Designing tasks to aid understanding of mathematical functions

Sinead Breen, St Patrick's College, Drumcondra, Dublin 9; Ann O'Shea, National University of Ireland, Maynooth

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The concept of a 'function' can be viewed as a threshold concept in mathematics. To properly understand functions and to work with them in diverse areas of mathematics, students should be able to conceive of a function as an action, as a process and as an object in its own right. In fact, some authors have claimed that successful mathematical thinking lies in moving flexibly from one interpretation to another. However, most students find the transition of thought involved in progressing to view a function as an object troublesome. Once such 'reification' has taken place, it is unlikely to be forgotten, previously inaccessible means of thinking about many different mathematical concepts are opened up, and students' conceptual understanding is transformed. The key role played by functions in mathematics justifies paying significant attention to the teaching of functions and to the types of tasks assigned to students to support their learning.

Traditionally undergraduate courses in mathematics tend to be described in terms of the mathematical content and techniques students should master and theorems they should be able to prove. Moreover, recent studies have shown that many sets of mathematical tasks produced for third-level students emphasize lower level skills, such as memorization and the routine application of algorithms or procedures, rather than endeavouring to develop students' understanding of the underlying concepts. In this paper, we will describe a set of tasks designed by the authors to help students develop a comprehensive understanding of functions and to move flexibly between different interpretations and representations of functions. The design of tasks drew on a number of frameworks, such as those of Swan (2008) and Mason and Johnston-Wilder (2004), adapted for use with undergraduate students.

Mathematical function as a threshold concept

The concept of 'function' is central to any Calculus course and indeed underpins many other areas of Mathematics. Much research has focused on the development of understanding of the function concept. For instance, Vinner (1983) considered the difference between *concept definition* and *concept image* in relation to the concept of function and found that students construct a variety of concept images that are not consistent with the definition (for instance, that a function should be given by one rule).

The concept of 'function' can be viewed as a threshold concept in mathematics (Pettersson 2012) –as it can be characterized as transformative, irreversible, integrative and troublesome (Meyer and Land 2003). To properly understand functions and to work with them in diverse areas of mathematics, students should be able to conceive of a function as an action, as a process and as an object in its own right (Dubinsky and McDonald 2001). Sfard (1991) discusses the complementary approaches of dealing with abstract notions such as functions: operationally as processes and structurally as objects. She introduced the term 'reification' to represent the transition of thought involved when a learner progresses to viewing processes as objects. She warns that reification is "an ontological shift, a sudden ability to see something familiar in a new light" (p.19) and a "rather complex phenomenon" (p.30), causing obstacles and frustration for learners — illustrating the transformative but troublesome properties of the concept. Gray and Tall (1994) maintain that the ability to think flexibly in this manner (operationally and structurally) is at the root of successful mathematical thinking. Thus, it can

serve as a marker of students' progress in learning mathematics. However, in mathematics teaching 'reification' often remains an implicit learning outcome, a form of tacit knowledge that is not explicitly articulated to learners.

Once such 'reification' has taken place, a previously inaccessible means of thinking about the mathematical concept is opened up and is unlikely to be forgotten or reversed. There is a permanent repositioning of the learner in relation to the concept and it is unlikely that there will be any 'conceptual decay' over time. Meyer and Land (2003) mention how expert practitioners in a field can have difficulties looking back over a threshold they have personally long since crossed in order to understand the difficulties faced by their students. Gray and Tall (1994) suggest the flexibility in thought achieved by those who have experienced 'reification' (e.g. with 'function') can explain why a mathematics expert may find it difficult to appreciate the difficulties of a novice. The integrative nature of the understanding associated with a threshold concept also highlights a distinction between the thinking of a novice and the community of practice within a discipline (Meyer and Land 2003). This aspect of integration is also a feature of comprehensive understanding of mathematical functions. Dubinsky and McDonald (2001) describe a further stage (beyond action, processes and objects) in the understanding of mathematical concepts. They use the term 'schema' to describe the collection of actions, processes and objects an individual associates with a particular concept (e.g. function) and links by general principles, to each other and to other concepts in the subject area, to form a coherent framework in the individual's mind. A schema outlines previously hidden relations between concepts. Dubinsky and McDonald (2001) suggest that a student who has reached the stage of constructing a coherent schema, integrating aspects and features of the concept in question (e.g. function), is more likely to be successful in using the concept and solving problems involving it. For instance, reaching a comprehensive understanding of the concept of 'function' can lend shape and structure to a student's concept image of Calculus.

Traditional approaches to undergraduate mathematics curriculum design

Cousin (2006) claims

> *a tendency among academic teachers is to stuff their curriculum with content, burdening themselves with the task of transmitting vast amounts of knowledge bulk and their students of absorbing and reproducing this bulk* (p.4).

Generally speaking, undergraduate mathematics courses have traditionally been defined in terms of mathematical content and the techniques students are expected to master or theorems they should be able to prove (Hillel 2001). Although the main goal of a mathematics lecturer is to foster mathematical understanding in their students, such an understanding is seldom specifically fostered by the mathematical tasks and assessments students are required to complete (Sangwin 2003), with many authors expressing the view that mathematics at third level suffers from an over-emphasis on procedures and memorisation. For instance, Dreyfus (1991) asserts that many students learn a large number of standardised procedures in their university mathematics courses and, although they end up with a considerable amount of mathematical knowledge, they lack the working methodology of a mathematician and therefore cannot use their knowledge in a flexible manner. This often leads to a reliance on shallow, superficial or rote learning (Sangwin 2003).

As an alternative, Cousin (2006) advocates teachers make refined decisions about what is fundamental to a grasp of their subject and focus on threshold concepts in curriculum design and in teaching. However, teachers must demonstrate that they can tolerate learner confusion and 'hold' their students through the liminal state they may find themselves in during the process of mastery of a threshold concept. It is important to create a safe environment which supports students and provides them with the opportunity to succeed. If teachers cannot devise appropriate activities to deal with the uncertainty experienced by students, and students

remain in a pre-liminal state for too long, Cousin (2006) describes how students may resort to mimicry and superficial learning to survive the course.

Task types & samples of tasks

Mason (2002) contends that

> *in a sense, all teaching comes down to constructing tasks for students…This puts a considerable burden on the lecturer to construct tasks from which students actually learn* (p.105).

However, Groves and Doig's (2002) assert that

> *insufficient attention is being paid to the critical role of the development of conceptually focussed, robust tasks which can be used to support the development of sophisticated mathematical thinking* (p.31).

Thus, our focus in the project reported here, in aiming to promote, develop and transform students' understanding of the concept of 'function' in the first year undergraduate Calculus courses we were teaching, was on the design and use of tasks.

Research has shown that the types of tasks assigned to students affect their learning. For instance, Boesen et al (2010) found that when faced with familiar tasks students employed imitative reasoning (that is, reproduced from memory or used well-rehearsed procedures) and, in contrast, used creative mathematically founded reasoning (that is, formulated mathematically well-founded arguments which were new to the students) to tackle unfamiliar tasks. They claim that the solutions to familiar tasks required little or no conceptual understanding and they conjecture that exposure to these types of tasks alone limits the students' ability to reason and gain conceptual understanding. Likewise, Selden et al (2000) recommend that lecturers should regularly assign non-routine problems to students in order to develop their mathematical thinking skills.

Many authors (e.g. Dreyfus 1991) agree that the mathematical practices and thinking to be encouraged in learners of mathematics should mirror the practices of professional mathematicians, and Bass (2005) describes these ways of thinking and practising as including experimentation, reasoning, generalization, and the use of definitions and mathematical language. Cuoco et al (1996) further propose that students need to conjecture, visualise, describe and invent. They claim the inclusion of such 'mathematical habits of mind' will 'give students the tools they will need in order to use, understand and even make the mathematics that does not yet exist' (p.376). Swan (2008) selected five task types which he believed would promote conceptual understanding amongst secondary school students: those were classifying mathematical objects; interpreting multiple representations; evaluating mathematical statements; creating problems; analysing reasoning and solutions. Tasks of these types encourage the development of mathematical skills such as classifying, interpreting, comparing, evaluating and creating. Mason and Johnston-Wilder (2004) and Sangwin (2003) promote the use of exercises in which students are required to generate or construct their own examples, as mathematicians would do.

Drawing on and synthesizing the advice from the literature described above, we identified the following types of tasks as being appropriate for Irish first year undergraduate Calculus students: tasks requiring students to generate examples, evaluate statements, analyse reasoning, conjecture, generalise, visualise, and/or use definitions. Our aim was to move away from a content-driven curriculum, and to provide students with an opportunity to actively engage with mathematical concepts and to enable them to experience and gain an understanding of the ways of thinking and practices of mathematicians. We designed a number of tasks and include samples of these types of tasks, which deal with functions, below.

Example Generation: Give an example of a function with natural domain R\{2,4}.

Conjecturing/Generalising:

(i) Sketch the graphs of $f_1(x)=x^3$ and $f_2(x)=x^3+4$ (using the natural domains).

(ii) Sketch the graphs of $g_1(x)=1/x^2$ and $g_2(x)=1/x^2+4$ (using the natural domains).

(iii) Sketch the graphs of $h_1(x)=3^x$ and $h_2(x)=3^x+4$ (using the natural domains).

(iv) What is the relationship between the functions in the pairs f_1 and f_2; g_1 and g_2; h_1 and h_2? Can you make a general conjecture regarding the graphs of functions from your observation of the graphs of these pairs?

Visualisation: Sketch a graph of a function, f, which satisfies all of the given conditions:

$f(0)=0$, $f(2)=6$, f is even.

Evaluating Mathematical Statements: Suppose $f(x)$ is a function with natural domain R.

Decide if each of the following statements is sometimes, always or never true:

(i) There are two different real numbers a and b such that $f(a)=f(b)$.

(ii) There are three different real numbers a, b, c such that $f(a)=b$ and $f(a)=c$.

The latter three tasks, as well as being of the type indicated in order to promote conceptual understanding, also provide the students with an opportunity to encounter a function as an 'object' rather than a 'process' or 'action', as an 'inner task' (Mason and Johnston-Wilder 2004). (Note, for instance, that in the third and fourth tasks the students are not given a formulaic representation of a function to work with and so must focus more on *structure* than *operation* to answer the question posed.)

In designing calculus tasks we were mindful of the need to create a rich variety of tasks as it was imperative that the coursework should not become predictable nor a particular type of task become over-familiar. Moreover, Mason and Johnston-Wilder (2004) advocate what they term a 'mixed economy' (p.6) of tasks as no single strategy or task type has proved to be universally successful in developing mathematical thinking.

Feedback from students

Both authors taught a first year undergraduate Calculus course in their home institutions in the 2011/2012 academic year. The tasks designed were assigned either as homework problems (for students to work on independently) which would subsequently be discussed in tutorials or as problems to be worked on during tutorials (for students to work on in small groups), for these courses. This arrangement provided tutors with the opportunity to listen to students' misunderstandings and to 'hold' them through liminal states, as advocated by Cousin (2006).

At the end of the respective courses, students were invited to participate in interviews with a research assistant to discuss various aspects of their experiences of the Calculus course, with a particular focus on their responses to the tasks assigned. Ten students volunteered and were interviewed. A preliminary analysis of the interviews indicates that the interviewees had a mature understanding of the purposes of the different types of tasks assigned. They also appeared to appreciate the benefits of the tasks. For instance, in relation to the example generation tasks assigned they reported improving their conceptual understanding and acknowledged the quality of these as learning tools, as can be seen from the following quotations.

> Student E: But you really have to think more about things and understand the concepts and the different - ahm - possible solutions that may be there and why one solution isn't going to work……That would be more to cement

> the whole concept, why it works this way and - you know to cement that into your mind.
>
> Student P: The conceptual ones [tasks] really get you thinking…the conceptual ones ah helped you to learn the topic better, because you really know why it works not just how it works
>
> Student A: You're kind of bringing together what you know from other things….you kind of actually are more thinking yourself.

A full analysis of the interview transcripts is currently being carried out.

Concluding Remarks

The concept of 'function' can be considered as a threshold concept in mathematics. As such it is a key concept that students must master and could be viewed as a 'jewel in the curriculum' (Cousin 2006) of Calculus courses and act as a focal point for teaching. Constructing tasks and activities for students that will engage them with the concept, and effectively develop and transform their understanding of functions and other threshold concepts then becomes a challenge for teachers and lecturers. Some suggestions have been made here as to the types of tasks suitable for first year undergraduate Calculus students in this regard. A key component of the successful understanding of functions involves the ability to move flexibly between different representations and characterisations of functions. Thurston (1990) terms this integration and encapsulation of ideas as a 'compression', indicates this 'compression' can actually have the effect of simplifying an idea, and speaks of the joy inherent in reaching such a level of understanding:

> *Mathematics is amazingly compressible: you may struggle a long time, step by step, work through some process or idea from several approaches. But once you understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics.* (p.847)

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References

Bass, H., 2005. Mathematics, mathematicians and mathematics education. *Bulletin of the American Mathematical Society*, 42, pp.417-430.

Boesen, J., Lithner, J, and Palm, T., 2010. The relation between types of assessment tasks and the mathematical reasoning students use. *Educational Studies in Mathematics*, 75, pp.89-105.

Cousin, G. 2006 An introduction to threshold concepts. *Planet 17*. Available online at http://www.gees.ac.uk/planet/p17/gc.pdf last accessed 20 June 2012.

Cuoco, A., Goldenberg, E.P. and Mark, J., 1996. Habits of mind: an organising principle for mathematics curricula. *Journal of Mathematical Behaviour*, 15, pp.375-402.

Dreyfus, T., 1991. Advanced mathematical thinking processes. In: D. Tall ed., *Advanced Mathematical Thinking.* Dordrecht: Kluwer Academic Publishers. pp.25-41.

Dubinsky, E. and McDonald, M.A., 2001. In: D. Holton ed., *The Teaching and Learning of Mathematics at University Level*. Dordrecht: Kluwer Academic Publishers, pp.275-282.

Gray, E. and Tall, D., 1994. Duality, ambiguity and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education*, 25(2), pp.116-140.

Groves, S. and Doig, B., 2002. Developing conceptual understanding: The role of the task in communities of mathematical inquiry, *PME26*.

Hillel, J., 2001. Trends in curriculum: A working group report. In: D. Holton ed., *The Teaching and Learning of Mathematics at University Level*. Dordrecht: Kluwer Academic Publishers, pp.59-70.

Mason, J., 2002. *Mathematics teaching practice: a guide for university and college lecturers.* Chichester: Horwood Publishing Ltd.

Mason, J., & Johnston-Wilder, S., 2004. D*esigning and using mathematical tasks*. St Alban's: Tarquin Press.

Meyer, J.H.F. and Land, R., 2003. *Threshold concepts and troublesome knowledge: linkages to ways of thinking and practicing*. Edinburgh: ETL Project (Occasional Report 4). Available from: http://www.etl.tla.ed.ac.uk/docs/ETLreport4.pdf [Accessed 20 June 2012].

Pettersson, K., 2012. *The Threshold Concept of a Function - A Case Study of a Student's Development of Her Understanding*. Sweden: MADIF-8. Available from: http://www.mai.liu.se/SMDF/madif8/Pettersson.pdf [Accessed 20 June 2012].

Sangwin, C., 2003. New opportunities for encouraging higher level mathematical learning by creative use of emerging computer assessment. *International Journal of Mathematical Education in Science and Technology*, 34(6), pp.813-829.

Selden, A, Selden, J., Hauk, S. and Mason, A., 2000. Why can't calculus students access their knowledge to solve non-routine problems? In E. Dubinsky, A. H. Schoenfeld and J. Kaput eds., *Research in collegiate mathematics educations IV*. Providence, RI: American Mathematical Society, pp.128-153.

Sfard, A., 1991. On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, pp.1-36.

Swan, M., 2008. Designing a multiple representation learning experience in secondary algebra. *Educational Designer*, 1(1), pp.1-17.

Thurston, W., 1990. Mathematical Education. *Notices of the American Mathematical Society*, 37, pp.844-850.

Vinner, S., 1983. Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology*, 14, pp.293-305.