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Example generation exercises in an introductory analysis course

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In this article, we describe how we systematically included example generation exercises into our introductory Analysis course and explain our motivation for doing so. Feedback on these exercises, collected from 39 students via course evaluation questionnaires, is presented. We also provide examples, taken from one-to-one interviews with seven students, of how engaging with the exercise caused two students to expand their concept images of divergent sequence, and resulted in three students developing techniques for generating certain types of sequences. The merits of incorporating these exercises into an advanced mathematics course are discussed.

Motivation

Definitions of mathematical concepts play a pivotal role in advanced mathematics courses at university. For a given mathematical concept, students are expected to engage with its formal definition and to ultimately validate and produce proofs based on this definition. Undergraduates often meet this mathematical practice for the first time in an introductory analysis course, and since we both have taught such courses, we are aware of the difficulties that students encounter. As a result, we are always interested in activities that can be incorporated into our advanced mathematics courses to assist students in successfully engaging with, and using, definitions.

Fortunately, there has been quite a bit of research carried out on students' understanding of, and reasoning with, definitions in advanced mathematics courses. Alcock and Simpson [1] have surveyed some of the literature and they highlight two problem areas that students have in relation to definitions. Firstly, a student may not be aware of the importance of definitions in advanced mathematics and the role they play in accurately defining concepts and producing proofs. Secondly, for a given concept, a student's *concept image* may not be well-developed and may not coincide with the definition. Here the notion of concept image is as described by Vinner and Tall [2]:

> ***"… the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (p. 152).***

In this article we discuss our attempts to address this second issue in our analysis course.

We would love if all our students on completing analysis were able to use definitions to validate and produce proofs, and indeed we would like to think that those who achieve an honours grade have made considerable progress in this direction. However, we personally are not sure that *mastery* of these skills is possible in an introductory course. What we do believe however, is that *all* students who pass our course should possess rich, accurate concept images of the main concepts of analysis, and demonstrate the ability to reason "informally" about these concepts even if they cannot always frame their reasoning to the appropriate mathematical standard. Thus a key question for us is: *How can we encourage our students to develop rich concept images that accurately portray, and include, the definition of the concept?*

Watson and Mason [3] provide us with an answer. They believe that students should be encouraged to generate examples of a concept in order to construct an example space. They suggest that:

"As learners repeatedly construct example spaces associated with a concept, they are exploring and building a concept image by relating things that come to mind to a definition or instructions" (p. 97).

They also give several suggestions for example generation exercises. One which we particularly like is the exercise of generating *boundary examples* [4].

We decided to systematically incorporate example-generation exercises into our analysis course, and we did so by asking students to keep a "*portfolio of examples"* which we now describe.

Portfolio of examples

Each week, for six weeks, students were given example generation exercises to complete as homework. These required the generation of, on average, fifteen to twenty examples per week of the topical concepts that were being discussed in class. Students were asked to keep the examples in a portfolio which had to be submitted for grading towards the end of the semester. This exercise constituted 20% of the course assessment mark. All portfolios were returned before the final examination as some students were keen to have them as revision aids.

Students were allowed to work on the exercises by themselves or in groups. However, if they worked in groups they had to declare who they worked with and no-one in the group was permitted to submit the same example for a given exercise (unless there was a unique example). Thus if three people were working together, they had to generate three examples for a given exercise – one for each individual's portfolio.

A number of the exercises were in the general spirit of Mason and Watson's [4] technique for generating boundary examples. Some examples from the "*portfolio of examples*" exercise are as follows:

Example 1. Please give an example of a sequence (a_n) that has:

- $(1/n)$ as a subsequence;
- $(1/n)$ and $(-1/n)$ as subsequences;
- $(1/n)$ and $(-1/n)$ as subsequences and converges to 0; and,
- $(1/n)$ and $(-1/n)$ as subsequences and doesn't converge to 0.

Example 2. Please give an example of each of the following:

- a function g: $N \to Z$ that is injective;
- a function g: $N \to Z$ that is injective and surjective;
- a function g: $N \to Z$ that is injective but not surjective; and,
- a function g: $N \to Z$ that is neither injective nor surjective.

Example 3. Please give an example of each of the following:

- an infinite set that is bounded above by 10;
- an infinite set that has supremum 10;
- an infinite set that has maximum 10;
- an infinite set that has supremum 10, but does not have maximum10;
- an infinite set that is bounded above by 10, but does not have supremum 10; and,
- an infinite set that is not bounded above by 10.

In some instances we specifically required that graphical examples were given. By doing this we hoped that students might develop flexibility in representing mathematical concepts.

During interviews with seven students from the analysis class of 2007, we asked them whether they consulted the definitions when completing the "*portfolio of examples*". We were surprised when two said they did not and one said they did so only sometimes. As the aim of the exercise was to encourage students to develop rich concept images that accurately portray, and *include*, the definition, we attempted to address this by including exercises of the following form the next year:

Example 4. Please give an example of a sequence (a_n) such that:

$$\forall \varepsilon > 0,\ |a_n| < \varepsilon,\ \ \forall n \in N.$$

$$\forall \varepsilon > 0,\ |a_n - 2| < \varepsilon,\ \ \forall n \in N.$$

$$\forall \varepsilon > 0,\ |a_n - 2| < \varepsilon,\ \ \forall n \geq 4.$$

$$\forall \varepsilon > 0,\ \exists N \in N \text{ such that } |a_n - 2| < \varepsilon,\ \ \forall n \geq N.$$

We hoped that these types of exercises would not only encourage students to make a connection back to the formal definition, but also help them engage with mathematical notation and sensitise them to how small changes in notation can change an example.

Student Feedback

From the analysis class of 2008, 39 students completed a course evaluation questionnaire. A number of questions were aimed at eliciting students' views on the "*portfolio of examples*" exercise. When asked how long it took to complete the entire portfolio, answers ranged from 4 to 24 hours. The average length of time to complete the portfolio was 10.9 hours. They were also asked to rate the difficulty of the "word" and "notation" examples, where one was "*not at all difficult*" and five was "*very difficult*". The averaging ratings were 2.8 and 3.2 respectively. In response to the question on how useful they felt the "*portfolio of examples*" exercise was to their understanding of analysis, where one was "*not at all useful*" and five was "*very useful*", the average rating was 4.5. All surveyed agreed that the exercise should remain as part of the course assessment.

The following question was also included in the survey: What did you learn (if anything) from completing the "*portfolio of examples*"? All but one person gave a response to this question. We examined the 38 responses for themes and are were able to place each response into at least one of five categories.

The first category, with a total of eighteen responses, was: "helped better understand the course/aspects of the course". Among these, six indicated that it helped in developing a better understanding of the concepts, and a further six specifically mentioned topics that they understood better as a result of the exercise.

"It helps you gather your knowledge of different concepts, and you can notice patterns or similarities between different sequences and the rules governing them – enables you to understand them in your own way."

The second category, "link examples with theory", had five responses. These students felt that completing the exercise had helped them make a theoretical course like analysis more concrete. As one student commented:

"I learned to apply the theory of analysis in order to find examples. By moving from the theoretical to reality is very helpful."

The third category, "ability to give examples and/or counterexamples", was in the same spirit as the second. Seven students commented that they felt they had the ability to generate examples or counterexamples.

"I now know how to give examples of a lot of the concepts used in analysis (not just written down but also on a graph, which is good when trying to visualise what I was being asked for)."

There were eleven responses in the fourth category, "helped to read/write mathematics", with six of these specifically stating that the exercise had helped them better understand mathematical notation. The final category, "study/revision aid", contained seven responses of students who commented that the "*portfolio of examples*" acted as a good study or revision aid.

Boundary examples – the case of divergent sequence

We also have data relating to the "*portfolio of examples*" exercise from one-to-one interviews conducted with seven students from the analysis class of 2007. In these interviews, students were given three statements about concepts from analysis, and were asked to decide whether each statement was true or false and to provide a justification. On completing these tasks, participants were then questioned on the "*portfolio of examples*" exercise. They were asked to rate the difficulty and usefulness of the exercise and were questioned on whether they had worked alone or in a group; whether they had methods for coming up with examples; whether they had found any parts of the exercise particularly difficult; whether they referred to the formal definitions when completing the exercise; and whether they ever experienced an "aha moment" in generating an example.

If an aim of the exercise is to encourage students to develop rich, accurate concept images, there is certainly evidence that two of the students struggled to incorporate a boundary example into their concept images while completing the exercise. Brendan and Colm both mentioned the exercise requiring the generation of two divergent sequences (a_n) and (b_n) such that $(a_n b_n)$ converges, as one that they (independently) experienced difficulty with. They both explained that they thought of divergent sequences as sequences that must tend to plus or minus infinity. Colm describes his difficulty:

"I couldn't think of divergent sequences because I kept thinking of sequences that were, eh, increasing… […] The natural numbers, sequences of natural numbers, one going right up to infinity, is a divergent sequence."

Both Brendan and Colm had to expand their individual concept images of divergent sequence to accommodate bounded, divergent sequences. As Brendan puts it, it was about "seeing divergent as not convergent".

Techniques for generating examples

There is also evidence of three of the students having developed a technique for generating certain types of examples. Adrienne, Aoife and Brendan all mentioned that they (independently) had difficulty generating an example of a sequence that had a subsequence with a particular property. We take Brendan as an example. He talks about how he realised that you could generate a sequence where the even and odd terms could each follow a different rule.

"I say that a_n equals n if n is an even number, and a_n equals a half n say if n is an odd number, and in that way that gives you a lot. I thought that was cheating myself, but it does give you a lot more room to come up with different kinds of series [sic], come up with like subsequences when one part of it diverges and another part converges. It can get you out of a lot of tight situations so I use that quite a lot and it helped."

Earlier in the interview Brendan had been asked if the following statement is true or false: If a_n has a convergent subsequence, then a_n is bounded. After some initial thoughts he successfully generated a counterexample using the technique described above.

"In my head I need to find, to work out if it is true or false or rather if it is false I need to find one way in which a_n would have a convergent subsequence but not be bounded. I suppose what came into my head first was a sequence where the terms would alternate. Every second term would be one, two, three, four, five … the natural numbers, the even terms would be say zero. One, zero, two, zero, three, zero, four, zero, five, zero. The subsequence of every second term converges because it is just zero. But then the other one doesn't, eh, because it goes one, two, three, four, five - it will go on for ever."

Adrienne and Aoife also successfully generated similar types of counterexamples on this task, and both spoke later in the interviews of how they had initially struggled in generating examples of this type. Adrienne spoke of having an "aha moment" when she realised you could have "alternating" sequences like these, and Aoife described an exercise that she had particular difficulty with until she realised how to "put a sequence back in" with another sequence.

This technique may not seem particularly sophisticated to a mathematician, but we have noted that students regularly have difficulty generating a sequence that satisfies the second part of Example 1 above.

Conclusion

It is difficult to say whether the "*portfolio of examples*" exercise encourages our students to develop rich concept images that accurately portray, and include, the definition of the concept. One might argue that if the exercises are chosen judiciously then a student who completes the portfolio successfully, should benefit in the desired way. And although it is hardly overwhelming evidence, it is clear from the interviews with Brendan and Colm, that the exercise encouraged them to expand their individual concept images of divergent sequence.

Even if the exercise does not have the intended effect, the fact that our students have indicated that it helps them in their understanding of analysis, makes a theoretical course more concrete, improves their ability to generate examples and counterexamples, assists them in interpreting mathematical notation, and acts as a useful study/revision guide, certainly makes it a worthwhile assessment component.

It is also important to mention that to encourage students to further generate examples we also incorporate True/False tasks into our course. Exercises where students are asked to decide if statements are true or false may encourage them to engage with examples, and hence build on the work started by the "*portfolio of examples*".

Finally, we note that as lecturers, we too have benefited and learned so much from this exercise. It may be fanciful to say this, but we feel the portfolios have given us a mirror with which to examine our students' concept images. In particular, we have become very familiar with what Tall and Vinner [2] describe as *potential conflict factors*, and feel that an awareness of, and sensitivity towards, these issues can only make us better teachers.

References

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