility of implementing such mechanisms. While some felt that group work has the potential to be "super, super helpful" (Sullivan), the graduates appeared dubious as to whether it would be possible to assuage their concerns in relation to assessment. Quinn summarised this sentiment, saying "if there is a way of kind of preventing that, then I think it is really good. But I don't know what that would be".

It is possible to argue that many of the aforementioned student concerns regarding assessed group work seem to stem from assigned, heterogeneous student groupings. The issue of group composition forms the cornerstone of the theme 'Navigating freedom and fairness in relation to group composition'. While the informal collaboration that students voluntarily engage in is beyond the scope of this paper, some graduates did contrast their experiences of assessed group work with a certain sense of freedom experienced in their informal collaborations. Lynch discussed the flexibility, in the informal case, of being able to work with others if you wish but "if you don't then it's not like affecting anyone else either. It's just your grade". Others discussed a sense of ease in working with peers who you know well:

Reilly: [Assessed group work] was more kind of stilted because you didn't know the people. When it was informal, you knew the people, everything flowed pretty easily. There was kind of an unwritten routine.

In spite of this, only two graduates reported being in favour of letting students self-select their own groups in the context of assessed group work. A number of participants deemed the prospect unfair, their reasons falling broadly into two categories. Some graduates felt that students were likely to pick their "smartest friends" (Nolan) and believed that it did not "seem like a very fair approach to [. . .] group people together by levels and tell them to work together on something they are going to be graded on" (Brennan). Others worried about potential social implications, and how "people in the module who didn't know anyone else"

(Reilly) may be unfairly disadvantaged. It was also felt that assigned, heterogeneous groups were reflective of the nature of collaboration that students may be expected to engage in later in life. These beliefs are seemingly at odds with difficulties encountered when working in assigned groups, and the freedom experienced when working with peers in an informal setting. This apparent tension is captured by Lynch who reflected:

Lynch: It's a funny thing because [. . .] if you don't get to pick your groups then you're put with people who want to work differently. But if you just pick like, in essence I guess we kind of just picked our own group that was quite strong and then did well because of that so [. . .] yeah, I'm not sure.

7.5 Discussion

For those seeking an answer to the question of whether assessed group work has a role to play in undergraduate mathematics education, this paper offers no clear answers. However, our findings do suggest that any implementation of assessed group work requires careful thought, consideration, and an evaluation as to the purpose it is to serve. We, along with Iannone and Simpson (2015b), advocate that any such evaluation “should take into account the views of students” (p. 983), on which this paper does shed some light.

As previously outlined, the interviewed graduates questioned whether there are direct benefits of assessed group work to students' mathematical learning. Adopters of a developmental perspective assert that “interaction among students on learning tasks will lead *in itself* to improved student achievement” (Slavin, 1996, p. 49). They emphasise learning through the resolution of cognitive conflicts which arise in students' discussion of content. This diverges from our participants' depiction of the process of dividing problem sheet questions among group members and the role of checking each member's contribution, in some cases, falling to one student. Research on students' assessment preferences has highlighted the importance of not automatically accepting findings from the general education literature as valid in a mathematical context. While not explicitly addressed in this paper we, along with Slavin (1996), highlight the need for “research at the intersection of cooperative learning and curriculum” (p. 62) in order to examine whether there is something different about assessed group work in the context of mathematics, as opposed to in other subjects.

Some graduates suggested that a potential purpose of implementing assessed group work could be encouraging students to engage in peer-teaching. This aligns with a cognitive elaboration perspective (Slavin, 1996) on collaborative learning. Slavin asserts the importance of rewarding a group based on the learning of its individual members in order to encourage stu-

dents to engage with the group and to help one another. He argues against rewarding groups on the basis of a product which “could theoretically have been done by one group member” (Slavin, 1996, p. 54), as is the case with students working together to submit a problem sheet, and stresses that each group member should see their own personal success as inseparably linked to the success of the group. Relevant here is the graduates’ discussion of the perceived inequity of a student feeling compelled to do additional work in order to compensate for differences in individual member’s goals in relation to assessment. This raises interesting points to be considered by lecturers with a view to implementing assessed group work. Do they consider the purpose of assessed group work as, in the words of one participant, allowing for the delegation of some of the teaching and, if so, how do they perceive the applicability of Slavin’s thesis to an undergraduate mathematical context?

This paper also contributes to our understanding of students’ preferences in relation to the assessment of undergraduate mathematics. Complementing the work of Iannone and Simpson (2015a, 2015b, 2017), it demonstrates that students’ views in relation to assessment are thoughtful and nuanced, and ought to be considered in the evaluation and implementation of alternative assessment methods. As reported by Iannone and Simpson (2015a, 2017), participants in our study also indicated a preference for the assessment of their *individual* mathematical ability. They additionally exhibited concern regarding the allocation of groups and the assessment of any potential implementation of assessed group work. We therefore advocate for these two areas of concern to be addressed explicitly with students in any implementation of assessed group work in undergraduate mathematics degrees.

Finally, although beyond the scope of this paper we note some participants’ favourable discussions of their engagement in informal collaboration with peers, outside the context of assessed group work. Of particular interest is the apparent alleviation of concerns regarding assessment and unfamiliar, heterogenous groups. Noting Burton’s (1998) mathematicians’ depiction of mathematics as an increasingly collaborative discipline, we hope to examine mathematics students’ engagement in informal collaboration further in our wider study.

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Chapter 8

Discussion and conclusion

In this chapter, we provide a summary of our research findings and examine how, collectively, they address our two thesis-level research questions. This is followed by a discussion of the empirical, theoretical, and methodological contributions of our work. Limitations of the research are also outlined. We conclude by discussing the implications of our findings for the practice of undergraduate mathematics instructors, as well as for future research.

8.1 Summary of research findings

The broad aim of our research was to examine teaching and learning in upper-level, lecture-based mathematics courses. We adopted a fundamental approach (Herbst & Chazan, 2023), with the aim of gaining further insight into the nature of students' and lecturers' activity in these courses, rather than trying to understand how this activity can be optimised. In particular, we were interested in mathematicians' pedagogical decision-making and students' out-of-class collaboration on their mathematics work. These points of focus are captured by our two thesis-level research questions:

- **RQ1:** How do mathematicians make pedagogical decisions about their instructional design of upper-level, lecture-based courses?
- **RQ2:** How do upper-level students collaborate on their mathematics work outside of their lectures?

We designed and conducted two qualitative studies to address each of our research questions. The lecturer study consisted of four mathematicians participating in a series of three, one-on-one interviews about their design of an upper-level, lecture-based course that they had recently taught. In the student study, ten recent graduates of degrees in the mathematical sciences were interviewed about the collaboration on their mathematics work they had undertaken as students. In Chapters 3-7, we detailed the findings of these two studies in a series of papers. We now review these findings and explain how they collectively address our two thesis-level research questions.

In relation to our first research question, we saw that mathematicians' pedagogical decisions are influenced by both individual and socio-technical factors (Herbst & Chazan, 2011, 2023). Individual decision-making factors include beliefs, orientations, goals, knowledge, and experience (Herbst & Chazan, 2023; Schoenfeld, 2010, 2011). Experience as a student appeared influential in participants' course design. Participants did not aim to completely replicate the teaching methods they experienced (cf. Woods & Weber, 2020), but in some cases identified aspects of their experience as students that they believed could be improved. Beliefs about how mathematics is learnt were also influential. For instance, problem-solving was considered essential to students' learning. Participants specified that students should attempt and struggle on problems before checking the solution and this factored into their approach to the provision of homework solutions in their courses. Mathematicians can also have individual goals that shape their approach to course design. For instance, they may want to cover particular content that they believe is interesting or important.

Knowledge is another individual decision-making factor. We found that there are differences in the nature of the pedagogical content knowledge (Shulman, 1986) enacted when teaching upper-level, lecture-based courses versus in the K-12 context. For instance, while teaching in the K-12 context might require knowledge of an externally developed curriculum, mathematicians typically have the autonomy to plan their own curriculum in upper-level courses and require the knowledge to do so. Additionally, lecturers must also decide how and on what to assess their students, an issue which is typically decided by state examination boards in an Irish and UK K-12 context. We also found that lecture-based courses offer limited opportunities for interaction with students. Consequently, instructors may gain less insight into student learning and understanding than they would in a more interactive classroom environment. This reflects the nature of university teaching where instructors have limited oversight of student learning which happens largely outside of class.

Mathematicians' pedagogical decisions are also shaped by socio-technical factors including norms and professional obligations (Herbst & Chazan, 2011, 2023). Norms of teaching undergraduate mathematics include lecturing (Johnson et al., 2018) and assessing students through closed-book examination (Iannone & Simpson, 2011, 2022). Participants in our study adhered to both these norms. We also found that participants' professional obligations impacted their decisions about the design of their courses. Given the upper-level nature of the courses, disciplinary obligations were particularly influential. Participants were eager to represent the branch of mathematics that their course covered accurately. Additionally, they wanted to ensure that students were adequately prepared to progress to postgraduate study in mathematics. The significance of institutional obligations was also evident. For example, participants reported that having to adhere to the university policy of semesterisation led to

them being constrained in terms of the time available in their courses. Institutional obligations are potentially less relevant in upper-level courses than in first-year courses where practices such as having multiple sections, common final examinations, and assigned textbooks are more common (Ko et al., 2024).

We found that when various decision-making factors are in conflict it can give rise to tensions. Two such tensions that we identified were the content and problem-solving tensions. The content tension centred on the pressure to cover certain content due to disciplinary obligations, which conflicted with a lack of time in the course and the need to accommodate diverse student backgrounds. The problem-solving tension was rooted in participants' goal of engaging students in authentic problem-solving, which was constrained by institutional assessment norms and limited control over students' out-of-class work. While these two tensions were experienced by multiple participants, we documented how they attempted to manage them in a variety of different ways. This reinforces the importance of considering both individual and socio-technical factors when examining mathematicians' pedagogical decision-making.

In terms of our second research question, we found that upper-level students reported collaborating with friends, identifying trust as important to the success of their collaboration. Participants contrasted their experience in the latter years of their degree with their first year when they did not know their peers well. While they were initially reluctant to be open and honest about their understanding of material, they became more comfortable and confident collaborating as they established more solid friendships. Participants also specified that the peers they worked well with were similar to them in terms of academic ability and motivation. For this reason, they were hesitant about the prospect of being assigned to groups in an assessed group work context, believing it unfair to group students with different academic goals together. Most of students' out-of-class collaboration focused on solving homework problems, rather than other tasks such as reading lecture notes. Collaboration on homework revolved around one or more students getting stuck on a problem and reaching out to their collaborators. Sometimes, if a collaborator had solved the problem, they would provide a hint or tip. However, this exchange was only momentarily asymmetric and students would reciprocally help their collaborators according to their strengths. If multiple students were stuck on a problem, they would typically all brainstorm together. Collaboration was generally followed by a period of individual reflection and consolidation of learning.

Participants attributed benefits with cognitive, social, and affective dimensions to their collaboration. They felt that working with peers benefitted their learning. Additionally, they found working with others to be more fun and less isolating. The companionship of peers was particularly valued when stuck on problems. Being stuck alone on a problem was described

as lonely and frustrating and could cause students to question their own ability. Participants valued collaborating as an important aspect of their learning experience and appeared to particularly appreciate its organic, informal quality. For this reason, they expressed misgivings about the implementation of group work as an assessment component of undergraduate mathematics degrees. That said, participants did note other ways in which collaboration could be encouraged. At times, vague messages from instructors on whether collaboration was allowed caused them to doubt the legitimacy of the practice. They advocated for up-front messaging from lecturers regarding the distinction between collaboration and copying. For some students, their impressions of research mathematics as an individual endeavour led them to doubt whether they should be working with peers. This issue was particularly pertinent for students with the ambitions of pursuing a career in research. For this cohort, their perceptions of the nature of research had the potential to tangibly impact their approach to learning.

8.2 Contributions of the thesis

In this section, we discuss the contributions of our research. We structure our discussion by outlining the empirical, theoretical, and methodological contributions of the thesis in turn.

8.2.1 Empirical

In each chapter of this thesis that contained empirical findings, we discussed the contributions of our results. We now take the opportunity to look across our findings and identify common themes. In particular, we will discuss four themes. In Section 8.2.1.1, we explore the common focus of both studies on examining teaching and learning *beyond the lecture*. This is followed, in Section 8.2.1.2, by a discussion of the importance of getting stuck on problems to students' collaboration and learning more generally. We also identified the importance of considering context as a theme running throughout our findings. This is explored in Section 8.2.1.3. Finally, in Section 8.2.1.4, we talk about the intertwined nature of cognitive, social, and affective factors that were evident in our research.

8.2.1.1 Looking beyond the lecture

In Chapter 1, we mentioned that mathematics education research originated by studying teaching and learning in the K-12 context. In the K-12 setting, student learning is largely guided by in-class activities (Ellis et al., 2015). However, the transition from school to university entails a shift in the didactic contract (Brousseau, 1997), with greater emphasis

placed on out-of-class learning (Ellis et al., 2015). To date, research in undergraduate mathematics (RUME) has paid limited attention to teaching and learning that occurs outside the classroom, with the majority of extant research focused on in-class instruction (Rupnow et al., 2021). A common goal of both of our empirical studies was to look *beyond the lecture*, to gain a more complete insight into teaching and learning in upper-level courses.

An overarching finding of our research is that out-of-class learning is very important in upper-level courses. In Chapter 3, we reported that the lecturer-study participants viewed out-of-class learning as essential. They believed that in upper-level courses, most of students' learning happens independently, guided by the assignment of homework problems. The emphasis our participants placed on the importance of students' engagement with homework problems underpins Rupnow et al.'s (2021) finding that homework plays an important role in undergraduate mathematics courses. An appreciation for the importance of homework was also reflected in the views of the student study participants. In Chapter 5, we discussed how participants were reluctant to become overly reliant on their collaborators and stressed the importance of maintaining a balance between individual and collaborative work. Additionally, in Chapter 7 we mentioned that some participants were opposed to the idea of divvying up homework problems among group members as they felt it important that they individually understood every problem. These findings indicate that students can appreciate the learning opportunities associated with homework and understand the importance of independent learning outside of class.

Student and lecturer study participants appeared to have a common understanding regarding the importance of homework and out-of-class learning more generally. This can be considered part of the didactic contract (Brousseau, 1997) that exists between teachers and students and governs expectations regarding learning. Pepin (2014) proposed that the transition from school to university mathematics can be thought of in terms of a transformation or rupture of the didactic contract (Brousseau, 1997). The nature of the mathematics content is different, there are new ways of learning, and students are expected to take greater responsibility for their education (Pepin, 2014). Pepin suggested that when students are not provided adequate support to facilitate them “learning to learn mathematics” (p. 654) in this new context, the transition can be complicated and stressful.

While Pepin's (2014) study provided insight into students' experiences of first-year university mathematics, our student study primarily focused on the final years. In contrast to Pepin's depiction of a ruptured didactic contract, our participants appeared confident and settled in their approach to learning undergraduate mathematics. This is illustrated by the routines participants had developed around collaboration. Participants' well-established collaborative practices contrasted with their experiences in first year when they didn't know

their peers well and were reluctant to be vulnerable around them. Lecturer study participants seemed to expect students to mature and become more capable of managing their learning between first and final year. This was illustrated in Chapter 3, when participants reported that they expected third- and fourth-year students to understand the importance of working on homework. Consequently, they appeared to allow upper-level students more autonomy to manage their own learning than they might in a first-year course.

Looking *beyond the lecture* was a common theme in both empirical studies included in this thesis. Our findings indicate that out-of-class learning, in particular homework, is very important in upper-level courses. Both student and lecturer participants appeared to agree on this. This agreement suggests that the didactic contract (Brousseau, 1997), which Pepin (2014) depicts as ruptured in first-year courses, can function effectively in the later years of undergraduate mathematics degrees. However, as noted in Chapter 5, participants in the student study were likely highly motivated and high achieving, as evidenced by their progression to postgraduate study. Therefore, we do not wish to suggest that all students *learn to learn* mathematics in the manner depicted by our participants. We discuss the importance of considering the influence of specific attributes of our study samples in Sections 8.2.1.3 and 8.3.

8.2.1.2 The importance of getting stuck on problems

In both the student and lecturer studies, getting stuck on problems emerged as an important aspect of learning mathematics. In Chapter 5, we explained that participants' collaboration on homework revolved around getting stuck on a problem. Collaboration occurred most frequently on difficult problems and was considered unnecessary if students were able to complete homework independently. Students would initially attempt their homework problems individually and would only come together if one or more of their collaborators were stuck. It was at this stage that the majority of their collaborative work occurred. Lecturer study participants also considered getting stuck on problems very important. In Chapter 3, we talked about the disciplinary significance of problem-solving (Burton, 2004; Halmos, 1980). Consequently, participants perceived it important that upper-level students learn how to struggle and persevere on problems, with one participant stating that "you couldn't become a mathematician" otherwise. Aside from the disciplinary significance, struggling on problems was also considered essential for students' learning. Participants felt strongly that viewing a solution to a problem prior to properly attempting it would have negative implications for students' learning.

We identified getting stuck and struggling on problems as an important factor in both studies. Students' collaboration on homework revolved around getting stuck on a problem and instructors considered struggling on problems to be essential for students' learning. In light of these findings, we question the impact that the growing usage (Stöhr et al., 2024) of increasingly mathematically proficient (Hubert et al., 2026) artificial intelligence (AI) tools will have on students' learning in undergraduate mathematics. Both lecturer and student study participants talked about how being stuck on problems can be an uncomfortable and potentially stressful experience. As AI tools become increasingly able to solve undergraduate mathematics problems, will students recognise the importance of independently attempting problems before seeking solutions? In Section 8.4.2.4, we discuss the potential for future research to investigate the influence of the increasing availability of AI tools on the teaching and learning of university mathematics.

8.2.1.3 The importance of considering context

A common theme running through our findings is the importance of considering context. One advantage of adopting a qualitative approach to our two empirical studies was that it allowed us to conduct research with participants in their natural context (Creswell, 2013). Existing research on undergraduate mathematics students' learning has largely been conducted in classrooms or in clinical interview settings (Lew & Zazkis, 2019). At the university level, most of students' learning happens outside of class (Ellis et al., 2015; Rupnow et al., 2021). In Chapter 5, we provided initial insight into the nature of undergraduate mathematics students' learning in this context. We documented the collaborative routines that students can organically establish when working outside of their lectures. While research has indicated the importance of homework (Rupnow et al., 2021), our findings advance our understanding of what students actually do when they are assigned homework problems. Our study of lecturers' course design also prioritised studying participants in their natural context by basing interview questions on participants' personal pedagogical practices. This approach contrasts with prior research on mathematicians' instructional practices that has employed task-based interviews (e.g., Moore, 2016; Rupnow et al., 2021; Woods & Weber, 2020). While these studies have undoubtedly generated valuable insights, researchers have acknowledged the potential for unnaturalistic settings to influence research findings (Moore, 2016; Rupnow et al., 2021). Our research has the advantage of being grounded in what lecturers actually do and think when planning their courses.

Adopting a qualitative approach to our research, with relatively small sample sizes, also allowed for us to account for the individual context of participants. For instance, in Chapter 5

we found that participants expressed a preference for collaborating either *as a group* or *one on one*, and illustrated both modes of collaboration with a vignette. Additionally, students can have varying goals, motivations, and academic strengths. In Chapter 5, participants reported collaborating with peers who were similar in terms of academic ability and motivation. They deemed these commonalities important to the success of their collaboration. This was also reflected in their views on assessed group work, which were outlined in Chapter 7. Participants expressed concern about the prospect of working with students who may have different goals in terms of academic achievement. Lynch illustrated this tension explaining: “some people [...] are happy enough to get a certain grade and that’s completely fine but it’s not fair throwing them in with someone who wants to do better”. We also found that participants were united by factors other than their academic ability and aspirations. For instance, in Chapter 6 Sullivan and Lynch explained that their group of collaborators bonded over their shared goal of becoming research mathematicians. Collectively, these findings illustrate the fact that students are not a homogeneous group. Individual students have different goals, academic strengths, motivations, and learning preferences. All of these factors influence their approach to learning and should not be overlooked in research studies.

The influence of individual context was also evident in the lecturer study. In Chapter 3, we saw how the differing beliefs and experiences of the four participants influenced how they managed tensions in their course design. We presented a vignette of each participant to provide insight into their personal pedagogical practices and the variation in their approaches to course design. Similarly, in Chapter 4 Dr Doyle’s personal beliefs and goals influenced his pedagogical decisions. We documented how he reflected on his experience of learning measure theory as a student and identified aspects that he wanted to improve. This illustrates the fact that individual instructors can adopt different approaches to course design, even when the material is the same. As was the case with the student study, we believe the implication of these findings is that mathematicians should not be treated as a homogeneous group and the influence of their personal experiences, beliefs, and orientations should be accounted for.

In addition to considering participants’ personal context, we also found it important to consider the broader context in which the studies were conducted. The research in this thesis focused on upper-level courses which likely influenced the findings of both studies. In first-year university mathematics courses, you are likely to encounter a broad range of students including those who will progress to major in mathematics, as well as students for whom mathematics is a pre-requisite of their chosen specialism. In the student study, we interviewed graduates of degrees in the mathematical sciences. As these are students who have chosen to major in mathematics, they are perhaps more likely to be interested in the subject and motivated to succeed in their mathematics courses. This was the case for our participants all

of whom appeared highly motivated. We discussed this factor in Chapter 5 and will return to it in Section 8.3. Additionally, the upper-level course context also influenced the findings of the lecturer study. In Chapter 3, we discussed how participants' disciplinary obligations shaped their pedagogical decisions, as they sought to accurately represent the branch of mathematics that their course focused on and prepare future mathematicians. More broadly, the university, lecture-based context was a key consideration in Chapter 4 in which we adapted and combined two existing pedagogical content knowledge theoretical frameworks (Carlson et al., 2019; Magnusson et al., 1999) for use in undergraduate mathematics education. Additionally, our research was conducted in Ireland and was likely influenced by national educational norms. We discuss this further in Section 8.3.

Across our research, the importance of considering context emerged as a key theme. Both of our studies sought to examine participants in their natural contexts by examining students' work outside of class and lecturers' personal pedagogical practices. Additionally, we saw the importance of considering individual participant's context across both studies which illustrated the influence of personal experience, goals, beliefs, and orientations. Finally, the significance of considering broader context was also evident, with factors such as the upper-level nature of the courses considered proving significant.

8.2.1.4 The intertwinement of cognitive, social, and affective factors

In Chapter 1, we noted that early research in undergraduate mathematics education (RUME) had a strong cognitive orientation, focusing on examining students' conceptions of, and difficulties with, various mathematical topics. However, in line with the broader social turn in mathematics education (Lerman, 2000), RUME has begun to consider social and affective issues (Dreyfus et al., 2025). The intertwined nature of cognitive, social, and affective factors is evident across our research. In Chapter 5, participants identified benefits of collaboration with cognitive, social, and affective dimensions. Participants felt that their work with peers benefitted their mathematical learning. They also perceived social benefits in deepened friendships between them and their collaborators. Additionally, there were affective elements to students' collaboration. Participants reported that being the only one stuck on a problem can be lonely and isolating. Even if their collaborators could not help them solve a problem, they valued the emotional comfort they derived from not being alone in their struggle.

We provided further insight into affective issues associated with learning undergraduate mathematics in Chapter 6. Although Sullivan and Lynch were undoubtedly high-achieving students as evidenced by their progression to postgraduate study, we saw that, on occasion,

both students questioned their mathematical ability. For instance, Sullivan explained that it “never felt good” to be reliant on your peers. Similarly, Lynch reported that working with other high-achieving students could “stress you out sometimes” and cause you to “feel a bit stupid”. This aspect of collaboration contrasts with the depiction provided by participants in Chapter 5 who spoke about the value of collaboration in terms of emotional support and helping to alleviate stress. This contrast serves as a reminder that affective issues are nuanced and highly contextual.

In Chapter 3, we saw that instructors also appreciate that there are affective issues associated with the teaching and learning of undergraduate mathematics. Our participants understood that being stuck on problems in time-limited conditions can be a stressful experience for students. We saw that they made accommodations in their approach to homework and provision of solutions to account for this. Additionally, when choosing content for their courses, lecturer participants were mindful that they were not overloaded with content.

Collectively, these findings illustrate the intertwined nature of cognitive, social, and affective issues in undergraduate mathematics education. The benefits of collaborating on homework identified by participants in Chapter 5 had cognitive, social, and affective dimensions. The case study of Sullivan and Lynch in Chapter 6 illustrated the complex and nuanced nature of affective issues. Our findings also indicate that instructors do not only consider cognitive factors when designing their courses. Participants were aware of affective issues such as the emotional impact of being stuck or overloaded with content which also influenced their pedagogical decision-making.

8.2.2 Theoretical

Theory and theoretical frameworks play an important role in conducting research in undergraduate mathematics education (RUME) (Florensa et al., 2025). In a recent paper identifying priorities for the next decade of RUME, Florensa et al. called for greater attention to be paid to the selection and use of theoretical frameworks in the field. In particular, they reflected: “while some researchers view theory as a personal choice shaped by training, institutional expectations, and research communities, others call for greater transparency in how and why theoretical frameworks are selected and employed” (p. 411). One issue is that the current diversity of theoretical frameworks has made it difficult for the synthesis of research findings within RUME (Melhuish et al., 2022) and across disciplinary boundaries (Andrews et al., 2022). Suggested ways of trying to combat this issue include connecting and integrating multiple theoretical frameworks for use in a study (Florensa et al., 2025) and

considering using theoretical frameworks from other disciplinary traditions (Andrews et al., 2022).

In Chapters 3 and 4 of this thesis, we drew on and connected two pre-existing theoretical frameworks. In Chapter 3, we drew on Herbst and Chazan's (2011, 2023) theory of practical rationality as our overarching theoretical framework and connected it to Schoenfeld's (2010, 2011) theory of resources, orientations, and goals (ROGs) which we used to specify the nature of individual decision-making factors. Utilising the theory of practical rationality allowed us to capture the "broader, social, long-term and institutional dimensions of teachers' thinking" (Petropoulou et al., 2020, p. 352) which is not considered by Schoenfeld's theory of ROGs. However, drawing on the theory of ROGs allowed for greater specification of the nature of individual decision-making factors than that which is provided in the theory of practical rationality. Similarly, in Chapter 4 we drew on two pedagogical content knowledge (PCK) theoretical frameworks (Carlson et al., 2019; Magnusson et al., 1999) which each offered different advantages. Carlson et al.'s (2019) refined consensus model allowed us to distinguish between three realms of PCK: enacted, personal, and collective. We considered this distinction important in the undergraduate context where teachers typically receive little pedagogical training and are likely to develop their PCK mainly through experience. However, the refined consensus model did not specify components of personal and enacted PCK. For this reason, we connected it to an adapted version of Magnusson et al.'s (1999) framework which allowed us to specify different dimensions of PCK.

The two theoretical frameworks that we drew on and connected in Chapter 4 (Carlson et al., 2019; Magnusson et al., 1999) were originally developed in the K-12 science education context. By looking across disciplinary boundaries we were able to arrive at a conceptualisation of PCK that offered several advantages for studying PCK in the university setting over the PCK theoretical framework by Ball et al. (2008) that is most regularly used in mathematics education. Additionally, utilising theoretical frameworks developed in a science education context might facilitate greater connection to researchers considering PCK in the broader university STEM setting (Andrews et al., 2022).

We hope that our research serves to illustrate to other RUME researchers the advantages of connecting different theoretical frameworks and considering the use of theoretical frameworks developed in other disciplinary contexts. In the context of our research, doing so allowed for broader conceptualisations of concepts like PCK and influences on pedagogical decision-making. More widely, it is one way of facilitating the synthesis of research findings within RUME and across disciplinary boundaries (Andrews et al., 2022; Florensa et al., 2025; Melhuish et al., 2022).

8.2.3 Methodological

Both the lecturer and student studies were interview based and data analysis involved the application of qualitative coding techniques which “often form the basis of qualitative studies in mathematics education and therefore play a considerable role in knowledge generation in our field” (Czocher & Melhuish, 2024, p. 148). In recent years, it has become increasingly recognised within the mathematics education community (de Freitas et al., 2017), and more generally (Berger, 2015), that engaging in reflexivity is integral to conducting interviews and qualitative data analysis. Within the community of those working in research in undergraduate mathematics (RUME), the increasing awareness of the importance of reflexivity has manifested in researchers explicitly acknowledging the influence of their positionality (e.g., Darragh, 2022) and the increasing prevalence of the inclusion of positionality statements in published research articles (e.g., Dobie & MacArthur, 2022; Shultz et al., 2024; Tremaine et al., 2022).

While these are undoubtably positive steps, there are still very limited accounts of the practical realities of engaging with reflexivity. This absence is not unique to RUME, and it has also been noted in other academic fields such as sport and exercise psychology (Trainor & Bundon, 2021) and international human resource management (Collins & McNulty, 2020). The lack of discussion about engaging in reflexivity in published research can be partially attributed to journal-imposed word limits (Trainor & Bundon, 2021), an issue which we faced in preparing our research for publication. In Chapter 2, I took the opportunity to discuss my engagement in reflexivity. In particular, I reflected on the impact my status as an insider researcher in the student study and an outsider researcher in the lecturer study had on the research process. Additionally, I outlined my engagement in three strategies for engaging in reflexivity and illustrated that qualitative data analysis is an active process which is inherently shaped by the researcher.

We propose that these kinds of accounts in which researchers expose “the ‘messy’ details of research practice” (Collins & McNulty, 2020, p. 204) are particularly important in RUME for three reasons. Firstly, RUME is generally considered a social science (Nardi, 2015) and the qualitative coding of interview- or observation-based data is a common form of knowledge generation in the field (Melhuish & Czocher, 2022). However, most researchers come to the field with academic backgrounds in mathematics. For instance, many active RUME researchers have PhDs in mathematics (Darragh, 2022). Additionally, it is common for doctoral students researching undergraduate mathematics to have bachelor’s degrees in mathematics (Nardi, 2015). For this reason, many novice RUME researchers have limited experience of qualitative research. Therefore, reports in which researchers demonstrate their

engagement in reflexivity, in addition to discussing its importance, are particularly useful in terms of facilitating the transition to qualitative research. Additionally, as well as having personal experience of learning mathematics at the university level, RUME researchers frequently teach mathematics to undergraduate students (Winsløw et al., 2018). Therefore, RUME researchers are likely to have personal experience of the topics they are studying and thus would be considered insider researchers (Berger, 2015; Braun & Clarke, 2022; Hill & Dao, 2021). Status as an insider researcher has particular implications for engagement in reflexivity (e.g., Berger, 2015) and accounts that address these are likely to be of value to RUME researchers. Finally, RUME is a relatively young academic field, with a history of systematic research that can be traced back to the 1990s (Winsløw et al., 2018). As the field develops, it is important to ensure that it is not built upon “methodologically shaky foundations” (Melhuish & Czoher, 2022, p. 840). Promoting thorough engagement in reflexivity when conducting qualitative research is one way of ensuring that research in the field is methodologically sound.

8.3 Limitations

We now take the opportunity to discuss some limitations of the research presented in this thesis. Statistical-probabilistic generalisability is not a relevant goal in qualitative research (Smith, 2018). For this reason, qualitative researchers are not concerned with ensuring that their sample is ‘representative’ in the same way that quantitative researchers might be. Indeed, “small samples purposefully chosen are . . . unique strengths of qualitative research, not weaknesses” (Smith, 2018, p. 139). Qualitative research may generalise in other ways such as through transferability which refers to the situation where a reader feels that research findings are applicable to their personal context (Smith, 2018). While obtaining a ‘representative’ sample is often not a goal in qualitative research, it is important to consider how the characteristics of samples may influence the transferability (Smith, 2018) of qualitative research findings. A common limitation of interview-based studies with mathematicians is that those who agree to participate may be more thoughtful and reflective about their teaching than those who were not invited or declined to participate (Melhuish et al., 2022). This limitation applies to the lecturer study and we must, therefore, be cautious about generalising our findings to the population of mathematicians. While our findings may not generalise to all mathematicians, we contend that if a goal of the mathematics education community is to collaborate with mathematicians (e.g., Dawkins & Weber, 2023; Johnson et al., 2025; Stewart, 2023), then understanding the perspectives of mathematicians who are interested in their teaching is an important and relevant goal.

There are also specific attributes of the sample in the student study that must be considered when evaluating the transferability (Smith, 2018) of our findings. In Chapter 5, we discussed how the participants in the student study appeared to be highly motivated as evidenced by their progression to postgraduate study. Additionally, some participants specifically identified themselves and their collaborators as high-achieving students and reported that they felt while it was possible to help other students, they needed to work with other similarly high-achieving students in order to properly collaborate. In Chapter 6, we presented a case study of two participants, Sullivan and Lynch, who shared the ambition of becoming research mathematicians. We examined the influence of their perceptions of the nature of research mathematics on their collaborative practices as undergraduate students. However, we acknowledge that some of their concerns are likely not relevant to students who do not wish to progress to a career in research mathematics. As discussed in Chapter 2, for ethical reasons we interviewed graduates as opposed to current undergraduate students. This meant that, at the time of interview, up to three years had passed since the participants had engaged in the practices that we were interviewing them about. This may have led to some inaccuracies in the data, however we specified that the participants be recent graduates to try mitigate this.

Another contextual factor that must be considered in relation to the transferability of our research findings is that the research in the thesis was conducted in Ireland and thus our findings are likely influenced by certain national norms. The influence of national contexts on research findings was illustrated by Kinnear et al. (2024) who examined how differences in university teaching traditions between the United Kingdom and United States influence mathematics lecturers' questioning practices. We posit that there are certain norms of university mathematics teaching in Ireland that may have impacted our findings. For instance, anecdotally, it is common for instructors in Ireland to create their own course notes and homework assignments, as opposed to selecting a course textbook. This is a contextual factor that likely influenced our findings. For instance, in Chapter 4 we exemplified Dr Doyle's KISR by examining his design of his course notes. This would be less relevant to instructors who select a course textbook as opposed to designing their own notes.

An additional limitation of the research was that both studies relied on self-report data. In the student study, we decided to interview graduates as opposed to current students to try to ensure that participants felt comfortable speaking openly about their experiences. In the lecturer study, we were interested in gaining insight into participants' pedagogical decision-making. However, it is worth noting that studies designed to examine mathematicians' rationale for decisions are likely gaining insight into the "rationality that mathematicians will use to account for their decisions" (Melhuish et al., 2022, p. 8) rather than the cognitive processes themselves. This limitation applies to the lecturer study. However, we did request

teaching artefacts from the participants' courses such as the notes, homework problems, and final examination to assist in triangulating the provided rationale with participants' actual practices. Additionally, by situating our questions in the context of participants' personal pedagogical practices we mitigate the influence of unnaturalistic settings which has been identified as a limitation of other research on mathematicians' instructional practices (e.g., Moore, 2016; Rupnow et al., 2021).

Finally, we feel that the research in this thesis could have benefitted from explicitly exploring the connection between mathematicians and students' perspectives on teaching and learning in upper-level, lecture-based courses. We view this as an opportunity for future research and elaborate on it further in Section 8.4.2.

8.4 Implications

We now take the opportunity to discuss the implications of our research. This section is divided into two subsections. In the first, we discuss the implications of our findings for the practice of instructors of undergraduate mathematics. This is followed by a discussion of the implications of our research findings for future research in undergraduate mathematics.

8.4.1 Implications for practice

In Chapter 1, we described how we adopted a fundamental approach (Herbst & Chazan, 2023) to the research in this thesis, with the aim of understanding more about the nature of current teaching and learning practices in upper-level courses. Therefore, our goal was not to evaluate the practices of our participants or to prescribe any particular 'desirable' method of teaching. However, the ultimate aim of understanding the nature of teaching and learning is to use these understandings to improve instruction (Schoenfeld, 2000). In this section, we reflect upon the findings of the research included in this thesis and identify three ways in which our findings could inform the practice of those involved in undergraduate mathematics instruction. In Section 8.4.1.1, we question whether lecture-based instruction constitutes a sensible choice for mathematicians teaching upper-level courses. Reflecting on the findings across the thesis, we wonder whether research efforts are better focused on collaborating with mathematicians to find ways in which lecture-based instruction can be improved, rather than advocating for its replacement. In Section 8.4.1.2, we note that our, and other, research has indicated that collaborating with peers can be an important part of students' learning experiences (Brown et al., 2005; Iannone & Nardi, 2005; Johnson & Hanson, 2015; Solomon et al., 2010). We discuss the implications of our findings in terms of how students' out-of-

class collaboration can be best supported. Finally, in Section 8.4.1.3 we consider potential ways in which a stronger connection between research and undergraduate mathematics can be fostered.

8.4.1.1 Lecture-based instruction in upper-level courses: A sensible choice?

In the mathematics education literature, dissatisfaction has been expressed with lecture-based instruction (e.g., Leron & Dubinsky, 1995; Lew et al., 2016). However, across the research in this thesis we saw a number of indications that led us to consider whether, in the context of upper-level courses, lecture-based instruction is, in fact, a sensible choice. In Chapter 3, we saw that disciplinary obligations (Herbst & Chazan, 2011, 2023) are important influences on pedagogical decision-making in advanced courses where students may progress to further mathematical study. For instance, in previous research the pressure to cover certain content has been linked to institutional obligations (e.g., Johnson et al., 2013, 2025; Woods & Weber, 2020). However, our participants indicated that their content goals were primarily linked to disciplinary obligations to represent the relevant branch of mathematics accurately. Given the apparent disciplinary significance of upper-level courses, we echo other researchers who have questioned whether mathematicians will be willing to implement forms of instruction that may not align with their goals and disciplinary obligations (Woods & Weber, 2020).

In addition to a potential lack of alignment with mathematicians' goals and disciplinary obligations, research has demonstrated that student-centred instruction requires considerable pedagogical content knowledge (PCK) (Johnson & Larsen, 2012; Speer & Wagner, 2009; Wagner et al., 2007). In Chapter 4, we proposed that mathematicians can develop PCK through lecturing, but that the nature of it likely differs from that which is required to effectively implement student-centred instruction. While it is possible for mathematicians to develop their PCK over the course of multiple implementations of student-centred instruction (Andrews-Larson et al., 2019), doing so requires considerable time and effort. We note that mathematicians have been described as tending to *satisfice* (Simon, 1972), that is, “rather than exerting considerable time and cognitive resources in trying to find the best form of pedagogy, many mathematicians will select the first acceptable form of instruction that they encounter” (Dawkins & Weber, 2023, p. 109). With this in mind, we question whether mathematicians would consider it sensible to devote considerable time and energy to developing a new skill set to implement a form of instruction that they may not necessarily be convinced is beneficial in upper-level courses.

The final indicator in our research of the suitability of lecture-based instruction in upper-level courses is the emphasis that both lecturers and students placed on out-of-class learning. Participants in the lecturer study stressed that lectures serve to introduce students to the content, but that ultimately most of student learning will happen outside of class, guided by the assignment of homework problems. This is illustrated by Dr Doyle in Chapter 4 who remarked: “the amount of time you get to spend with students is fairly small in comparison to the amount of time it takes a student to learn something like measure theory”. This was also reflected in the views of student participants who reported needing time to work independently on a problem before being ready to collaborate on it. Given the common belief of lecturer and student participants that learning mathematics in upper-level courses requires significant time, we question the feasibility of having students work on exercises and problems in class. Instead, we wonder whether the approach of introducing students to content in class and providing them with the time and support to consolidate their understanding through working on homework problems is, in fact, better suited to the advanced nature of the content covered in upper-level courses.

For the outlined reasons, we do not advocate for the replacement of lecture-based instruction in upper-level courses. Instead, we feel that research efforts may be better focused on collaborating with mathematicians to find ways in which lecture-based instruction can be improved and augmented (see Section 8.4.2.3).

8.4.1.2 Supporting students’ out-of-class collaboration

In Chapter 5, we outlined the various benefits that our participants attributed to collaborating on homework with peers. These benefits align with those identified in other research (Brown et al., 2005; Johnson & Hanson, 2015; Seymour & Hewitt, 1997; Solomon et al., 2010), suggesting that out-of-class collaboration can be an important aspect of students’ learning experiences. A natural question that this finding raises is how this practice can be best supported. Our findings reinforce that the provision of physical spaces for social learning is key to fostering community among undergraduate mathematics students (Brown et al., 2005; Iannone & Nardi, 2005; Johnson & Hanson, 2015; Solomon et al., 2010). Participants in our study reported that if they and their peers had been stuck for some time, they may then approach their instructor. As others have previously advocated, we recommend that instructors “have availability to answer student questions and respond to students’ needs” (Mesa et al., 2015, p. 89) whether it be in tutorial/recitation classes, during office hours, or through adopting an open-door policy. Locating tables for students to work at close to faculty

offices is a simple, cost-effective way of facilitating greater student-instructor interaction out of class (Johnson & Hanson, 2015).

In Chapter 5, we reported that one factor that constrained students' out-of-class collaboration was their instructors' generally vague stance on students working together, particularly on assignments submitted for credit. We recommend that, at the beginning of the semester, instructors should explicitly address whether, and how, they want students to work together. Our findings, as well as those of (Johnson & Hanson, 2015), indicate that students appreciate when they are encouraged by their instructors to collaborate. However, we are aware that instructors may deliberately avoid being explicit about students collaborating. As evidenced in Section 3.5.3.3, instructors may have legitimate concerns regarding copying or taking credit for work they do not understand. Students in our study appeared to have developed nuanced understandings of the difference between collaboration and copying and appeared willing to work within the parameters set by their instructors. However, as noted in Chapter 5, our participants were highly motivated, upper-level students and instructors should assess whether our findings are transferable (Smith, 2018) to their own context. Nonetheless, we hope that this research can make a valuable contribution in terms of providing mathematics instructors insight into the way in which upper-level students can work together productively outside of their lectures.

Given that participants found collaborating on homework informally with peers to be a positive aspect of their mathematical learning experience, a natural reaction from instructors might be to instigate some form of assessed group work in their courses. In Chapter 5, we noted that participants found the organic and informal nature of their collaboration important to its success. We also recall Iannone and Nardi's (2005) study in which participant mathematicians expressed the view that while students could be supported and encouraged in their collaboration, any major form of external intervention would only serve to detract from the quality of students' interactions. We tend to agree with this view. In Chapter 7, we outlined participants' views on assessed group work. Based on these findings, we recommend that if instructors are to include group work as an assessment component in their undergraduate mathematics courses, they must reflect carefully on its purpose and communicate this explicitly to students. We would encourage instructors to consult the collaborative learning literature (e.g., Slavin, 1996) for guidance on how to implement collaborative learning in a manner that is consistent with their goals. Based on the perspectives of our participants, we suggest that particular attention is paid to how students are assigned to groups and how the group work is assessed.

While recognising the importance of the organic quality of students' collaboration, we also note the contrast between participants' initial apprehension about working with

unfamiliar peers in the earlier years of their studies and their accounts of collaborating with trusted friends in the latter years. We suggest that instructors, particularly of first-year mathematics courses, may be able to help expedite the process of students getting to know one another by providing opportunities for students to meet each other and sample working together. We feel that activities in class or in tutorial/recitation sessions could be an ideal way for students to trial working with others, increasing the chances that they find peers with whom they can successfully collaborate. We note that the *MAA instructional practices guide* (Abell et al., 2018) advises that “taking time to build a classroom community allows students to form and broaden their support networks outside of class” (p. 2) and offers practical guidance to instructors on helping students connect and engage with one another.

8.4.1.3 Fostering connection between undergraduate and research mathematics

In Chapter 3, we saw that the preparation of future mathematicians was a central concern of participants when designing their upper-level courses. It influenced their decisions about which content to include in their courses as they felt obligated to ensure they were representing the discipline accurately and covering sufficient content to facilitate students’ progression to postgraduate study. Additionally, it shaped their approach to problem-solving, with one participant describing learning to persevere and struggle on problems as essential to becoming a mathematician. Furthermore, some participants reported including a difficult, unseen problem on their examinations with the purpose of identifying students who could go on to further mathematical study. These findings align with previous research (Woods & Weber, 2020) which also suggested that preparing future mathematicians is an important goal of mathematicians, particularly in upper-level courses.

In Chapter 6, we presented a case study of two participants from the student study which provided insight into the student perspective on preparing for the transition from undergraduate to research mathematics. In particular, we saw how their impressions of the nature of research mathematics strongly influenced their learning practices as undergraduates as they sought to align their practices with their conception of what it means to be a research mathematician. In Chapter 5, we also saw the broader influence of students’ perceptions of the nature of research mathematics with some participants questioning whether their collaboration was reflective of the nature of research mathematics. This is in spite of the fact that published research (Burton, 2004; Grossman, 2005) and our own review of the authorship of published mathematics journal articles (see Appendix 5.A) indicate that mathematics is becoming an increasingly collaborative discipline. These findings underpin the importance of accurately representing the nature of research mathematics to undergraduate students.

Additionally, providing students with insight into the nature of research can also help further mathematicians' goal of preparing future mathematicians.

One way of fostering awareness of the nature of research mathematics is through the facilitation of capstone undergraduate research projects. In Chapter 6, we saw the influence that engaging in such research projects can have, with Sullivan stating that he went from thinking of a mathematician as a “man on his own in a room writing on a piece of paper” to understanding that, within certain research groups, mathematics can be very collaborative. Another way of facilitating such insight might be inviting upper-level students to relevant research seminars or presentations in the department. Additionally, mathematicians may take the opportunity in lectures to provide some insight into their own research experiences. Artigue (2021) identified a lack of “connection between the living field of mathematics so dynamic and diverse, and undergraduate mathematics education, both in terms of content and practice” (p. 15) as a challenge facing the field. We hope that our outlined suggestions may assist mathematicians in fostering a stronger connection between their teaching and research activities.

8.4.2 Directions for future research

The findings in this thesis point to several interesting directions for future research. In this section, we outline four potential avenues for further study. The two empirical studies reported on in the thesis separately examined mathematicians' and students' perspectives on teaching and learning in upper-level courses. We propose that a direction for future research is to explicitly investigate the connections between these perspectives. We elaborate on this in Section 8.4.2.1. Our research findings indicate that homework is a key learning resource in upper-level courses. However, we know less about mathematicians' instructional design of homework than lectures. Additionally, our research indicates that mathematicians' approaches to homework may vary more than their in-class instructional practices. In Section 8.4.2.2, we call for future research to examine mathematicians' homework practices through large-scale quantitative surveys. In Section 8.4.1.1, we stated that, based on our research findings, we do not advocate for the replacement of lecture-based instruction in upper-level courses and instead call for collaborative research between mathematicians and mathematics educators examining the ways in which lecture-based instruction can be improved. We discuss how such research could be pursued in Section 8.4.2.3. Finally, in Section 8.4.2.4 we note that data collection for this research took place prior to the availability of generative artificial intelligence tools becoming a widely recognised issue in higher education. We propose that an interesting direction for future research is examining the implication of the

increasing usage (Stöhr et al., 2024) and mathematical ability (Hubert et al., 2026) of artificial intelligence tools for students' and mathematician' practices in relation to homework.

8.4.2.1 Investigating the connection between mathematicians' and students' perspectives

In this thesis, we reported on findings from two empirical studies that separately examined instructors' and students' views on teaching and learning. This approach is in line with the general tendency of research to examine issues from either the student or lecturer perspective (e.g., Iannone & Simpson, 2015a, 2015b). In this chapter, we have examined themes across the findings of our two studies to provide some insight into the connections between students' and mathematicians' perspectives on teaching and learning. However, a direction for future research would be to explicitly investigate these connections. Woods and Weber (2020) asked mathematicians to give their views and reflections on summaries of key findings from research on lecturers' in-class practices. We propose that a similar study design could be used to investigate the connection between mathematicians' and students' perspectives. For instance, mathematicians could be presented with a summary of findings about students' out-of-class collaboration (Chapter 5) or alternatively, students could be asked to reflect on the tensions mathematicians experience when designing their courses (Chapter 3).

We believe that fostering such connection between mathematicians' and students' perspectives could be beneficial for both groups. Our findings indicate that there are aspects of students' learning practices that mathematicians have limited insight into. For instance, in Chapter 4 Dr Doyle acknowledges the limited insight he gains into student learning as a result of the lecture-based nature of the course. In some cases, lecturer study participants appeared to make assumptions about student learning practices. For instance, in Chapter 3 Dr Liffey stated that he believed that “ideal state of affairs [of equitable student collaboration] often isn't there”. However, our findings in Chapter 5 indicate that in some cases there is the potential for students to contribute fairly to collaborative efforts. Previous research on students' assessment preferences (Iannone & Simpson, 2015b) has demonstrated that they have reflective and thoughtful views on educational issues with potentially important implications for instruction. With this in mind, we believe that mathematicians may appreciate greater insight into students' perspectives on learning, particularly in relation to their out-of-class practices.

There are also potential benefits to student learning from pursuing this line of research. Melhuish et al. (2022) noted that we now know a good deal about mathematicians' instructional practices, but much less about the influence of these practices on student learn-

ing. Asking students to give their perspectives on research on mathematicians' instructional practices may be one way of generating insight into the relationship between instruction and student learning. Additionally, our findings build on other research that indicates that instructors are very intentional in their course design (e.g., Rupnow et al., 2021; Woods & Weber, 2020). However, relatively little is known about the extent to which students understand their instructors' goals and how these goals are reflected in the course design. Investigating students' understandings of their instructors' intentions is a worthwhile direction for future research and also may encourage mathematicians to be more explicit in outlining their goals to students.

8.4.2.2 Examining mathematicians' instructional approaches to homework

Research into mathematicians' in-class instructional practices means that we now know a good deal about the nature and commonalities of lecturing as a pedagogical genre (e.g., Artemeva & Fox, 2011; Fukawa-Connelly et al., 2017; Johnson et al., 2018; Melhuish et al., 2022). Our findings suggest that mathematicians' approach to the assignment of homework may vary more than their lecturing practices. For instance, in Chapter 3 we saw how participants adopted different approaches to homework in their courses in an attempt to manage the problem-solving tensions. Dr Dargle had students submit problems for grading and provided students with partial solutions, where as Dr Boyne had students complete problems in preparation for a midterm test and provided students with full solutions after the relevant tutorial class had taken place. The variation in our participants' approaches to homework in their courses aligns with the findings of Ellis et al. (2015) who highlighted that instructors must make a number of decisions about homework in their courses, including decisions regarding the content, structure, and feedback systems associated with homework.

Additionally, our research suggests that both instructors and students perceive working on homework as an important learning opportunity in upper-level courses. In Chapter 5, we outlined how participants reported individually reflecting on a problem after collaborating on it, indicating that they valued the learning associated with homework. Furthermore, in Chapter 7 we mentioned that some participants took issue with the prospect of divvying up homework problems in the context of assessed group work, feeling that it was important that they understand all the problems. The importance that the student participants attributed to homework was reflected in the perspectives of the lecturer study participants, who identified students' work on homework problems as essential to their learning. This aligns with Rupnow et al.'s (2021) study which also identified homework as playing a major role in student learning in upper-level courses.

Given that homework is a key learning resource in upper-level courses, an important direction for future research is investigating mathematicians' instructional practices in relation to homework. Compared to lectures, we know relatively little about the nature of homework. Is it common practice for lecturers to grade students' homework submissions? Do instructors allow students to complete homework in groups? Do mathematicians provide homework solutions and, if so, what format do these homework solutions take? Are homework problems typically discussed in tutorial/recitation classes? How are these classes structured? These and other questions could be addressed through a survey-based study like that which Johnson et al. (2018) conducted to investigate mathematicians' in-class instructional practices. We posit that, as is the case for lectures and instructors' questioning practices (Artemeva & Fox, 2011; Kinnear et al., 2024), mathematicians' approaches to homework are likely to be influenced by cultural and national norms. Future research should account for and attend to this contextual variable.

8.4.2.3 Collaborating with mathematicians to investigate ways in which lecture-based instruction can be improved

An increasing number of mathematics education researchers are coming to the conclusion that in order for pedagogical innovations to have impact on the practices of mathematicians they must be designed with meaningful consideration of their goals and concerns (Artigue, 2021; Dawkins & Weber, 2023; Johnson et al., 2025). For instance, Artigue (2021) reflected “we must not think in terms of dissemination of research results, but in terms of collaborative projects, building and negotiating, jointly with mathematicians and other university teachers” (p. 15). Mindful of the reasons why mathematicians choose to lecture (e.g., Johnson et al., 2018; Woods & Weber, 2020), there has been increased focus on improving and augmenting lecture-based instruction, rather than advocating for its replacement (Alcock, 2018; Dawkins & Weber, 2023; Johnson et al., 2025). As discussed in Section 8.4.1.1, based on our findings we believe this is a valuable direction for future research, particularly in the case of upper-level courses. We now take the opportunity to highlight two implications of our findings for how this research could proceed.

In Chapter 4, we noted that despite Dr Doyle's limited engagement with the mathematics education research literature or other forms of professional development, he reported engaging in practices that had been discussed in the research literature. Using pedagogical content knowledge (PCK) terminology, Dr Doyle had developed enacted PCK through experience that in some cases mapped onto collective PCK that is documented in the research literature. We suggest that a potentially productive direction for future research might be to work with

mathematicians to connect their existing teaching knowledge to relevant research. Such an approach would have the benefit of building on mathematicians' existing practices and knowledge, rather than adopting a deficit perspective as some previous studies on mathematicians' PCK have done (e.g., Johnson & Larsen, 2012; Speer & Wagner, 2009; Wagner et al., 2007). In Chapter 3, we proposed another approach to collaborative research with mathematicians. We highlighted that our participants were not entirely happy with their current approaches to instruction and experienced tensions in their course design. We suggest that these tensions could constitute a valuable starting point for collaborative research aimed at improving lecture-based instruction. For instance, some participants reported wanting a greater level of student interaction in their tutorial classes but appeared unsure as to how to achieve this. An interesting project for mathematics education researchers might be to investigate ways in which tutorial classes can be best optimised for student learning in upper-level courses.

Both of these suggestions for pursuing collaborative research with mathematicians aimed at improving lecture-based instruction have the commonality of being rooted in mathematicians' personal practices, knowledge, and concerns (Artigue, 2021; Dawkins & Weber, 2023; Johnson et al., 2025). Although such collaborative research must be conducted on a reasonably small scale, we would hope that the results might resonate with other mathematicians who could relate to the concerns and experiences of participants. Woods and Weber (2020) identified an orientation of mathematicians of tending to teach in the way in which they were taught. While our participants' experiences as students did seem to be an important influence on their pedagogical decision-making, we also saw that they were interested in improving upon their experiences as students. For instance, in Chapter 4 Dr Doyle talked about his experience of learning measure theory as a student and the various changes he wanted to make in his own teaching of the subject. Therefore, we contend that there are some mathematicians who are interested in improving the teaching of undergraduate mathematics and that collaborating with these mathematicians is one way in which research has the potential to have a tangible impact on instruction.

8.4.2.4 Investigating the influence of the availability of artificial intelligence tools

The interviews for the student and lecturer studies were conducted in spring 2022 and summer 2023 respectively. Open AI's ChatGPT was launched in November 2022 (OpenAI, 2022) and the usage of it and other generative artificial intelligence (AI) tools has become increasingly prevalent in higher education in the subsequent years (Stöhr et al., 2024). Given these timings, the influence of AI tools was not as relevant a topic in our interviews as it might be if they were conducted today. However, we posit that the availability of AI tools

has and will continue to have a large influence on the teaching and learning of undergraduate mathematics courses, particularly in relation to homework. In Chapter 3, we reported that stuckness was an important catalyst for students collaborating on homework problems. In Chapter 5, we gained insight into the instructor perspective on stuckness, with our participants identifying struggling and persevering on problems as an essential aspect of student learning. Consequently, the decision of whether or not to provide homework solutions was viewed as consequential as instructors sought to balance supporting students with wanting them to engage in productive struggle.

As the ability of AI tools to solve advanced mathematical problems improves (Hubert et al., 2026), we question the implications for students' engagement with homework. Will students be less likely to struggle and persevere on problems if they know that they can use AI to access the solution? What are the implications for students' engagement in collaboration with peers? Additionally, it would be interesting to consider if instructors are changing their instructional approaches in response to the availability of AI tools. Are they now less likely to give credit for students' homework solutions? Are they adapting their approach to the provision of homework solutions? We believe that these are interesting questions for future research to consider in the changing education landscape.

8.5 Conclusion

In this chapter, we provided a summary of our findings, addressing our thesis-level research questions, and discussed the contributions, limitations, and implications of our research. Our research has demonstrated that there is more to the teaching and learning of upper-level, lecture-based courses than what happens in class. We have highlighted promising avenues for future research and identified potential implications for the practice of undergraduate mathematics instructors. We hope that subsequent research continues to consider issues *beyond the lecture* to obtain a more complete understanding of the nature of teaching and learning in university mathematics courses.

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