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We thank the keynote speakers, paper presenters and all participants in the first MEI conference on research in mathematics education in Ireland. In keeping with the conference theme ‘Opening Doors’ we sought to bring together, in St. Patrick’s College, those who have an interest in mathematics education research in Ireland, to discuss recent research and developments; to consider future directions for such research in Ireland, and, most importantly, to improve linkages and encourage research collaboration among mathematics education communities within Ireland. We hope we succeeded in this and that the conference provided participants with an interesting and challenging programme of presentations, and with opportunities for discussion and making contacts and renewing friendships.

We also express our sincere gratitude to all those who supported the conference including: Dr. Pauric Travers, President of St. Patrick’s College, Maeve Fitzpatrick, Conference Secretary; our sponsors; Department of Education and Science, St. Patrick’s College Research Committee, Larry Quinn of Accuris Ltd, and, AIB Drumcondra; and, of course, the participants.

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Keynote Addresses

Research Mathematicians as Learners –

and what mathematics education can learn from them

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I recently undertook a study of the epistemologies of 70 research mathematicians (35 women and 35 men) in universities in England, Ireland, Northern Ireland and Scotland. The mathematicians were asked to explain how they came to know some new mathematics, when researching. The results of the study are both illuminating for practice in schools but also very surprising. In the Irish context, they reinforce an active, enquiry-based approach to learning mathematics as well as drawing attention to the continuing disadvantages faced by women in the community of mathematicians and calling for serious changes in the mathematical culture in classrooms. In this presentation, I outline the results of the study and draw out the implications for learning and teaching of mathematics in schools.

Background

Over a period of time, my mathematics education research interests have spanned problem-solving from two perspectives, the learner's and the nature of mathematics. I became convinced, and there was research evidence to substantiate that conviction, that learners reacted differently to mathematics if they had the opportunity to engage with interesting and challenging situations. The outcomes needed to be a feature of the thinking and the work that they did, rather than the expectations of textbook authors or teachers. I was also convinced that mathematics itself was an exploratory, research-based activity, not the kind of closed, handle-turning and reproductive subject so many of us, and our pupils, have experienced in schools and so many refer to as "boring"!

But I also had another research interest in that I was concerned about social justice in mathematics learning. These concerns began with the international findings that females lost interest in, and performed less well at, mathematics than males. But they expanded to ask what kind of beast is mathematics that it seems to appeal most to a very limited constituency of people and how does this happen? The genetic explanation "Mathematicians are born not made" did not seem reasonable and is more and more discredited by research findings. (See Murray, 2001.) I became convinced that, to explain differences in choice and performance in mathematics, the literature was focusing on pedagogical experiences, what happens in classrooms, while ignoring the processes of coming to know, the epistemology of the discipline. While, of course, pedagogy is very important, for me it was insufficient to explain what I saw as a divisive subject, favouring some, discriminating against others, by processes of coming to know that were based on a socially-devised epistemology.

So, in 1995, I looked for evidence in which to root these convictions, I explored the literature in the history, philosophy and sociology of mathematics, out of which I developed a theoretical model. (See Burton, 1995.) The model, while reflecting that literature, describes how mathematicians come to know mathematics. The model has five categories. They are:

person- and cultural-social relatedness; that is, mathematics is part of people, their culture and their society, not something which is abstractly disconnected from those who develop it; **aesthetics**; mathematics is sensually experienced by those learning it in terms of its beauty to which pattern, shape, etc. all contribute;

intuition (and insight); the process of coming to know mathematics appeared to be one that involves intuitive reactions and responses that might not, in the first instance, be explicable;

styles of thinking; at the time of the generation of the model, the literature described two different styles of thinking in mathematics, the visual and the abstract. I conjectured that, to be a research mathematician, you would have to move freely between both;

connectivities; it seemed from the literature references to mathematics that it was not understood in the fragmented way in which it is frequently taught but as a subject which was internally connected as well as connected to other subjects and to the outside world.

In 1997, I undertook a study with 70 research mathematicians, 35 females, 35 males, primarily to find out how well the model matched what they had to say about their processes of coming to know through research. But I was also interested to know if female and male mathematicians differed on the dimensions of the model, or in any other ways, and, if so, what these differences were. Why did I think it was important to find out about mathematicians' epistemologies if my foremost interest was in learning? First I believed that, when researching, mathematicians are learners. Second, I believed that the processes of learning, how mathematicians come to know, are the same no matter what is the level of sophistication of the learners. So, possibly somewhat provocatively, I take the position that coming to know is the same for a five year old, as for a fifty-five year old, for a naïve and unsophisticated infant as for a mathematician working at the boundaries of the discipline (for a development of this theme, see Chapter 3, Burton, 2004). My concerns, therefore, were not with mathematics as a collection of knowledge objects, nor with their acquisition. I wanted to know how mathematicians would describe the changing state of their knowing when they were researching and this reflects my understanding of epistemology as a framework for explaining knowing, not as a theory of knowledge.

In this paper, I outline the study to give you a context then talk about how well the theoretical model was matched by what the mathematicians described of their beliefs and practices. I will draw implications throughout for the teaching and learning of mathematics and outline a distinction that became important, for me, between the culture of mathematics and the mathematical culture in order to help in understanding what influences the experiences of different learners of mathematics.

The Study

I based the study on data, tape-recorded plus detailed notes, collected from face-to-face interviews, and six telephone interviews, with the 70 mathematicians. The interviews averaged an hour and a half in length and were conducted in either office or home, at the mathematician's preference. I interviewed in 22 universities in England, Scotland, Northern Ireland and the Republic of Ireland. Prior to the interview, participants were provided with a one-page outline of the topics I wished to discuss. These related to their "history" as mathematicians, their current research practices and, of course, how they come to know mathematics. They were offered the option, which none took, of deleting anything that they did not wish to discuss. Notes of the interviews were returned to them so that they could agree content, amend, change or delete. They were guaranteed anonymity. The data were entered into NUDIST, a qualitative, analytical computer-based tool. Quantitative data were collected, computed and tabulated in EXCEL. The interviews were discursive in style and the participants were free to introduce and explore any issues of their choice. In the course of the interview when they described their experiences, participants inevitably spoke of themselves as learners and as teachers. This gave me data on teaching and learning which had not been the subject of the study. The distribution of the participants by sex, and status, is shown in the Table:

	Postdoc.	Lecturer	Senior Lecturer	Reader	Professor	Senior Res. Officer	Research Fellow
Females	1	19	7	3	3	1	1
Males	1	17	9	2	6		

Sex and status of participants

How did the model match the mathematicians' descriptions of their practices?

All five categories of the model were meaningfully discussed as appropriate to the research practices of the mathematicians. Most frequently, the categories occurred naturally in the discourse but, if not, I would note that nothing had been said about, say, aesthetics, and that would be enough of a stimulant to gain a response. As individuals, the mathematicians positioned themselves very differently with respect to each category except for the importance of connectivities on which they all agreed, even where they said that finding connections was not specifically important to them. On the other categories, their responses lay along continua between opposites: "there is no such thing" to "it is the most important thing". Differences became apparent between whether the mathematician was in pure or applied mathematics, or statistics but gender did not appear to be an influence.

Personal- and Cultural-Social Relatedness: My expectation, at the start of the study, was that enculturation into mathematics would be more powerful than gender, race or class, in influencing how the mathematicians understood and practised mathematics. Given the predominantly Platonist view of mathematics presented both in the literature and in public discourse, I was surprised to find the variation between individuals. As one said:

> *Some of the time I am a Platonist. As mathematicians we want to believe that it is objective but I guess rationally I think it is socio-cultural, and emotionally, I want to believe it is objective.*

Applied mathematicians emphasized utility: *You can do things with it, you can model real things, you can make predictions, you can compare experiments.* They were mainly concerned about describing physical things using formulae that work, modelling the real world. But such a practical approach did not preclude one mathematician from acknowledging the socio-cultural basis by *having a set of rules, a language, which allows you to do that.*

A mathematician summed up the dilemma in which some of them found themselves:

> *There is a desire to be a Platonist, to accept these ideas as 'truths' but there is a contradiction that I cannot really do that. On the one hand, I agree with the cultural dependency argument, but on the other hand, I do find appealing the idea that it might be possible to build something which would be meaningful to some other race of people or creatures.*

Given that 10% took a socio-cultural stance and that there were a number who expressed the kind of ambivalence described in the above quotes, it seems to me not before time to shift our stance in classrooms from a dominance of the abstract, objective position, to an embracing of mathematics as a cultural phenomenon influenced by social factors. This would open the door to a more human, holistic and variable approach to the teaching and learning of mathematics than, formerly, has been seen, an approach that many students say they like and they miss in their mathematics classes (see Boaler, 1997; Burton, 2001).

Aesthetics: The continuum with respect to aesthetics ran from statements such as *I think words like beauty and elegance are over used* to *Mathematics is closely related to aesthetics*. Although the applied mathematicians tended towards rejecting aesthetics as relevant to them, they acknowledged its importance for pure mathematicians. In the same interview, a mathematician contradicted herself in first saying *beauty doesn't matter*, and then, later:

This is a three-dimensional generalization of the two-dimensional problem I have already done. I expected it to be the same and it is not the same. It is annoying because if it were the same, it would be beautiful.

A female mathematician said: *I think beauty probably requires a culture before you can appreciate it*, a sentiment with which I entirely concur and which reinforced, for me, the importance of the personal and cultural-social relatedness of mathematics as well as underlining how, as mathematicians, we are encultured into a very particular set of beliefs and practices. But people experience aesthetics differently and find value in different aspects of mathematical practice and outcomes. However, they all expressed delight and motivation from the pleasure of teaching perceived beauty. Why should mathematics learners be any different? But we make an assumption in classrooms of homogeneity amongst our students – that they can and will learn things in a similar way, with similar responses and accord similar meanings. From listening to the mathematicians, I became very conscious of just how damaging and distorting this is both for students *and* for mathematics. Furthermore, why should we think that lack of mathematical sophistication precludes an appreciation of aesthetic pleasure – anyone who has worked with young children knows this not to be the case. So the argument in favour of teaching 'the basics' before permitting learners to touch complexity, seems to me extremely shallow and a misunderstanding of the nature of learning *and* of mathematics. Philip Davis and Reuben Hersh put it well:

> Blindness to the aesthetic element in mathematics is widespread and can account for a feeling that mathematics is dry as dust, as exciting as a telephone book, as remote as the laws of infangthief of fifteenth century Scotland. (1983: 169).

Intuition/Insight: A few participants said something similar to one who asserted: *I don't think intuition plays a part*. However most, whether they called it intuition or insight, recognized when *a light switches on when I look at a problem*. But this was not a mystical thing – they were clear that experience helped to show *what works and what doesn't*. For the majority, their intuitions (or insights) played a major role in how they conducted their research. Without being able to describe what these were, or how they came by them, the mathematicians *knew* they were important. Yet, despite this conviction with respect to their research, they did not appear to respect intuition in their students. Nor did they deliberately nurture intuitions or help their students to establish or recognize links between their intuitions and the power and function of argument. Consider the comment of one mathematician from the perspective of its implications for teaching:

One of the things I find about students, undergraduates in particular, is that they seem to have very little intuition. They are dependent on being spoon-fed. The ability to look at a problem from different angles is crucial.

If intuition is as important to mathematics as so many claimed (and see Hersh, 1998) and as important to mathematics education as Efraim Fischbein (1987) demonstrated, why is it not acknowledged and nurtured in classrooms? It seems to me that a teaching style which encouraged learners to explore their intuitive reactions to a situation, whether apparently consistent with "required" knowledge or not, is far more likely to provoke learning than denial of such possibilities (and see Burton, 1999 and Lehner *et al*, 1998).

Thinking Styles: I had two conjectures at the outset of the study, neither of which was confirmed. The first was that I would find the two different thinking styles recorded in the literature, the visual and the analytic (for example, see Hadamard, 1945, p.86). The second was that research mathematicians would move flexibly between the two. When talking through how they solved problems, I found that the participants used three, not two, different thinking styles. I have called these:

Style A: Visual (or thinking in pictures, often dynamic);
Style B: Analytic (or thinking symbolically, formalistically); and
Style C: Conceptual (thinking in ideas, classifying).

Differentiated between pure, applied mathematics and statistics, my participants spread across the three styles as shown in the Table:

	Pure	**Applied**	**Statistics**	**Totals**
Style A	21 (11/10)	15 (5/10)	9 (5/4)	45 (21/24)
Style B	11 (7/4)	11 (5/6)	6 (2/4)	28 (14/14)
Style C	10 (6/4)	16 (6/10)	7 (4/3)	33 (16/17)
Totals	42 (24/18)	42 (16/26)	22 (11/11)	106 (51/55)

Thinking style by area of mathematics

The numbers do not sum to 70 since 45 mathematicians claimed to make use of more than one style. But for the 25 single style thinkers, 15, used Style A (9/6), 3 used Style B (2/1) and 7, Style C (4/3). The majority of mathematicians, 42, used a combination of two out of the three styles and only 3 claimed to use all three styles (2/1). Some 60% of the 70 participants claimed to be using Style A alone, or in combination, 37% style B and 47% Style C so visual thinking did dominate but the breakdown between female and male does not justify an assertion of difference and certainly does not substantiate the stereotype of the visual male.

Those who claimed not to be using a visual style of thinking appeared to be aware that there are alternative ways of thinking about mathematics. However, many of the visual thinkers appeared to hold an assumption that everyone thought about mathematics in the same way as they did. Additionally, most of my participants did not appear to realize that thinking and learning styles are very closely related and that, consequently, a lecturer with one dominant thinking style might be failing to communicate with students whose thinking style differs. (See Cowan, 1975). Also, teaching materials are not constructed to exploit differences between thinking styles nor to offer alternative views or pathways for teaching a particular mathematical goal. Teachers need to be alerted to differences in thinking style and provide opportunities for learners to exploit their dominant style and learn from its distinctiveness.

Connectivities: The mathematicians accorded great importance to the making of connections in mathematics. This was whether or not they themselves were able to make connections. The making of connections might be done entirely within mathematics itself or between the mathematics on which they worked and disciplines and practices external to mathematics. One mathematician said: *Sometimes areas overlap and that is important. I am very interested in connecting up areas. Applicability across fields is important.* Another linked categories from the epistemological model in saying: *The aesthetic is the aesthetic of the connections.* Yet another pointed out that:

> *Your work is not geared immediately to applications although it would be excellent if one could establish the bridgehead which allowed that. One is always keen to try that.*

A lecturer summed up:

> *It is very exciting when you make a connection, put a piece in a jigsaw, see how some ideas that were understood by one group of people were exactly what another group of people needed to make something work.*

Despite the commitment of the mathematicians to connectivities, they, and we, continue to work with a fragmented curriculum, teaching mathematics in disconnected blocks of knowledge objects. The value of exploiting, developing and incorporating connectivities seems to me to be a most important lesson to take from the mathematicians. I can see how a classroom that was determined to make connections, would also be a classroom where intuition was fostered, heterogeneity was celebrated and students would feel a sense of connection to the mathematics that they are learning.

The mathematical culture

The mathematicians, particularly the women but also some of the men, were very unhappy about attitudes, behaviours and values that I am calling, together, the mathematical culture. In particular, they drew attention to three: hierarchy, competition and isolation. In their discourse they demonstrated discomfort with many of the ways in which these three descriptors constrained and influenced their participation in the mathematical community. Indeed, we know from research done with postgraduate students (Herzig, 2002; 2004), that many abandon or express disaffection from their postgraduate studies in mathematics for precisely these reasons. But even those who do not, still express their discomfort with the kinds of attitudes, behaviours and values being discussed here.

Hierarchy

Hierarchies are common in academe, often role-legitimated (such as, in the UK, doctoral student, post-doc., lecturer, senior lecturer, reader, professor). But mathematicians appear to be skilled at developing many other hierarchies:

> *The biggest hierarchy is the one that mathematicians are always putting themselves in. People rank each other.*

A female mathematician mentioned this ranking drawing a distinction between 'real' and others:

> *I consider myself a mathematician because I lecture and research in mathematics and I suppose that by definition makes me a mathematician. But I have met people whom I consider to be 'real' mathematicians and I am not one of those people.*

And Margaret Murray found much the same in her study of female mathematicians:

> "she is careful to distinguish her work from 'real research' […] which is the work of 'top mathematicians' such as 'Gauss and Archimedes and Newton', who reside 'somewhere near heaven' ". (2001: 214).

However, even a senior mathematician with an established international reputation managed a similar differentiation:

> *There is a level above the standard level at which we work. The top mathematicians can transcend these different fields and know enough. Often the dramatically new things are accepted at that level and then the smaller communities have to adapt themselves and, for example, learn the new language. Recently in complex dynamical systems very original ideas from probability theory have come in and have been accepted now by the world's top mathematicians and the complex dynamicists have to get on with it and accept it.*

However, it was not only that people were labelled into hierarchies. The mathematicians were also quick to invoke a hierarchy of judgment when speaking of the quality of published work. Work of quality was 'significant', work could be 'important' or 'interesting'.

> *Interesting is a polite way of saying it hasn't done very much. Significant means that it has made a major contribution whereas important indicates a contribution.*

The kiss of death was to declare something 'trivial':

> *Trivial – they aren't really saying anything different; it is a re-hash that might have been said before, and possibly better.*

Apprenticeship, as a model for learning, is not discussed from a gender perspective despite the fact that many studies point to the gender implications of its use (see Etzkowitz et al., 1992; Herzig, 2002). One particularly poignant way in which apprenticeship operates is to institutionalise hierarchies, ensuring that the learners are at the bottom of the hierarchy. Elena Nardi and Susan Steward identified some effects of this in schools:

> "The hierarchy inherent in the above outlined elitist situation alters the nature of the classroom experience from one that focuses on catering for the individual learner's needs to one that focuses on establishing and assessing each learner's position in this hierarchy. The students express their alienation from this depersonalised, deterministic mathematical experience (2000: 359).

So, the making of hierarchies is part of the mathematical culture. From school into university and then into the community of practice of mathematicians, one learns of the importance of hierarchies, not only within the actual discipline, but within the practices as well. Establishing such hierarchies invokes competitive practices since moving up and down the hierarchy is subject to the judgments of your peers and those in positions of power.

Competition

Pat Rogers pointed out: "lack of competition is not usually associated with the mathematics classroom" (1995:184). A female mathematician said: *The competitiveness of mathematics is institutionalised from the beginning. So people are made to feel stupid if they don't achieve in mathematics.* A male mathematician reflected: *There can be tremendously savage competition in our field and that is not something I feel particularly comfortable with.* Thinking back to her undergraduate studies, a female said: *The thing I hated about doing mathematics was that the boys were so competitive.* It is the institutionalisation of this competition that has been observed as starting in school classrooms and continuing throughout undergraduate and postgraduate studies into the practices of the community of mathematicians.

I am labelling such competition as constituting part of the mathematical culture. It can be found in practices at both school and university level. Competitive practices emanate from many different assumptions but are implicitly part of institutionalised hierarchies. Jo Boaler and Jim Greeno pointed to a dominant view held by those within mathematics that the subject is cognitively difficult. They challenged the effects of this stance as "unusually narrow and ritualistic…[producing] environments in which students must surrender agency and thought in order to follow predetermined routines" (2000:171). The result that they observed in their research was that "by emphasising drill and practice of procedures, they [mathematicians] create a rite of passage that is attractive only for received knowers" (Ibid: 190). Not only does this exclude the very type of learner who might be able to benefit from learning mathematics and, possibly, contribute to the development of the discipline, a learner interested in generating questions, ideas, and pursuing challenges, but it establishes a climate of discrimination and consequent competition. Furthermore, as Candia Morgan has shown, because assessment is also a social practice, it should be understood as "a process by which a student may gain or be denied access to particular forms of privilege or power" (2000:231). Discriminatory outcomes have been well charted by Cooper and Dunne (2000)

As a teacher, once one has adopted an assumption about some students having 'ability', the next step is to classify those in the class into a hierarchy which is helped by a cultural assumption that mathematicians are 'born not made'. This was certainly not substantiated by the mathematicians in my study and was refuted by Margaret Murray (2001) who referred to "the myth of the mathematical life course" (p.16). Of the 70 mathematicians in the study, only 18 (approximately 26%) spoke of early influences, 26 (approximately 37%) chose mathematics during their secondary school years and for 5 (approximately 7%), their choice was only confirmed at university. One said: *I certainly cannot say that I ever set out to be a mathematician. It was more a drifting into mathematics.*

My research shows that competition features within the community, even though many mathematicians express dislike of it (see Burton, 2004). With others, I have also recorded the dislike of competition expressed by school students. There can be little doubt that competition flourishes inside mathematics and that its effects are not welcome to many. One of these effects is to isolate, particularly women, so that they feel alienated from the discipline and from the mathematics community of which, as mathematicians, they are entitled to be a member.

Isolation

Many female mathematicians spoke about the sense of isolation that they experienced within the world of mathematics. Sometimes this was because fewer women study the discipline. This was the case for a female mathematician who said:

> *One thing I have learnt from my own experience of undergraduate and postgraduate work was the isolating experience of being the only one.*

But this kind of personal isolation was not the only kind to which they referred. One female mathematician explained her movement into statistics because:

> *a statistical problem is never as isolated, it has all sorts of things impinging on it; it is always in a context. Whilst mathematics can be given a context, it can be very isolated.*

And a number of mathematicians cited collaborative work as being a good way to overcome isolation.

> *If you are collaborating with mathematicians, you feel much less isolated. Because mathematics is an isolating experience, you often feel burnt out, dried up, you have no skills and everyone else does. When you start working with someone else, you discover that there are things that you can do, perhaps better than they can, and things that they can do that you can't.*

This is, of course, one of the reasons for collaborative group work given by those of us who advocate it in mathematics classrooms. But in the above quotation, I would like to draw attention to the intensity of the feelings (*burnt out, dried up, no skills*) expressed by this female mathematician and the degree to which they coincide with descriptions to be found in the literature from research with school pupils as well as with university staff and students. In one of my own studies with 16-18 year old pupils specialising in Advanced Level mathematics, I noted:

> "The students wanted their teachers to facilitate discussion, teamwork, a light-hearted approach, a relaxed classroom environment where you are not afraid of making errors. They told anecdotes to support the fact that they did not want to be put down, persistently asked the same questions, made to look a fool or feel patronised, be put into a position where others laugh at you, be thrown 'in at the deep end'." (Burton, 2001: 67)

These are amongst the features of the mathematical culture to which I have been drawing attention. But I would dispute that they are an inevitable part of the mathematical culture. They are constructs that have come about for particular socio-cultural reasons and should be changed because they no longer match or contribute positively to the conditions of practice of mathematics. While students at every level persistently call for collaboration, a discursive environment in the classroom, freedom to make errors without criticism or loss of respect, they equally persistently report experiencing competition, being laughed at or made to feel a fool, being put down or feeling isolated. These features of the mathematical culture are both widespread and destructive. I believe that they help to explain the loss of interest in mathematics as a discipline of study in the university and the composition of the student community who enter mathematics, as representative of power, i.e. predominantly male and of the dominant culture. Such negative behaviours not only affect those who experience them but they also impact upon the discipline itself.

I have not spoken of those aspects of the culture of mathematics that have, historically, been defined as integral to mathematics and seen as part of what students are expected to acquire in the process of becoming mathematicians. To clarify the difference, I am using the term 'culture of mathematics' to describe, for example, structure or pattern which were discussed by the mathematicians in uncontentious language that was free of judgmental, hierarchical or competitive statements. 'Rigour', on the other hand, was heavily imbued with judgments. My point is that it is a chicken and egg question as to whether our attitudes to such apparently integral features of mathematics as structure, pattern and rigour are learnt with our mathematics or grow out of the mathematical culture which encourages and facilitates the application of competitive judgments. However, I am claiming that it is the mathematical culture which frequently dictates the ways in which the culture of mathematics is described and used.

Conclusions

While the mathematicians that I interviewed spoke very positively of the excitement, satisfaction and, indeed, euphoria that they experienced from their engagement with mathematics, as has been seen from the discussion above, this was not in the absence of some very negative feelings. And we know that students reflect these negative feelings to such a degree that many who could study mathematics, are failing to choose to do so. We might be concerned about the loss of these potential mathematics students, or of the mathematicians they might have become, or, more simply, by the social experience and reputation of mathematics as a discipline which is inaccessible, unattractive and difficult. Either way, I believe that much negativity can be explained by focussing, as I have done, on the mathematical culture. It is to the mathematical culture that I think we should turn when trying to explain and change rejection of the discipline and failure to succeed at it. I cannot feel at ease within a discipline that is described in terms of the misapplication of power through the maintenance of hierarchies, competitive practices and isolation of those who do not appear to match some mythical prescription of a mathematician. Nor do I feel at ease with the unthinking acceptance of those aspects of the culture of mathematics that, themselves, have been affected by the mathematical culture.

The evidence of heterogeneity from the study was, for me, one of the most positive features to emerge from the research. Building in acceptance, even celebration, of heterogeneity would not only change the experiences of those within the discipline, student, researcher, mathematician, but would open the discipline itself to the promotion of variation and to ways of encouraging learning that valued differences and sought means to compare and contrast them. This is how many of the mathematicians I interviewed are applying their research practices but, unfortunately, is rarely found in classrooms. The next major question is how to shift an acceptance of these research practices into the classroom so that learners can feel, behave and be accepted as researchers.

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The Increasing Role of Teachers as Co-Researchers: The Case of Design Research

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This paper relates developments in educational research to an Irish mathematics reform context, particularly as it pertains to the growing importance of teachers in the research process. Of the new approaches to research, what is now being called *design research* would appear to have value for those involved in the development and use of innovative teaching practices (see Kelly, Lesh & Baek, in press).

Overview of Design Research Methods

Effective teachers continually experiment with new ways to teach subject matter. In the last decade, researchers have begun to focus on what can be learned about teaching, learning and processes of design by focusing on the design and use of instructional objects and processes. In the main, the focus has been on the development and use of educational software and related supporting technologies.

In design research, there is an attempt not just to develop an effective instructional intervention, but also to explicitly study and answer the question, why is it being effective, and what does that tell us about learning and teaching? As in all design and invention processes, no innovation is final, especially on a first attempt. Thus, the iterative nature of design and real-world feedback combine in such a way as to provide additional sources of data to advance instructional and learning theories.

The innovative and developmental aspects of design may be seen as a stage in a multi-stage program of research that moves from speculation, to observation, to identification of variables and processes via prototyping, to models, to more definitive testing of those models, to implementation studies, to scaling studies, and ongoing diffusion of innovations (Bannan-Ritland, 2003).

There is no single design research method; rather the approach prizes mixed methods that support innovation and discovery (Holton, 1998; Holton & Brush, 2001) via trial and error, iteration, focusing on emerging criteria. In character, it is exploratory and ambitious. It does not shy away from ambiguity, ill-defined problems, open systems that are socially multi-level, multi-timescale (Lemke, 2001), multi-phasic in learning, and subject to attrition and outside interference. Putatively the results and products of design research are more "adoptable" products for teachers since they grow out of practice (see Lagemann, 2002a, 2002b, 2003).

Design research activities in mathematics education and technology studies, lead not just to software products, but are quite varied and can provide:

- Existence proof of the new intervention
- Teacher professional development – theories of change; stages of conceptual and dispositional growth; instructional materials
- Student development – theories of change; stages of conceptual and dispositional growth; instructional materials
- Greater construct validity around the research conceptual goal – varied assessment practices both formative and summative
- Assessment task redesign (e.g., Lesh & Kelly, 2000)
- Reformulation of the original research problem, iteratively, during research
- Explanatory frames or interpretive frameworks future research efforts. For example, "ontological innovations" (ways of conceptualizing the complex milieu – "sociomathematical norms" or "meta-representational competence") "theoretical constructs that empower us to see order, pattern, and regularity in complex settings" (diSessa & Cobb, 2004, p. 84).
- Working models of practice
- Principles of design in complex social settings
- "Hypothetical learning trajectories" – proposed conceptual "pathways" employed by students and teachers navigating complex concepts.
- Curricular materials ["Japanese Lesson Study" see www.lessonresearch.net/]
- Software and associated curricular supports (e.g., GenScope/Biologica; Worldwatcher)
- "Learning environments" (e.g., WISE and TELS, see www.telscenter.org)
- New or hybrid research methods (e.g., Integrative Learning Design Framework – Bannan-Ritland, 2003; McCandliss et al., 2003).
- Growth of a database of principles of research/artifact/ "theory" design (Zaritsky/Fuson DVD) use of a pre-designed artifact (e.g., a statistical display tool) to better understand process
- Innovative Learning Environments

Examples include:

Progress Portfolio, Loh, Radinsky, Russell, Gomez, Reiser & Edelson, 1998; *Worldwatcher*, Edelson, Gordin & Pea, 1999; *LiteracyAccess Online*, Bannan-Ritland, 2003; *Genscope/Biologica*, Hickey, Kindfield, Horwitz, & Christie, 2003; *SimCalc*, Roschelle, Kaput & Stroup, 2000; and *Jasper Series*, Cognition and Technology Group at Vanderbilt, 1997. WISE, Linn, Davis & Bell, 2003.
www.letus.org; LIFE Science of Learning Center at Stanford

Not too surprisingly, design research is still being developed and debated (Educational Researcher, vol 32, 2003, special issue; Journal of the Learning Sciences special issues (e.g., Barab & Squire, 2004; Barab & Kirshner, 2001), and the Educational Psychologist, special issue (2004) with commentary by O'Donnell.

Design Research in an Irish Context

Design research involves first negotiating the goal of the design. In the Irish context:

- What is the vision for a mathematically proficient student?
- How can this definition be operationalized among the various stakeholders (policymakers, researchers, parents, students, others…)?
- How does it account for a number of mathematical factors: knowledge, facts, skills, dispositions, beliefs, emotional intelligence, communication, etc. (see US National Council of Teachers of Mathematics; US National Research Council report: "Adding it up")

Any design has to be implemented with the cooperation of teachers. What are the parameters in the Irish context that influence teacher participation?

- Curriculum design
- Resources
- Cost
- Time horizon for "success"
- Openness in the system to change incentive structures
- Tests
- Structural aspects of life in classrooms (scheduling, co-curricular activities)
- Political support, etc.
- Teacher training
- Other

In any system, realistically, some aspects can be changed in a reasonable amount of time for reform; whereas, others are problematic.

- What parts of the problem are malleable?
- Which are policy-tractable?
- Which of these parameters can be expressed as research-able questions?
- Which are outside the scope of (current) science?

Some reform is straightforward, other change may require substantial efforts in innovative directions, such as new conceptual goals for learning by teachers or students. Design research is appropriate if the goal is to:

- Explore new assessments, new indicators of progress
- Improve teacher professional practices toward an innovative goal
- Describe (at least initial) students' learning trajectories toward this new goal
- Document the "means of support" necessary during the implementation (school leadership buy-in, technology, teacher dispositions, parental support, etc.)
- Learn about implementation/diffusion issues

Three examples for the Irish context could include assessment redesign (following Tatsuoka or Lesh). The introduction of transformative pedagogy such as dynamic geometry – e.g., using Cabri. Or the introduction of new mathematics topics: e.g., fractal geometry or chaos theory. The Lesh problem is reproduced here. The examples from Cabri and fractal geometry were presented at the conference only. Online materials may be found at
www.math.fau.edu/kasia/Cabri/ and
www.math.fau.edu/Teacher/Teacher_homepage.htm.

A typical Leshian Model-eliciting Problem would be:

> You must decide in 15 minutes if news reports are a hoax:
>
> "A bank robber walked out of a bank with 1 million euros in small denomination bills."
>
> Should you run this story? Is it likely to be true? Give your mathematical reasoning.

Once a goal is adopted, various design research approaches for instructional intervention can be chosen from (Cobb et al., 2003; Kelly & Lesh, 2000):

- One-to-one teacher experiments
- Classroom teaching experiments (including multi-tiered and transformative versions)
- Preservice teacher development studies
- Inservice teacher development studies
- School/district restructuring experiments

The importance of the role of teachers flows from the fact that reform is a diffusion of innovations problem, which involves:

- Social aspects of adoption, adaptation, and institutionalization
- Local "means of support" realities
- Professional knowledge and experiences
- Real-time "repairs" with students who have varied backgrounds
- Proximity to the problem for 'action research' reflections supporting design research efforts
- Teachers and students are not "subjects to be studied," but co-participants in the research.
- Teachers are seen as local experts with crucial observational, practice, implementation and diffusion skills and experiences
- Rich, on-going documentation of learning links the worlds of researchers to teachers

Concluding Remark

It is difficult in this short paper to express the potential for impact of design research iterative activities on teaching and learning. Some sense of the excitement of the method was communicated in the live presentation at the conference demonstrating the lived aspect of this work. Design research is best experienced as a part of an ongoing research team.

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Japanese Teachers' Lesson Study in Mathematics Education as an exemplar of Design Research

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The case is made that Lesson Study in mathematics education as pursued by Japanese school teachers serves as an exemplar of Design Research. Faced with the difficult question of improving mathematics instruction what should one do? The Japanese solution in the primary grades is Lesson Study. This solution focuses on the central role of teachers in collaborating in designing, reflecting on, and delivering, mathematics lessons that positively affect student achievement. If you want to improve your mathematics instruction, what could be more obvious than collaborating with fellow teachers to plan, observe, and reflect on lessons? While a simple idea, lesson study is a complex process, supported by collaborative goal setting, careful data collection on student learning, and protocols that enable productive discussion of difficult issues. This "simple idea" is explored in the context of teaching mathematics in Irish primary and certificate classrooms.

1.0 Introduction

The goal of this presentation is to engage the audience in a simulation of Japanese Lesson Study and ask what the implications of such practice would be for the teaching of mathematics in Ireland. To that end I initially describe what Lesson study is, I provide a specific example (documented by Fernandez and Yoshida, 2004), and then I engage the MEI conference attendees in a series of questions around this example with both instructional and policy issues in mind. Finally, I close with comments about schooling in Japan and how it differs from schooling in Ireland. Throughout the presentation and interactions with the attendees I leverage the extant research literature on student learning of mathematics that has direct implications for instruction including for example: Cognitively Guided Instruction (see Fennema, Carpenter, and Peterson, 1989), Model Eliciting Problems (Lesh and Kelly, 2000), Variation Theory (Marton, 1998), and new theories of assessment (NRC, 2001).

2.0 Understanding classroom practice in mathematics: the case of Japanese Lesson Study

In this section we describe what has in the last 20 years become known as "Lesson Study," a central component of Japan's major effort of teacher professional development (kounaikenshuu) in elementary and middle school teaching (infant through second year teaching in Irish schools). We begin by outlining the processes associated with lesson study and how they are developed and utilized. Next we describe the meta-context for education and schooling in Japan. We note that, given the number of students attending evening cram schools in Japan, Japan's international performance in mathematics draws on the combined effects of lesson study and Juku (Japan's cram school system). We then highlight a finding from the US's National Academy of Science report (see Snow, date?) on the current debate in reading circles about phonics versus whole language learning. The Academy notes that reading with competence and comprehension is likely due to both strategies. We suggest that this may likely be the case for the learning of mathematics. However, this should not take from the fact that the quality of instruction found in Japanese classrooms is something that should be emulated where possible worldwide. In an effort to better link research and practice we ask if the process of Lesson Study could be improved by drawing on current research knowledge (in the first iterate of the lesson study process).

In their now classic book *The Teaching Gap* Stigler and Hiebert (1999) contrast mathematics instruction in three countries: Germany, Japan, and the United States. The authors draw on randomly sampled lessons from the TIMSS 1995 video study of eight-grade mathematics instruction (the rough equivalent of second year maths in the Irish school system). The countries were picked for a number of reasons professional, financial, and political. Japan was chosen because of its high performance in SIMS (the second international study of mathematics) and expected high performance in TIMSS, Germany was selected because of anticipated changes in performance due to the merging of East with West Germany in period between SIMS and TIMMS. The US wanted to participate because it was felt that a lot could be learned through the contrasting of the three systems and because the US was underwriting a large proportion of the study.

Stigler and Hiebert (1999) use their video analysis to highlight instructional differences across the participating countries asking why teachers in each country teach differently and what is it about the educational systems of each country that supports the types of instruction that can be seen nation to nation. Additionally, they outline the policy issues raised by their work for the teaching and learning of mathematics in American schools. It is in this regard that they provide insight into the processes underlying Lesson Study in Japan.

In Lesson Study groups of teachers meet regularly over long periods of time (sometimes ranging from several months to a school year) to work on the design, the implementation, testing and improvement of one or more "research lessons." Lesson study is a deceptively simple idea. If you want to improve instruction, what could be more obvious than getting together with your peers to collaborate on the development and implementation of lessons? While a simple idea, Lesson study is quite complex in practice with many component parts. Stigler and Hiebert (1999) focus our attention on what they believe to be the eight common parts of Lesson Study, these include: Defining the problem; planning the lesson; teaching the research lesson; Evaluating the lesson and reflecting on its effect; revising the lesson; teaching the revised lesson; evaluating and reflecting again; and sharing the results. Lewis (2002), on the other hand emphasises the meta-structure in what she labels the lesson study cycle. She attends to four critical structures: goal setting and planning; researching the lesson; lesson discussion and consolidation of learning. In the discussion that follows we combine the two renderings as appropriate. While we present the general features of Lesson Study we include, where appropriate, what teachers likely to do. For more detail see Lewis (2002), Fernandez (2004), Lewis provides rich examples of Lesson Study for the learning of primary school science whereas Fernandez explicitly attends to the learning of mathematics.

2.1 *Goal Setting and Planning*

1. Defining the problem. At its essence lesson study is the act of problem solving (problem solving by teachers about teaching and instruction). Consequently, the first step in the process is to define the problem to be resolved. What is the central problem that motivates the work of the lesson study group? Here the teachers identify the goals for student learning and for long term development. These goals can be general or specific in nature focusing on student motivation or on the learning of a specific mathematical topic. Normally, teachers pick a topic that arises from their daily practice. Occasionally, topics come from the Ministry of Education. When this arises they are of the form "how can we help students learn x." The Ministry then invites a sample of schools throughout the country to this question using lesson study as a vehicle for generating this knowledge. The schools are then expected to report back on their findings. Additionally, the Ministry will issue recommendations in a top down fashion. The interplay between the bottom up and top down mechanisms for change is unique to Japan. This feature of educational policy allows teachers direct input into national policy in a manner not available in other countries. This is not a common policy feature in Ireland.
2. Planning the Lesson. When the learning goal has been set teachers meet regularly to plan the lesson. The end goal here is twofold: to produce an effective lesson and also to understand why and how the lesson promotes (or doesn't) improved mathematical understanding on the part of students. Often teachers begin the planning process by looking at books and articles produced by other teachers who have studied a somewhat similar problem. This initial planning process can occur over weekly meetings for periods of months. In

general, the lesson begins with the statement of a mathematical problem Consequently, teachers engage in detailing the following:

a. The mathematical problem, its exact wording, and the numbers to be used;
b. The materials students would need and be given to address and solve the problem;
c. How the materials are likely to be used by the students;
d. The possible solutions that students will likely generate as they struggle with the problem; the likely misconceptions;
e. The questions the teacher will use to promote student mathematical thinking during the lesson;
f. The types of guidance that teachers could provide to students who showed one or another misconceptions in their thinking;
g. The use of the blackboard by the teacher;
h. The allocation of the finite amount of time devoted to the lesson and its component parts (the introduction of the problem, student work, etc.,);
i. How to handle individual differences in the level of mathematical preparedness of the students in the class;
j. How to close the lesson. Lesson endings are considered crucial and involve teachable moments to advance student understanding.

2.2 T*he Research Lesson*

3. Teaching the lesson. The team, that is, all teachers prepare the lesson together including role playing. One teacher teaches the lesson. The others attend the lesson with specific tasks in mind. When the students are asked to work at their desks, the other team members collect data on student thinking, what is being learned, student engagement, and student behaviour. They do this by observing and taking careful notes. Occasionally, the lesson is video-taped for further discussion.

2.3 *Lesson Discussion*

Here the teachers share and analyze the data they have collected at the research lesson. They carefully examine evidence as to whether the goals of the lesson for student learning and development were fostered (or met). Finally they consider what improvements to the lesson specifically, and to instruction more generally, are necessary.

4. Evaluating the lesson and reflecting on its effect. On the day the lesson is taught the teachers will meet to discuss what they observed. In general the teacher who taught the lesson speaks first generating their own personal evaluation. Then the other teachers contribute. Comments tend to be quite critical in nature but focus

on the group product, the lesson itself, and not on the teacher. All team members fell responsible for the final group product. This is an important activity because the team places attention on the possibility and opportunity for lesson improvement.

5. Revising the lesson. Next the teacher group, as a group revises the lesson. They do this based on their observational data and on shared reflections. They may change any number of things in the lesson including the materials, the specific activities, the opening problems posed, the questions asked, or on occasion all of these components. The teachers base their changes on misunderstandings demonstrated by the students they have observed.
6. Teaching the revised lesson. When the revised lesson is formulated it is then taught to a new classroom of students. It may be taught by the first teacher but it is often, and more than likely, taught by another team member. It is the lesson that takes on central focus not the teacher. During this iterate other teachers may be invited to observe. Often the full complement of teaching staff is invited to the second or third iterate of the lesson. The goal is simple it is one of slow deliberate iterative revision in search of perfection.
7. Evaluating and reflecting. Again, the teacher who taught the lesson comments first. The teacher will describe the goals of the lesson, his or her own assessment of how the lesson went and what parts of the lesson the teacher feels need re-thinking. Observers then critique the lesson and suggest changes based on their observations. An outside expert may be invited to these deliberations. The lesson conversations vary in focus from the specific issues of student learning and understanding, but also with respect to more general, or meta-issues, raised by the initial hypotheses used to guide the design and implementation of the lesson.

2.4 *Consolidation of Learning.*

If desired, the teachers refine and re-teach the lesson and study it again. When the iterative process is complete they write a report that includes the lesson plan, the student data, and the reflections on what was learned by the students, and by the teachers about the process of learning.

8. This description has focused on a single lesson. But Japan, like Ireland and unlike the US for example, has a well articulated national curriculum. Consequently, what the teachers has learned have relevance for other teachers locally and nationally. Teachers (and the school system) need to embrace the opportunity to share results. This can be undertaken in a number of ways. We highlight the ways that learning is consolidated locally and more globally in Japanese schools. First, as noted above, teachers write a report. The report documents the lesson, the reason for the lesson, what is learned by students and by

teachers, what the beginning assumptions were, and how these assumptions changed through the process of teaching, observation and lesson revision. The report is bound and is available to all teachers in the local school. They are read by faculty. Consider, for example, the power of this resource for the new or neophyte teacher. They are read by the principal. They are shared within the prefecture if considered interesting enough. If a university professor collaborated with the group the work may find an even broader audience.

Another way of consolidating and sharing the lesson study results occurs when teachers from other schools are invited to a rendering of the final version of the lesson. Sometimes, Lesson fairs are conducted at schools and teachers from a neighbouring geographic region are invited to watch research lessons in many subjects produced by a school over an extended time period. These are considered festive occasions, and a critical part of ongoing teacher development.

2.5 Some Reflections

Lesson study is of course culturally based, and we discuss the culture of Japanese schooling below. We believe that the process of lesson revision is a powerful one and one that can be legitimately adopted and adapted to fit the Irish school system. Lesson study will not bring quick rewards but it is a deliberate and concentrated effort to improve the process and products of teaching in mathematics.

3.0 Planning to Teach: An interactive session.

Given the outline of Lesson Study provided above how would we rethink the teaching of subtraction. Specifically, how might we teach children to solve the following problem 12-7=?

4.0 Lesson study re-situated: The meta-context of schooling in Japan

Understanding the Japanese people and culture requires understanding the factors that mould them. Particularly important are those components which influence them in their formative years. The Japanese education system is one of the most influential agents moulding Japanese youth. Given the large amount of time that Japanese students spend in schools, it is little wonder that the education system plays a tremendous role in determining the fabric of Japanese society. An examination of the "typical" high school experience illuminates the function of the education system in Japanese society.

4.1 Getting to School

Japanese high school students do not drive cars. Many either walk or ride bicycles if the distance is not too great. In other cases, students must take public buses and trains often changing public transportation services several times in order to reach their destinations. It is not uncommon for students to spend two or more hours each day on public transportation. After junior high school, students attend schools based on standardized high school entrance examination scores. As a result, some students travel a great distance to attend the school determined by their test scores. The school day begins at 8:30 am, so students may leave home as early as 6:30 am. While some students sleep or study during their long commute, public transportation also provides a chance for socializing with peers. Student behaviour on the way to school is regulated by school policies. These policies may prohibit certain activities in public--chewing gum, consuming snacks, reading books while walking--anything that might reflect badly on the reputation of the school. Each school has a unique uniform that makes its students easily identifiable to the public. School policies often require students to stand on buses and trains, leaving seats open for other passengers in order to demonstrate consideration. In practice, however, the behaviour of students tends to relax as they move farther away from school.

4.2 At School

Once at school, the students usually enter an area full of small lockers in which they place their street shoes and don school slippers. These slippers may be colour coded: pink for girls and blue for boys. Many schools have a weekly school-wide assembly. Then students assemble in their homeroom classes for the day's studies. The school day starts with classroom management tasks, such as taking attendance and making announcements. These activities usually are conducted by the students themselves on a rotating duty schedule called *toban*. Each homeroom has an average of 40-45 students. Students stay in their homeroom classrooms for most of the school day while the teachers move from room to room, operating out of a central teachers' room. Only for physical education, laboratory classes, or other subjects requiring special facilities do students move to different parts of the school. Between classes and at lunch time, classrooms can be noisy, and pretty lively places. Some schools may have a cafeteria, but most do not. Even in schools where a lunch is prepared and provided to the students, they usually eat together in their homeroom classrooms. In most schools, students bring a box lunch from home, almost always consisting of foods prepared by the mother in the early morning hours, such as rice, fish, eggs, vegetables, and pickles.

Japanese students spend 240 days a year at school, approximately 60 days more then their Irish counterparts. Although many of those days are spent preparing for annual school festivals and events such as Culture Day, Sports Day, and school excursions, Japanese students still spend considerably more time in class than their Irish counterparts. Traditionally, Japanese students have attended school for half a day on Saturdays; however, the number of required Saturdays each month is decreasing as the result of Japanese educational reforms. Course selection and textbooks are determined by the Japanese Ministry of Education. Schools have limited autonomy in their curriculum development. Students in academic high schools typically take three years each of the following subjects: mathematics, social studies, Japanese, science, and English. Other subjects include physical education, music, art, and moral studies. All the students in one grade level study the same subjects. Given the number of required subjects, electives are few.

At the end of the academic day, all students participate in *o soji*, the cleaning of the school. They sweep the classrooms and the hallways, empty trash cans, clean bathrooms, clean chalkboards and chalk dusters, and pick up litter from the school grounds. After *o soji*, school is dismissed and most students disperse to different parts of the school for club meetings.

4.3 Extracurricular Activities

Club activities take place after school every day. Teachers are assigned as sponsors, but often the students themselves determine the club's daily activities. Students can join only one club, and they rarely change clubs from year to year. In most schools, clubs can be divided into two types: sports clubs (baseball, soccer, *judo*, *kendo*, track, tennis, swimming, softball, volleyball, rugby) and culture clubs (English, broadcasting, calligraphy, science, mathematics, and yearbook). New students usually are encouraged to select a club shortly after the school year begins in April. Clubs meet for two hours after school each day and many clubs continue to meet during school vacations. Club activities provide one of the primary opportunities for peer group socialization.

Most college bound students withdraw from club activities during their senior year to devote more time to preparation for university entrance examinations. Although visible in the general high school experience, it is in the clubs that the fundamental relationships of *senpai* (senior) and *kohai* (junior) are established most solidly. It is the responsibility of the *senpai* to teach, initiate, and take care of the *kohai*. It is the duty of the *kohai* to serve and defer to the *senpai*. For example, *kohai* students in the tennis club might spend one year chasing tennis balls while the upperclassmen practice. Only after the upperclassmen have finished may the underclassmen use the courts. The *kohai* are expected to serve their *senpai* and to learn from them by observing and modelling their behaviour. This fundamental relationship can be seen throughout Japanese society, in business, politics, and social dealings.

4.4 Juku: Cram Schools

An interesting component of Japanese education is the thriving industry of *juku* and *yobiko*, after school "cram schools," where a large proportion of Japanese high school students go for supplemental lessons. *Juku* may offer lessons in non-academic subjects such as art, swimming, abacus, and calligraphy, especially for elementary school students, as well as the academic subjects that are important to preparation for entrance examinations at all levels. *Juku* for high school students must compete for enrolment with *yobiko*, which exist solely to prepare students for university entrance examinations. Some "cram schools" specialize in preparing students for the examination of a particular school or University. Although it would seem natural for students to dread the rigor of additional lessons that extend their school day well into the late evening hours and require additional homework, many students enjoy *juku* and *yobiko*, where teachers often are more animated and more interesting than some of the teachers in their regular schools. Remember that the role of the teacher as portrayed by Stigler and Hiebert (1999) is to draw the mathematics out from the student. This requires that the teacher listen to the student more than is typical in Irish classrooms.

Juku and *yobiko* are primarily private, for profit schools that attract students from a wide geographical area. They often are located near train stations, enabling students to transport themselves easily to *juku* directly from school. *Juku* and *yobiko* thrive in Japan, where it is believed that all people possess the same innate intellectual capacity, and it is only the effort of individuals, or lack thereof, that determines their achievement above or below their fellows. Much like Ireland, in Japanese schools, there is the tendency to pass students up to the next "year" with their entering cohort. Therefore, without the supplemental *juku* lessons, some students could fall well behind their classmates. *Yobiko* also exist to serve *ronin*, "masterless samurai," students who have failed an entrance examination, but who want to try again. It is possible for students to spend a year or two as *ronin* after graduating from high school, studying at *yobiko* until they can pass a university entrance examination or until they give up. "Cram school" tuition is expensive, but most parents are eager to pay in order to ensure acceptance into a selective junior high school, high school, or university, and thus, a good future for their children.

4. 5 Entrance Examinations

In addition to university admission, entrance to high school also is determined by examination, and the subjects tested are Japanese, mathematics, science, social studies, and English. Private high schools create their own examinations, while those for public high schools are standardized within each prefecture. Students (and their parents) consider each school's college placement record when deciding which examinations to take. Success or failure on an entrance examination can influence a student's entire future, since the prospect of finding a good job depends on the school attended. Thus, students experience the pressure of this examination system at a relatively early age. The practice of tests at school and *juku* help teachers to direct students toward institutions whose examinations they are most likely to pass.

4.6 Free Time

Japanese students devote approximately two hours per weekday to homework, and about three hours on Sunday. They spend an average of two hours per day watching television, half an hour listening to the radio, an hour reading casually, and less than half an hour in social relations with peers outside of school. Japanese adults tend to perceive high school students in many ways as large children instead of young adults. And, while opposite sexes are interested in each other, parents and teachers strongly discourage teenage dating. Most young people do not begin to date until after high school. Finally, for a variety of reasons not explicated here, there are few drug problems among Japanese adolescents.

5.0 What do Lesson Study and the context of learning in Japan mean for mathematics teaching and learning in Ireland?

Le Métais (2003) warns policy makers that they should be careful when they simply copy one system of education with the vain hopes that such a system will work in the context of another culture. To that end we raise a number of salient questions to increase our capacity to generate informed local policy for Ireland that draws on cross national educational study. To summarize Le Métais (2003), sensible policy making that results from cross national study when it helps us to (1) build informed self review and (2) improves our capacity to link progress to purpose.

Japanese performance in international comparisons is based not only on high quality Lesson Study but also on a system of cram schools. First, there is the anomaly of no homework being assigned in primary and the early years of secondary schooling. Stigler and Hiebert (p. 30, 1999) note that "no homework is typical in Japan" in their study of eight grade. As recently as July 25, 2005, in the Japan Times, Minako Sato writes: "Cram schools cash in on failure of public schools." Further, she notes that "according to a 2002 survey by the Ministry of Education, Culture, Sports, Science and Technology, 39 percent of public elementary school students, 75 percent of public middle school students, and 38 percent of public high school students attend juku." It is difficult if not impossible to disentangle the possible confounding effects of this dual system of education. One part of the system emphasises conceptual understanding while the other part leverages skills and practice. This bares an interesting resemblance to the NRC finding regarding the interplay of phonics and whole language approaches to childrens' development of reading comprehension (Snow,).

Second, is the fact that teacher development (kounaikenshuu) tends to be more idiosyncratic in Japanese secondary schools. Instruction in the later secondary school years is not centrally linked to the process of Lesson Study. This may be due, in part, to departmental specialization in Japanese secondary schools somewhat parallel to the Irish setting. It is also likely due to the pressure imposed by university entrance examinations and a focus on examination preparation (Yoshida, 1999). This is again quite similar to the Irish setting.

Third, some view Lesson Study as a way to decease primary school teacher autonomy (anonymous presenter at a meeting funded by the National Science Foundation, June 2003). This rendering of the use of Lesson Study is in stark contrast to the manner in which Lesson Study is presented by most of not all authors.

In sum, Lesson Study is culturally bound and needs to be understood in the context of the Japanese culture. Some might argue that before we can fully understand and use the lessons of Lesson Study we likely need know what fundamental social values are reflected in the education systems of the Ireland and Japan? Further, we should also know what are the intrinsic and extrinsic incentives motivating Irish and Japanese students? Additionally, it would be wise to acknowledge the different definitions of democracy as applied to education in both Ireland and Japan. In the Republic of Ireland, recognition of different talents is consonant with democracy. In Japan, "equal access" based on standardized scores on entrance examinations is the implied culturally held definition of democracy. So what does this mean for teaching and learning in Ireland?

Finding answers to these and a myriad other questions are quite daunting and take us away from a central tenet of this document. That is, we believe that Irish teachers can learn a lot from the engaging in the process of Lesson Study. When used appropriately this is a very strong professionalizing activity that improves both teaching and learning. However, we do not believe that Lesson Study alone will move Irish education forward, nor should it. In embracing Lesson Study we also believe that the iterative process can be shortened if the mathematics education research literature paid more attention to issues of practice and teachers in turn attended to this "new" literature (see for example recent work in Variation Theory approaches to the learning of mathematics, or the work by Lesh and others on model eliciting mathematics problems). Embracing Lesson Study provides an unique opportunity for improvements in mathematics instruction and in the profession of teaching in Ireland while also building bridges between research and practice in Irish schools.

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Early Primary Mathematics Education

Does Estimation Make Sense to Four-Year-Old Children?

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The Primary School Curriculum (1999) places considerable emphasis on the development of children's estimation strategies. Indeed, it recommends that prediction and estimation skills should be taught from junior infants onwards. While it is generally agreed that estimation skills are necessary for developing a good number sense, there is some debate amongst researchers as to whether young children can make sense of what is involved in estimation. This paper explores the responses of young children, at the point of entry to school, to a number of tasks designed to elicit quantitative thinking. In particular the responses to the estimation items are considered. The fourteen four-year-old children, eight boys and six girls, were presented with the tasks in the course of individual discussions related to various aspects of number. Data reported here are part of a wider study related to young children's number sense. Findings suggest that, while a few children did appear to understand what was involved in the estimation task, many of them did not seem to understand the difference between approximation and counting. The relationship between children's ability to estimate and the various quantification strategies that they might have at their disposal is discussed. Implications are drawn for both pedagogy and curriculum planning as they pertain to children in the first year of school.

Background

In the *Primary School Curriculum: Mathematics* (National Council for Curriculum and Assessment (NCCA), 1999a) it is proposed that children's understanding of how numbers relate to each other should be developed through a range of content objectives which are comprehensively outlined for teachers. These objectives are generally uncontroversial and similar to those found in early primary school curricula in a number of countries. In the Irish context a major new emphasis at infant level is the introduction of activities related to *subitizing*, and what is referred to in the *Primary School Curriculum: Mathematics: Teacher Guidelines* (NCCA, 1999b) as sensible 'guessing' or *estimation* (NCCA, 1999b: 32). Sowder (1992) observed that until recently few primary school curricula included any mention of estimation for very young children. When or how estimation should be introduced to young children are issues for debate among researchers, cognitive psychologists and practitioners. The NCCA takes a strong pro-stance on the introduction of estimation activities to very young children at school. For example, the *Primary School Curriculum: Mathematics* advises that estimation is something that should be emphasised right from the beginning of school mathematics (NCCA, 1999a: 47). Others such as Sowder (1992) take a rationalistic standpoint and suggest that educators should not be in too much of a rush to teach estimation skills since children in the early years at school may be too young to master them in anything but a superficial manner. As part of a wider study related to number sense in young children, I sought to explore how they handled estimation tasks by engaging them in relevant activities. In this paper I present findings pertaining to estimation and related aspects of quantification.

Quantitative thinking

For most people quantitative thinking is an essential element of mathematics. Young children construct their knowledge of quantification in the course of their everyday experiences and this leads them to quantify sets of objects reliably. The elements that I deem important in relation to quantification in the four- year-old age group are subitizing, counting and estimating. Children's responses to phrases incorporating the words 'how many' are used in this paper to provide data related to their quantitative thinking and their estimation skills.

Perceptual quantity

Nunes and Bryant (1996) summarised research pertaining to the genesis of mathematical experiences of young children. They locate it's beginning in the ability of infants to distinguish between sets of one to three objects on the basis of numerosity and the suggestion that infants also realise when the set size changes when one object is added or taken away. Steffe and Cobb (1988) however, take the position that such discriminatory ability is unrelated to number words, counting, or any kind of numeral system. Fuson concluded that one- and two-year-old children possess the ability to label small numerosities with distinctive 'words'. For instance, a diary entry related to her daughter (aged 1 year 8 months) read '*You know two items and use it correctly in new situations: two means one in each hand.'* (1988: 18) This she describes as a perceptual process. Subitizing implies *'The immediate correct assignment of number words to small collections of perceptual items . . .'* (von Glasersfeld 1987: 303) Clements uses the term '*perceptual subitizing'* to describe this early ability and to differentiate it from '*conceptual subitizing'* (1999: 400). The former, according to him, is a process of recognising a quantity without using mathematical processes.

Where quantities are not easily or accurately differentiated by perceptual means children who are able to count can use their cultures counting system to determine how-many. Fuson's (1988) seminal study demonstrated the ways in which children first relate the counting and cardinality meanings of number words. Her research showed that by the age of 4 and a half most children can quantify using an incremental counting procedure starting from one. They count large numbers of objects in rows (to 20) with considerable accuracy. This confirmed earlier studies which concluded that most five-year-old children in the United States had considerable experience and competence in counting sets of up to thirty objects on occasions (e.g. Gelman and Gallistel, 1978; Saxe *et al.*, 1987). However, research also consistently demonstrates a wide variation in counting related activity amongst children at the age of entry to school (Fuson, 1988; Wright, 1991; Aubrey, 1997).

It appears that overt counting behaviour, such as pointing and saying number words aloud, undergoes progressive internalisation across the age range 3 to 6 years (Fuson, 1988). Most three-year-old children in the Fuson study counted aloud, while most five-year-old children counted silently, thus indicating that silent counting is a more mature counting strategy than overt counting. Fuson also observed that the first internalisations of counting may result in less accurate counting. She provided an overview of the ways in which counting enables children between the ages of 2 and 8 to become more proficient in handling numeric relations. It appears that once children appreciate the ordering of number words in the number word sequence this permits equivalence and order relations to be established on the sequence of number words and after/before and just after/just before can then be derived.

Relationship between subitizing and counting

Fuson (1988) reviewed various positions on the relationship between (perceptual) subitizing and counting. She sought to clarify the developmental relationship between labelling a subitized situation with the correct number word, and counting in order to label that situation. Some authors argued that the realisation that the last number word used when counting an array is the same as the subitized word for that set, is what enables children to learn and generalise that the last word used answers the question of how-many (e.g. von Glasersfeld, 1987). Similarly, Steffe and Cobb's (1988) review of the literature led them to conclude that children count perceptual patterns at a very young age (c. three years) in order to form the semantic links between patterns and number. The position of others was that such learning does not depend on subitizing but rather on accurate counting (e.g. Gelman and Gallistel, 1978) which decreases markedly with increasing set size. Fuson (1988) found many of the two- and three-year-old children in her study did not seem able to or did not spontaneously use subitizing in a counting situation. She interpreted this as undermining the subitize and count position above. She concluded that children follow different routes to early relations between counting and cardinality. In her study the subitize and count account applied for some children, but not for others.

The perception of composite figural patterns i.e. spatial patterns, is seen by some as fundamental to children's understanding of number (von Glasersfeld, 1987) and indeed perceptual patterns have been shown to play a key role in children's construction of arithmetic meanings and strategies (Steffe and Cobb, 1988). It is claimed that four-year-old children begin to recognise and visualise sets of quantities from five to ten as special patterns that they can see when looking inside numbers (Fuson *et al.,* 2001). Such patterns appear to be the key to 'conceptual' subitizing as described above by Clements (1999). Indeed he argues that children use counting and patterning abilities to develop conceptual subitizing, which he describes as '*an ability to group and quantify sets quickly'* (p. 401). He relates this type of subitization to a situation where people [his term] 'just know' how many on, for example, an eight-dot domino. He suggests that in such a case, they just know the domino's number because they recognise the number pattern as a composite of parts i.e. two groups of four, and as a whole i.e. eight.

Estimated quantity

Two decades ago Fuson and Hall (1983) reviewed the research evidence in relation to estimation. This included a study, which claimed that where time permits, adults will count (mentally), sets of up to six objects even when asked to estimate. Above this size they tended to estimate. They found very little research on estimation in young children and indeed they speculated that because of the likely lack of experience amongst young children, and thus of stored mental representations of numerosities, the estimating abilities of young children would be quite weak. Sowder (1992) also expressed the view that estimation may be too difficult a process for young children since they lack experiences that permits the establishment of mental pictures of quantities or a feel for size. Both of these, in her view, are prerequisites for being able to estimate. MacNamara (1996) cites evidence that a seven-year old child in her study could quantify sets of up to ten items without overt counting. When asked how he did this he described a strategy that involved subitizing some of the items and them counting on the remainder. This compression of the counting process (Gray, 1997) in order to efficiently generate a solution to the quantification problem would appear then to be one important strategy that some children can use to estimate quantity. Fuson also identified what she termed a '*subitize and add'* (1988: 349) strategy amongst children (aged five- and six-years) on sets of four to six objects. Clements (1999) suggests that conceptual subitizing, together with counting is essential for estimating.

In her review of research related to the question of whether young children estimate, Sowder (1992) cites a study by Dowker (1989) that suggests that children (aged five years) offered estimates that appeared to make no sense when related to the numbers in the problem presented to them. Aubrey offers further evidence that estimation tasks make little sense to children of four-or-five-years of age. She found that in general the children in her study, except those without counting skills, immediately started to count the items when asked to 'guess' and furthermore ' . . . *were confused when prompted again to guess when the items were removed.'* (1997: 86) ?Practitioners too have observed similar reluctance on the part of some young children to make a guess. Andrews (1995) reported that children in her kindergarten class often attempted to count the number of items in response to an estimation task and furthermore she noted that they frequently became frustrated when they were precluded from doing so because of the arrangement of the objects in a jar. So while the word ‘guess’ is sometimes suggested as a way of clarifying what is involved in estimation (See *Primary School Curriculum: Mathematics: Teacher Guidelines,* 1999b), Andrews speculated that the actual effect for some children may be to trivialise the mathematical process of estimation. It does so, she argues, by suggesting that there is no information available against which to draw a logical conclusion. Sowder (1992) suggests that confusion between guessing and estimation may be a characteristic of poor estimators. If she is correct in this then encouraging children to guess may actually be working against developing estimation abilities. Sowder’s (1992) review of the literature indicated that even older children (aged approximately 9-14 years) did not understand estimation. Indeed Andrews (1995) suggests that children need structured experiences that enable them to understand the differences between a guess, an estimate and an exact answer. It also appears that they need to be able to use counting in a flexible way and in different kinds of situations.

Affective factors are seen as important in relation to estimation and Sowder (1992) argues that children are more likely to use estimation if they see the value of it. Indeed she argues, as did Silver (1989) before her, that what is being asked of older children when estimation is requested flies in the face of what they have generally learned at school about the primacy of one right answer. It is possible that younger children may face a similar dilemma when they are asked how-many without counting i.e. to estimate. Such tasks may not yet make any sense to them in the light of the enormous importance and power that counting assumes for them at this stage of their numerical development (Fuson, 1988). Also, many everyday experiences with number emphasise ‘exactness’ or ‘correctness’, for example just enough spoons are required for everyone to be able to eat ice-cream and too many or not enough responses are likely to draw negative comment from children’s perspective.

The how-many question

It has been observed that the question of how-many seems to function for some young children as a request for the counting act rather than as a request for information gained from the counting act (Fuson and Hall, 1983). The use of the how-many question appears to predispose some children to overt counting and so to ignore or fail to see the utility of other related strategies. We saw that many children do not seem to be able to use subitizing in a counting situation (Fuson 1988), or at least do not do so spontaneously. Also, subtle differences between the ways in which questions related to how-many are posed can effect whether or not children re-count a set in response, and consequently to different inferences about children's understanding of the relationship between counting and quantification. The weight of the evidence seems to suggest that for many children counting is the key skill in relation to solving quantification problems. Nunes and Bryant (1996), in their review of how children learn to count, concluded that four and five-year-old children often fail to connect their use of counting with other aspects of the logic of number, such as transitivity. This led them to argue that knowing how to count is one skill, but knowing when counting is a good problem-solving strategy and presumably the converse i.e. when it is not, is an entirely different and more advanced skill.

In summary, young children use a number of enumerative strategies to quantify sets. These include subitizing, counting and estimation. Research suggests that counting is not dependent on subitizing. Different children use different strategies at different times. However, when children can count, then they may combine this skill with their ability to perceive numerosity in small sets and their ability to create and use patterns related to quantity. Thus they develop a more powerful way of quantifying sets containing more than three or four items. In this way, subitizing may be important for estimating for some children in some contexts. At four-years-of-age counting is very important to children (Fuson, 1988) and for pedagogic reasons, extensively practised in the home (Aubrey *et al*., 2000). However, adult purposes for counting are generally not well understood by children of this age (Munn, 1994). During the early years quantitative thinking becomes increasingly more complex, as a result of developing relational understanding between subitizing, counting and estimating. Counting is then used not just to quantify, but children can learn to make inferences based on this counting in order to reason quantitatively (Fuson, 1988; Nunes and Bryant, 1996).

The Study

The questions that this paper seeks to answer are:
How do four-year-old children respond to questions related to how-many?
How do four-year-old children appear to understand estimation tasks?

Method

The sample

A variant of clinical interviewing, which I refer to as experience-based flexible and focused interviewing (Dunphy, 2005), was used to explore the number sense of eight boys and six girls (age-range 4 years 1 month to 5 years 1 month), all of whom were just beginning their first term at primary school. The study was carried out with careful consideration to issues such as gaining access, establishing relationships with the participants, and ethics.

The interviews in the boys' school were carried out over the first two weeks of the new school year in September. The interviews in the girls' school were carried out during the third and fourth weeks of September. Each child was interviewed twice. On average the first interview lasted about twenty minutes and the second interview about thirty minutes. Interviews were fitted around the events of the day. The children were interviewed in a room separate from the classroom.

The Tasks (See Appendix)

The purpose of this interview was to ascertain children's number sense in relation to specific tasks. I decided on the tasks after research and review of the concept of number sense and a preliminary analysis of the curriculum content for this stage of schooling (NCCA, 1999b). The tasks I used were based on a number of successful ones found in the literature (e.g. Hughes, 1986; Saxe *et al.,* 1987*;* Munn, 1994; Aubrey, 1997; Cook *et al.* 1997*;* Clements, 1999; Van de Walle, 2001, 4th edn*).* My teaching experience, together with the research I did in preparation for the interviews led me to the conclusion that 'birthdays' are really powerful triggers for number and other early mathematical 'talk' (See also Carr, 2001). Consequently, I built almost all the tasks around a birthday theme (i.e. that of the fourth birthday of a toy character, Coco, from a popular breakfast cereal). Some tasks revolved around 'games' with Coco, though I was aware that this strategy was not without its effects in that the discourse associated with such games might not do justice to children's understandings and skills (Walkerdine, 1988). However, I felt from my teaching experience with young children that overall these themes/strategies offered potential for exploring the four key aspects of number sense identified for this study.

Findings

In seeking to establish how children responded to selected tasks (See Appendix), I found that sometimes their quantitative strategies were overt and discernible directly from what they said or did in response to a task. At other times children's strategies were more covert and I sought to discern these by interpreting their responses. In order to assist me in making these interpretations I derived a categorisation system very loosely modelled on the more extensive one to be found in Ginsburg, Pappas and Seo (2001).

The categories I use to describe children's strategies are as follows:

Relating Numbers: The child focuses on subgroups within a quantity as a way of finding a solution, or in some other way relates numbers to other numbers;

Guessing: The child's response is a wild guess, apparently unrelated to the numbers in the task;
Instant Response: A response to the question is offered immediately but it is incorrect;
Counting: The counting words are used in response to a task that includes the words how-many;
Instant Solution: The child instantly recognises the cardinal value of a set of objects. The child subitizes i.e. immediately recognises the amount without overtly counting;
Estimating: The child offers a solution based on logical approximation. The child's response makes sense in relation to the numbers in the task.

Table 1: Children's responses to selected tasks designed to elicit quantitative thinking

Child	*Task 4* *Counting objects*	*Task 5* *How many now? (4 Items)*	*Task 7* *How many dots? (5 Items)*	*Task 9* *About how many pieces? (5 Items)*
Bob (M) 4y 8m	Counts Relates numbers	Counts (4)	Instant Response (4) Instant Solution (1)	Guesses/Estimates
Shay (M) 4y 11m	Counts Relates numbers	Instant Solution (4) Relates numbers	Instant Solution (4) Relates numbers (1)	Estimates
Terence (M) 4y 3m	Counts	Counts (3) Instant Solution (1)	Instant Response (1) Instant Solution (4)	Estimates/Guesses
Tom (M) 4y 11m	Counts	Counts (1) Instant Solution (3)	Counts (1) Instant Response (1) Instant Solution (3)	Estimates/Counts
Jamie (M) 4y 9m	Counts Relates numbers	Instant Solution (4)	Instant Solution (5)	Counts
Con (M) 4y 9m	Counts	Counts (1) Instant Solution (3)	Counts (4) Instant Response (1)	Counts
Owen (M) 4y 6m	Counts Relates numbers	Counts (4)	Count (1) Instant Solution(4)	Counts
Jerry (M) 5y 1m	Counts Relates numbers	Counts (2) Instant Solution (2)	Instant Response (1) Instant Solution (3) Relates numbers (1)	Counts
Maura (F) 4y 5m	Counts	Counts (3) Instant Solution (1)	Instant Response (3) Instant Solution (2)	QNP
Sonia (F) 4y 11m	Counts	Counts (1) Instant Solution (3)	Instant Solution (5)	Estimates
Sile (F) 5y	Counts Relates numbers	Counts (2) Instant Solution (2)	Counts (1) Instant Solution (4)	Counts
Mary (F) 4y 6m	Counts Relates numbers	Counts (4)	Counts (4) Instant Solution (1)	Estimates
Kate (F) 5y	Counts	Instant Solution (3) Counts (1)	Counts (1) Instant Solution (4)	Estimates
Lara (F) 4y 1m	Counts	Counts (4)	Counts (2) Instant Response (1) Instant Solution (2)	Guesses/counts

QNP denotes Question Not Put

Task 4 Counting objects within 10 (Groups of 2,6,7,5,4 presented in sequence). One array at a time, the children were shown a number of arrays of candles that Coco had arranged in linear form. They were asked to count the presented array of candles to see if that was the amount that should go on the cake for Coco's fourth birthday.

The purpose of this task was to explore children's ability to count to find out how-many and all of the children displayed competence in relation to this. They all subitized i.e. recognised instantly, the quantity two. Most children counted the arrays of 6, 7, 5 and 4 candles as was suggested to them but on a few occasions I noted a subitizing response in relation to the array of 5 candles and even more so in relation to the array of 4. Several children talked about a quantity under consideration by relating it to the target quantity of 4. Different children used different types of 'relating' strategies to enable them to solve the problem. For instance, Shay appeared to use an additive strategy in solving the problem:

'He needs two more.' In relation to the array of six candles he remarked: '*That's not enough . . . so you only have to take out two and two . . . that's four.'*

Bob in contrast appeared to use a subtraction strategy, reducing the excess to produce the quantity required: '*Take two out and that's four.'*

The array of seven candles provoked the following response from Terence: '*Look four and three.'*

This suggests to me that he used his knowledge of composition of number to solve the problem. This was similar to the strategy invoked by Shay who commented: *'Now there's four . . . there's three . . . so that makes . . . there's seven.'*

Four children commented on the array of five candles and related it to four. Owen suggested that we take one away and similarly Mary suggested that we should 'get' one away.

Sile's suggestion was *' . . . so take away the fiveth one . . . so then he has four.'*

Terence articulated his way of 'seeing' four by removing one from the array with the comment: '*Not enough . . . there's one in the middle there is . . . Look, ah that is enough it is . . . that's four it is.'*

The children's comments bear out Fuson's (1988) contention that children follow different routes to the early relationships between numbers. The task above proved to be an enabling context within which some children could display different strategies for solving the quantification problem. Thus they could display the very beginning of their understandings of how numbers relate to each other and their understandings of how they can be manipulated. Steffe and Cobb (1988) suggest that the creation and use by children of the above type of patterns, helps them to develop abstract number and arithmetic strategies. In relation to this particular task, several children displayed their use of analytic skills as opposed to counting skills to solve the problem, the skills identified by Marton and Neuman (1990) as necessary for solving quantitative problems.

Task 5 How many now? (Groups of items of 4,7,3,5 presented sequentially).

The children were asked to count out an array of birthday presents (e.g. 4) and say how-many. They were then asked to add/remove one and say how-many then.

The purpose of this task was to explore whether children overtly counted each time, or whether they used a variety of strategies based on perceptual processes such as subitizing or perhaps visual or auditory patterns rather than incremental counting to respond to how-many-now tasks. In Fuson's (1988) terms the concern then is with the extent to which children utilize information gained from counting to solve the task. I found that there were two categories of responses to Task 5, the first category was one where the children responded by counting again, and the other was one where children straight away offered the solution (See Table 1). Certainly, children in this study appeared to give more instant solutions than the younger children in the Fuson (1988) study, perhaps indicating the role of experience and practise in developing quicker responses in a quantification situation. There is the possibility also that the children were drawing on other aspects of their number knowledge, for example *one before/one after*. Four children always counted the items in response to the question of how-many-now, two children never appeared to count while the remaining eight children used both counting and subitizing strategies to varying degrees (See Table 4.4). This suggests that some children were perhaps moving from an overt counting strategy to a covert one (Fuson, 1988). Or perhaps they were subitizing or combining counting with subitizing (MacNamara, 1996; Clements, 1999).

From Fuson's perspective, as children attempt the task above they can respond only with the conceptual unit items they have available. If they view the set as a group of unit items rather than as a single set then their response to the question of how-many-now will be to count again. The children who counted again after an item had been added/removed do not appear to respond to the quantity of the initial set as a number. According to Greeno (1991), children who count in response to this type of task don't appear to be able to objectify numbers i.e. to see numbers as conceptual objects in their own right i.e. as objects that can be manipulated and with which to construct and reason in the domain of number. The question of why some children count in response to *'How-many now?* would appear, then, to be comprehensively dealt with in the literature. However Sonia's remarks below gave me pause for thought:

'*Liz: Well let's pretend he only got that many* [I showed four]. *How many has he got there now? [She counted four]*

Liz: Right . . . and he has four presents there and let's pretend that one got lost [I removed one]

Sonia: Three

Liz: Three . . . okay!

Sonia: I don't need to count that

Liz: Why do you not need to count?

Sonia: Because . . . I know that's three . . . one, two, three . . . I don't need to count but I'm practising.'

Perhaps Sonia is 'practising' as von Glasersfeld suggests (1987) to verify for herself the subitizing result by now counting the set,. Sonia seemed to think that it was necessary to explain her ability to solve the problem without overtly counting. When Shay's subitized in response to the initial set of four items I asked him if he was sure. He then counted the items before '*I was just making proof for you, it's easy for me to do it*.' He appeared to be using counting to verify his rapid response, *for me*. In fact, by asking him if he was sure, it seems very likely that I prompted him to overtly count the array. These comments by Shay and Sonia appear to substantiate Fuson's speculation that children may perceive certain strategies to be 'acceptable' in certain instances (1988: 349).

Previous research that focused on children's counting skills early in their first school year identified social pressures on children to behave ' . . . *in a certain manner to conform to particular kinds of teacher expectations, particularly . . . to count where counting was not required.'* (MacNamara, 1996) The literature suggested that developmental progress in relation to counting may well include the relating of covert counting and perceptual subitizing. The findings here certainly show that the majority of children in this study do use processes other than overt counting to solve problems related to quantification. However my findings also suggest that where children overtly count in response to the task, there may be other reasons besides a lack of alternative strategies for their use of counting in such contexts. Such reasons may include a desire to be co-operative and helpful and to behave in ways that they perceived to be socially desirable.

Task 7 How many dots?

Children were shown a dice with dot patterns, my birthday present for Coco. As an introduction to this task the children and I discussed their experiences with die. I then explained what I wanted them to do. Each face was displayed for only about 3 seconds to avoid giving an opportunity for children to count. The prompt *'Look quickly. You'll need to be quick here!'* was used in this instance to prompt the children to subitize. Fuson (1988) had speculated that where children assume that a quick answer is required, this may result in the use of rapid perceptual strategies for quantification. The purpose of this task was to explore the extent of children's ability to subitize.

Table 1 shows that eleven of the children responded rapidly to four or five of five items. In these instances no overt/covert counting behavior was displayed by the children. It appears then, that the majority of children can respond very rapidly when quantities are presented in a spatial pattern, especially when urged to do so. Furthermore, two of the children explicitly demonstrated the ability to group objects to quantify sets. For instance Gerry looked at the six pattern on the dice and then remarked *'What's three plus three . . . oh six'.* While he appears to be using a known fact to solve the problem, he is also drawing on his ability to subitize three items. Shay in relation to the six pattern also 'sees' something similar to Gerry as communicated in his response of '*Three and three. I have to count on my fingers.'* Shay subitized the set of three but to solve the problem he actually counted incrementally from one. Both of these children display the beginnings of a flexibility with the number six. Marton and Neuman (1990) observed that the seven-year old school starters in their study frequently used this type of 'seeing' strategy. My results indicate that it may be discernible in some children at a much earlier age. However, neither Gerry nor Shay showed any evidence of having integrated the strategies of subitizing and counting in the way suggested by MacNamara (1996) or Clements (1999).

Task 9 How many pieces?

Children were presented with a succession of small plates of 'food' items for the party (8 lemons, 5 oranges, 6 apples, 7 bananas, 9 strawberries) using 'Fruity Fun' counters. Each plate was displayed for only about 3 seconds to avoid giving the opportunity for children to count. The prompt '*Look quickly. You'll need to be quick here!'* was again used, in this instance to prompt the children to estimate. Note was taken of whether they appeared to estimate or whether they attempted to count the items. The purpose of this activity was to explore children's ability to estimate quantity without resorting to overt incremental counting.

Table 1 shows the responses to this task. We saw that, in relation to Task 7, most children responded quickly. However, the response pattern on Task 9 was somewhat different. About half of the children appeared to estimate/guess in response to some/all of the items, while the other half of them counted. The estimates/guesses offered by the seven children are presented in Table 4.5. Some of the responses of both Bob and Terence appeared to be guesses rather than estimates and they were categorised as belonging to both/either categories. Lara offered instant responses, but I categorised her behaviour as guessing because her solutions were considerably off the target number.

Some studies of young children's number understanding reported on the correctness of their estimates (e.g. Aubrey, 1997). The children in my study rarely quantified exactly, though I would argue that the point of estimation is not exactness. Table 2 shows that in many cases children's estimates are within an acceptable range i.e. within one or two of the target number

Table 2: Children's estimates/guesses in response to Task 9

Size of set	***8***		***5***		***6***		***7***		***9***
Bob (M) 4 y 6m	3		4		4		5		14
Shay (M) 4y 11m	6		5		8		6		11
Terence (M) 4y 3m	7		7		4		4		5
Tom (M) 4y 11m	6		5		8		6		9
Sonia (F) 4y 11m	9		5		9		8		10
Mary (F) 4y 6m	7		3		6		6		10
Kate (F) 5y	6		4		4		6		7

Of the seven estimators, all of them save Mary had also responded rapidly to Task 7. On reviewing the recording of the interview and the relevant notes of my discussion with Mary I noted that while her responses to Task 7 were very rapid, I had recorded in my field notes her explicit counting behaviour during this task. Was her counting behaviour now more covert than in the previous task? Or did she use some other strategy in relation to Task 9?

About half of the thirteen children who were presented with the task responded by counting. Four of these (Jamie, Owen, Gerry, and Sile) had generally subitized in relation to Task 7. Perhaps the arrangement of the objects in Task 9 and/or the number of items could have been a factor in relation to how they responded. Research has shown that randomly-arranged sets prohibit rapid counting, and that as the number of items in sets increase it becomes more difficult for children to quantify them (Fuson, 1988). Owen, Jamie and Gerry were all very explicit in relation to how they perceived Task 9 and their comments provide some evidence that for them at least, at this point in time, covert counting is the key to quantification. Of course, this does not rule out the possibility that they also use perceptual processes when quantifying. Owen's response to my suggestion that he tell me how many without counting was

'But I don't know . . . I have to count them . . . I think I have to put them in a line.'

Gerry's comment was *'But if I don't count them I won't know how many!'*

Jamie's forthright response when I asked him why he had counted the objects in the first item in Task 9 was as follows:

Jamie: I counted them very low.

Liz: Well the next time don't count them. Just tell me how many you think.
Jamie: I think nine. I didn't count them. I just counted them in your hand.
Liz: Without counting them tell me how many about (strawberries). [He then counted silently].
Jamie: EIGHT!
Liz: And you counted them didn't you?
Jamie: I counted them in my head.
These responses suggest to me that some of the children just didn't understand the difference between an exact answer and an approximation. In fact this is not so surprising when you consider that it is very likely that most four-year-old children will have had very few experiences of quantitative estimation. Most experiences of quantification are likely to be ones in which children are learning to count to quantify, or perhaps are prompted to count to verify how many. It is as a result of these latter experiences that that the '*absolute reliability'* (Fuson and Hall, 1983: 68) of correctly executed counting for determining the quantity in a set is established.

Pike and Forrester (1997) suggest that using the phrase *how-many* in a context that requires children to approximate rather than quantify exactly, introduces a tension and thus confusion for some children. Aubrey (1997) had also noted presentational difficulties with a similar task and she observed that most four-year children in her study preferred to count when presented with the estimation task. If we include Shay, because of his initial statement '*I have to count them',* then seven of the thirteen children presented with Task 9 either counted or indicated a preference for overt counting rather than any other strategy.

However, about half of the children appeared to be happy to indulge my request that required them, perhaps, to put aside previous experiences that provoked them to count in response to how-many. My feeling is that these children were confident about their responses because most of them had, in fact, covertly counted the items *to their satisfaction.* Fuson (1988) observed that, in general, evidence suggested that when children begin to internalise counting this can result in less accuracy than overt counting.

In summary, the data presented here shows that

- While overt counting was an important quantification strategy, the four-year-old children in this study, also solved quantification problems by relating numbers to each other using a variety of relational strategies, thus demonstrating that they understand that numbers are not just the product of counting but are things that can be manipulated and reasoned with;

- Covert counting was a key quantification strategy for some children, while others were in the process of internalising counting and were adopting its use to varying degrees;
- Most children responded very rapidly when quantities were presented in spatial patterns. Children who subitized spatial patterns were more likely to estimate, but a number of children who subitized spatial patterns did not estimate and one child (Mary) who didn't subitize spatial patterns appeared to estimate.

In addition the data suggests that

- Children may perceive that counting is the expected/acceptable response to problems related to quantification;
- Children may quantify larger numbers (e.g. six) by seeing patterns (e.g. three and three) in these arrays;
- Contextual factors may determine what strategy children use to solve quantification problems;
- The differentiation between an exact answer and an approximation appears to have little meaning for children in this study.

Discussion

Looking across the tasks in Table 1, it is clear that one of the issues in terms of children's responses is the ability to interpret the question how-many flexibly, i.e. an understanding that these words don't always have to imply an overt counting process. Research with three-and –four year olds showed that when some children hear the words how-many their reaction is to think in terms of counting and to develop responses based on overt incremental counting processes (Fuson, 1988). My data shows that, in response to Task 5, eleven of the fourteen children choose to recount all/some of the items when asked how-many-now, after an item had been added or removed. However, a counting response to this task is not surprising given way that the directions as presented did suggest counting as appropriate. However, in Task 7 eight of these children responded instantly. It is possible that the removal of the dice after a few seconds in Task 7 may have had the effect of forcing children to abandon a counting strategy to determine how-many, and to subitize. Indeed, we saw that hardly any of the children overtly counted in response to Task 7. But the removal of the items in Task 9 did not seem to prompt most children to estimate in that case. In fact we have seen that most preferred to count and almost half explicitly did so.

In Task 9 the sets are too big for most children to count quickly. It seems likely that while some children may have recognised the quantity perceptually in Task 7, some of those labelled as subitizers on that task actually counted to quantify, but in a covert way. Jamie's comments above certainly raise such a possibility. Also, Tom's overt behaviour in relation to some items in the estimation task also pointed to the use of covert counting in such tasks, although interestingly he didn't consider that he was counting. The dice with the arrangements of dots (Task 7) was relatively easy to count quickly, or merely recognise. For instance, children were very successful with quick responses in relation to the pattern of 2 and 3, relatively successful in relation to 4 but there were more quantification errors in relation to the pattern of 5 and 6. This suggests that when there is no time/opportunity to count, as in Task 9, it is very difficult for some four-year-old children to come up with reasonable responses in so-called 'estimation' tasks. Those who do come up with appropriate responses could be using any of a variety of strategies: internalised counting (Fuson, 1988); compressed counting (Gray, 1997), perhaps involving subitization and counting on (MacNamara, 1996); and relating numbers to other numbers (Steffe and Cobb, 1988; Marton and Neuman, 1990). The responses of the majority of the children in this study strongly suggested that they were operating with these strategies to varying degrees. However, we can only speculate about this since I didn't ask children about their strategies and most of the mainly four-year-old children in my study didn't articulate their approach in the way that the seven-year-old child in MacNamara's study did.

Clements (1999) position is that conceptual subitization is a developmentally more mature quantification strategy than perceptual subitization or counting, and that it often depends on accurate enumerative skills. I interpret this to mean that children who can count and who can also subitize as in Task 7, will be in a position to estimate in response to sets of 6-10 items, ones most likely to be just beyond their subitizing range. According to his argument those children with high levels of accuracy (100% or 80%: See Table 4.4) on Task 7, should estimate on Task 9. However, Jamie very clearly told us why he counted in response to the items in Task 9. Owen, Gerry and Sile also counted. Sonia, Shay, Terence and Kate all estimated with ease. At the other end of the scale, Mary counted covertly in response to Task 7, but estimated with ease in relation to Task 9. While MacNamara (1996) and Clements (1999) may be correct in speculating that accurate enumeration combined with perceptual processes is an important route to estimation, the data here shows that it is not a route followed by all children. The fact that a number of children provide instant solutions in relation to Task 7, but do not estimate in response to Task 9 suggests that the ability to enumerate rapidly and accurately doesn't mean that estimation follows. It is impossible to say, in relation to the four-year-old children in my study, the extent to which they depend on perceptual processes or the role such processes play in their quantitative reasoning. What is clear is that some of them can /do estimate but some have difficulties with estimation tasks. These may relate to one/more factors including the size of the set; the arrangement of the set; the wording of the task; and the purpose of the task.

With some children particular tasks seemed to call forth particular strategies. For example, Jamie, Con and Sonia all appeared to give instant solutions to items in Task 5. In relation to Task 7, Con gave mostly counting responses while Jamie and Sonia gave instant solutions. However Jamie and Con both insisted that a counting response was appropriate in relation to Task 9, while Sonia appeared to estimate. It was often difficult to infer the exact strategies some children were applying to the quantification problems presented. Children of this age may reason mathematically/numerically but they may have some difficulty articulating that reasoning (See Pound 1999 for a review of research in this area).

Conclusions

Fuson (1988) described the internalisation of counting as a feature of most children's behaviour at around five- or six-years of age. The children in this study are clearly focusing on this process, as is evidenced by the nature of their responses to some of the tasks discussed here, in particular Tasks 7 and 9. The use of covert counting strategies varies from child to child (See Table 4.4), depending no doubt, as Fuson (1988) observed, on their experiences and practise with counting. However, it is also clear from the findings here that contextual factors can determine what counting strategy (i.e. overt or covert) children choose to use. It is possible that at a certain point of development some children experience a tension between, on the one hand the utility and ease of covert counting, and on the other hand the perceived social desirability to count aloud to quantify. Some children in this study certainly appeared to indicate such confusion. In thinking about why children might persist with overt counting we should also consider the possibility that there is an important affective dimension at work and that some children may enjoy the pleasure of counting in the way implied by Munn (1994). This may be reinforced by adult approval of the process. Some of them then, may be loathe to leave behind the affective benefits of overt counting.

Implications

What all of the above implies is that estimation cannot make sense to many four- or five-year-old children simply because they have not yet internalised counting. Children's experiences to date are generally ones in which overt counting is practised. Some children in this study were observant of people who 'did it in the head' and such children seemed to have a metacognitive awareness of their own/other peoples covert/mental activity with number. To illustrate my point I cite Shay's comments, in discussion about magnetic numerals. The discussion begins with my comment:

Liz: If you put them all up there and tidy them up a bit we can have a good look at them
Shay: We'll have to . . . Is this a one? I just want to remember
Liz: That's a one

Shay: And I need to have a two
Liz: Okay . . . oh I see what you're doing
Shay: I'm trying to count Is this a two?
Liz: What do you think?
Shay: It's kind of like it . . . oh no it isn't . . . it's a seven
Liz: Is it? Okay
Shay: I thought it was . . .because . . . no this is a seven
Liz: Is it?
Shay: Because I got my Mum's thoughts

Such children may be in a better position to understand the estimation process, than those who do not appear conscious of such metacognitive issues. From a personal perspective, this possibility reminds me very much of my son's unsolicited comments (at about five years of age) about a story which I was reading him (for maybe the second or third time) from a book with no pictures 'Oh! I get it now. You have to make the pictures in your head'. Could it be that young children also have to 'get' it in relation to number and understand that you can 'make' the numbers/quantity in your head before they can estimate? What this suggests is that children will need to participate in estimation activities with more skilled partners ' . . . *in order to allow them to internalize tools for thinking and for taking more mature approaches to problem solving*.' (Rogoff, 1990: 14)

Teachers will need to consider very carefully the form of words they use when discussing estimation tasks with young children. I suggest, based on my analysis here, that children need to practise counting sets of increasing size, both covertly and overtly. They also need to be encouraged to estimate by use of a prompt such as '*Count in your head, not out loud…'*

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Appendix: Tasks

Task 4 How Many? Counting objects within 10

Do you remember earlier you showed me how you could count? Let's pretend it's Coco's birthday. Could you help Coco count the birthday candles for his cake?

One array at a time, the children are shown a number of arrays of candles that Coco has arranged in linear form, and asked to count them to see if that is what should go on the cake. A small cake (made from playdough) is then placed on the table and the child arranges the candles on it around the edge (i.e. in a circle). The child was asked again if there are four candles on the cake. The purpose of this task was to see the extent of children's understandings of counting and of cardinality.

Task 5 How many now?

The children were asked to count out an array of birthday presents (e.g. 4) and say how many. They then were asked to remove one and say how many now. They were also asked to add one to another array (e.g. 5) that they had counted and state how many now. One purpose of this task was to explore children's count-to-cardinal and cardinal-to count transitions. A second purpose was to ascertain if children needed to count again each time.

Task 7 How many dots?

Children were shown a dice with dot patterns (1-6). This was one of Coco's presents. They were questioned on what he might use it for and then we discussed it's role in board games. We examined the dice and children were encouraged to say how many dots on each face. Each face was displayed for only about 3 seconds to avoid giving the opportunity for children to count. The prompt '*Look quickly. You'll need to be quick here!'* was used in this instance to prompt the children to subitize.The purpose of this task was to explore the extent of children's ability to subitize.

Task 9 How many pieces?

Food for the fruit party! Children were presented with a succession of small plates of 'food' items for the party (8 lemons, 5 oranges, 6 apples, 7 bananas, 9 strawberies) using 'Fruity Fun'counters. Each plate was displayed for only about 3 seconds to avoid giving the opportunity for children to count. The prompt '*Look quickly. You'll need to be quick here!*' was again used here to prompt the children to estimate. Children were shown collections of various sizes from 2 to 10. Note was taken of whether they appear to estimate or attempted to count the items. The purpose of this activity was to explore children's ability to estimate quantity without resorting to counting.

The Challenge and Responsibility of Providing Optimum Maths Opportunities to Young Children During Their Years of Promise

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This project seeks to build on two features of children's early mathematical knowledge: the intuitive, informal nature of this knowledge and its complexity. Vygotskian style clinical interviews will be used to track children's ongoing construction of mathematical knowledge. The children will be exposed to a combination of two programs during the year: the Number Worlds program and the Big Math for Little Kids program. It is hoped to involve parents in playing traditional board and card games with the children at home. A key focus of this project is the concept of metacognition: children learning to think and talk about their own thinking.

"[T]he developmental seeds of underachievement may be sown before impoverished children enter school" (Ginsburg, Klein and Starkey 1998).

Background

In March 2004 I received a grant from the Centre for Early Childhood Development and Education (CECDE) to research the assessment of learning in young children. This topic posed a number of fundamental questions including why assess, how to assess, what to assess and whom to assess.

Why Assess

Assessment may serve a number of purposes including the support of learning, the identification of special needs, program evaluation and accountability (Shepard et al., 1998). The most important function, the support of children's learning, is best served by viewing assessment and teaching as inseparable processes. According to this view, an important first step is to assess the child's current knowledge. This raises the issue of the nature of such assessment.

How to assess young children

Although useful for other purposes, traditional methods of standardised testing are unlikely to be helpful in informing instruction. Nevertheless, it is worth considering, the origins and rationale of such testing, as Ginsburg (1997) has done.

Standardised testing has been justified on both scientific and on ethical grounds. The scientific principle involved is a simple one: if we wish to compare individuals along a given dimension, all other dimensions, such as test format or instructions, should be held constant. Equivalence of test administration was also seen as important in terms of fairness. Horace Mann listed a number of reasons for introducing standardised testing to schools including impartiality and the absence of favouritism (Wainer, 1992). Also listed was the elimination of the "officious interference of the teacher".

The psychological trait of interest to schools in the early 20th century was intelligence. At that time in France, assessment specialists called "alienists" were employed to identify schoolchildren unable to profit from mainstream instruction. Once identified, such children were placed in special classes. The alienists' admirable goal, however, was badly served by their subjective and arbitrary methods. Inaccurate diagnoses, resulting in withdrawal from mainstream education, had implications for individual children: "to be a member of a special class can never be a mark of distinction and such as do not merit it should be spared the record ….(T)he precision and exactness of science should be introduced into our practice wherever possible" (Binet and Simon, 1916). Thus, the concept of standardised testing was based on the noble goals of scientific accuracy and impartiality.

The basic approach of standardised testing is to make inferences based on children's performance on a series of tasks. Clear instructions are provided in terms of what the child is required to do. A fundamental aspect of such testing is that the child's brief responses can be easily scored and recorded in an objective manner. Standardised testing, therefore, is based on the assumption that it is reasonable to infer underlying cognitive processes from behaviour. However, as Ginsburg and Pappas (2004:173) note: "if cognitive developmental psychology from Piaget onward has taught us anything it is that overt behaviour, or performance, does not necessarily provide a good indication of underlying knowledge". Standardised testing, per se, is based therefore on an obsolete assumption. The focus of standardised testing is on the static products of learning, reflecting the traditional view that learning consists of the passive accumulation of facts. Constructivist theory suggests otherwise. Originating with Piaget, constructivist theory indicates that children actively construct their own knowledge from a variety of inputs. Traditional and constructivist approaches result in contrasting implications.

Firstly, standardised testing views competence as a dichotomous variable something the child "has" or "has not". In contrast, the constructivist approach suggests a continuum of competence, along which the child progresses. According to the latter approach, change is viewed as an integral part of the development of competence and not "an undesirable lack of reliability which must be eliminated" (Ginsburg 1997: 20). Recent research points to the complexity of this competence, particularly in the area of mathematics. Children's competence is linked both to context (Vygotsky 1978) and to the manner in which the task is presented (Donaldson 1978). Secondly, children construct meaning as well as knowledge, so children from different backgrounds may respond to the same standardised test with varying levels of motivation. Ginsburg (1997) notes that standardised testing may suit some children. Middle class, mainstream students who are competent and confident may delight in the opportunity to show what they know. The same test instrument may not due justice to the abilities of children with a different set of characteristics. Thirdly, young children's cognitive development is uneven. Their knowledge is initially fragile, so that what they appear to know today they may have forgotten tomorrow. Any single test is unlikely to reflect accurately this nascent state of the child's cognition. The combination of emergent self-regulation abilities and a complex continuum of cognitive abilities displayed by young children further undermine the usefulness of standardised testing with this population.

A number of alternatives to standardised testing exist, including the clinical interview, dynamic forms of assessment and performance or authentic assessment.

The Clinical Interview, developed by Piaget, refers to a range of techniques designed to identify children's underlying thought processes. The interviewer begins with a menu of tasks and it is the child's responses to these that determine the direction of the interview. These responses inform the interviewer's hypotheses regarding the child's underlying thinking, hypotheses which undergo repeated testing. During the interview, the child is also asked to reflect on and articulate her thinking processes. Piaget developed the Clinical Interview to accomplish three goals. He wanted to identify the issues of interest to the child and her methods for dealing with these, ie child's "natural mental inclination". He wished to access the "reasoning processes underlying the child's right but especially (her) wrong answers". Finally Piaget was interested in the mental context of the child's behaviour, including the child's motivation. His interest in affective issues like motivation was purely cognitive: his concern was that lack of motivation could undermine the validity of his findings.

Vygotsky described the Clinical Interview as a "truly invaluable tool" for studying children's cognitive development, but believed that Piaget had not exploited the tool to its full potential. Piaget' used the Clinical Interview to establish the child's current, unaided level of cognitive development, her "actual developmental level". In contrast, Vygotsky's focus was on the Zone of Proximal Development (ZPD), defined as "the distance between the actual development level, as determined by independent problem solving, and the level of potential development, as determined through problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky,1978: 86) Vygotsky was interested in the dynamic nature of children's cognition, in particular their ability to profit from adult assistance. In contrast, Piaget was interested only in the way the child constructs knowledge without help from adults. Dynamic, Vygotskian style assessment is particularly useful as a means of supporting learning, since the goal of such assessment is to identify the child's ZPD. Once identified, the child can be presented with optimally challenging opportunities, namely, tasks that are just beyond her unaided capabilities but with assistance, within reach.

Advocates of performance assessment take the view that the best way to assess competence is by observing, recording and evaluating children's performance. When based on realistic tasks in the context of the child's ordinary classroom, such assessment can be classified as authentic. Performance assessment requires multiple observations of behaviour before meaningful conclusions can be made about individual children. The aim of these observations is to uncover "the patterns of success and failure and the reasons behind them" (Wiggins, 1998:705).

It is worth noting that whilst traditional, standardised testing of ability and achievement is summative in nature, all three alternatives are formative: they provide information useful both for tracking children's progress and for altering the type of educational opportunities provided. It is hoped to combine these three forms of alternative forms of assessment in the current intervention.

What to assess

I have chosen to assess learning in the domain of mathematics in young children for a number of reasons. Firstly, there has been little research into the way young children learn mathematics, particularly in the Irish context. Secondly, the contrast between the mathematical competence displayed by very young infants and the fear of maths expressed by many adults is intriguing. Thirdly, there is evidence that children are acquiring mathematical concepts that are superficial and procedurally based, rather than meaningful. This gap, between children's intuitive mathematical understandings and formal school learning, is detectable at the very start of primary school. Such a gap is particularly worrying in the case of children from low SES backgrounds, whose average level of mathematical achievement is persistently lower than that of their more advantaged peers.

Since the processes of assessment and teaching are inseparable, the decision to focus on mathematic assessment raises the issue of what mathematics to teach. The decision was based on three main considerations. One was a desire to apply the findings of cognitive science, the study of the mind and mental functioning, to the classroom. Second, was the hope of harnessing the role of affective i.e. non-cognitive, factors in children's cognitive development. Finally, in light of previous interventions that included a parental component, (eg Starkey, Klein and Wakely, 2004), it was decided to include parents in the current project.

The Number Worlds program represents a Vygotskian approach to mathematics, based on findings from cognitive science. Four year olds possess two types of mathematical knowledge: numerical knowledge that they can use only with small sets of objects and non-numerical, maths knowledge. Children use the latter knowledge to solve problems that require judgements of the type "a lot" versus "a little". Research indicates that four year olds cannot integrate these two types of knowledge. However, by age six, most children have combined these two types of mathematical knowledge into a coherent mental number line. This line consists of two extreme values, and a continuum of equidistant values in between. According to Griffin and Case (1997) this "central conceptual structure" is a prerequisite for mathematical success. They hypothesise that the absence of this structure, in some children, stymies mathematical progress. The Number Worlds program is designed to teach this fundamental structure to young children.

What is of interest, given the cognitive focus of the program, is the games based methodology chosen to teach this central conceptual structure. These games are played in small groups, thereby emphasising the social and emotional (ie non-cognitive) aspects of learning. Also of interest is the program's exclusive focus on number.

This relationship between affect and cognition is fundamental to a program called Big Maths for Little Kids (BMLK). The program calls itself a "joyful mathematics program" and like Number Worlds is firmly focussed on the mathematical potential of "children born to poverty". BMLK is more comprehensive than Number Worlds in that it covers the four areas of shape, measurement, patterns and logic, and spatial concepts, as well as number. It is also more comprehensive in that it explicitly addresses the domain of language and literacy through its maths games and stories. Related to this focus on language is the BMLK emphasis on learning to think and talk about one's thinking (metacognition), a concept firmly rooted in cognitive science. Viewing thinking as a skill has important implications for education, namely, that the characteristics associated with skill learning in general can be usefully applied to the development of children's thinking. Such characteristics include being explicit about components of the skill, learning by observation and modelling and the importance of practice and feedback in realistic contexts (McGuinness, 1999).

Like Number Worlds, BMLK uses a games format for its mathematical activities, but where Number Worlds begin with children's intuitive, incomplete understandings (Greenes et al., 2004), the authors of BMLK, take the complexity of children's early knowledge as their starting point. Growing evidence suggests that we may have seriously underestimated the mathematical competence of young children.

Number Worlds has been used at least once in Ireland (Mullan, 2004). To my knowledge, BMLK has not been used in this country before now. I plan to combine these two programs, Number Worlds and Big Math for Little Kids, and add a parental component featuring commercial and traditional games (e.g. Snakes and Ladders, Ludo, Guess Who and simple card games). I intend to implement this combination in one or more Junior Infant classes in the coming academic year, 2005-2006. For assessment purposes, I hope to use assessment components from both programs in conjunction with Vygotskian style clinical interview.

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Mathematics and Educational Disadvantage: the Impact of the *Number Worlds* Programme on the Number Knowledge of Children in a Junior Infant Class

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Number Worlds (Griffin and Case, 1997) is based on the Rightstart intervention (Griffin, Case and Siegler, 1994) which was used successfully to close the number knowledge gap between children in schools in low-income, high-risk communities and their more affluent peers in the United States. During the school year 2004/05 a quasi-experimental study was carried out to assess the impact of Number Worlds in an Irish context. The results point to the value of the Number Worlds intervention and the need for more curricular interventions that are based on cognitive science evidence. The research highlighted the need for more emphasis on counting in the early mathematics curriculum, more adult support in Junior Infant classrooms and more Learning Support at Junior Infant level in disadvantaged schools.

Introduction

Irish students appear to perform relatively well in terms of mathematical achievement in studies that facilitate comparison with their international peers. Ireland was one of twelve countries that performed above the international average at the fourth grade and one of the countries that scored at the international average at the eighth grade in the Third International Mathematics and Science Study (TIMSS) (Mullis, Martin, Beaton, Gonzalez, Kelly and Smith, 1997). Irish students were also found to be well within the average range for achievement in mathematics in the OECD Programme for International Student Assessment (PISA) 2003 (Sheil, Cosgrove, Sofroniou and Shortt, 2004). Distributions of achievement in Ireland were narrower than in most OECD countries, indicating greater equity between students in terms of learning outcomes. It may be comforting to know that Ireland has greater equity in learning outcomes than many other countries, but more in-depth examination of national learning outcomes in Ireland are less reassuring. Last year, 7,284 (13.7 %) students who attempted the Higher Level Leaving Certificate Mathematics Examination achieved a grade C or higher. In the same year, 6,907 (13%) of the students who attempted the Ordinary Level Examination achieved a grade D or lower and a further 1,432 (2.7%) of students who attempted Mathematics at Foundation Level achieved a grade D or lower (State Exams

Commission, DES, 2005). At Primary level in Ireland, the most recent 1999 National Assessment of Mathematics Achievement (Sheil and Kelly, 2001) results showed that pupils achieved a significantly higher mean score than Irish pupils who took part in the TIMSS (Mullis et al, 1997), indicating a possible improvement in achievement since 1995. However, the results also highlighted the fact that the mean mathematics achievement score of pupils attending schools that were designated disadvantaged was significantly lower, by about one half of a standard deviation, than that of pupils attending non-designated disadvantaged schools. The report recommended that the mathematical achievement of pupils in disadvantaged schools should be addressed as a matter of urgency. There has been heavy investment in initiatives to address the problem, but there is little evidence of improvement in the average literacy and numeracy standard of pupils in disadvantaged schools (Eivers, Sheil and Shortt, 2004; Weir, 2003). The question remains therefore why, despite the initiatives, is there little evidence of any improvement in the average literacy and numeracy standard of pupils?

Counting in early number development

Piaget (1952) is regarded by many as one of the leading authorities on the question of how young children learn mathematics. His work is clearly evident in the educational thinking and research that underpins the Primary School Curriculum (DES, 1999) now in use in Irish primary schools. Piaget prioritised classification and seriation abilities as the basis of number concepts (Lovell, 1961). For some years, researchers on both sides of the Atlantic have reported findings that appear to cast doubt on aspects of Piaget's theory (Bryant, 1974; Donaldson, 1978; Gelman and Gallistel, 1978). Many researchers share the belief that children's failure on some Piagetian tasks might be due a lack of clarity about what the children were supposed to do and that Piaget underestimated the intuitive number knowledge of children in the pre-operational period (Hughes, 1986). Studies have been carried out which suggest that young children *can* succeed on the class inclusion and conservation tasks, under certain circumstances (McGarrigle and Donaldson, 1974; Dockrell, Campbell and Neilson, 1980). Further evidence that children in the pre-operational stage are much more competent than Piaget allows for, comes from the work of Gelman and Gallistel (1978) who claim that children as young as three years of age understand the invariance of small numbers and they appear to understand that displacing objects in an array does not affect numerosity in the way that adding or subtracting does (Gelman and Gallistel, 1978). Hughes found clear evidence that pre-school children can carry out simple addition and subtraction particularly when the numbers are small (Hughes, 1981). In a study of 60 children in Nursery school, of whom only three succeeded on a standard Piagetian number conservation task, 75% were able to work out the result of adding two bricks to one brick or taking two bricks from three bricks. Over 25% succeeded when the initial set was larger than five (Hughes, 1981, p. 30).

It is also thought that Piaget might have underestimated the role of counting in the conceptual development of number. Recent analytical research has demonstrated, not only the complexity of counting, but also its centrality in the development of number concepts, addition and subtraction (Gelman and Gallistel, 1978; Carpenter and Moser, 1983; Fuson and Hall, 1982). Counting is considered by many to be the most fundamental of all early number skills (McLellan, 1993). An important step in learning to count is the recognition that the last word said in counting a set of items gives the numerosity (the cardinal word) of the set. This has been variously referred to as the 'Cardinality Rule' by Schaeffer, Eggleston and Scott (1974), the 'Cardinal Procedure' by McLellan (1990) and the 'Cardinality Principle' by Gelman and Gallistel (1978). The Cardinality Principal is one of five principles which Gelman and Gallistel (1978) advocate as necessary for counting. The One-to-One Principle states that children must be able to map each item in a set with or onto one and only one item in a second set. The Stable–Order Principle states that a child must be able to use verbal tags to correspond to items in a stable or a repeatable order. The Order-Irrelevance Principle states that the order of enumeration is irrelevant; a child must understand that the order in which items are tagged, and hence which item receives which tag, is irrelevant. The Abstraction Principle states that all of the preceding principles can be applied to any array of entities (Gelman and Gallistel, 1978).

Counting-on marks an important step in the child's development of flexibility in handling numbers in a more abstract sense (Dickson, Brown and Gibson, 1984). In particular, counting-on is thought to be a crucial step in the development of two central concepts necessary for school arithmetic, addition and subtraction. Fuson (1982) proposed that the ability to count-on depends on understanding basic principles involving the relation between cardinality and counting and on recognising that an addend can play a double role in that it is both an addend and a part of a sum. A child for example, has to be able to think about the 'three' of three objects in a box as a cardinal three, and then, if more objects are added to the box, the child has to be able to think of the three as an ordinal three in order to continue the count, …three, four, five.... Fuson concluded that children need to have the ability to begin a counting sequence at any number, the ability to maintain a double count and in the case of subtraction, the ability to count backwards (Fuson, 1982).

Recently, models have been constructed that attempt to characterise children's internal cognitive processes when they count, count-on, count back, add and subtract. One such structure is proposed by Resnick (1983) in her description of stages in the development of the procedural knowledge necessary for mental addition or written calculation. In Resnick's first stage, children in the late pre-school years use a kind of mental number line as a schema to direct elementary counting and quantity comparisons. Resnick

(1983) compared the use of fingers when counting to traversing such a mental number line. According to Resnick, this mental number line can be used both to establish quantities by the operations of counting and to directly compare quantities. By combining counting and comparison operations a considerable amount of arithmetic problems can be solved (Resnick, 1983)

Central conceptual structure theory

The findings of Griffin, Case and Siegler (1994) and their colleagues (Case, 1985; Case and Sandieson, 1987; Case and Griffin, 1994; Griffin, Case and Sandieson, 1982) led them to propose that four-year olds and six-year olds represent number in different ways. Four-year olds tend to represent all possible variables in a polar fashion whereas six-year olds tend to represent variables in a continuous fashion, i.e. as having two poles with a number of variables in between. Griffin and Case (1994) proposed that as children move from age four to age six they gradually become capable of merging these two types of knowledge with the result that they can then construct a central conceptual structure such as that illustrated in Figure 1.

They argued that the knowledge implied in the structure is central for successful performance on a broad array of tasks and that the absence of such knowledge constitutes the main barrier to learning.

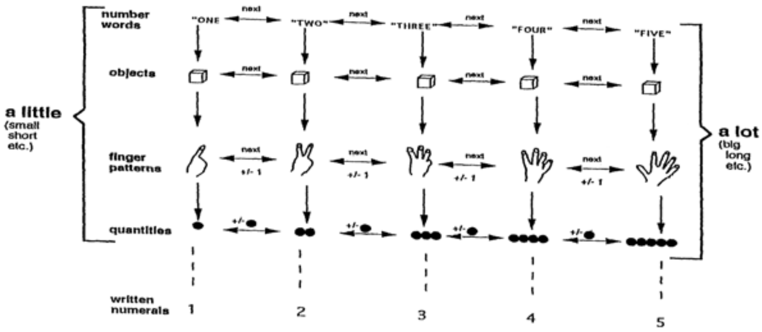

Figure 1. Central Conceptual Structure (Griffin, Case and Siegler, 1994)

The top row of the structure is the *verbal labelling* line- children can recognise and generate the number words one… two… three… . The second row is meant to capture the idea that children have a routine for pointing at or tagging a set of objects as they say the number words in such a way that each object is tagged only once in the process. Initially this is a sensory-motor process, which can only be applied to real objects, but it becomes a purely cognitive one, which mentally simulates the more primitive sensory motor activity (Case and Okamoto, 1996). The third and fourth rows represent *conceptual interpretation* through which children understand that each number label has a set size associated with it and that this set size has a certain canonical perceptual form. The children also understand that movement from one of these set sizes to the next involves the addition or subtraction of one unit. The bottom row indicates that children can recognise the written numerals from one to five. The horizontal arrows indicate transformations or rules for movement that allow children to move forwards and backwards from one item to the next in any row. The vertical arrows signify that children understand that there is a one-to-one mapping between each row and the next. The big brackets at the end indicate that the child knows that movement forward or backwards along these rows is a movement towards more or less (Griffin and Case, 1997, p.8).

Griffin, Case and Siegler (1994) discovered that a number of children in low-income communities who had achieved a chronological age of six were performing at the three- to four-year-old level on a measure they had designed to test children's understanding of number (Number Knowledge Test, Griffin, Case and Siegler, 1994). According to Griffin and Case (1997) when children come to school without the competencies implied in the central conceptual structure, it can mean that there has not been a heavy emphasis on counting and quantity in their early environments. Griffin and Case (1997) advocated that the knowledge implied in the *central conceptual structure* should be the core focus of the early school curriculum, so that children who have not yet acquired the knowledge in the conceptual structure will have an opportunity to do so and children who already have the knowledge will have an opportunity to strengthen it. They designed an intervention programme called *Rightstart* to enable children to acquire the knowledge implied in the central conceptual structure. The programme was used over a period of three years with small groups and whole classes of kindergarten children attending schools in Canada, Massachusetts and California (Griffin and Case, 1997). The children were all attending inner city schools with large minority populations and were from low-income communities with school populations considered to be at risk of school failure. The results of the study indicated that the Rightstart programme was effective in enabling children to acquire the conceptual knowledge that it was designed to teach. Following the success of Rightstart, Griffin and Case (1997) developed a more comprehensive programme called *Number Worlds*. They were guided by feedback from teachers who taught the programme and by observation of the difficulties encountered by teachers who tried to implement the underlying principles. The *Number Worlds* programme uses small group games which are specifically designed to teach the knowledge implied in the central conceptual structure theory.

Method

The primary approach to addressing the research question on the effectiveness of the *Number Worlds* programme was a quasi-experimental design. During a 12-week period in the school year 2004/05 the *Number Worlds* Preschool Programme was used in a Junior Infant class in a school in an area designated as disadvantaged. The mean pre-test and post-tests scores on the Number Knowledge Test (Griffin and Case, 1994) of the experimental group were compared with those of a matched control group in the same school and with two other control groups in a non-disadvantaged school. Analyses of data from three qualitative sources were also included in the evaluation of the *Number Worlds* programme. The teacher of the programme was interviewed, notes were taken from participant observation and a video recording of one of the classes was prepared and observed in detail in order to obtain a qualitative analysis of the intervention.

Participants

The groups that were chosen for the research were pre-existing Junior Infant class-groups in two schools in which the researcher worked. The *Number Worlds* (experimental) group and the matched control group were in the same disadvantaged school and thus were matched in terms of age, class, and community. Two further control groups, ND (non-disadvantaged) Control 1 and ND Control 2 were matched with the *Number Worlds* group in terms of age and class but not in terms of community as they were in a non-disadvantaged school.

Limitations of the Study

As is the case with any research study, the conclusions drawn must be viewed within the context of the study's limitations. Foremost of the limitations was the non-randomised nature of the sample. The best way to achieve comparability between experimental and control groups is to randomly allocate members of the target population to each group (Rossi et al., 1999). Furthermore, the generalisability of findings to other geographic areas and students should be investigated.

Further research into the effect of the *Number Worlds* Programme should be done with a greater sample to allow more definitive and more advanced statistical analysis. Additionally, further research should be done over a longer time period and should include teacher training. In the original research, the programme was implemented for four and five months and teachers were offered training. In the current research the programme was implemented over a 12-week period. The teacher did not receive any additional or specific training and implemented the *Number Worlds* Programme by following the *Number Worlds* Manual.

Results

Table 1. NKT Pre-and Post test Group Mean Raw Scores and Standard Deviations (maximum score =33)

	Group	Mean	Standard Deviation	N
Pre-test	Experimental	6.10	3.872	21
	Matched Control	7.33	4.778	21
	ND Control 1	10.60	5.030	20
	ND Control 2	8.58	4.042	24
	Total	8.14	4.653	86
Post-test	Experimental	11.38	3.217	21
	Matched Control	10.19	4.718	21
	ND Control 1	14.65	4.332	20
	ND Control 2	12.67	4.469	24
	Total	12.21	4.462	86

ND Control 1 and 2 are the control groups in the non-designated school.

An analysis of variance followed by a post–hoc test (Tukey HSD) revealed the differences between the four groups before and after the *Number Worlds* intervention. The pre-and post-test differences had an F-value of 3.85 ($p = .012$) and post-test differences an F-value of 4.17 ($p = .008$). The difference between the pre-test mean scores of the experimental group and ND Control 1 was statistically significant ($p = .009$). A post hoc test (Tukey HCD) confirmed that this difference between the pre-test mean score of the experimental group and ND Control 1 was unlikely to have arisen by sampling error (Table 2).

Table 2. Differences in group mean pre-test scores on the NKT between the experimental group and the control groups and between the matched control groups and the non-disadvantaged control groups and their significance levels.

		Mean Difference (I-J)	Std. Error	Sig.	95% Confidence Interval	
(I) GROUP	(J) GROUP				Lower Bound	Upper Bound
Experimental	Matched Control	-1.24	1.369	.802	-4.83	2.35
	ND Control 1	*-4.50	1.386	.009	-8.14	-.87
	ND Control 2	-2.49	1.325	.246	-5.96	.99
Matched Control	Experimental	1.24	1.369	.802	-2.35	4.83
	ND Control 1	-3.27	1.386	.094	-6.90	.37
	ND Control 2	-1.25	1.325	.782	-4.73	2.23

* The mean difference is significant at the .05 level.

After the intervention, the mean post-test score of the experimental group was not significantly different from the mean score of any of the other groups. Thus the significant difference between the experimental group and ND Control 1 had been reduced and was no longer significant. However, this was not the case with the matched control group who made less progress and whose mean post-test score was lower than that of all other groups and was significantly lower than that of ND control 1 (p=.006) (Table 3).

Table 3. Differences in group mean post-test scores on the NKT between the experimental group and the control groups and between the matched control groups and the non-designated disadvantaged control groups and their significance levels.

		Mean Difference (I-J)	Std. Error	Sig.	95% Confidence Interval	
(I) GROUP	(J) GROUP				Lower Bound	Upper Bound
Experimental	Matched C.G.	1.19	1.306	.799	-2.23	4.61
	ND Control 1	-3.27	1.322	.072	-6.74	.20
	ND Control 2	-1.29	1.264	.740	-4.60	2.03
Matched control	Experimental	-1.19	1.306	.799	-4.61	2.23
	ND Control 1	*-4.46	1.322	.006	-7.93	-.99
	ND Control 2	-2.48	1.264	.212	-5.79	.84

* The mean difference is significant at the .05 level.

The differences between the improvement in the pre- and post-test mean scores can be seen clearly in the Figures 2 and 3. At pre-test the experimental group was weaker than all other groups. The mean score of the experimental group was 17% lower than the mean score of the matched control and almost 40% lower than the ND Control 1. After the Number Worlds intervention the mean score of the experimental group had increased by 87%. This contrasts with the 39% improvement in the mean score of the matched control group, 38% in ND Control 1 and 48% in the ND Control 2. As a result, the mean score of the experimental group was higher than that of the matched control group and the significant gap between the mean scores of the experimental group and ND Control 1 had decreased from 74% at pre-test score to a 29% difference at post-test.

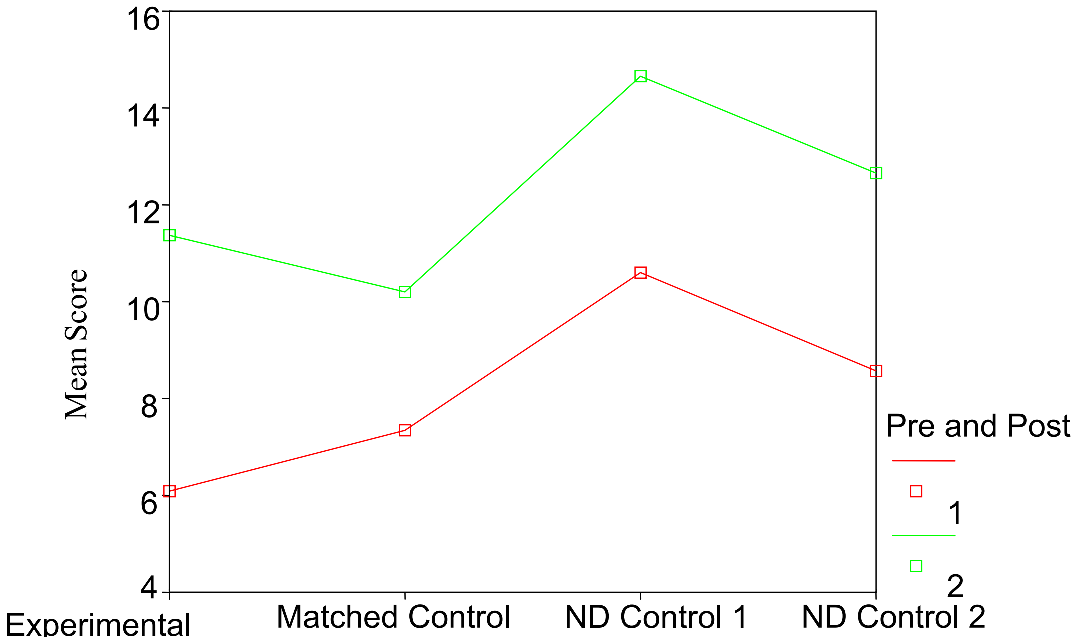

Figure 2

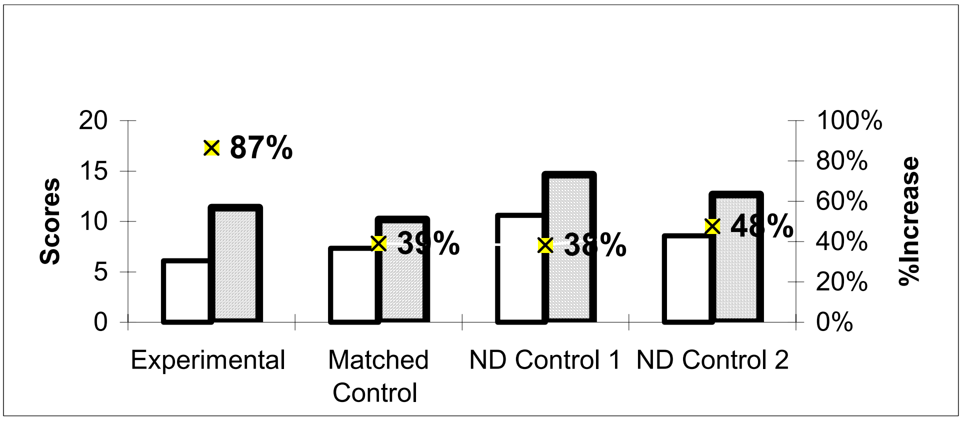

Figure 3. Pre-and post-test mean scores and % increase in mean score of each group.

The Number Knowledge Test results indicated that the lowest performing pupils in the experimental group made substantially more progress than the lowest performing pupils in the matched control group. The progress of low attaining pupils in the experimental group and in the matched control group can be seen in Figure 4. The mean low score of the experimental group went from 2.33 at pre-test to 9.44 at post- test while the mean low score of the matched control group went from 1.71 at pre-test to 5.14 at post-test. Means and standard deviations can be seen in Table 4.

Table 4. Mean and standard deviation and number of low attaining children experimental and matched control groups (raw score <6)

Group		Mean	Standard deviation	Number of children
Experimental	Pre-test	2.33	1.58	9
	Post-test	9.44	2.96	1
Matched Control	Pre-test	1.71	1.70	7
	Post-test	5.14	3.80	4

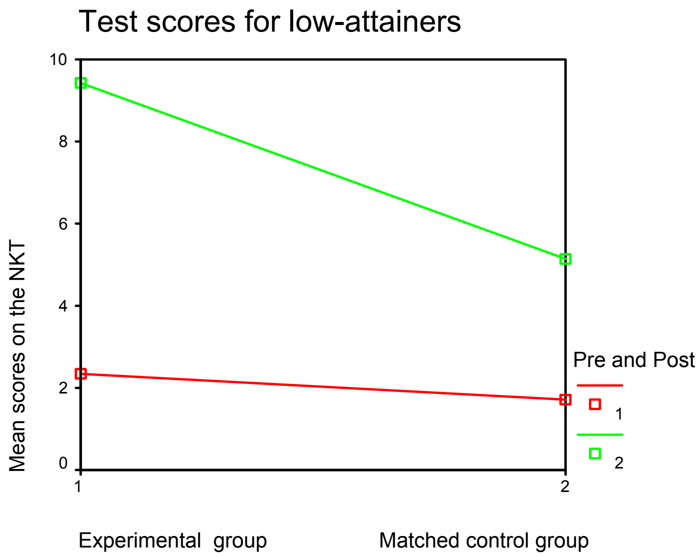

Figure 4. Differences in mean scores between pre and post test on the NKT for low scoring students (raw score <6) in experimental (n=9) and matched control group (n=7).

Discussion

Pre-test results of the Number Knowledge Test indicated lower mean scores for the groups in the disadvantaged school than for the groups in the non-disadvantage school. One of the aims of Irish disadvantage initiatives has been to ensure equality of benefit from education Initiatives in place in the disadvantaged school in this study had served level the playing pitch somewhat: There was evidence of classroom materials and books which would not have been available had it not been for enhanced capitation funding. Positive home school linkages had been fostered that were impacting positively on the schoolwork of the children. Class sizes in each school were similar and the curriculum in each school was the same. Children in both schools were due to be taught mathematics in an almost identical way, despite the obvious fact that half of the children were starting school with less number knowledge. Levelling the playing pitch is only a necessary first step; the second step should be the provision of different mathematical experiences for children with differing levels of number knowledge.

The *Number Worlds* intervention was designed to increase pupils' knowledge of the number sequence from 1 to 10 and to help children towards an understanding of the knowledge in the *central conceptual structure (*Figure1). One of the most noteworthy aspects of the post–test was the improvement in the test scores of children in the experimental group whose pre-test scores had indicated a developmental delay of up to two years. The lowest attainers in the experimental group at pre-test made substantially greater gains than the lowest attainers in the matched control group. It could be argued that this greater increase in score of the low attainers in the experimental group and in the increase in mean score of the experimental group was due to the provision of so many opportunities for counting.

This distinctive feature of the Number Worlds programme, the emphasis on counting was made possible by a second distinctive and integral feature of the programme, small group work. Small group work afforded children many opportunities to count, and perhaps to construct their own knowledge about number. However, during small group activities when groups were not supervised adequately or professionally, some children did not engage, some became disengaged and others had difficulty playing games correctly. The task of supervising several small groups was reported by the teacher to be the most difficult aspect of implementing the programme. Evidence from field notes and the video recording highlighted the need to have another adult working alongside the teacher in the classroom. Concessionary staffing has been provided to disadvantaged schools but has in the main had little impact on class sizes as most of the concessionary appointments have been to non-class teaching roles thus failing to impact on class size. Some countries have addressed the need to reduce class sizes by the provision of a trained classroom or teaching assistant in every infant classroom. The focus then falls not on class size but on the adult to teacher ratio in the classroom. This was one of the key recommendations of the OECD (2004) report on Early Childhood Education carried out for the Department of Education and Science last year.

Counting is not included in the *Early Mathematical Activities* in the Mathematics Primary School Curriculum (DES, 1999) but it is one of the strand units in the Number strand of the Junior Infant Curriculum. There is a professional freedom implicit in the Primary School Mathematics Curriculum that allows teachers to meaningfully adapt and change teaching methods, content and learning experiences to meet children's needs. However, there is a dearth of information about what type of adaptation is possible or desirable. The in-service education that has accompanied the introduction of the Mathematics Primary School Curriculum has focused on the provision of information in relation to content and methodology and there has been a lack of focus on the specific needs of disadvantaged children and children with learning difficulties. This need for further professional development on how to adapt the curriculum for pupils who are not ready for a particular class curriculum is recognised in the recent DEIS (DES, 2005) and LANDS reports (DES, 2005).

Number Worlds is just one example of an intervention based on cognitive science evidence that has been shown to successfully bridge the number knowledge gap between children in disadvantaged schools and their better off peers. There may be many more. However, it is not acceptable to encourage schools to innovate and to experiment with interventions without providing them with a variety of options along with solid research evidence about the strengths and limitations of different interventions. Little attention has been paid to date to exploring specific early curricular interventions that have been shown to make a significant difference to the educational experiences of disadvantaged children.

The Learning Support Guidelines (DES, 2000) specifically mention the value of implementing intensive early intervention programmes in the early primary classes "senior infants to second class" (DES, 2000, p.46). However, previous research findings and evidence from the analyses of qualitative data in this study indicate that some children need learning support when they start school in Junior Infants. It would seem logical to direct learning support at the foundations of learning and to help children who have difficulty with pre-number concepts rather than wait for children to develop difficulties later on in school. In disadvantaged schools the caseloads of Learning Support tend to be dominated by children who need support for literacy leaving no room for children who need learning support in mathematics. In non-disadvantaged schools, a lower incidence of serious literacy difficulties means that learning support in mathematics can be offered to a greater percentage of pupils. Thus, there is a fundamental inequality of access to learning support for pupils in disadvantaged schools in the area of mathematics.

Finally, if as Case and Griffin (1997) suggest, less conceptual number knowledge is due to the lack of emphasis on counting and quantity in the early environment then the sooner that there is such an emphasis, the better it will be for those who do not get the benefit of such experiences at home. The lasting and consistent benefits from pre-school education (Da Sylva, 1999) have been well documented but it is still not always accessible or affordable for families on low income. It was not available in the disadvantaged school in this research. Free quality pre-school education should be available to all children in low-income communities. This recommendation is in line with established government policy to develop a comprehensive system of early childhood education, the current draft report by the National Social and Economic

Forum (NESF 2005) and with the proposal in the DEIS (DES, 2005) action plan to provide pre-school or early childhood educational opportunities to all children served by the 150 most disadvantaged schools in the country.

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Primary Mathematics Education

'Litres can't go into millilitres': The Effects of Standard Algorithms on Children's Thinking about Division

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Current research on the mathematical operations of addition, subtraction, multiplication and division shows how standard pen and paper procedures for these operations (known as algorithms) impede children's thinking. In this paper, an account is given of interviews that were held with some 5th class pupils in an Irish school about their choices of operations in division word problems. In general it was found that algorithmic feasibility rather than problem structure was a strong determinant of their selection of operation. It was also found that standard algorithms reinforce misconceptions about division such as 'a divisor must be less than a dividend'. In the 1999 curriculum greater emphasis is placed on mental calculations estimation, appropriate use of calculator and problem-solving skills than on complex pen and paper calculations. This paper shows that we cannot afford to prevaricate on this issue.

Background

Algorithms are the finite step-by-step procedures used to complete mathematical tasks. Those that are mainly associated with primary school mathematics are column addition, subtraction, multiplication and division. The algorithm takes its name from an Islamic mathematician, al-Khwarizmi (ca. 780 – 850) who was central to the transmission of the decimal system and associated algorithms from India to Western Europe. It was from his work that Fibonacci and other Western European medieval mathematicians began to learn algorithms some time in the late 12th century. There was some reluctance to adopt these algorithms and heated battles ensued between those who supported the continued use of an abacus for computational purposes and those who favoured the Hindu-Arabic system. The chief reason for the eventual acceptance of the Hindu-Arabic system was the reliability it afforded. Whereas the abacus did not allow the maintenance of a record of the steps taken, algorithms allowed verification of a result and it was this factor that made them attractive to the merchant classes. By the sixteenth century the modern Hindu-Arabic numeration system was in place throughout Europe (Barnett, 1998).

The teaching of pen and paper algorithms has long been the mainstay of the mathematics curriculum. (Reys and Nohda, 1994). Brown (2001) gives an account of how basic arithmetic skills and careful, accurate calculation were essential in a pre-technological age. For example, shopkeepers had to present written accounts to their customers and bookkeepers. In the society of the 19th century what was required of most workers was the ability to do repetitive tasks (and this included arithmetic) and 'real thought' was left to those in charge. We are now in an age where technological tools such as the calculator and the computer are ubiquitous and so there is no longer a need for people to be proficient with paper and pencil procedures to cover all instances. Electronic devices pervade today's commercial life. The changing demands of society have been reflected in various national curricula. For example in the U.S. (NCTM, 1989), Australia (Australia Education Council, 1991), Netherlands (TAL Team, 1998), England (DfEE, 1998), and Ireland (Government of Ireland, 1999), the official policy is one of increased emphasis on mental calculation and on appropriate use of calculators with a concomitant decreased focus on written computation. Despite this, there is a marked reluctance to relinquish complex algorithms for whole number and fraction algorithms in the primary school system in all of these countries with the exception of the Netherlands where the Realistic Mathematics Education (RME) curriculum has been received with enthusiasm in classrooms (Menne, 2001). Reasons cited for the continued use of algorithms include reliability, efficiency, the creation of a written record, the establishment of a mental image and their use as substeps in other algorithms (Plunkett 1979, Usiskin, 1998). The hand-held calculator has been with us since the 1970s. Will it suffer the same fate as the algorithm and wait several centuries before it reaches full acceptance in our schools?

Literature Review

Aside from the fact that this focus on pen and paper calculations ill prepares children for the society in which they will live and work, many mathematics educators and researchers have questioned the impact of standard algorithms on children's thinking. Haylock (1987) contends that, in problem solving situations, algorithms can lead to a 'mental set' and that this is manifested by pupils' continued use of previously successful algorithms even when they are inappropriate or less than optimal. He suggests that many children fail to experience 'insight' in mathematics because of this algorithmic fixation. Constance Kamii suggests that algorithms are harmful because they encourage children to give up their own thinking and because they 'unteach' place value (Kamii and Dominick, 1998). Anghileri (2001) argues that standard pen and paper procedures are often incompatible with children's intuitive approaches and that this leads to erroneous calculations. Rowland (1999) coined the phrase 'vertical algorithm (or arithmetic) syndrome' to describe the situation where a student uses a vertical algorithm, often inaccurately, when a mental method could be more appropriate and efficient.

Another possible outcome of the use of standard algorithms is erroneous understandings about the effects of operations. Multiplication situations have been categorized by Greer (1992) as follows:

- Equivalent groups (3 bags each with six sweets)
- Multiplicative comparison (3 times as many sweets as apples)
- Rectangular arrays (3 rows of 6 sweets)
- Cartesian product (the number of possible sweet-wrapper combinations)

Each multiplication situation leads to a variety of division problem. The models of division that relate to equivalent groups are partition and quotition. Partitive division arises when the number of subsets is known but the number of elements is unknown, e.g., 20 sweets, 5 children, how many each? It is often described as 'sharing'. In quotitive division, the size of each subset is known but the number of subsets is unknown, e.g., 20 sweets, how many children can receive 5 sweets each? It has been found that multiplication as repeated addition (which corresponds to 'equivalent groups') is the model which children find easiest (Hart, 1981; Nesher, 1988; Kouba, 1989). Fischbein, Deri, Nello and Marino (1985) contend that the repeated addition model of multiplication and the partitive and quotitive models of division are intuitive or tacit models that exercise a very strong influence on children's thinking even when different models of multiplication and division are presented. For example, implicit in these intuitive models are the notions that 'multiplication makes bigger and that division makes smaller', a phenomenon commonly referred to as MMBDS. This idea has been found to have powerful influence on choice of operation in word problems, (Hart, 1981; Bell et al., 1981; Af Ekenstam et al., 1983; Fischbein et al., 1985, Bell et al., 1989; Tirosh and Graeber, 1994). Also implicit in the partitive and quotitive models of division is the belief that in a division situation, a ÷ b, the divisor, 'b' must be less than the dividend 'a'. This was found in the 'Concepts in School Mathematics and Science' (CSMS) study which was a research programme conducted in England between the years 1974 and 1979 wherein 10,000 children aged between 11 and 16 from fifty different schools were tested on their understandings of number operations (Hart, 1981). When asked to divide 16 by the number 20, 51% of 12 year olds and 23% of 15 year olds claimed that there was no answer; 8% of 12 year olds and 37% of 15 year olds gave the correct response. In a study conducted by Fischbein et al. (1985) with 628 pupils, aged 10 -15 years, in schools in Pisa, Italy, the operation 5 ÷ 15 was identified correctly by 20% of pupils aged 10/11 years and by 41% of pupils aged 12/15 years. In general, there was a tendency to reverse the roles of divisor and dividend (that is, to regard 5 ÷ 15 as 15 ÷ 5) for such an operation. This tendency to regard division as commutative is less frequent when the dividend is a decimal. The authors of the study suggest that 'the correct mechanism by which a division like 0.75 ÷ 5…becomes intuitively feasible (in terms of the partitive model) consists in merely neglecting the decimal point (and thus seeing 0.75 as 75).' (Fischbein et al., 1985: 13). They had felt that for the quotitive model, the divisor might conceivably be a decimal provided that that it was smaller than the dividend. This was confirmed only for those students in grade 9 (14/15 years). Pupils in grade 5 (10/11 years) had difficulty with the decimal divisor regardless of the underlying model and those in grade 7 (12/13 years) appeared to be in a transitional phase. Bell et al. (1989) suggest that, of the many factors that influence choice of operation in word problems, numerical misconceptions (such as MMBDS) and numerical preferences (such as multiplying and dividing by an integer and dividing a larger number by a smaller number to the extent of ignoring decimal points when convenient) have the most powerful influence on solution strategy.

It is likely that these notions of MMBDS and that a divisor must be less than a dividend are further entrenched by the use of standard algorithms because pen and paper procedures for decimal multiplication and division embody the procedures for whole number operations. In the primary school curriculum in Ireland and elsewhere, there is a hierarchical arrangement of procedures to be taught so that sophisticated algorithms are built upon more basic ones. Hiebert and Lefevre (1986:7) describe this in terms of superprocedures:

> Procedures are hierarchically arranged so that some procedures are embedded in others as subprocedures. An entire sequence of step-by-step prescriptions or subprocedures can be characterized as a superprocedure. The advantage of creating superproceures is that all subprocedures in a sequence can be accessed by retrieving a single superprocedure. For example, to apply the superprocedure 'multiply two decimal numbers' (e.g., 3.82 x .43) one usually applies three subprocedures; one to write the problem in appropriate vertical form, a second to calculate the numerical part of the answer and a third to place the decimal point in the answer. The second of these is itself made up of lower level procedures for (whole number) multiplication.

There is much to recommend such a hierarchy. It leads to a logical, structured programme that allows primary school pupils to move with relative ease from single digit to multiple digit operations and from whole number to decimal operations. On a negative note, however, it may reinforce ideas such as MMBDS as the procedures allow scant attention to be paid to the relative magnitude of the solutions obtained. For example, the algorithm for decimal multiplication is the same as that for whole number multiplication, the main difference being the insertion of a decimal point at the end of the operation. The algorithm for 6 ÷ 0.3 involves the 'removal' of the decimal point so that the operation becomes 60 ÷ 3 where the solution, '20', although bigger than the original dividend (0.6), is smaller than the 'new' dividend, 60. The procedure for division of a number by one bigger than itself supports the belief that a divisor must be less than a dividend. For example, 2 ÷ 5 becomes 2.0 ÷ 5 where children are encouraged to use language such as '5 into 20'.

There follows an account of interviews with a group of children which shows that, for many of them, choice of operation in division word problems was usually determined by the numerical preference that a divisor must be less than a dividend.

Study

This study was conducted in 1995 when the calculator was not officially part of the primary mathematics curriculum (Government of Ireland, 1971) but its imminent introduction in the new mathematics curriculum (Government of Ireland, 1999) had been heralded. The main aim of the study was to determine the extent to which guided instruction with a hand-held calculator caused a decrease in numerical misconceptions such as MMBDS and numerical preferences such as 'a divisor must be smaller than a dividend' (Dooley, 1996). To this end a pretest was administered to a group of 32 children in a 5th class in boys' school in Dublin. This test consisted of 66 problems which were read aloud and in which the pupils were required to choose but not to implement the correct operation. A variety of number types was used, e.g., integers, decimals greater than 1 and decimals less than 1. In some items the solution involved multiplication or division by a decimal less than 1 or division of an integer by an integer larger than itself, e.g.,

(a) Cotton costs £2.74 a metre. Mrs. Byrne needs 0.45m of cotton. How much does it cost her?[12]

Ring the correct operation

2.74 ÷ 0.45	2.74 – 0.45	2.74 + 0.45	2.74 x 0.45	0.45÷ 2.74
0.45) 2.74				2.74) 0.45

One point was awarded for each correct response to problems for which multiplication or division were involved, of which there were fifty items in total. Addition and subtraction problems had been included in the test as 'distracters' and thus responses to

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1

2 The pilot test showed that many children interpreted 6 ÷ 3 as '6 into 3'. It was therefore decided to give both options, 6 ÷ 3 and 6)3 to children in the main test.

these items were not considered in the final analysis. On the basis of results achieved in the test, pairs of pupils were matched and randomly assigned to treatment or comparison groups. Sixteen pupils in the treatment group received twelve lessons, each of approximately one hour's duration, on calculator usage. The foci of the lessons were decimal multiplication and division and, also, division of a number by one larger than itself. Sixteen pupils in the control group practiced algorithms for the same operations through the use of games. After the treatment period, a posttest, the same as the pretest was administered to both groups. A statistical analysis of the pretest-posttest results showed that the experimental group did improve in their ability to choose the correct operation. This improvement pertained mainly to multiplication where operators were integers or decimals greater than 1. Items where the multiplier was a decimal less than 1 achieved few correct responses. With regard to division word problems, items involving division of integers where the divisor was greater than the dividend showed the greatest improvement. However, this improvement did not generalize to the domain of decimal division. Statistical analysis of the posttest scores of the experimental group (mean score = 27.87) and of the control group (mean score = 23.47) showed that the difference between mean scores was statistically significant ($p = 0.5$).

Interviews

Individual interviews were held with seven pupils from the calculator group and seven pupils from the non-calculator group who had been matched on the basis of the pre-test results. Each interview was audio taped. An observer, who took notes of the interviewee's reactions to questions, was also present in the room. During the interviews, pupils were asked to choose operations for selected problems and, in some cases, to calculate solutions using either pen and paper methods or a calculator. In the interview, ten word-problems from the post-test were chosen for further questioning. Some subtraction problems were included to prevent automatic selection of multiplication and division as the correct operations. The multiplication and division problems were chosen to cover a range of structures and number types. The problems that were chosen to investigate the numerical preference that a divisor should be less than a dividend were the *lotto* and *cola* problems:

(a) 24 people do the lotto. Last Saturday they won £18. How much will they each receive?

Ring the correct operation

24 x 18   18 ÷ 24   28 + 14   24 ÷ 18   24 + 18

(b) There are 2 litres of coca-cola in a bottle. Each glass holds 0.32 litres. How many glasses of coca-cola can be filled?

Ring the correct operation

2 x 0.32 2 ÷ 0.32 2 – 0.32 0.32 ÷ 2 2 + 0.32

In the post-test it was found that the majority children reversed the order of division for the lotto problem and chose 24 ÷ 18 as the correct operation rather than 18 ÷ 24. However, while it had been anticipated that the majority of children would overlook the decimal point and choose 0.32 ÷ 2 rather than 2 ÷ 0.32 as the correct operation for the *cola* problem, this tended not to be the case. A higher number of children chose the correct operation for the cola problem than for the lotto problem. This may have been due to the different structures of the two problems (the *lotto* problem is an example of the partition model of division and the *cola* problem is an example of the quotition model of division) or to the number types involved. Some of the responses to these items will be presented in order to probe the factors which influenced children's choice of operation for these problems. Pseudonyms are used for interviewees. These names will be followed by C or NC to indicate whether the pupil was a member of the 'calculator' group or the 'non-calculator' group. The number following the C or NC indicates the child's score on the post-test. Each problem was read aloud to the pupil being interviewed and he was then asked to choose the correct operation and to give reasons for his choice.

Responses to *Lotto* and *Cola* Problems

In general it was found that pupils who attained high scores in the post-test tended to refer more to problem structure and context than to numbers in justifying their choice of operation. For example, Ernest (C:44) discussed his choice for the *lotto* problem as follows:

P: 24 into 18
I: Why would you divide24 into 18?
P: Because each of them was doing it, so they had to get an equal number…to find out how much they got each.
I: And do you think they would get an amount that is less or more than a pound?
P: Less

Emmett (NC: 39) rationalised his choice of operation for the *cola* problem in a similar manner:

P: You divide 0.32 into 2 litres.
I: Why would you divide 0.32 into 2?

P: Cos that would give you the answer. If you divide that (*points to 0.32*) into that (*points to 2*), you'd find out how many times it goes in and that would be how many glasses.

However, for many of the children interviewed, the numerical preference that a divisor should be less than a dividend was a strong deciding factor for choice of operation in the *lotto* problem. Some children who were able to infer correctly that the answer should be less than a pound were initially more concerned with the viability of the operation than with the logic of the solution obtained. For example, Fergus (NC: 32) and Hugh (NC: 21) chose 24 ÷ 18 the correct operation for the lotto problem. Both were unhappy with the solution they obtained on implementing the algorithmic procedure but were reluctant to change the order of division. Hugh was adamant that a divisor had to be less than a dividend:

P: I think I done that wrong *(pupil points to algorithm)*
I: Why do you think it's wrong?
P: Because eighteen doesn't go into twenty four thirteen times
I: So do you think twenty four into eighteen will give you a better answer?
P: No
I: Why not?
P: Because you can't divide twenty four into eighteen
I: Are you sure about that?
P: Yes

Initially Fergus was reluctant to accept that twenty four could be divided into eighteen but accepted the idea on the basis of 'adding on noughts',

P: Divide
I: Which division would you do?
P: I'd do 18 into 24
I: Can you show me?
Pupil points to 18)24
I: Why would you choose that particular one?
P: Cos it says there's twenty four people, so you need to find out how much, split it into how much each person got.
I: So if twenty four people won the lotto, and won £18, would they each receive more or less than one pound?
P: Less
I: Do you think you would get that by dividing 24 by 18?

P: No…em, yeah
I: Do you want to try it on paper?
Pupil works out solution using short division algorithm
P: One remainder four
I: Are you happy with that answer?
P: *pauses...*um
I: So how much do you think each of them would get?
P: *pauses...*fourteen pence
I: Do you think that's the right answer for the problem you did
P: I'm not exactly sure
I: Tell me, would you think at all about doing this one (*interviewer points to 24) 18)*
P: It wouldn't go
I: Why not?
P; Because 18 is less than 24
I: So you don't think it's possible to do that?
P: Oh, sorry, yes, two noughts, you add on two noughts.
I: So you think that if you add on two noughts, you can divide 18 by24?
P: Yes
I: Do you think that might be a better way of finding the answer?
P: Yes
I: Why?
P: Because there'd be two noughts and…it would be more of a division sum
I: Do you want to try it?
Pupil works out solution using long division algorithm
I: What answer did you get there?
P: Seventy four pence
I: Do you think that's a better answer than the other one?
P: yes
I: So which operation do you think?
P: That one *(pupil points to 24) 18)*
I: Are you sure about that?
P: Yes

Harry (C: 32) was also initially more concerned with the viability of the operation than the problem conditions and only began to think about the sensibleness of his answer after he had decided how the algorithm could be executed:

P: *long pause* You divide 18 into 24
I: What answer do you think you would get if I asked you to work that out?
P: One pound fifty or something

Student is given opportunity to try working out answer using calculator
P: One pound thirty three
I: So if twenty four people won eighteen pounds, each would receive one pound thirty three? Are you sure about that?
Pupil looks doubtful
I: Do you think you could try this one (*interviewer points to 24) 18*)
P: Yes, that would work out better
I: Why would it work out better?
P: Cos eighteen pounds would be won, and you could put two noughts at the back, the bottom, after eight … and twenty four into eighteen hundred
I: And why do you think that would work out better now?
P: Because it would be giving you the amount of the eighteen pounds.
I: So why did you say this one first?
P: I thought it would be better because it's a smaller number than twenty four.
I: But now you think this one *(interviewer points to 24) 18)*?
P: Yes

Although these children correctly identified that the lotto problem represented a division situation, they were then prepared to suspend their reasoning about the solution in favour of algorithmic viability. While Hugh could not accept the possibility of dividing an integer by one larger than itself, Fergus and Harry changed their minds about their initial choices on the basis that 'twenty four could be divided into eighteen hundred'. So strong is the conviction that the divisor should be less than the dividend that it led to the correct choice of operation for the *cola* problem. Rather than choosing the more easily executable 0.32 ÷ 2, many pupils converted litres to millilitres or 'added on noughts' in order to reconcile any ambiguity. Ernest, who obtained the highest score in the post-test, was able to support his choice by reference to the glasses and the 'two litres' but did allude to the algorithm and the need to add on 'noughts',

P: 0.32 into 2.000
I: You have to add on the noughts, do you?
P: Yes
I: Would that give a better answer than 2 into 0.32?
P: Yes
I: Can you tell me why?
P: Because if you divide 2 into 0.32, you are asking how many two litres are in 0.32 litres and 0.32 into 2 litres, you are asking how many glasses are in the two litres.

For other children, it seemed that the underlying structure of the problem focused their thinking on the smaller of the two numbers, 2 and 0.32. For example, the following interview took place with Jack (C:32)

P: Divide 0.32 into 2 litres
I: Why would you divide 0.32 into 2?
P: Because 0.32 is smaller than one litre and that's two litres of coke, so you divide 0.32 into 2 litres

The underlying structure of the problem also seemed to influence the thinking of Jason (NC: 18). However, he was determined that the divisor should be less than the dividend.

P: (Pause) Multiply
I: Why would you multiply?
P: Divide, I mean. It is two litres of coke and each glass holds thirty two ... point three two litres, how many glasses of coca-cola can be filled, you have to divide two litres into that.
I: Which division would you do?
P: 32 into that
I: 32 into?
P: 32…into 2.00 litres
I: And why would you not choose 2 into 0.32?
P: Cos litres can't go into millilitres

Fred (C: 72) supported his choice by reference to glasses and litres and also feasibility of operation:

P: You would divide
I: Which would you divide?
P: This one (*points to 2 ÷ 0.32*)
I: Why?
P: Because, if there was a decimal point, it would be behind the two and over here there's one before the thirty two, so you would divide that in, because that's the thirty two glasses and there's the thirty two glasses and there's the two litres of coke.
I: So you think that's (*interviewer points to 2 ÷ 0.32*) better than this one (*interviewer points to 0.32 ÷ 2*), which is 2 into 0.32?
P: Yes, because then it's 200 into that and it wouldn't go.

Interestingly, Harry (C: 32) made the correct choice by converting two litres to millilitres but chose a different operation for the calculator:

P: You'd divide 32 into 2000 millilitres
I: Why would you divide 32 into 2000?
P: You have to find out how much one glass holds ... and 0.32 ... and miss, it says how many can be filled out of a bottle that holds 200 ... or 2 litres, so you divide 0.32 into 2000 millilitres
I: And how would you do that if I asked you to do that on a calculator?
P: I'd do that one which is 2 into 0.32
I: You would do this one which is 2 into 0.32?
P: Yes
I: But isn't that one different to that one (*interviewer points to 2 ÷ 0.32*)?
P: Yes
I: So why would you try this one (*interviewer points to 0.32 ÷ 2*) out on the calculator?
P: Because it's smaller than 0.32

In the *cola* problem the idea that the divisor must be less than the dividend seems to exert a stronger influence on these children's thinking than an easily executable algorithm. Indeed so strong is this influence that Harry suggests that that an operation such as *2 ÷ 0.32* can be solved algorithmically but cannot be executed on the calculator, which does not 'add on noughts'.

Discussion

In a time when pen and paper were the chief tools available for the calculation of complex number operations, there was logical sense in having a hierarchical arrangement of procedures in the primary mathematics curriculum. Indeed single digit operations were often presented in vertical format so that pupils would be able to transfer learning to the same operations for multi-digit numbers. This appears to have worked well in the Irish primary system. In national and international studies, Irish children have performed consistently well in the area of number and number operations. (Greaney and Close, 1989; Mullis et al., 1997). However, their performance in problem solving is a matter of some concern. A National Assessment of Mathematics Achievement was conducted in Ireland in 1999, and involved almost 5000 children in fourth class. It was found that pupils performed best on lower level processes such as understanding and recalling and using strategies and implementing procedures and least well on higher level processes such as integrating and connecting and analysing and solving problems (Shiel and Kelly, 2001). The 1999 curriculum retains a hierarchical structure for number operations but it 'places less emphasis on long, complex pen and paper calculations and a greater emphasis on mental calculations estimation, and problem-solving skills' (Government of Ireland, 1999: 7). Use of the calculator as a tool for the development of problem-solving skills in fourth to sixth classes is also recommended. The extent to which this is happening is questionable. A perusal of textbooks commonly in use in senior primary classes shows that there remains a focus on the practice of standard algorithms for the operations of addition, subtraction, multiplication and division of whole and decimal numbers. Recent surveys on the implementation of the 1999 curriculum were conducted by the National Council for Curriculum and Assessment (NCCA, 2005) and by the Inspectorate of the Department of Education and Science (DES, 2005). The DES survey of mathematics is based on visits to 61 classes. In almost a third of these classes it was found that there was an over-reliance on traditional textbook problems. In the NCCA survey a total of 702 teachers completed a teacher template (a booklet which contained questions and statements to prompt teachers to think about their teaching) and a collective case study of schools was designed and developed. Similar to the 1999 study mentioned above, it was found that lower order skills received more attention than higher order thinking and problem-solving skills. The calculator was used most often for predicting and checking results and least often for exploring the number system and discovering facts and relationships.

These interviews showed that, as has been found in other studies, children had great difficulty with the idea of a divisor being greater than a dividend. This was particularly true of the *lotto* problem but also emerged in *cola* where children tended to choose the algorithmically difficult 2 ÷ 0.32 as the (correct in this case) operation for that problem. This indicates that in a quotitive division situation, children's numerical preference seems to be for dividing a larger number by a smaller number than for dividing by an integer. Some children were at a high level of sophistication in their understanding of problem structure and content and made little reference to number type in their reasoning. For others, the conventional algorithms seemed to have affected their thinking about division in two main ways. First, the pen and paper procedures seemed to induce an 'algorithmic set' whereby any insight children had about a solution to a problem was impeded by the feasibility of algorithm (e.g., 'you can't divide twenty four into eighteen'). Second, it seemed that the division procedure itself reinforced the notion that a divisor must be less than a dividend. This was manifested by the extent of reference to 'adding on noughts' ('you could put two noughts at the back...and twenty four into eighteen hundred'). This ratifies the findings of Kamii and Anghileri that algorithms interfere with and even damage children's thinking.

The interviews also showed, while some children chose the correct operation by reference to problem structure ('Because each of them was doing it, so they had to get an equal number…to find out how much they got each'), others did so for superficial reasons ('litres can't go into millilitres'). This highlights the importance of not only focusing on 'correct solutions' in assessment of children's mathematical achievement but also on their reasoning and problem-solving skills. This can be done by the encouraging children to justify their reasoning orally and in written form. If this becomes an integral part of assessment, there is a strong likelihood that more attention will be given to the development of these skills in classroom mathematical activity.

Fryer (1996:5) suggests that 'most school curricula comprise received knowledge from the past but offer woefully inadequate preparation for the future'. She maintains that if students are to cope with the demands of the future, they will have to be flexible and creative in arriving at solutions to unfamiliar problems. Our children will not be prepared for the future if emphasis is placed on complex algorithms that impede thinking and hinder reasoning. It is essential that children develop a sound understanding of number so that they make sensible choices about method of calculation whether it is estimation, mental calculation, pen and paper calculation or calculator. Full advantage needs to be taken of available technology for the development of concepts and of problem-solving skills. Higher order processes need to be at the forefront of primary mathematics education. The time is nigh to go beyond algorithms.

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Challenging the children – are we expecting too little? A Longitudinal Study of Maths Attainment at Primary Level in Ireland.

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This paper looks at the mathematical performance of 200 Irish pupils at primary level in a longitudinal study over the period 2000-2004. This study looks at the pupils' performance in Number across four dimensions of understanding namely, Skills, Properties, Uses, Representations. Test 0 looks at only two of these dimensions while test 5 looks at all. It emerges that while Irish pupils are not performing at the level of some of their international counterparts, they are coping reasonably well in relation to their own opportunity and exposure of the material involved.

Background

The International Performance of Mathematics Attainment project was set up by Prof David Burghes of the University of Exeter with the aim of making recommendations on good practice in the teaching and learning of mathematics through research at an international level. Ireland was invited to join the IPMA project at the end of 1998. Other participating countries include Brazil, China, Czech Republic, England, Finland, Greece, Hungary, Japan, Poland, Russia, Singapore, South Africa, Ukraine and United States of America.

Sample

There are approximately 446000 children in primary education in Ireland. The Department of Education and Science has responsibility for 3278 primary schools, 2100 of which have 7 teachers or less or put another way, 2468 have less than 200 pupils. In choosing schools, attempts were made to include a range of schools which reflected the mix of primary schools in Ireland, but it was never intended to be a representative sample. Fifteen schools were involved in the initial cohort: 3 girls' schools, 3 boys' schools and 9 mixed schools. Two of these were rural schools, five were from towns/villages and the remaining eight were city schools. All schools involved are state-funded. This sample, the second cohort of children to be tested for the project was drawn from twelve schools from the original list of schools but for a variety of reasons e.g., local school amalgamation, three chose not be become involved in the second cohort.

At the time of sitting Test 0, the participants had attended formal primary school for over a year and were beginning Senior Infants. The tests being analysed in this paper are those completed by the second cohort tested early in the academic year of 2000. At that stage they were in the second year of their formal schooling, Senior Infants. Since then they have completed tests 1, 2 , 4 and 5. Test 6 will be administered in Autumn 2005. The analysis in this paper involves schools who were involved in all tests from test 0 through to test 5.

Method

The test items were designed in 1998 by Prof David Burghes at the Centre for Innovation in Mathematics Teaching (CIMT). They assess content areas across the following strands: Number, Algebra, Chance, Data, Geometry, Measures. They were slightly modified in terms of language for an Irish context. The tests were unique, designed for a longitudinal study. Test 1 includes all the items from test 0 and so on for all the tests. The tests were administered by the class teachers. No time limit was placed. Maximum scores for each of the tests were as follows:

Table 1

Test 0	10 marks
Test 1	20 marks
Test 2	40 marks
Test 3	60 marks
Test 4	80 marks
Test 5	110 marks
Test 6	140 marks

Summary Analysis of Tests 0,1,2,4,5

Scores on Tests 0 through to Test 4 are presented for pupils in the schools which sat all four tests. Table 1 shows the mean, median, mode, standard deviation and range on each test. Test 0 scores produced an average of 5.41 of a possible 10 marks; test 1 an average of 14.18 of a possible 20 marks; test 2 a score of 24.09 of a possible 40 marks, test 4 shows a mean score of 54.57 out of a maximum of 80 marks with test 5 showing a mean score of 76.94 out of a possible 110 marks.

Table 2

	Test 0 *(10)*	Test 1 *(20)*	Test 2 *(40)*	Test 4 *(80)*	Test 5 *(110)*
Mean	5.41	14.18	24.09	54.57	76.94
Median	5.00	15.00	25.00	55.00	79.00
Mode	5.00	16.00	25.00	59.00	79.00
St Dev	2.14	3.61	6.12	10.87	11.70
Range	10.00	20.00	36.00	72.00	62.00
Minimum	0.00	0.00	2.00	7.00	37.00

Results and Discussion

An analysis of the Irish pupils' performance in each of the strand areas over each of the tests reveals much about their progress in relation not only to their international counterparts, particularly those in England and Singapore, but also to the objectives as laid out in the Irish maths curriculum. The tables which follow show the performance of Irish pupils across tests 0 to 5 within the Number strand using the SPUR classification. The year on year progress of the group is shown in the tables which follow below.

Table 3 Number – Skills cluster of items

Items	**Mean**				
	Test 0 (Senior Infants)	*Test 1 (First Class)*	*Test 2 (Second Class)*	*Test 4 (Fourth Class)*	*Test 5 (Fifth Class)*
50, 51a, 51b, 51c, 51d, 52a, 52b, 52c (8 marks)					Irl:3.19 Sgp:5.57
37, 38 (2 marks)				Irl:1.18 Eng:0.37 Sgp:0.21	Irl:1.38 Sgp:1.9
20a, 22a, 22b, 22c, 22d, 28a, 28b (7 marks)				Irl:4.26 Eng:2.59 Sgp:4.34	Irl:5.25 Sgp:6.19
9a, 9b, 9c, 9d, 9e, 9f, 10a, 10b, 10c, 10d, 11a, 11b, 11c (13 marks)			Irl:5.37 Eng:5.39 Sgp:8.79	Irl:10.15 Eng:7.66 Sgp:11.46	Irl:11.23 Sgp:12.02
7a, 7b, 7c (3 marks)		Irl:1.39 Eng:6.87* Sgp:8.26*	Irl:2.32 Eng:8.86* Sgp:9.56*	Irl:2.86 Eng:9.66* Sgp:10.31*	Irl:2.88 Sgp:10.51*
3a, 3b, 3c, 3d, 3e, 3f, 4a, 4b, (8 marks)	Irl:3.74	Irl:5.85 Eng:6.87* Sgp:8.26*	Irl:6.55 Eng:8.86* Sgp:9.56*	Irl:7.26 Eng:9.66* Sgp:10.31*	Irl:7.66 Sgp:10.51*
Total (41 marks)					

* Neither England nor Singapore administered Test 0 and so analysed the items in Test 0 and Test 1 together

The above table shows the Irish pupils' mean performance relative to the total number of marks available for each category and to their counterparts in England and Singapore. The improvement through the years is evident in each area. An analysis of the items causing difficulty shows that in many cases the content has not been covered by the pupils at that point in their schooling. The following tables and discussion looks at each aspect of the Irish pupils' performance using the SPUR classification of the items within the Number strand.

Table 4 Number: Skills cluster of items (Test 0)

Number	Q3d	Skills	33%
Number	Q3e	Skills	26%
Number	Q3f	Skills	36%
Number	Q4a	Skills	39%
Number	Q4b	Skills	27%

Table 4 identifies the test items which proved most difficult for pupils. In test 0, items 3d, 3e and 3f show average attainments by the sample of scores of 33%, 26% and 36% respectively while items 4a and 4b also demonstrate a significant level of difficulty for the pupils. It is important to note that these items relate to curriculum objectives for later years in the Irish maths programme. Question 3d, 3e and 3f involve missing addends and missing subtrahends and items 4a and 4b relate to putting numbers to 20 in order of size and identifying the odd numbers.

Table 5 Number: Skills cluster of items (Test 1)

Number	Q4a	Skills	39%
Number	Q4b	Skills	39%
Number	Q7b	Skills	49%
Number	Q7c	Skills	31%

A slight improvement is recorded in this test in items 4a and 4b though the material has still not been covered formally. Items 7b and 7c cause significant problems. Again, this can be attributed to the pupils' lack of exposure to the topic at this point. In fact, they do not feature as part of the Irish programme for a further two years. Q7b asks the pupils to complete the pattern by subtracting 4 each time while Q4c also involves the completion of a pattern but by increasing by 4.

Table 6 Number: Skills cluster of items (Test 2)

Number	Q4b	Skills	22%
Number	Q9d	Skills	42%
Number	Q9e	Skills	44%
Number	Q9f	Skills	34%
Number	Q10a	Skills	50%
Number	Q10b	Skills	22%
Number	Q10c	Skills	17%
Number	Q10d	Skills	19%
Number	Q11a	Skills	38%
Number	Q11b	Skills	24%
Number	Q11c	Skills	1%

This table shows areas of poor performance in the skills dimension in test 2. None of the items above, with the exception of 4b, has yet been covered by the pupils in their maths programme. Q9d involves subtraction with renaming; Q9e involves a two-digit missing addend; while Q9f involves a missing subtrahend. Q10 involves multiplication and division. Q11a and b require the pupils to complete patterns in ascending and descending order. The cubing of numbers in 11c has still not been covered and is not a specific objective of the Irish curriculum at all.

Table 7 Number: Skills cluster of items (Test 4)

Number	Q11c	Skills	26%
Number	Q20a	Skills	37%
Number	Q22c	Skills	45%
Number	Q22d	Skills	40%
Number	Q28a	Skills	49%
Number	Q37	Skills	27%
Number	Q38	Skills	28%

Poor performance here cannot be entirely accounted for by lack of opportunity to learn, for example items 28a, 37 and 38. Q28a involves rounding to the nearest 10. Items 37 and 38 require renaming tenths as decimals and vice versa. At this point, 20a, 22c and 22d have not been formally and explicitly taught in class. The relatively good performance is probably accounted for by the more able pupils being able to infer from previous knowledge.

Table 8 : Number: Skills cluster of items (Test 5)

Number	Q11c	Skills	26%
Number	Q50	Skills	10%
Number	Q51a	Skills	40%
Number	Q51b	Skills	48%
Number	Q51d	Skills	24%
Number	Q52c	Skills	34%

At the time of testing, items 11c (cubing of numbers), 50 (estimation of a three-digit number divided by a two-digit number) and 51d (division of a four-digit number by a two-digit number) had not been covered by the pupils. Item 52c was not presented in a manner familiar to the pupils.

Table 9 Number – Properties cluster of items

Number – Properties cluster of items					
Items	**Mean (Standard Deviation)**				
	Test 0 (Senior Infants)	**Test 1 (First Class)**	**Test 2 (Second Class)**	**Test 4 (Fourth Class)**	**Test 5 (Fifth Class)**
32a, 32b, 33 (3 marks)	---	---	---	Irl:2.21 Eng:2.29 Sgp:2.73	Irl:2.77 Sgp:2.83
20b, 23a, 23b, 25 (4 marks)	---	---	---	Irl:3.01 Eng: 2.72 Sgp:3.22	Irl:3.50 Sgp:3.25
Total (7 marks)					

As can be seen from the table above, items pertaining to the properties dimension were quite well answered by the Irish pupils with an improvement from test 4 to test 5.

Table 10 Number – Uses cluster of items

Items	**Mean (Standard Deviation)**				
	Test 0 (Senior Infants)	**Test 1 (First Class)**	**Test 2 (Second Class)**	**Test 4 (Fourth Class)**	**Test 5 (Fifth Class)**
15 (1 mark)		---	Irl:0.42 Eng: 0.39 Sgp:0.77	Irl:0.86 Eng: 0.79 Sgp:0.95	Irl:0.93 Sgp:0.97
Total (1 mark)					

Irish pupils had not had an opportunity to cover item 15 until the middle of the year following the test. There was a marked improvement by the pupils in test 4 and almost complete success in test 5.

Table 10 Number – Representations cluster of items

Items	Mean				
	Test 0 (Senior Infants)	**Test 1 (First Class)**	**Test 2 (Second Class)**	**Test 4 (Fourth Class)**	**Test 5 (Fifth Class)**
43a, 43b, 53a, 53b (4 marks)					Irl:2.01 Sgp:2.54
36a, 36b, 42a, 42b (4 marks)				Irl:1.46 Eng:1.8 Sgp:1.59	Irl:2.62 Sgp:2.85
19, 27 (2 marks)				Irl:0.94 Eng:0.86 Sgp:1.33	Irl:1.46 Sgp:1.63
8, 16 (2 marks)			Irl:1.17 Eng:1.42 Sgp:1.24	Irl:1.70 Eng:1.55 Sgp:1.52	Irl:1.88 Sgp:1.7
5a, 5b, 6a, 6b, 6c, 6d (6 marks)		Irl:5.01 Eng:5.76* Sgp:6.53*	Irl:5.13 Eng:6.72* Sgp:7.43*	Irl:5.5 Eng:7.16* Sgp:7.59*	Irl:5.67 Sgp:7.75*
1, 2 (2 marks)	Irl:1.71	Irl:1.84 Eng:5.76* Sgp:6.53*	Irl:1.83 Eng:6.72* Sgp:7.43*	Irl:1.94 Eng:7.16* Sgp:7.59*	Irl:1.97 Sgp:7.75*
Total (20 marks)					

* Neither England nor Singapore administered Test 0 and so analysed the items in Test 0 and Test 1 together

The representation dimension of the Number strand is assessed using 20 items. Irish pupils proved to have fewer difficulties in this dimension. No significant difficulties emerge until test 2. Item 16 (representing a fraction of a set) had not been covered at that stage by the pupils.

Table 12 Number: Representation cluster (Test 4)

Number	Q19	Representation	29%
Number	Q36a	Representation	24%
Number	Q36b	Representation	26%
Number	Q42b	Representation	20%

Item 19 (reading a scale 1:20) is not a curriculum objective to which the pupils have been exposed at this point. The same is not true of the other items. Q36 asks the pupils to show a fraction and decimal involving tenths on the number line.

Table 13 Number: Representation cluster (Test 5)

Number	Q42b	Representation	43%
Number	Q43a	Representation	36%
Number	Q53b	Representation	33%

The poor performance across these items by Irish pupils is not due to lack of exposure to the concepts involved.

Discussion

Analysis of the test data raises certain issues which are beyond the scope of this paper in relation to the teaching and learning of mathematics in Irish primary schools.?

On an optimistic note, Irish pupils are comparing favourably, at least within the Number strand, with their English and Singaporean counterparts. In most areas, Irish pupils outperformed the English cohort while at the same time not lagging far behind the Singaporean pupils. This gives cause for hope considering the high international profile of maths attainment in Singapore.

The relatively poor performance of many children on core areas of the programme raises questions regarding the timing and structure of the curriculum. It is of particular note that with most of the items where the most significant difficulties emerged, the pupils had not had an opportunity to learn the material involved. While this is important in its own right, it is worth questioning the wisdom of the sequencing of the objectives of the maths curriculum. The data seem to indicate that many of the pupils are performing well where they have had an opportunity to learn the concepts or skill involved. There is evidence to suggest that the more able pupils would benefit from further challenge. Greater differentiation in the maths programme may meet this need. A closer look at the teaching of maths in Singapore would suggest, however, that ‘moving together’ is the preferred option.

The need to emphasise problem solving in the Irish mathematics programme is strengthened on the basis of the results so far. The reduced time allocation for mathematics in the revised programme also needs review.

The starting age of children in Ireland is lower than most, if not all, of the countries in the study. It does not appear to particularly benefit the children’s mathematics scores in the medium to longer term.

The dependency on textbooks in mathematics teaching in Ireland also needs further attention but it is fair to say that the majority of teachers use textbooks as standards?. It is apparent in several of the test items above that while the topics may have been covered in a particular way, when they appeared/sounded different they left the children unsure of how to tackle previously-learned concepts in an unfamiliar setting. Hence, a greater variety of real-life presentations of concepts and skills are necessary to bolster the pupils' performance.

There are many drawbacks to making international comparisons in educational attainment. One of the main advantages of such work is that it encourages reflection on the similarities and differences in approaches to teaching and learning, expectations and attainment of our pupils. In this sense, we can be hopeful but far from complacent.

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Mathematical Problem Solving in Senior Primary School – an Examination of the Beliefs and Experiences of Class Teachers

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Traditional mathematical classrooms have predominantly focussed on the facts and procedures that define the body of knowledge that is mathematics. This research into constructivist teaching and teacher beliefs and experiences in mathematics education provides valuable insight into the difficulties in actively involving children in their mathematical learning. It discusses survey and interview data of primary teachers' approaches to mathematical problem solving in the senior primary school. This survey followed clinical investigations into fourth class pupils' problem solving abilities. It emerged that students did not engage in significant classroom interactions with their peers in an effort to develop, construct or refine their own mathematical problem solving strategies

Introduction

Both TIMSS 1995 (Mullis, Martin, Benton, Gonzalez, Kelly & Smith, 1997) and *The 1999 National Assessment of Mathematical Achievement* (Shiel & Kelly, 2001), have shown that Irish students at primary and post primary levels have achieved levels of considerable competence in engaging with number facts and processes, yet worryingly their achievement with higher level mathematical processes are not comparable. The following research discusses survey results of 200 5th and 6th class primary teachers' approaches to mathematical problem solving in the senior primary school. The research was conducted following clinical investigations into 4th class pupils' problem solving approaches and the identification of pupils' needs and abilities in relation to open-ended problem solving (O'Shea, 2003). The clinical investigation into 4th class pupils' problem solving abilities employed co-operative group experiments with suitable non-routine mathematical problems relevant to the prior experiences of the children. These were selected to reveal students' approaches to mathematical problem solving. The results of these experiments are discussed.

Mathematical Problem Solving

Mathematical problem solving is a central focus of the Revised Primary Mathematics Curriculum. The Revised Primary Mathematics Curriculum (Government of Ireland, 1999a) encourages teachers to present children with "real problems related to their own experience encouraging them to develop strategies for solving them imaginatively" (Government of Ireland, 1999a:47). Current literature urges the use of problem solving activity to connect different ideas and procedures in relation to different mathematical topics and other content areas (Ernest, 1988). Mathematical problem solving is the vehicle through which students attain mathematical power (Romberg & Fennema, 1999). Although the pre-requisites to success in problem solving must lie in grasping basic skills, the Revised Primary Curriculum emphasises the use of constructivist teaching so that the child will construct new mathematical knowledge utilising mathematical problems.

The constructivist approach to mathematical learning requires students to acquire mathematical modes of thinking such as inferring and questioning which will enable them to combine domain specific knowledge with understanding in the current mathematical climate. Changing professional standards, new workplace demands and recent changes in learning theory demand mathematical students to utilise higher order thinking skills (Foshay & Kirkley, 1998). To illustrate the mathematical beliefs of students' regarding mathematics, Frank (1988) conducted a survey of mathematical beliefs of junior high school students. It revealed junior high students conceptions of mathematics

1. Mathematics is computation
2. Mathematics problems should be quickly solvable in just a few steps
3. The goal of doing mathematics is to obtain the right answers
4. The role of the mathematics student is to receive mathematical knowledge and demonstrate that it has been received
5. The role of the teacher is to transmit mathematical knowledge and to verify that students have received this knowledge.

Teaching Mathematical Problem Solving

Teaching problem solving can cause difficulties for the teacher. Carpenter, Fennema, Peterson, Chiang & Loef. (1989) emphasise the importance of posing problems to students and listening to how students describe the way they have solved such problems. However, Fosnot (1989) explains that teachers are unfamiliar with such teaching because they are products of a system that emphasised drill and procedure. She debates+- that teachers enjoy the safety provided by workbook pages, computation sheets and drill during instruction often because they themselves are products of a similar approach. Children must be enabled to interpret and develop an understanding of mathematical processes rather than simply just the ability to perform these mathematical processes. The ultimate responsibility for this resides with the teacher.

But, Ball (1996) contends that teacher confidence impedes the exploration of problem solving in this manner because,

> When we ask students to voice their ideas in a problem solving context, we run the risk of discovering what they do and do not know. These discoveries can be unsettling when students reveal that they know far less than the teacher expected and far more than the teacher is prepared to deal with. Ball (1996).

Mathematical Problem Solving in the Senior Primary School: An Investigation into the Beliefs and Experiences of Class Teachers (O'Shea, 2003a).

On analysis of data, O'Shea (2003a) found that 62% of the survey population used all problem solving lessons to facilitate the practice of number concepts and skills. It seems mathematical problem solving was utilised as a strategy in the practice of existing knowledge rather than in the acquisition of new mathematical knowledge.

The Primary Mathematics Curriculum (1999) demands problem solving activity play an integral role in a students mathematical learning. Ball (1996) and Taback (1992) found that the majority of teachers have not experienced mathematics as a discipline involving problem solving. They have rather experienced mathematics as a fixed body of knowledge to be learned. 82% of respondents to this survey indicated that they chose not to study either the subject mathematics or the teaching of mathematics to any specific extent at third level. Consequently, the researcher argues that Irish primary school mathematical explorations are not encouraging enough students to study mathematics because of the lack of investigative and experimental mathematical experiences. The mathematical experience of the teacher is an obstacle to the use of mathematical problem solving in teaching. It seems the mathematical experiences of teachers are related to their teaching of and teaching through problem solving. The survey indicated that a considerable amount of time is spent on the student acquiring mathematical knowledge through rote memorisation and computational exercises.

Shiel and Kelly (2001) established that a significant number of pupils are taught by teachers who have not engaged in professional development related to the teaching of mathematics in recent years. In my opinion, as a primary teacher, in-service related to the Primary Mathematics Curriculum in mathematics was insufficient in exploring mathematical problem solving in depth. However, where teachers have engaged in professional development, teachers have deemed the quality of such courses 'uneven' (Shiel and Kelly, 2001). This study revealed similar satisfaction levels with in-service education related to investigating mathematical problem solving with children at 5th and 6th class levels. Specifically, survey respondents indicated mathematical problem solving required a more in-depth focus on approaching mathematical problem solving utilising group activity. Indeed, Shiel & Kelly (2001) highlight that the matter of grouping pupils for mathematics continues to challenge schools and teachers. 25.4% of respondents to this study 'almost never' allow students to work in small groups/pairs during problem solving lessons without any significant teacher influence. The mathematics curriculum purports that students should "operate in pairs or small groups to solve problems co-operatively". It must be observed that the majority of teachers "almost never" or only during "some" lessons use any grouping strategy. In fact, 57.5% of survey respondents reported that students work together as a whole class with the teacher instructing the class during "most lessons".

A calculated mean of 2.13 on a scale of 0 (Never) -5 (Often) explains co-operative grouping arrangements are employed overall only during "some" problem solving lessons. In relation to the use of

1. The school environment
2. The local environment and
3. Concrete materials

in the facilitation of mathematical problem-solving lessons, results indicated that more respondents (34.6%) used concrete materials during "most/every lesson". This is in contrast with 8.9% and 14.9% of respondents respectively indicating the involvement of both local and school environments during "most/every" mathematical problem solving lesson.

During the problem solving process, students engage in various activities in an attempt to provide solutions to problems. The following table illustrates the research sample's allocation of lessons to various activities

Table 1: Activities Explored During Mathematical Problem Solving

	Never	Almost Never	Some Lessons	Most Lessons	Every Lesson
Explain the reasoning behind a problem	1.5%	6.0%	37.6%	39.8%	15.0%
Practice computational skill	0.8%	3.0%	28.0%	48.5%	19.7%
Restate the problems in their own words	2.3%	15.2%	40.9%	30.3%	11.4%
Discuss the meanings of words	2.3%	8.3%	36.8%	38.3%	14.3%
Students provide questions for problems you present	12.0%	34.6%	39.1%	10.5%	3.8%
Students provide stories for problems you present	9.8%	33.8%	48.1%	7.5%	0.8%
Students attempt problems using alternate strategies	2.3%	12.8%	64.7%	18.8%	1.5%

A mean result of 1.59 on a scale of 0 (Never) – 5 (Every Lesson) concerning student questioning, suggests that students are not challenged to extend the problem being explored. There is a clear lack of deep discussion and development surrounding the context of the problem situation.

In examination of survey respondents' use of novel problem situations in mathematical explorations, results indicate that children are not provided for sufficiently. 31.3% of respondents reported that they "almost never" explore problems where the pupil must decide upon an excess, sufficiency or deficiency of information. This suggests that the types of mathematical problems students explore are familiar to them. It is interesting to note that 56.3% of problems that children are tasked with come from their textbooks during "most lessons". Polya (1971) established a four step problem solving procedure. This procedure illustrates why novel problem solving situations need to be provided for children to have a holistic experience in problem solving. It is

1. Understand the problem – Decide on the relevant information and be clear about what is being asked.
2. Devise a plan/approach for attacking the problem. This represents a shift in emphasis from information found in the problem to information brought to the problem.
3. Carry out the plan. The student carries out his/her selected approach, taking care to monitor his/her progress and regulate his/her methods. Often a student may revisit the problem for clarification purposes.
4. Looking back. Reaching a conclusion is not an end in itself. One must consider the answer in terms of the information given.

Given the obvious reliance on textbooks and the very nature of a textbook, Irish students do not have sufficient opportunity for extensive mathematical problem solving based on the above.

Mathematical problem solving seems to play a supportive role in the refinement and assessment of mathematical knowledge acquired during significant student/teacher interactions. Mathematical problem solving situations are employed in the classroom to provide a context for a student's mathematical learning. From survey data, students do not engage sufficiently in significant interactions with their peers in an effort to develop, construct or refine their own mathematical problem solving strategies. The constructivist approach to mathematical learning appears restricted.

The majority of teachers of senior grades who responded to the study do not have a comprehensive mathematical background themselves. The mathematical experiences of teachers remain with their formative schooling mathematical explorations. Senior grade teachers do not seem to have engaged in significant mathematical problem solving situations themselves.

Prior to the above research, the researcher conducted clinical examinations of fourth class pupils' problem solving approaches. These were designed to allow pupils to experiment with the methods they chose to solve mathematical problems in small group situations. These investigations were closely monitored by the researcher and examined in detail focussing on the processes they employed to solve the problems involved. The situations were designed to reflect constructive learning. The investigation of teacher behaviours following this research was designed to examine teacher behaviours in relation to student problem solving.

Clinical Investigations into 4th Class Pupils' Problem Solving Approaches

All four phases of Polya's (1971) four step problem solving procedure were discussed with the target group of students involved in the above study. The researcher asked the children to describe what each stage required of them, to ensure a complete awareness of all four stages. This awareness contributed to a rich discussion amongst group members in relation to the background of the problem, methods of solution and analysis of subsequent solutions. It encouraged students to reflect on their work entirely.

During the initial stages of the experiment the researcher was frequently asked "When are we going to do real sums?" However, as the experiment progressed, the same students contended that the mathematical activity of experimentation was extremely "interesting" and "fun". As evidenced by Shiel & Kelly (2001), traditional Irish primary classrooms focus heavily on the performance of mathematical number operations. Students need to be exposed to a variety of problem types on a more regular basis to provide them with a more complete mathematical education.

The National Council of Teachers of Mathematics (NCTM) suggests that teachers facilitate problem solving by supplying children with good problems (Carpenter & Gorg, 2000). In conducting this research, good problems were those that exhibited familiar contexts to pupils. The problem became less of an inconvenience and more of a challenge when problems constructed around familiar contexts for children were explored. Pupils were engaged in significant conversation surrounding these problems, resulting in a more in-depth analysis of the problems and the application of appropriate problem solving strategies.

Non-routine problems encouraged students to invest in a significant amount of time in planning. Pupils realised that planning contributed towards the successful solving of the problem. It also encouraged students to revise and modify previously chosen strategies. Activities of this nature are embedded in the principle of constructivism that is so heavily central to the Primary Mathematics Curriculum (1999). English (1995) suggests children must first examine problems for clues to guide retrieval of information from memory. Children found it easier to solve problems following the completion of similar yet not identical problems. Problems that were related to earlier situations allowed pupils to examine and develop more sophisticated strategies for solving these problems. Pupils exhibited their creative and divergent talents through the discussion and evaluation of their problem solving strategies.

The operations of repeated addition, multiplication, subtraction and division were confidently applied by participants. This the researcher attributes to the continuous monitoring and evaluation of each others progress throughout the experiment. Students were circumspect about utilising trial and error, as they believed it inappropriate to mathematical activity. Towards the end of the experiments, pupils began to revise chosen approaches rather than disregard them. This was clear evidence of students attempting to construct mathematical knowledge by utilising prior knowledge.

Mathematical problem solving during the course of the research became a chore for participants when the context of the problem was alien to the student. Students were not keen to engage in the analysis, interpretation and solving of such problems. This hindered the application of appropriate and sophisticated strategies and approaches. Teachers need to foster positive attitudes and beliefs about problem solving amongst students, through the construction of stimulating mathematical problem solving situations for the child. Children exhibited confidence in problem solving having achieved a degree of success, be it either in the selection and use of an appropriate strategy or in the production of a correct answer. The goal of mathematics teaching must be to enable students to build upon and develop such confidence.

This research argues in favour of the constructivist approach to learning. Students encouraged one another, providing support when required, and displayed the ability to refine one another's strategies and suggestions. Problem solving allows teachers to judge the effectiveness of the learning or the level of understanding by a student. The clinical investigations required reasoning, selection and application of skill by the student which induced a profound interest amongst the students in mathematics. Children were "experiencing mathematics as an intellectual pursuit in its own right" (Government of Ireland, 1999a:48).

Problem solving according to Trafton (2001) is the vehicle by which students make sense of mathematics and learn content, skills and strategies. "It is natural to young children because the world is new to them and they exhibit curiosity, intelligence and flexibility as they face new situations". According to the *Principles and Standards for School Mathematics* (Carpenter & Gorg, 2000), the challenge must be to build on children's innate problem solving inclinations and to preserve and encourage a disposition that values problem solving. This study highlighted that Irish 4^{th} class students can successfully approach non-routine mathematical problems and apply domain specific mathematical skills and strategies to a significant extent.

Conclusion

The above research argues the benefits of constructivism and peer interaction in problem solving in the senior primary grades. Research into the beliefs and experiences of class teachers highlights a possible gulf between positive classroom interactions and current prevalent conventions. Whilst acknowledging the limitations of both pieces of research, the Irish primary mathematical classroom does not appear to operate with constructivist mathematical teaching methodology to a major extent. Mathematical problem solving appears to play a complementary role to student learning.

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The National Assessment of Mathematics Achievement in Primary Schools

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In both 1999 and 2004, the Educational Research Centre administered an assessment of mathematics achievement to a representative national sample of pupils in Fourth class (primary level) on behalf of the Department of Education and Science. In both years, over 4,000 pupils in over 100 schools completed a test of mathematics, while their teachers and parents completed questionnaires that asked for information about the teaching of mathematics in schools and classes, the support provided to pupils with difficulties in mathematics, and the support for mathematics learning at home. First, the purposes of the assessment (including its role in monitoring achievement in mathematics over time), the assessment framework, and the test and questionnaires administered in the assessment, are described. Second, the main outcomes for the 1999 assessment are summarised, including achievement outcomes, and associations between key school- and pupil variables and achievement. Third, modifications to the assessment framework and instruments in 2004 to accommodate changes arising from implementation of the Primary Schools Mathematics Curriculum (1999) are outlined, including changes in content, the level of emphasis allocated to different mathematical content areas and processes, and the use of calculators. The presentation concludes with a consideration of how the outcomes of the national assessment of mathematics achievement might be used to impact on policy.

In Ireland, five national assessments of mathematics achievement involving pupils in primary schools have been conducted since 1977 (see Table 1). This paper provides an overview of the national assessment programme, and, in particular, the 1999 and 2004 assessments, although the outcomes of the 2004 assessment will not be available until later this year.

Table 1: National Assessments of Mathematics Achievement 1977-2004

Year	Population Assessed	Report
1977	Second and Fourth classes (primary)	Department of Education (1980)
1979	Sixth class (primary)	Department of Education (1980)
1984	Sixth class (primary)	Department of Education (1985)
1999	Fourth class (primary)	Shiel and Kelly (2001)
2004	Fourth class (primary)	Shiel Surgenor & Close (in preparation)

National Assessments Before 1999

Following implementation of the 1971 curriculum (Curaclam na Bunscoile, Department of Education, 1971), three national surveys were conducted by the Department of Education between 1977 and 1984. The purpose of these surveys was to 'assess the level of achievement of pupils in mathematics specified in the primary school curriculum' (Martin, 1990, p. 27). Sets of content objectives were drawn up, based on the mathematics framework and content presented in Curaclam na Bunscoile, and criterion-referenced tests designed to assess mastery of each objectives were developed and administered. Performance was reported in terms of the percentages of pupils achieving particular objectives in each of several mathematics content areas. There seems to have been less interest in overall performance, although Martin (1990) found that, in the case of the 1984 assessment in Sixth class, overall scores (defined as the number of items answered correctly out of 125) exhibited the characteristics of a norm-referenced test, and therefore the test could be used to describe students' overall achievement in mathematics.

A feature of the earlier assessments was the strong emphasis on Number. In the 1977 assessment in Fourth class, for example, 18 of the 32 objectives assessed were in the areas of Operations with Whole Numbers, Whole Number Structure, and Fractions and Decimals, while just 3 were in the area of Spatial Experience (Geometry). The outcomes of the 1977 assessment at Fourth class showed that pupils were strongest in Operations with Whole Numbers (75% correct).[1] Performance was in the average range in three areas – Whole Number Structure (59%), Measurement (58%) and Fractions and Decimals (58%) – and relatively weaker in three – Spatial Experiences (Geometry) (50%), Graphs (53%) and Problems (41%) (Department of Education, 1980). The 1980 and 1985 assessments in Sixth class revealed again that pupils were strongest in Operations with Whole Numbers (85.7% correct in 1979, and 83.3% in 1985) and

[1] Detailed analyses of the outcomes of the earlier Department of Education assessments of mathematics at the Second, Fourth and Sixth classes may be found in the original sources (Department of Education, 1980, 1985), and in Shiel and Kelly (2001).

Operations with Fractions (70.0% and 75.2%), and weakest in the areas of Problems (52.2%, 50.8%), Metric Measure (44.0%, 48.8%) and Geometry (34.2%, 37.1%).

These earlier national assessments resulted in a number of recommendations on the teaching of mathematics. For example, the following recommendations emerged from the 1984 assessment:

- A greater emphasis should be placed on developing pupils' understanding of basic concepts
- A more structured approach to the teaching of problem solving should be adopted
- A higher priority should be accorded to metric measure in the teaching of mathematics
- Regular revision and assessment should form an integral part of teachers' planning and class work
- National attainment surveys should continue to be carried out at regular intervals
- Aspects of the mathematics curriculum for primary schools should be reviewed.

Although there were no national assessments of mathematics conducted between 1984 and 1999, Irish primary-level pupils participated in two international assessments between those years – the Second International Assessment of Educational Progress [IAPE II] in 1991 (see Martin, Hickey & Murchan, 1992), and the Third International Study of Mathematics and Science [TIMSS] in 1995 (Mullis et al., 1997).[1] At primary level, IAEP II involved an age-based sample (9-year olds in the Third and Fourth classes), while TIMSS involved two grade-based samples (pupils in the Third and Fourth classes). Although the tests used in these assessments were not based on the mathematics curriculum in Irish primary schools (though there was considerable overlap), the outcomes provide useful information on standards in Ireland relative to other countries, as well as aspects of mathematics in which Irish students are strong and weak.

The 1999 National Assessment

The Educational Research Centre was asked by the Department of Education and Science to conduct a national assessment of mathematics achievement in Fourth class in primary schools in 1999. The purposes of this assessment (referred to as NAMA 1999) included the following: (i)To generate baseline data that can be used to monitor achievement over time; (ii) To establish links between the study and the Third International Assessments of Mathematics and Science; (iii) To identify factors related

[1] Six percent of Irish 13-year olds who participated in the First International Assessment of Educational Progress in 1988 were in sixth class (primary level) (see Lapointe, Mead & Philip, 1989).

to the mathematics achievement of pupils; and (iv) To make recommendations with regard to the teaching and assessment of mathematics.

Framework for NAMA 1999

In September 1999, a new Primary School Mathematics Curriculum [PSMC] (NCCA, 1999a; 1999b) was published, with implementation planned for 2002. Nevertheless, it was decided to develop the test for the 1999 national assessment (to be administered in May 1999) around the revised curriculum, since it would be possible to obtain baseline data on the performance of pupils on the content and processes underpinning the revised PCMC, before it had been introduced to schools. Hence, a framework incorporating the strands of Number, Algebra, Shape and Space, Measures, and Data, was developed, and included the processes of Analysing Problems and Evaluating Solutions, Integrating and Making Connections between Mathematical Procedures and Concepts, Engaging in Mathematical Reasoning, Implementing Mathematical Procedures and Strategies, and Understanding and Recalling Terminology, Facts and Definitions. One mathematical process specified in the PSMC, Communicating and Expressing the Processes Used in Mathematical Activities and in Explaining Results, was not included in the framework, because it was felt that it could not be assessed using a paper and pencil test. It was decided not to assess pupil performance on items involving use of a calculator in 1999, since pupils would have had very little experience in working with calculators at that time. However, data on calculator usage in mathematics classrooms was collected.

The framework included a breakdown of the distribution of items across different content areas and processes. The relative weighting assigned to each dimension was ascertained by consulting with lecturers in mathematics education and representatives of the National Council for Curriculum and Assessment, who were involved in developing the revised PSMC. The actual weightings (which varied slightly from the proposed weightings) are given in Table 2. The content area weightings indicate a reduced emphasis on number (relative to earlier national assessments based on the 1971 curriculum) and an increased emphasis on Shape and Space.

Table 2: Weightings Assigned to Mathematics Content Areas and Processes - NAMA 1999

Content Area	Percent of Items	Process	Percent of Items
Number	36.8	Understand and Recall	13.6
Algebra	4.8	Using Strategies and Implementing Procedures	29.6
Shape and Space	14.4	Reasoning	21.6
Measures	35.2	Integrating and Connecting	6.4
Data and Chance	8.8	Analysing and Solving Problems	28.8
Total	100.0	Total	100.0

The 1999 Test of Mathematics Achievement

A test of mathematics achievement was developed around this framework. Over 150 items were tried out in a pilot study involving 873 pupils in 12 schools in March 1999. The final version of the test included 125 items, of which 20 were drawn from the Third International Study of Mathematics and Science (TIMMS). The item pool was divided into five blocks. One block (which included the 20 TIMSS items) was administered to all pupils, while the remaining blocks were counterbalanced within their respective forms, creating five booklets in all. Test booklets were translated into Gaeilge. Each pupil was asked to attempt 3 blocks.

The test items included equivalent proportions of multiple choice and short constructed response items. The new test included a number of items based on content objectives not in the 1971 curriculum, although the pilot test suggested that a majority of pupils would be able to attempt these items. The new objectives included the following:

- Explore, extend and describe sequences (relating to objects, shapes and number) (Algebra)
- Read time in one-minute intervals on analogue and digital clock (12-hour clock) (Measure)
- Express digital time as analogue and vice versa (Measure)
- Use the vocabulary of uncertainly and chance (Data)
- Order events in terms of likelihood (Data)
- Identify and record outcomes of simple random processes (Data)

Implementation of NAMA 1999

The assessment was administered by inspectors of the Department of Education and Science to over 5000 pupils in a representative sample of 120 primary schools in May 1999. 4747 pupils completed the test of mathematics achievement, while the pupils, their teachers and school principals, and their parents completed questionnaires that sought contextual information that might be used to interpret performance. Response rates exceeded 90% on the test and on all questionnaires.

The test was scaled using a three-parameter Item Response Theory (IRT) model (see O'Leary, 1999). The overall mean score on the test was set at 250, and the standard deviation at 50. Weights were computed and applied to analyses of the data to take the uneven representation of students in some sampling strata into account. The data were analysed using the WesVar statistical package, which takes the complexity of the sample into account in computing standard errors.

Main Achievement Outcomes

The primary purpose of 1999 National Assessment was to obtain baseline data on pupils' mathematical achievement so that the performance of pupils in 1999 could be compared with their performance in future assessments. To the extent that the test of mathematics achievement reflects the Primary School Mathematics Curriculum, this purpose was achieved.

On a four-point proficiency scale (advanced, proficient, basic and weak/inadequate), 15% of pupils in fourth class were rated by their teachers to be performing at a 'weak/inadequate' level, while just 11% were judged to be performing at an advanced level. The correlation between teacher judgements' of pupils' proficiency and pupil performance was 0.77. Teachers also indicated that 67.9% of pupils in Fourth class were achieving at a Fourth-class level in mathematics, while 18.4% were achieving at a Third class level and 5.5% at a Second class level or lower.

Percent correct scores were used to compare the performance of students across different mathematics content areas and processes assessed by TMA99. Across all schools, pupils in Fourth class were found to perform best on items dealing with Data, Number, and Algebra, and least well on items dealing with Measures and Shape and Space (Table 3). Performance on mathematical processes was strongest on items tapping lower-level processes such as Understanding and Recalling Terminology, Facts and Definitions and Implementing Mathematical Procedures and Strategies and weakest on items tapping higher-level processes such as Engaging in Mathematical Reasoning, Analysing and Solving Problems and Evaluating Solutions and Understanding and Making Connections between Mathematical Processes and Concepts.

Table 3 Performance on Mathematics Content Areas and Processes – NAMA 1999

Content Area	% Correct	Process Areas	% Correct
Data/Chance	68.6	Understand and Recall	63.0
Number	59.6	Implement	58.1
Algebra	58.4	Reasoning	55.4
Measures	54.1	Problem Solving	54.0
Shape/Space	45.3	Integrate and Connect	53.2

School / Classroom Variables

There was a significant difference in overall mathematics achievement between pupils attending schools designated as disadvantaged and those not attending such schools. The differences (in favour of the latter) was almost one half of a standard deviation (24.3 points). Furthermore 20.5% of pupils in disadvantaged schools (compared to 8.0% in non-disadvantaged schools) achieved scores at or below the 10^{th} percentile on the test, and many of these pupils were also judged by their teachers to be unable to read their mathematics textbooks. Just 4.1% of pupils in disadvantaged schools achieved scores at or above the 90^{th} percentile, compared with 11.2% in non-disadvantaged schools.

More than half (55.6%) of pupils attended schools in which learning support in mathematics was not provided by an officially-sanctioned learning support teacher. 7.4% of pupils were in receipt of learning support in mathematics, while 13.1% were in need of support according to their teachers, but were not in receipt. More pupils in disadvantaged (9.5%) than non-disadvantaged schools (7.0%) were in receipt of learning support in mathematics.

Seventy-five percent of pupils attended a school in which there was a clearly defined plan for the teaching and learning mathematics. Among the topics included least frequently were: organisation for teaching mathematics; provision for pupils with learning difficulties in mathematics; engagement of pupils in practical activities; strategies for teaching problem solving; and provision of enrichment activities for advanced pupils. Most pupils attended schools where provision was made in the plan for pupil assessment, and for maintaining records of achievement.

According to school principals, the most serious problems that inhibited the teaching and learning of mathematics in schools were

- Lack of learning support in mathematics (65.1% of pupils)
- Inadequate in-career development (53.3%)
- Multi-grade class arrangements (32.6%)

Just under 30% of pupils were taught by teachers who had attended in-career development on the teaching/learning of mathematics since completion of initial teacher training, while just 16.4% had attended in-career development on this topic in the five years prior to the survey. Just 4.1% of teachers who had completed initial training between 1990 and 1998 had attended any in-career development in mathematics.

Teachers indicated placing most emphasis during instruction on topics related to number (with place value, and basic operations receiving most attention). Problem solving (especially problems involving operations applied to money, and problems involving fractions) also received considerable attention, while geometry, and data received relatively less attention. Among the specific topics not emphasised strongly by teachers was estimating the length, weight and capacity of objects.

Over one-half (55.5%) of pupils in Fourth class were taught by teachers who administered standardised tests in mathematics at least once a year. Teacher-made tests and tests accompanying mathematics schemes were administered more frequently.

Calculators were used relatively infrequently during mathematics classes, with just 4.5% of pupils using them at all. The most common purposes for which calculators were used were checking answers (3.0% of all pupils) and routine computation (1.1%).

Across all classes, pupils received an average of 250 minutes of mathematics instruction per week (50 minutes per day). This ranged from 229 minutes in multi-grade classes to 268 minutes in single-grade classes. However, the average achievement of pupils in multi-grade classes was not significantly different from that of pupils in single grade classes.

Individual/ Family Variables

No gender differences in overall mathematics achievement were observed – a finding that also emerged for Irish pupils in Fourth class in the TIMSS (1995) study. However, the performance of boys scoring at the 10th percentile was significantly lower than that of girls scoring at this level. Similarly, teachers rated more males than females to be performing at a Second class, or First class or lower, level in mathematics.

The correlation between parents' highest educational level of educational attainment and mathematics achievement was 0.34, indicating a moderate association. The achievement of pupils whose parents had a medical card was significantly lower, by over one-half of a standard deviation (27.5 points), than that of pupils whose parents did not have a card. Other family variables associated with achievement in mathematics were family size (with children with 5 or more siblings doing less well than children with fewer siblings), and family structure (with children in lone-parent families doing less well than children not in such families).

Ninety percent of children received mathematics homework three or four times per week. Most children (59%) spent approximately 15-30 minutes on their maths homework. There was a negative correlation (-.27) between time spent on homework and achievement, suggesting that lower achievers spent longer on their homework. Pupils in disadvantaged schools received more parental support with their homework than pupils in non-disadvantaged schools.

Relatively few parents (9.2% in disadvantaged schools and 4.4% in non-disadvantaged schools) had participated in a programme aimed at helping their child's mathematics development. Parents of pupils with very low achievement were more likely to participate in such programmes than parents of higher achievers.

Recommendations

A number of recommendations for the teaching and learning of mathematics emerged from the 1999 assessment, including the following:

- Greater emphasis should be placed on the mathematics content areas of Measures and Shape and Space, and on higher-order mathematical processes such as Integrating and Connecting, Reasoning, and Analysing Problems and Evaluating Solutions
- Schools should organise programmes to familiarise parents with the content and processes underlying the mathematics curriculum and the school development programme in mathematics, and suggest ways in which parents can help in the development of children's mathematics knowledge.
- Teachers in schools and clusters of schools should work co-operatively to plan learning experiences and assessment activities in mathematics, in the context of preparing for the implementation of the revised Primary School Mathematics Curriculum.
- The Department of Education and Science should support schools in developing and implementing strategies to meet the needs of pupils with learning difficulties in mathematics, and in providing supplementary teaching in mathematics to pupils with very low achievement.
- The apparent decline in achievement in mathematics among Irish pupils between fourth class (primary level) and second-year (post-primary), revealed in the TIMSS 1999 study, should be investigated, and, if possible, addressed.

The 2004 National Assessment

The 2004 National Assessment of Mathematics Achievement of Mathematics Achievement in Fourth class took place in May 2004 – some five years after the revised Primary School Mathematics Curriculum had been introduced to schools, and almost two years after implementation had begun (in September 2002). The terms of reference for the study were similar to those for the 1999 assessment:

- Compare the performance of pupils in 2004 against the benchmarks established in the 1999 National Assessment of Mathematics Achievement in Fourth class.
- Examine the use of calculators by pupils in Fourth class
- Examine ways in which the teaching and assessment of mathematics have evolved since the introduction of the new Primary Schools Mathematics Curriculum

Hence, the assessment provided a first opportunity to look at the achievement in schools following initial implementation of the revised curriculum, bearing in mind that a longer time-frame may be needed before the full effects of the curriculum can be considered.

It is also relevant to note that a number of other studies, also conducted in 2004, might be expected to provide information on the teaching and assessment of mathematics in schools. These include a review of the implementation of the primary school curriculum (that included mathematics) by the National Council for Curriculum and Assessment (NCCA, 2005), and a study of numeracy (and literacy) in designated disadvantaged schools by the Inspectorate of the Department of Education and Science (DES, 2005).

Framework for 2004 Assessment

The 2004 assessment is based on the same framework that was used in the 1999 study. 114 of the 125 items used in the 1999 assessment were retained, and a further 11 items, intended to address aspects of the Primary School Mathematics Curriculum not fully covered on the 1999 test, were added to the test following a pilot study in February 2004. These covered topic areas such as: Place Value; Properties of Multiplication; Fractions; 2-D & 3-D Shapes; Lines and Angles; Measures; and Chance. The addition of items (and indeed blocks of items, see below) is facilitated by the use of IRT scaling procedures.

An additional block of 25 items was developed to assess the performance of pupils when they had access to a calculator. Items in the new block, also piloted in February 2004, were categorised according to the same content/process framework implemented in 1999. The breakdown of all 150 items used in the 2004 assessment is given in Table 4. The breakdown is broadly similar to that of the 1999 assessment (Table 2), though there is a small increase in the proportion of items that assess Analysing and Solving Problems and Evaluating Solutions, and a small decrease in the proportion on that assess Understanding and Recall.

Table 4: Weightings Assigned to Mathematics Content Areas and Processes – 2004 assessment

Content Area	Percent of Items	Process	Percent of Items
Number	38.7	Understand and Recall	12.7
Algebra	4.7	Using Strategies and Implementing Procedures	28.0
Shape and Space	14.0	Reasoning	20.7
Measures	32.0	Integrating and Connecting	6.7
Data and Chance	10.7	Analysing and Solving Problems and Evaluating Solutions	32.0
Total	100.0	Total	100.0

The 2004 Test of Mathematics Achievement

As indicated earlier, the test used in 2004 was broadly similar to the 1999 test, with the addition of a calculator block. It was decided to add a calculator block to the test, as a number of content statements for Fourth class in the Primary School Mathematics Curriculum (NCCA, 1999a), particularly in the Number strand, focus on the use of the calculator. They state that the child should be enabled to:

- Check estimates (of sums and differences) with and without a calculator
- Use a calculator to develop problem-solving strategies and verify estimations
- Use a calculator to check estimates (multiplication and division)

The curriculum states that calculators help the development of problem solving skills by allowing children to focus on the structure of problems, and possible means of solutions (NCCA, 1999a, p. 7). It views calculators as being useful for performing long and complex computations, and providing exact results to difficult problems.

They types of items included in 25-item calculator block on the 2004 test included the following:

- Identify the number in a number sentence that should be left out to make the number sentence correct (e.g., 175 + 236 + 318 + 240 = 733)
- Supply a missing digit to make a number sentence correct (e.g., 4_5 ÷ 9 = 45)
- Indicate the missing operations to make a number sentence correct (e.g., 27 __ (31 ___ 11) = 550)

- Identify the next number in a sequence (e.g., 4.2, 8.4, 16.8, ?)
- Find the perimeter of a field where the length and width are decimal numbers
- Find the cost of a toy, if its price is reduced by ¼ in a sale.
- Compare the performance of athletes over two rounds of a high-jump competition, where heights are presented as decimal numbers (e.g., 1.95 m).
- Solve one- and two-step problems where measures (money, length, weight, capacity) are presented as decimal numbers (e.g., given the combined weight of two identical boxes of biscuits and a box of sweets, and the weight of a box of biscuits, find the weight of the box of sweets).

In analysing performance on the test, it is intended to pool the calculator items with the other 125 items to arrive at overall scores, while also looking at performance on the calculator items themselves. An important issue in this context is the extent to which pupils benefit from the availability of a calculator, which may, in turn, be related to the level of practice they have had with calculators in mathematics classes, and their teachers beliefs about the value of using calculators in mathematics classes.

The six item blocks (five from 1999, with minor adjustments, and the new calculator block) were assembled into five booklets, as shown in Table 5. As in 1999, all pupils were asked to attempt a common block (B). Apart from the common block (which was always in the middle), each block appeared at the beginning and end of a booklet. As in 2005, pupils were asked to attempt 75 items, as each block contained 25. Since each pupil attempted one booklet (assigned at random), two-fifths of the sample attempted the calculator block. Calculators were available only to pupils attempting the calculator block, and only for the duration of that block (30 minutes). Total testing time was 90 minutes.

Table 5: Structure of Test Booklets – NAMA 2004

Booklet	**Blocks**
1	A, B*, C
2	C, B, D
3	D, B, E
4	E, B, F**
5	F, B A

*Common block – administered to all pupils

*Calculator block – included in Booklets 4 and 5 only

Questionnaires Administered in Conjunction in the 2004 Assessment

As in 1999, a suite of questionnaires was administered in conjunction with the 2004 assessment. Of particular interest in the context of implementation of the revised

Primary School Mathematics Curriculum is the Teacher Questionnaire, which asked teachers about the following topics:

- Allocation of time to the teaching of mathematics
- Frequency of usage of computers, calculators, and concrete materials in mathematics classes
- Purposes for which calculators and computer software are used in mathematics classes
- Emphasis on different mathematics content areas and processes during instruction
- Frequency of using various assessment tools, including standardised tests

Additional questionnaires administered in conjunction with the study included:

- a School Questionnaire, completed by school principals
- a Pupil Questionnaire, completed by pupils
- a Parent Questionnaire, completed by pupils' parents
- a Pupil Rating Form, completed by pupils' teachers; and
- a Questionnaire for Inspectors, completed by members inspectorate with responsibility for primary schools

Sample Design and Survey Implementation

Unlike 1999, the 2004 assessment was administered in conjunction with assessments of reading in the First and Fifth classes in the same schools. Prior to sampling, all schools on the database of primary schools were categorised according to whether they were large (35 or more pupils in Fifth class[1]), medium (21-34 pupils) or small (fewer than 21 pupils). Schools were also classified according to whether they had pupils in the First, Fourth and Fifth classes, or whether they had pupils in the Fourth and Fifth classes, but not in First class. Hence, six strata were established: large, medium and small schools with First, Fourth and Fifth classes; and large, medium and small schools with Fourth and Fifth classes, but not First. Within these strata, schools were sorted by designated disadvantaged status, area/language of instruction (Gaeltacht, Scoil lán-Ghaeilge, Ordinary School), proportion of female pupils, and measure of size, to ensure a representative mix of school types.

The second stage of sampling involved the selection of pupils within participating schools. All Fourth Class pupils in schools which had two or fewer Fourth classes were

[1] School size was based on number of Fifth class pupil in a school, as the same sample of schools was drawn for the 2004 National Assessment of Reading in Fifth class.

invited to participate. For schools which had more than two Fourth classes, two of these were selected at random, and the school was informed of the classed selected.

All pupils were asked to participate, except pupils with a moderate to severe general learning disability, a severe physical disability, or very low proficiency in the language of the test (English or Gaeilge) and less than one year of instruction in that language. Pupils in receipt of learning support in mathematics and pupils with a mild general learning disability were expected to participate.

Of the 136 schools invited to participate in the mathematics component (involving Fourth class), 130 agreed to so, giving a response rate at the school level of 95.6%. Within participating schools, 93.1% (4171 pupils) completed the Test of Mathematics Achievement. Response rates for the questionnaires within participating schools were uniformly high: School Questionnaire: 99.2%; Teacher Questionnaire: 99.5%; Parent Questionnaire: 93.8%; Pupil Rating Form: 99.6%; and Pupil Questionnaire: 96.3%.

Looking Ahead

Scores were scaled using Item Response Theory methodologies. The overall performance of pupils was placed on the same scale as the 1999 survey, allowing for a comparison of achievement with 1999.

Comparisons by content areas and mathematics process were made on items common 1999 and 2004 assessments only. Mean percent correct scores for both years were compared, using tests of statistical significance.

In addition to comparing overall achievement, and achievement by content area / process across the two assessments, the report on the study looks at the performance of children in at-risk groups between 1999 and 2004, including pupils attending schools designated as disadvantaged, members of the travelling community, and pupils who are not native speakers of English.

In the report, the responses of principal teachers, class teachers parents and pupils to key questionnaire items around the teaching and learning of mathematics are addressed, and, where appropriate, are linked to achievement.

Discussion

The results of the 2003 OECD Programme for International Student Assessment (PISA) indicates that the overall performance of Irish post-primary students (15-year olds) is average by international standards, while performance is above average on two subscales (Change and Relationships, Uncertainty), about average on one (Quantity) and below average on one (Space and Shape) (Cosgrove et al., 2005). Overall, performance in mathematics on the assessment can be considered disappointing, since Ireland performed well above the OECD average in reading and science in the same assessment.

While national assessments of mathematics achievement at primary level may not tell us very much about the reasons underlying the disappointing performance on PISA mathematics, they nevertheless serve an important function in highlighting strengths and weakness in pupil performance, as well as aspects of teaching and assessment that may need to be examined in further detail. It is instructive in this context to note that the recent study of implementation of the revised Primary Schools Mathematics Curriculum (NCCA, 2005) did not look directly at the effects of the curriculum on achievement. It is to be hoped that the programme of national mathematics assessments will, in the future, chronicle the effects of curriculum change and other changes in the educational system on pupil achievement in mathematics.

The national assessment programme also serves to highlight groups of pupils who are particularly at-risk of mathematics difficulties. The performance of pupils in designated disadvantaged schools in the 1999 assessment is a case in point. In addition to highlighting the performance of these and other at-risk pupils (e.g., members of the traveller community, pupils for whom English is not a first language) at specific points in time, the national assessment programme can track the performance of these pupils over time. However, in future assessments, it may be necessary to increase the representation of at-risk groups in order to generate meaningful data about their performance.

It has to be acknowledged the national assessment programme has not, to date, been successful in assessing pupils' ability to 'communicate and express the processes used in mathematics activities and explain the results'. Clearly, this important skill cannot be assessed using a paper and pencil assessment such as the tests employed in the 1999 and 2004 assessments. There may be some value in considering ways in which this important skill can be measured in future assessments, perhaps through the use of performance-based assessments (see, for example, the performance assessments implemented as part of TIMSS 1995 – Harmon et al., 1997).

Finally, it would seem important to identify ways in which schools and class teachers can engage with the outcomes of the national assessments of mathematics achievement, so that they can begin to impact on teaching and learning in classrooms.

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Children's Understanding of Place Value

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The importance of number sense has been highlighted in mathematics literature and research around the world. An understanding of the concept of place value has been identified by many as the key to children's success in developing number sense. The area of interest of the current research project is 9 & 10 year-old Irish children's understanding of place value. This research is particularly relevant at present with the recent introduction of the Primary School Curriculum –Mathematics (NCCA 1999) that emphasises children's understanding of mathematical concepts. This paper will focus on a review of international research related to the topic, and look at the Irish context. It will conclude by raising questions that need to be addressed in relation to the teaching of place value in Ireland and report on the proposed Irish-based research project.

Introduction

Good number sense has been identified as an important factor in the development of an adult who is numerate. The importance of number sense has been identified in mathematics literature around the world. In the U.K., *Mathematics Counts* describes number sense as 'at homeness with numbers' (DES 1982, 39). In the United States, The National Council of Teachers of Mathematics (NCTM 1989) stresses number sense as an essential outcome of school curriculum. An understanding of the concept of place value is the key to children's success in developing number sense. The NCTM states that "Understanding place value is another critical step in the development of children's comprehension of number concepts" (NCTM 1989, 39). Sharon Ross contends that "Understanding place value is important to achieving good number sense, estimating and using mental mathematical skills, and understanding multi-digit operations" (Ross 2002, 423). In countries such as Ireland, England and the United States, the mathematics curriculum places considerable emphasis on the teaching of place value, while countries such as The Netherlands make little explicit reference to place value in their curriculum, but look at numbers in a more holistic way.

Extensive research has been carried out around children's understanding and development of place value. Many different perspectives have been considered worldwide. The inconsistency of the English language system of written and spoken words, as well as the more consistent Asian language systems has been examined both in large-scale comparative international studies and in small-scale qualitative studies. Teaching methodology, analysis of place value development, curricular placement, use of materials and textbook presentations have also been studied in the attempt to better appreciate children's understanding of place value.

Language Systems

In the late eighties and early nineties, Karen Fuson and Diane Briars (1990) looked at the structure of the English language's spoken system for number words ('named value') in contrast to the written system ('positional base-ten'). They point out that in the English language system when numbers are written the value of each digit is implied by its position, where as when it is spoken the value is stated, 2, 679 as opposed to two thousand, six hundred and seventy nine. They also refer to the fact that some two-digit numbers do not follow this named-value structure. In the number 16 the unit value is said first and then the tens value. These aspects make it difficult for English speaking children to grasp the concept of place value. Fuson and Briars argue that children need classroom help to facilitate their understanding of this inconsistent system. They recommend the use of Dienes base-ten blocks for teaching addition and subtraction, but suggest that they be used only after children can work with number bonds to 18.

Fuson (1990a) also suggests that English speaking U.S. children may need substantial and extended support for "constructing multiunit meaning based on ten" due to the English language and U.S. culture's lack of adequate support for such meanings. She cited the use of the metric system of measurement and the use of the abacus as cultural advantages children from Asian countries have over English speaking U.S. children Thompson (1998) further investigated Fuson & Briar's recommendations using the dataset of the mental calculation strategies of 103 children from Year 2 and Year 3. His findings agree with Fuson and Briars, but he suggests that they ignored the strategy of partitioning and he recommends that when dealing with calculations of two-digit numbers we focus on "the 'partitionable' aspect of our system of counting words" (Thompson 1998, 6). He argues that the model of teaching place value needs to relate, "more closely to the language used to express two-digit numbers in the English spoken system of number words" (Thompson 1998, 1).

Miura et al (1994) compared the cognitive representation of number of children from China, Japan and Korea with children from France, Sweden and the United States. They suggest that Asian students' success over others in mathematics achievement may be, in part, affected by the "numerical language characteristics" of the Asian languages where "the spoken numerals in these Asian languages correspond exactly to their written form." (Miura et al. 1994, 403). The number 15 is named one ten five and number 34 is named as three ten four. They found that Asian-language speakers represented numbers in different ways than non-Asian speakers and were also better able to represent numbers in different ways using Base 10 blocks, which they claim "suggests greater flexibility of mental number manipulation". (Miura et al. 1994, 411)

Yang and Cobb (1994) compared the development of place value concepts of children in Taiwan and the United States and found similar results to Miura et al., but attributed the results to "deep-rooted cultural beliefs that extend beyond the classroom" (Yang & Cobb 1994, 30)

Saxon and Towse (1998) conducted a study with six and seven year olds (some English-speaking and some Japanese-speaking) and they found that the influence of language is not as important as suggested by previous studies. They discovered that slightly adjusting the task instructions and demonstrating the use of ten cubes in practice trials eliminated the differences between Japanese-speaking and English-speaking children.

While this study would need to be replicated with other Asian languages to support their findings that it might be the students' understanding of the research question that is in question here rather than his or her understanding of multi-digit numeration. However the inconsistencies in the English number words is certainly confusing for children trying to understand the number system especially with teen numbers. While the difference in language structure may be a contributing factor in explaining Asian's children's superiority in international mathematics surveys, many other factors may also be at play here as Asian students also excel in other subject areas (TIMSS 1995).

Teaching Methodology and Materials

Other researchers investigating children's learning of the concept of place value have adopted a teaching methodology approach. Fosnot and Dolk (2001) investigated the teaching of computation through an in-service project in New York City. Working in teachers' classrooms they studied children's working with numbers. They argue for the need for placing numbers in context and describe how "as they encounter larger groups of objects, they begin to find ways to organize their counting with landmark numbers such as five and ten" (Fosnot & Dolk 2001, 64).

Kamii and Joseph (1989) advocate not teaching place value as a separate activity from operations and not teaching any standard algorithms but allowing children to develop their own and encouraging children to agree or disagree with each other. They argue that understanding place value requires mental construction rather than the construction of groups or bundles of ten with materials. " The system of tens has to be constructed by children, in their heads, on *their* system of ones, through constructive abstraction" (Kamii & Joseph 1989, 30). Using further studies, Kamii and Livingston (1994) provide evidence that children forget what they have learned about place value and develop poor number sense when they are taught traditional algorithms. They contend that "children who are allowed to do their own thinking thus strengthen and extend their knowledge of place value by using it" (1994, 35). Many would agree with allowing children to develop their own algorithms but would disagree with Kamii and her colleagues about the use of manipulatives. The mental construction that they contend that children need can be facilitated by children building groups with many different materials themselves as researched by Fuson & Briars (1990), Ross (1999) and numerous other researchers.

Hiebert & Wearne (1992) researched the links between methods of teaching and children's understanding of place value in first grade. They compared the effects that conceptually based instruction and textbook-based instruction had on children's learning of place value and two-digit addition and subtraction without regrouping. They found differences in how teachers used materials, the time spent per problem, how children responded and the coherence of the lesson. They found that even though less time was spent on conventional procedural drills by the conceptually based instruction groups it "did not adversely affect students' proficiency on routine problems".

Earlier studies carried out by Ross (1989) had shown that at 4th & 5th grade only half of the students were able to complete digit-correspondence tasks successfully. In 1999 Ross looked at 3rd – 5th grade students' understanding of two- and three-digit numbers and the effects on this of digit-correspondence lessons in which children were encouraged to talk and write about their ideas. In the pre-assessment tests only 19% of the 69 students were successful. 70% of those who were unsuccessful in the pre-assessment were successful after experiencing the digit-correspondence lessons, which included group work and whole class discussion.

Textbooks and Curricular Placement

Fuson (1990) looked at the characteristics of textbook presentations and curricular placement of place value and multidigit addition & subtraction. She argued that they were partly to blame for U.S. children's inadequate multiunit conceptual structures and she proposed five new characteristics to replace the old ones. Her first two suggest that multidigit conceptual structures are not necessary for the reading and writing of two-digit numerals or for the addition and subtraction of numbers less than twenty and so this learning can be based on unitary conceptual structures. Her third characteristic suggests that much of the understanding of the concept of place value can be built within the multidigit addition and subtraction work instead of prior to as textbooks recommend. She goes on to say that it may be that some of the mature conceptual structures cannot be appreciated until multidigit multiplication is understood. Fuson's fourth characteristic states that multidigit addition and subtraction work should be started in second grade or whenever children are ready to start building multiunit conceptual structures and that all "possible combinations of trades are done from the beginning" (Fuson 1990, 274). Finally her fifth characteristic recommends that children be afforded adequate support for constructing multiunit conceptual structures. Arthur Baroody (1990) agrees with Fuson that textbook presentations of place-value don't help children build multi-unit concepts and he accepts most of her new characteristics. However, he argues that it is not clear if it is necessary to have all addition and subtraction of single digit numbers precede addition and subtraction of multidigit numbers.

Mental Arithmetic

Thompson and Bramald (2002) investigated the relationship between young children's understanding of the concept of place value and their competence at performing mental addition. They adapted tasks used in place value studies in the U.S. and Australia by several researchers. Thompson and Bramard compared the results of this task from their study with the results from studies carried out by Ross (1989) and Kamii (1988) in the U.S. and Price (1998) in Australia. British children were substantially more successful in this task than their American and Australian peers having a 77% success rate in Year 3 and 79% in Year 4. Thompson attributes this partially to the introduction of the National Numeracy Strategy and its emphasis on mental arithmetic and the delay in introducing formal written calculations until children reach Year 4 (Grade 5) and also to the use of Gattegno tables and place value cards in U.K. classrooms. Thompson and Bramard identify the need for further research to explain UK children's apparent superiority. As the American studies used as comparisons in the U.K. study are quite dated it doesn't seem as if Thompson and Bramard are comparing like with like. If the American studies were replicated today different results might be achieved. With the 1989 publication and implementation of the NCTM's *Curriculum and Evaluation Standards for School Mathematics,* classroom practices in Mathematics changed in many American schools with emphasis on mental calculations and discussion similar to those in the National Numeracy Strategy. This premise would be supported by the results of the classroom study carried out by Ross (1999) referred to above. While the Australian study was more recent it was based on a sample size of 16 students as opposed to 144 students in the U.K. study. The Australian study would need to be replicated on a larger scale to do a fair comparison. These results would not be consistent with Australia's superiority over England in the third grade international mathematics surveys. It would be interesting to compare the results of a replica study carried out in Ireland given Ireland's performance on these international surveys (TIMSS 1995).

Irish Context

One of the broad objectives of the Primary School Curriculum Mathematics (NCCA 1999) is that children "understand, develop and apply place value in the denary system (including decimals)" (NCCA 1999, 13). Place Value is first introduced in first class with the objective that "The child should be enabled to explore, identify and record place value 0-99" (NCCA 1999, 41) It is suggested that children "group and count in ten and units" using a variety of materials. In second class it is extended to 199. It suggests that children should be enabled to rename numbers and represent numbers in different ways. In third class it is extended to 999 and one place of decimals, and in fourth class to 9999 and 2 places of decimals. The significance of 0 as a placeholder is not emphasised until 3rd class. Much emphasis is placed on the use of groups of objects, base ten materials, notation boards, money, number lines, hundred squares and the abacus.

There is little emphasis in the curriculum of the use of calculators in helping children understand numbers and place value as it is recommended that calculators be used from fourth to sixth classes "by which time the child should have a mastery of basic number facts and a facility in their use" (NCCA 1999, 7). There is no mention of the use of calculators in discussing the teaching of place-value; in fact it is not until operations that the calculator is suggested. This is in contrast to the recommendations of the NCTM (2000, 81)) which states that "Place-value concepts can be developed and reinforced using calculators" and suggests activities for use with 2nd grade (2nd class) children. This is echoed by Williams and Thompson (2003, 165) who suggest that the "calculator is invaluable in developing the young child's understanding of place value".

In the recent *Primary Curriculum Review-Phase 1* the majority of teachers at 3rd to 6th class cited 'operations' and 'place-value' as the strand units they found most "useful in their planning for, and teaching of Mathematics" (NCCA 2005, 136). While it is not clear what exactly was meant by 'useful' in this context it perhaps suggests that operations and place-value are the strand units that teachers see as most important. In light of this it is interesting to note that many teachers also identified number as the area they would like to prioritise in furthering their own implementation of the Mathematics Curriculum. When asked how often children in their class were enabled to use estimation strategies in number, approximately 60% of teachers responded less than once a week for front-end strategy, approximately 37% responded less than once a week for rounding strategy and approximately 57% responded less than once a week for clustering. These are all strategies that build on and help develop understanding of place-value and number (NCCA 2005, 139).

Irish primary mathematics textbooks put considerable emphasis on place value with pictorial representations of grouping into tens of different materials. This is extended into pictorial representation of base 10 materials (Dienes blocks) and money. There are many examples of the use of an abacus and hundred squares in representing place value. However number lines were rarely used other than to demonstrate rounding up and down.

Both the *Primary School Curriculum* and the textbooks examined (Folens 2002, Fallons 2002) emphasise what Thompson (2000) calls ‘column value’ as opposed to ‘quantity value’ in place value. He relates column value to seeing 64 as 6 tens and 4 units and quantity value as 60 and 4. Column value is emphasised in the use of notation boards, abacus and standard algorithms while quantity value is more identified with mental arithmetic, informal written strategies and empty number lines. Thompson (2003) argues that both are necessary and that teachers need to “ensure that connections are made between quantity value and column value” (Thompson 2003, 188).

So what questions does the worldwide research pose for the teaching and learning of place-value in Ireland? How would Irish children compare in the international studies to Asian speaking children? Are Irish children given adequate time to explore and develop their understanding of place value using materials? Should we follow Kamii’s (1994) suggestions and move away from materials or the Dutch example that places little emphasis on materials and formal place-value teaching. Do we introduce place-value too early? Should we place more emphasis on mental arithmetic that encourages children to be more flexible with numbers? Do we rely too much on the textbooks and formal algorithms?

The starting point would seem to be to ascertain whether Irish children have a good understanding of the concept of place-value or not. It is only then that we can look at our teaching and learning and ask these questions.

Research Question and Research Methodology

Building on previous research, this study aims to investigate the question ‘Do children in 4th class in Ireland have a good understanding of the concept of place-value?’

Fourth-class children were selected for this study, as they would have explored the concept of place-value after the introduction in schools of the *Primary School*

Curriculum (NCCA 1999). As it will be qualitative research a small sample of children will be taken from 12 schools selected nationwide to reflect urban, rural and suburban populations as well as different socio-economic backgrounds. Students will be selected at random. Clinical interviews will be used to gather information on children's understanding of the concept of place-value. Each child will be given a series of tasks designed to probe their understanding of place-value. Some tasks will be adapted from those used by other researchers (Ross 1986; Kamii & Joseph 1989; Fuson & Briars 1990; Thompson 2002). All interviews will be videotaped for further analysis. The activities will be created with reference to the expected outcomes for place-value at 3rd class level in the *Primary School Curriculum – Mathematics* (NCCA, 1999).

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An Exploration of Self-Reported Strategy Use for Mathematical Word Problems by Sixth Class Pupils in Irish Schools

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The purpose of this pilot study is to explore the self-reported strategy use for word problems in mathematics of sixth class pupils (aged 11-13) in Irish schools. Possible differences in self reported attitudes and practices between pupils in terms of gender and school location (disadvantaged area or not), access to technology are examined. Statistically significance differences were apparent between the 6th class boys and girls in their reported use of strategies with girls reporting more persistence; and between the schools, with the pupils in the disadvantaged area reporting less use of meta-cognitive strategies. All pupils reported low use of the calculator, concrete materials and drawing diagrams, with trial and error the most common strategy used.

Introduction

The purpose of this pilot study is to explore the self-reported strategy use for word problems in mathematics of sixth class pupils (aged 11-13) in Irish schools. Possible differences in self reported attitudes and practices between pupils in terms of gender, school location (disadvantaged area or not), and access to technology will be examined. The focus of the study is on cognitive and metacognitive strategies used to solve word problems. Firstly, operational definitions of cognitive and meta-cognitive strategies are presented. This is followed by a review of the literature in terms of what strategies have been validated as aiding successful word problem solving in mathematics. The methodology section presents appropriate measures for a self-reporting format of data gathering for such strategies. Finally, the results are presented and analysed.

Cognitive and meta-cognitive strategies

Lerner defines cognitive strategies "as an individual's approach to a task; it includes how the person thinks and acts when planning, executing and evaluating performance on a task and its outcomes" (Lerner, 1997, p507). Children are encouraged to talk to themselves and to ask themselves questions about mathematical problems. It is closely related to metacognition, defined as, "emphasising self-awareness of how one approaches a task in order to plan, evaluate, and monitor progress" (Bley & Thornton, 1995, p. 21). Montague (1997, p165) distinguishes between the two as follows:

"metacognition facilitates the selection and allocation of techniques and strategies for successful task completion." It includes self-questioning, self-instruction and self-monitoring (Montague, 1997).

Recent developments in assessment stress the importance of pupils engaging in self – assessment as an aid to internalising the beliefs and practices associated with successful performance in a particular area of learning: "Making learning targets clear to students and improving student achievement by inviting them into the assessment process- are not a flash in the pan. This is the future of assessment in education" (Arter and McTighe, 2001, p36). This is closely linked to the development of metacognitive learning strategies where pupils become more aware of how they learn. Traditionally we have stressed content oriented learning targets. This study seeks to induce pupil reflection on their use of process strategies. The revised primary curriculum in mathematics in Ireland as in other countries stresses the key role of developing problem solving skills in all pupils and the importance of learning strategies (NCCA, 1999).

Mathematical problem solving

Xin and Jitendra (1999) in a meta-analysis of word-problem-solving intervention research with samples of pupils with learning problems examined 25 outcome studies across student characteristics, instructional features, methodological features, skill maintenance and generalisation components. Computer –assisted instruction was found to be most effective in terms of effect size. These studies mostly entailed strategy or representation techniques presented via tutorial or videodisc programs. These were found to be particularly effective "when empirically validated strategies and curriculum design principles are incorporated" Xin and Jitendra (1999, p218).

Representation techniques were the next most effective intervention. Not all of these studies were as effective as each other. Simply telling students to draw a diagram to represent the problem failed to make explicit the key relationships between the various parts of the problem. Xina and Jitendra (1999, p218) argue that "it is important that students develop conceptual understanding in making representational links to be successful problem solvers."

Strategy training was found to be a moderately effective intervention. This entailed a combination of approaches including direct instruction or self –regulation of particular strategies, such as, paraphrasing, visualising, hypothesising, or estimating. These were categorised as relating to how to solve a problem. In addition, metacognitive strategies relating to *knowing how* to solve a problem, such as self-instruction, self-questioning, and self –regulation were also included under strategy training.

Other studies found to be the least effective in aiding problem solving included those which focused on attention to key words only. This approach is often advocated in mathematics textbooks (Xin and Jitendra, 1999). The difficulties with this approach are very well illustrated by Haylock (1991) when he recounts how attention to key words often cues pupils to press the wrong key on the calculator. For example, focusing on the word "more" in the following question: There are 11 boys in the class and 10 girls. How many more boys than girls are in the class? Haylock argues that the problem "is that children tend to respond in a stimulus-response fashion to key words in questions rather than to the logical structure" (Haylock, 1991, p142). Askew (2003, p78) comments that "there must be a lot of wrong keys and rusty locks out there."

The use of diagrams, the calculator and attention to language structure

A study by Hegarty and Kozhevnikov (1999) distinguished two types of visual-spatial representations in relation to mathematical problem solving. The first is pictorial representations that encode the visual appearance of the objects in the problem. The second is schematic representations that encode the spatial relations outlined in the problem. Such a representation is usually a diagram showing the spatial relations between the objects in a problem as against a picture of the objects or persons referred to in the problem. In their study, the use of schematic spatial representations was associated with success in mathematical problem solving while the use of pictorial representations by children was negatively correlated with success. This is particularly interesting in the light of the frequently cited advice of "draw a picture of the problem" as aid to its solution.

Booth and Thomas (2000) in a study on the role of visualisation in problem solving also draw a distinction between pictures and diagrams and they highlight the role of spatial ability as being " at the heart of meaningful interpretation in arithmetic problem solving and its presence in individual students should not be taken for granted"(p186). The use of concrete or cybernetics materials on their own may not be enough to compensate if spatial deficits are present. They argue that explicit verbal explanations and descriptions may be necessary so students can make the required links (Booth and Thomas, 2000).

Haylock (1991) argues that the most important objective for numeracy, particularly for low attainers, is to know what calculation should be entered on a calculator in order to solve a practical problem, for each of the four operations, using any of the models of the operation, and to be able to interpret the answer. The key role of the calculator in problem solving is the opportunity it allows the student to focus on the structure of the problems and not on the mechanics of the arithmetic. In this way it is a real aid to understanding (Haylock, 1991). The revised mathematics curriculum advocates the use of the calculator from fourth class upwards (NCCA, 1999).

This emphasis on the structure of the problem draws attention to the importance of pupils understanding the language used. Westwood (2003) contends that this is basically a reading comprehension problem. Newman (1993) found that 50% of errors in word problems in mathematics were due to language factors. These included the pupil being unable to read the words; reading them but not knowing what they meant; knowing what they meant but not able to translate them into the action required to solve the problem or not knowing how to express the answer. The necessity for pupils to have strategies to deal with such issues then arises.

The above research has led to curriculum developers stressing the necessity of pupils being exposed to and using a wide variety of problem solving strategies: "Problem-solving strategies must be varied and the children given ample opportunity to try them out concretely, orally or in a written task" (NCCA, 1999, p36). This study seeks to elicit the self-reported use of the above problem solving strategies by sixth class pupils.

Methodology

A self-reporting format was chosen for the study as it was felt that all of the strategies could be presented in understandable pupil-friendly language. In addition, recent approaches to mathematics and assessment stress pupil reflection on strategies, justification of strategies, sharing of strategies and engaging in self-evaluation (Arter and Mc Tighe, 2001). Such expectations are now part of the revised mathematics curriculum (NCCA, 1999). It is acknowledged that a more comprehensive picture could be gleaned by combining a self-report format with observations of pupil problem solving together with a probing diagnostic interview. However, a self-reporting questionnaire gives a useful insight into what pupils actually think they do and of their ability to engage in self –reflection.

In order to gain measures of the different approaches and pupil behaviours in relation to word problem solving it is necessary to categorise these and outline a list of statements that attempt to measure each one. Cognitive strategies associated with positive achievement in problem solving were gathered from an examination of the literature in this area. Suggested validated strategies include rereading the problem, attention to the language structure, drawing a diagram, estimating, use of concrete materials, making a table, using a calculator, simplifying numbers, trail and error and checking your answer (Westwood, 2003; NCCA, 1999; Xin and Jitendra, 1999; Haylock, 1991). Given the complex nature of problem solving this list cannot claim to be exhaustive but it does incorporate the suggested strategies advocated in the revised mathematics curriculum (NCCA, 1999).

In addition, the metacognitive strategies found to be successful in problem solving were collated. These were grouped under the general idea of self-questioning and self-regulation during the problem solving process (Xin and Jitendra, 1999; Montague, 1995, 1997).

Appropriate measures for each of these were then devised to gather data incorporating the self-reported use of these strategies by sixth class pupils (aged 11-13) in a convenience sample of Irish primary schools. The above mentioned cognitive and metacognitive behaviours associated with problem solving were written in pupil-friendly language. Each strategy has at least two measures except for making a table.

The following is a list of the strategies and the measures used:

1. reading the problem at least twice: *I read the problem at least twice; I read the problem a few times to make sure I know what to do.*
2. looking at the logic of the language of the problem: *I ask myself 'what does this mean?'; I ask myself 'do I need to add, subtract, multiply or divide?'; I make sure I understand all the words in the problem; I try and put the problem in my own words; I underline the important information; I underline the key words in the problem.*
3. drawing a diagram: *I draw diagrams (example shown to distinguish from a picture) to help me solve word problems; I put numbers in the diagrams I draw.*
4. simplifying the numbers: *I replace the numbers with simpler numbers first; I round the numbers up or down.*
5. making a table: *I make a table (example shown) to help me solve some word problems.*
6. estimating: *I estimate what the answer might be;*
7. trial and error: *I use a rough work page first to solve the problem; I make lots of attempts to try and solve the problems; If my answer doesn't make sense I try again.*
8. using concrete materials: *I use materials like blocks and counters to help me solve the problems; I use my fingers to help me solve problems*.
9. checking the answer against the estimate: *I check my answer to make sure its right; I check my answer against my estimate.*
10. use of the calculator for computation to free up the pupil to focus on the logic of the problem: *I use a calculator when I am solving word problems; I think that solving word problems is easier when I use a calculator.*

11. self- questioning and self-regulation: *I ask myself questions as I solve problems; I ask myself 'does this make sense?'; I ask myself 'what do I need to do first?'; I ask myself 'what do I need to do next'?; I ask myself 'what am I being asked to find out?'; I ask myself 'does the diagram fit the problem?'; I ask myself 'Is my answer right?' I ask myself 'is there another way of doing this?' I ask myself 'what does this mean?'; I ask myself 'do I need to add, subtract, multiply or divide?'; Before I estimate I ask myself "will the answer be bigger or smaller?";(last three in more than one category)*

Because of their role in successful intervention studies, information on access to computers and use of mathematical software and calculators was sought. Finally information on pupil gender, age and location of the school was also sought.

These were then presented in random fashion on a questionnaire as statements to respond to in terms of how frequently pupils used certain strategies while solving word problems in mathematics (Appendix A). Following piloting some adjustments were made. There was some confusion in relation to the use of mathematical software, as to whether this related to home, school or both. This question was changed to allow all of these options in responding.

A convenience sample was used for this pilot study consisting of two classes in a school in an area of socio economic advantage and two classes in an area of socio economic disadvantage. All of the pupils were in sixth class, the top grade of primary school in Ireland. Each of the schools was co-educational. Permission was sought and received from the school principal, class teacher and parents. Pupils were also invited to participate.

The questionnaire was administered orally by the researcher to each class to ensure all pupils understood the statements and that pupils with reading difficulties were not precluded from responding. A sample of a word problem was first put up on a chart so that everyone was clear as to what it meant. All pupils expressed immediate recognition of the type of problem. Pupils were reminded that it was not a test and that there were no right or wrong answers only honest answers as to how they went about solving mathematical word problems. Pupils were also encouraged to ask if they did not understand a question. Feedback from pupils suggested they understood what was involved. Two charts of a diagram and a table were also used as examples of each and to distinguish them from a picture.

Analysis of the pupil responses centred around finding patterns of strategies which pupils reported using very often and often, and those which they reported using only a few times or never. In addition, the data were analysed in terms of possible differences in self-reported strategy use by pupils based on the location of their school (an area of socio economic disadvantage or not) and in terms of gender. Finally a factor analysis was conducted on all of the statements to help identify any underlying clusters of self-reported pupil behaviour.

Findings

The questionnaire was completed by 92 pupils across two schools, being all of the pupils present in sixth class on the day of administration. Fifty one percent of the pupils were from a school located in an area of socio-economic advantage and 49% from a school located in an area of socio-economic disadvantage. Fifty one percent were girls, leaving 49% boys. Ninety nine percent of the pupils were aged between 11 and 13 (Appendix A).

In terms of access and use of technology, 80% have their own calculator and 85% have access to a computer at home. Sixty percent say they do not have mathematical software, while 70% use mathematical software in school with 16% doing so at home (Appendix A).

Looking at the frequencies of overall self-reported strategy use, the trial and error strategy of “trying again if my answer doesn’t make sense” was the most frequented cited with 75% of respondents indicating that they use it often or very often. Also very high in terms of cited usage, combining the frequencies for “often” and “very often” include: I check my answer (74%); I make sure I understand the words (64%); I read the problem a few times (63%); I ask “is my answer right?”(62%); I ask myself “what does this mean? (62%); and I ask myself “what do I need to do first” (61%) (Appendix A).

Strategies which were cited as being used least often, combining “ a few times” and “never” include: I use materials like blocks and counters to help me solve problems (95%); I make a table (94%); I draw diagrams (90%); I use a calculator (89%); I underline the key words in the problem (82%); and I underline the important information (80%) (Appendix A).

Figure 1. Graph showing the different responses of pupils based on school location to use of fingers as a strategy when problem solving

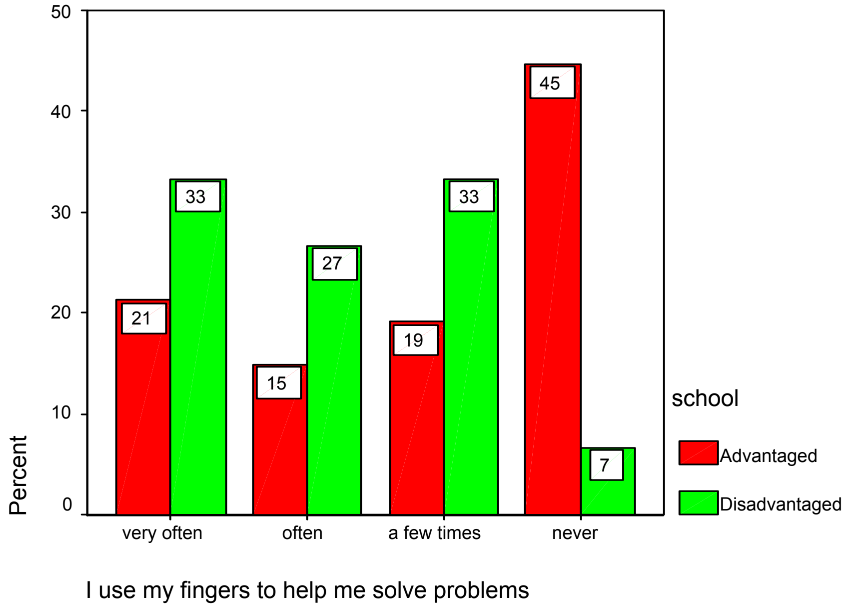
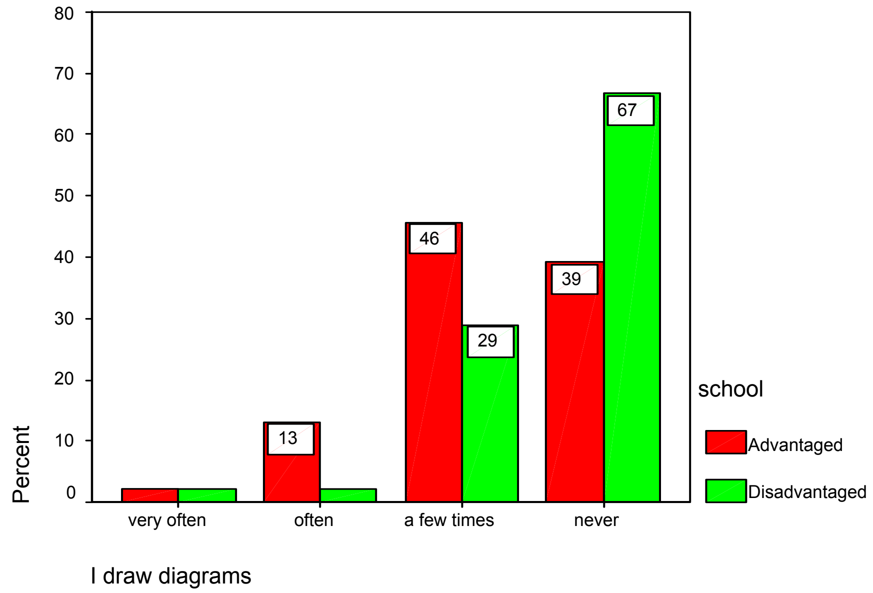


Figure 2. Graph showing the different responses of pupils based on school location to drawing diagrams as a strategy when problem solving



Possible differences between pupils based on location of school and gender were then explored. Differences in relation to use and access to technology were not appropriate as the sample size was too small for those without access. Tables 1 and 2 show the results of the Mann-Whitney test for areas of difference that were statistically significant. Pupils from the school in the area of socio economic disadvantage were more likely to report using a rough work page and their fingers while problem solving (figure 1). They were also less likely to draw a diagram (figure 2) or engage in self-questioning. The variables associated with self-questioning were transformed to a new variable called self –questioning with an interval scale from 11 to 44 (see factor analysis below). The scores being normally distributed on a histogram satisfied the assumptions of an independent t-test. The differences between the self –reported use of self-questioning for the pupils based on their school location were found to be statistically significant ($P < 0.003$, $t [88] = -3.002$), with pupils in the more advantaged area reporting more usage of such strategies.

Figure 3. Graph showing the different responses of pupils based on gender to use of the calculator as a strategy when problem solving

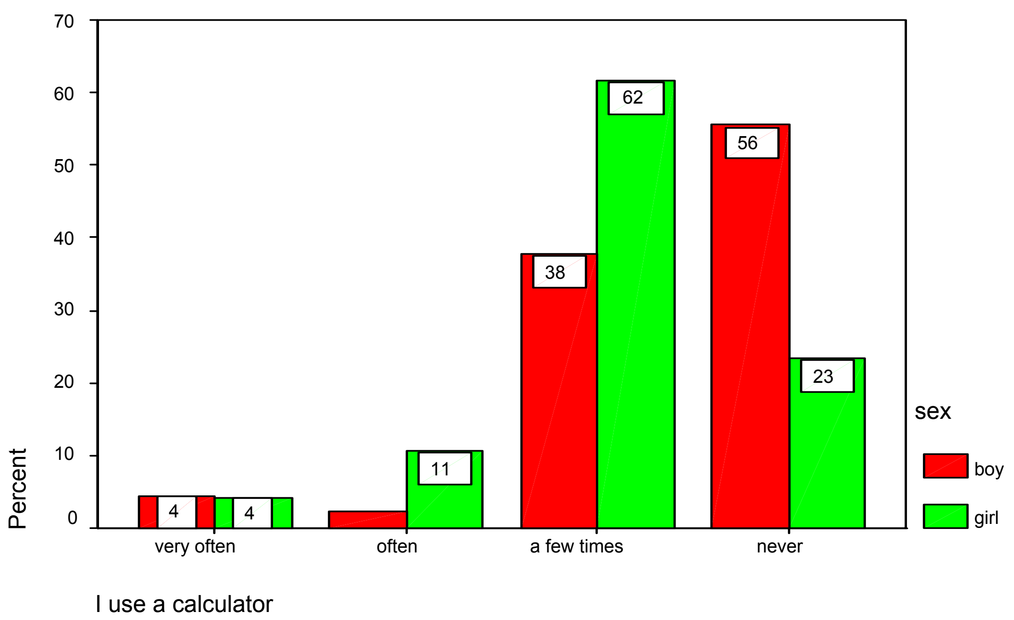
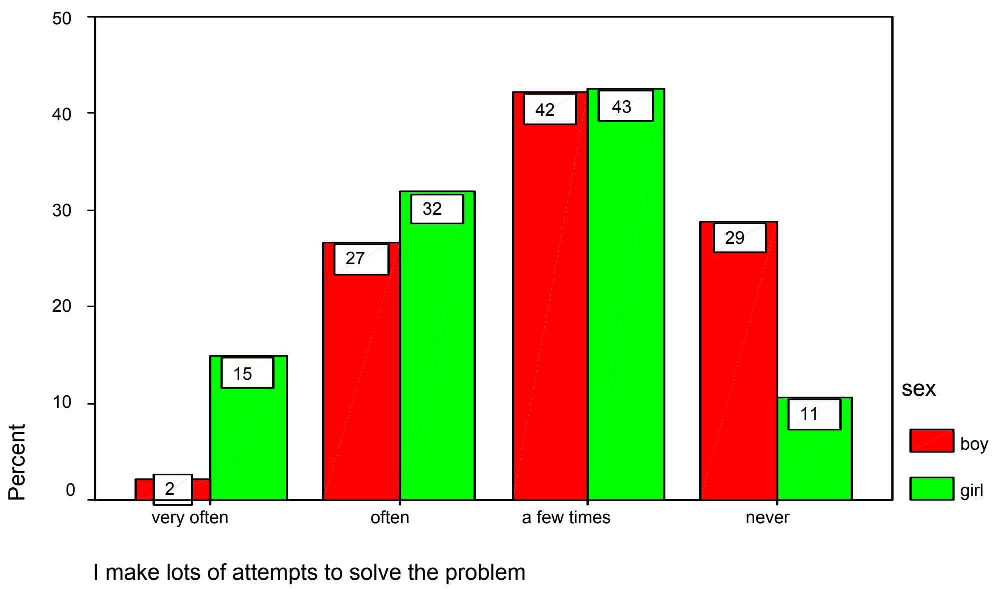


Figure 4. Graph showing the different responses of pupils based on gender to use of lots of attempts when problem solving



In relation to gender, differences between boys and girls on this new variable were not statistically significant. However, as can be seen from figures 3-6, girls reported greater use of a calculator, use of their fingers and persistence in terms of making lots of attempts and trying again if their answer didn't make sense. As seen from table 2, these differences were all statistically significant.

Figure 5. Graph showing the different responses of pupils based on gender to use of fingers when problem solving

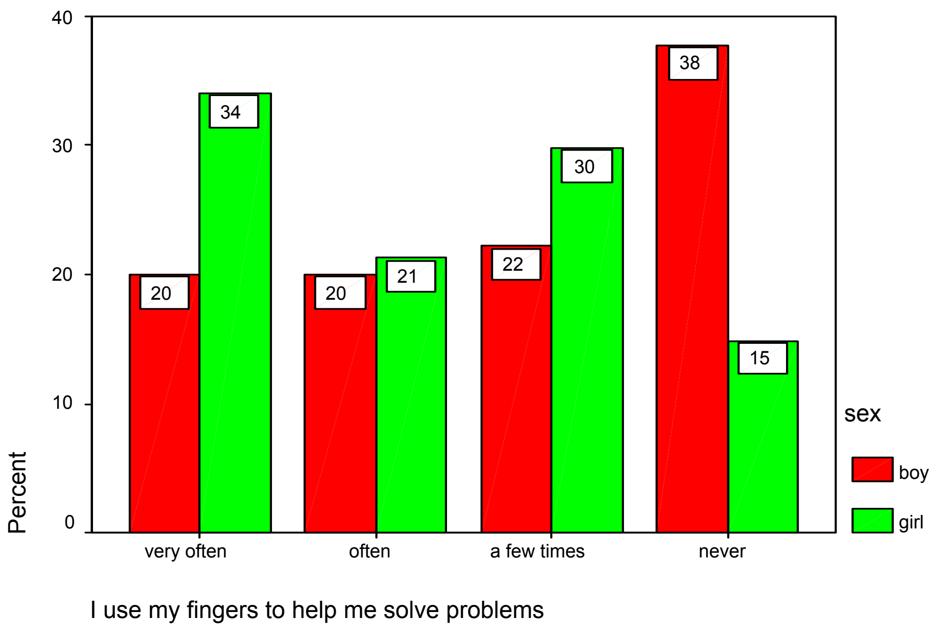
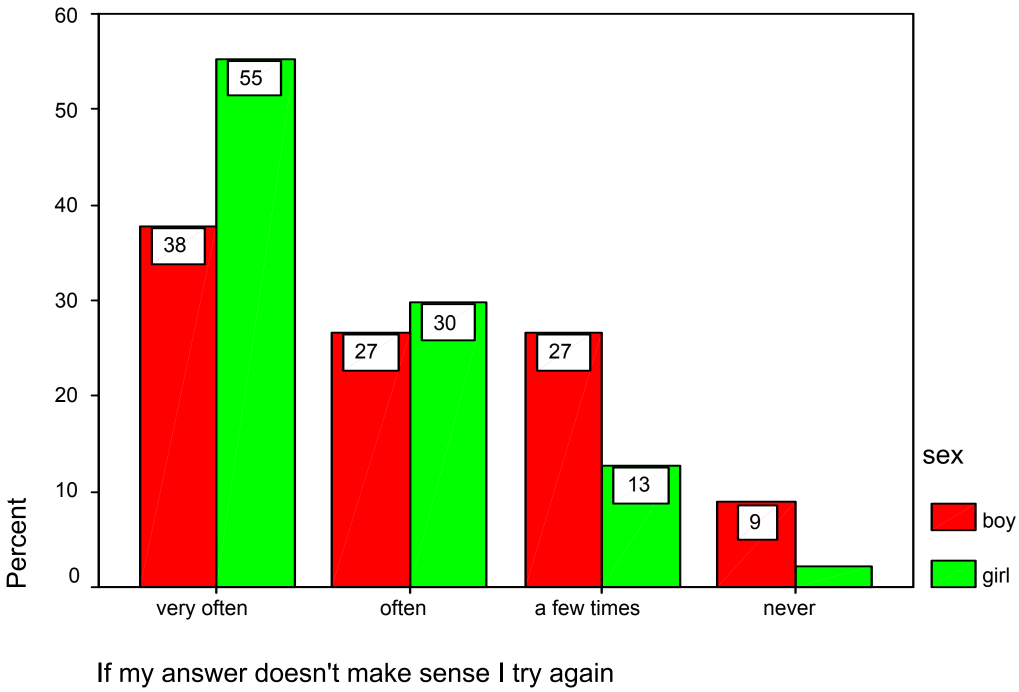


Figure 6. Graph showing the different responses of pupils based on gender to trying again if the answer doesn't make sense



A factor analysis was carried out on the variables with 11 components extracted. The first component explained 21% of the variance in the pupils' responses. Twenty- two of the variables had loadings on this factor of 0.4 or above. Interestingly, they include all 11 of the self-questioning variables. The other variables tie in very closely around the process of self-questioning: rereading the problem, rephrasing the problem, making a diagram or table, estimating and making numerous attempts to solve the problem. The second component explaining a further 8% of the variance had five variables with loadings of 0.4 or above. These related to the use of concrete materials and the underlining of key information in the problem. The third component explaining a further 7% related to the use of a calculator, constructing a table, and asking the question "do I need to add, subtract, multiply or divide?" All other components had few factors loading above the 0.4 level and accounted for a very small degree of variance.

Table 1. The results of the Mann-Whitney U test, effect size and associated two-tailed probabilities for the differences between pupils based on location of school (disadvantaged area or not) for the following variables

Variable	*Test of independence Mann-Whitney U test*	*Significance*
I use a rough work page	U= 561.5, z = - 4.013	*P* < 0.001
I ask, "what am I being asked to find out?"	U =622.5, z = - 3.546	*P* < 0.001
I use my fingers to help me solve problems	U =669.0, z = - 3.137	*P* < 0.002
I draw diagrams	U =728.0, z = - 2.723	*P* < 0.006
I put numbers in the diagrams I draw	U =811.5, z = - 2.023	*P* < 0.043
I ask, "does the diagram fit the problem?"	U = 743.5, z = -2.670	*P* < 0.008
I ask, "will the answer be bigger or smaller?"	U = 736.0, z = -2.652	*P* < 0.008

Table 2. The results of the Mann-Whitney U test, effect size and associated two-tailed probabilities for the differences between pupils based on gender for the following variables

Variable	*Test of independence Mann-Whitney U test*	*Significance*
I use a calculator	U= 705.5, z = - 3.046	*P* < 0.002
I make lots of attempts to solve the problem	U =751.0, z = - 2.537	*P* < 0.011
I use my fingers to help me solve problems	U =784.5, z = - 2.204	*P* < 0.027
If my answer doesn't make sense I try again	U =796.0, z = - 2.193	*P* < 0.028
I ask, " do I need to add, subtract, multiply or divide?"	U =789.5, z = - 2.031	*P* < 0.042

Discussion of findings

Before analysing the findings of this exploratory pilot study, it is important to stress the limitations of the study. The major limitation with this study is the convenient nature of the sample. How much the differences detected could be attributed to other sources outside of school location and gender is unknown. In addition, relying solely on a self-reporting format has limitations in terms of the ability of the pupils to report the actual strategies they use. Also, as the format required self-reflection, it may have disadvantaged those pupils not familiar with such a process.

However, it is worth highlighting some of the findings, which if they were replicated in a larger study, would present challenges to be tackled by the wider educational community. The popularity of trial and error as a strategy is confirmed in the international literature (Askew, 2003). However, there were differences between boys and girls in the self-reported use of making continuing efforts if a solution was not found, with girls reporting more persistence.

The low use of calculators as an aid to problem solving is surprising as it is formally advocated in the revised mathematics curriculum from fourth class upwards (two grades below the pupils in the present study). Interestingly, girls reported more usage and 20% of pupils reported that problem solving is never easier with a calculator. In addition, more pupils reported having a computer than a calculator. Given the reliance on the use of fingers as a strategy, particularly reported by girls, this raises the question of whether such pupils would make greater gains if freed up from the mechanics of computation by use of a calculator.

Chard and Kameenui (1995, p27) argue that “low performers may be reluctant to give up strategies that are initially useful in solving problems even though their efficiency may not hold up over time”. Kirby and Becker (1988) claim that “a lack of automaticity in basic mathematics operations and strategy use is responsible for the majority of math difficulties that children encounter” (cited in Chard and Kameenui, p28).

The low reported usage of concrete materials is also contrary to recent efforts to extend the use of such materials to the upper end of the primary school (NCCA, 1999). A study by Marsh and Cooke (1996) examining the effects of using manipulatives in teaching students with learning disabilities to identify the correct operation to use when solving mathematical word problems concluded that “ an examination of the data paths for their students suggests a functional relation between instruction with manipulatives and increased accuracy in identifying the correct operation needed to solve the word problems”(p63).

Likewise the self-reported low use of constructing diagrams presents a challenge. This strategy linked to the use of schematic spatial representations is associated with success in mathematical problem solving (Hegarty and Kozhevnikov, 1999). Perhaps it could be linked with the use of rough work pages, which 57% reported using very often or often (Appendix A). Greater attention may need to be paid to the differences between drawing a picture and a diagram, and the role of teacher modelling of the process of interpreting word problems with a diagram and its effect, could usefully be investigated.

One underlying cluster of self-reported strategy, use arising from the factor analysis, seems to centre around the process of self-questioning during the problem solving process. Differences in use of self -questioning were found between pupils based on school location. If replicated in a larger study, this also would have implications; as such strategies are an essential component of successful learning across the whole curriculum (Lerner, 1997).

Conclusion

The above self-reported patterns are often associated with difficulties in problem solving as evidenced from the literature. For example, impulsivity, anxiety, lack of attention to the language of the problem, problems choosing the method of calculation, lack of effective strategies and knowledge of tables are cited as characteristics of pupils who encounter difficulties in this area (Westwood, 2003, Haylock, 1991).

Recent evidence from an evaluation of mathematics achievement of sixth class pupils in schools in areas of socio economic disadvantage suggests a sharp fall in standards (Weir, 2003). The reasons behind this fall are unclear. As the ability to solve problems lies at the heart of mathematics, some of the issues raised here could be further explored as possible barriers to success (DES, 1982).

In relation to appropriate teaching strategies Behrend (1994) argues that:
A model of instruction which involves posing problems, allowing students time to solve the problems in their own way, listening to students' strategies, assisting only when necessary, and discussing similarities and differences between strategies provides several advantages over other forms of instruction. Teachers are able to make assessment an integral part of instruction, students are given more control over their learning, and mathematics is seen as a process of making sense of number relationships. Instruction becomes less a matter of following directions, or imitating what has been modelled, and more a way of making connections to what is already known (Behrend cited in Thornton, Langrall and Jones, 1997, p148).

However, this begs the question: does this work for all pupils? Recent research by Kroesbergen and Van Luit (2005) with students with mild general learning disabilities on strategy learning for multiplication in mathematics found that students who received directed instruction showed greater improvement than students who had received guided instruction. They argue for a "more structured variant of constructivism that provides skill development and guided practice" (Kroesbergen and Van Luit, 2005, p114).

Westwood contends that "the conclusion to be reached is that a balanced approach to the teaching of mathematics must include a significant measure of explicit teaching, as well as the valuable 'hands-on' activities and situations which typify constructivistic programmes" (Westwood, 2003, p.184). He further argues that "the amount of explicit instruction required varies from student to student, with direct teaching often being of most benefit for students with learning difficulties and disabilities" (Westwood, 2003,p.184). Dixon, Carnine, and Kameenui (1992) offer a number of principles for designing instruction for a broad range of students including those with disabilities. One of these they term "conspicuous strategies." They argue that explicit strategy instruction "broadens the range of students who can strategically solve math problems" (p31). They cite numerous studies that show how explicit strategy instruction is most effective for teaching specific strategies. They suggest that strategy instruction is critical to help most students solve new mathematical problems easily and fluently. Students with learning disabilities will benefit when strategies are explained in clear, concise, accurate, and comprehensible language (Chard and Kameenui, 1995, p31). Montague (1997) outlines an excellent example of such an approach. In her intervention study students were taught successfully specific strategies for reading, paraphrasing, visualizing, hypothesizing, estimating, computing, and checking mathematical word problems. In addition, they were inducted into a process of self-instruction, self - questioning and self- monitoring while engaging with the word problems.

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Appendix A : Pupil questionnaire and frequency table for valid percentages (n=92)

Maths Word Problems: How do I solve them?

1. I read the problem at least twice	Very often 23.1	Often 35.2	A few times 36.3	Never 5.5
2. I ask myself ‘do I need to add, subtract, multiply or divide?	Very often 24.2	Often 28.6	A few times 36.3	Never 11.0
3. I make sure I understand all the words in the problem	Very often 25.3	Often 38.5	A few times 25.3	Never 11.0
4. I try and put the problem in my own words	Very often 19.8	Often 27.5	A few times 24.2	Never 28.6
5. I underline the important information	Very often 3.3	Often 17.6	A few times 24.2	Never 54.9
6. I use a calculator when solving word problems	Very often 4.3	Often 6.5	A few times 50.0	Never 39.1
7. I ask myself ‘what does this mean?	Very often 22.8	Often 39.1	A few times 27.2	Never 10.9
8. I make a table (example shown) to help me solve some word problems	Very often 0.0	Often 6.5	A few times 20.7	Never 72.8
9. I draw diagrams to help me solve the problem (example)	Very often 2.2	Often 6.5	A few times 20.7	Never 72.8
10. I use materials like blocks and counters to help me solve problems	Very often 0.0	Often 5.5	A few times 23.1	Never 71.4
11. I replace the numbers with simpler numbers first	Very often 5.5	Often 15.4	A few times 44.0	Never 35.2
12. I ask myself ‘does this make sense?’	Very often 22.0	Often 28.6	A few times 37.4	Never 12.1
13. I read problems a few times to make sure I know what to do	Very often 35.9	Often 27.2	A few times 32.6	Never 4.3
14. I put numbers in the diagrams I draw	Very often 13.0	Often 19.6	A few times 26.1	Never 41.3
15. I estimate what the answer might be first	Very often 12.0	Often 15.2	A few times 41.3	Never 31.5
16. I check my answer	Very often 43.5	Often 30.4	A few times 18.5	Never 7.6
17. I round the numbers up or down	Very often 9.8	Often 19.6	A few times 47.8	Never 77.2
18. I use a rough work page first to solve the problem	Very often 28.3	Often 28.3	A few times 26.1	Never 17.4
19. I ask myself questions as I solve the problem	Very often 8.7	Often 30.4	A few times 38.0	Never 22.8
20. I underline the key words in the	Very often	Often	A few times	Never

problem	1.1	16.5	23.1	59.3
21. I ask myself 'what do I need to do first?'	Very often 27.2	Often 33.7	A few times 34.8	Never 4.3
22. I ask myself 'what do I need to do next'?	Very often 20.9	Often 35.2	A few times 38.5	Never 5.5
23. If my answer doesn't make sense I try again	Very often 46.7	Often 28.3	A few times 19.6	Never 5.4
24. I ask myself 'does the diagram fit the problem?'	Very often 8.7	Often 13.0	A few times 27.2	Never 51.1
25. I ask myself 'Is my answer right?'	Very often 32.6	Often 29.3	A few times 31.5	Never 6.5
26. I ask myself 'is there another way of doing this?	Very often 16.3	Often 30.4	A few times 41.3	Never 12.0
27. I make lots of attempts to try and solve the problems	Very often 8.7	Often 29.3	A few times 42.4	Never 19.6
28. I ask myself 'what am I being asked to find out?	Very often 21.7	Often 32.6	A few times 32.6	Never 13.0
29. Before I estimate I ask myself will the answer be bigger or smaller	Very often 6.5	Often 22.8	A few times 39.1	Never 31.5
30. I check my answer against my estimate	Very often 7.6	Often 26.1	A few times 29.3	Never 37.0
31. I use my fingers to help me solve problems	Very often 27.2	Often 20.7	A few times 26.1	Never 26.1
32. I think that solving word problems is easier when I use a calculator	Very often 26.1	Often 26.1	A few times 27.2	Never 20.7

About myself

I am a	Boy 49	Girl 51
I have my own calculator	Yes 80	No 20
I have my own computer	Yes 85	No 15
I have maths software	Yes 40	No 60
I play maths games on my computer	yes at home 4.3 yes at school 69.6 Both home and sch 12	No 14.1
My date of birth is		
What class are you in?		
My school is called		

The Role of Calculators in Teaching Whole Number and Decimal Numeration

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This paper outlines an exploration of the impact of the calculator, as an instructional tool, in enhancing place-value skills involved in the interpretation of whole and decimal numbers. The calculator place-value teaching experiment was conducted over a three-week period, and included the administration of pre and post calculator place-value teaching experiment tests in the interpretation of numbers. The study reports on both its introduction into a classroom and its usefulness as an instructional tool. Results from the pre and post-calculator place-value teaching experiment tests indicated an improvement in the interpretation of both whole and decimal numbers. Additionally, the use of the calculator proved particularly influential for pupils with less confidence or less enjoyment of *number*. Apart from the improvement in interpretation of numbers, the integration of calculator activities into the Mathematics class promoted greater peer co-operation, communication, exchange of ideas, and an increased level of comfort with technology. The use of calculators also provided access to a broader range of mathematical ideas in the area of place-value.

Literature Review

The National Council of Teachers of Mathematics (NCTM) first expressed its support for calculators in 1974. In 1980, it called for schools to introduce calculators and computers into the classroom at the earliest grade practicable (Walton, 1983). In its 1990 report, *Reshaping School Mathematics*, the National Research Council urged the replacement of most paper-and-pencil drills with calculator-based instruction, arguing that the time spent on routine computation could be better spent allowing children to be given activities with calculators that emphasised discovery and exploration.

Data from international surveys (Lapointe *et al.*, 1992); Office for Standards in Education, 1994) suggested that the planned use of calculators was not well developed, with few schools having a carefully considered policy for the use of calculators in mathematics. One of the factors cited as inhibiting the development of calculator use in schools was the public's, not least parents', employers' and politicians' strong reservations about the educational use of calculators. Amongst these concerns were feelings that children might become dependent on the calculator, that their use might encourage laziness, that availability might inhibit the acquisition of number facts and that these factors might handicap children when they came to take public examinations. Bright, Waxman and Williams (1992) point out that there are two obstacles to incorporating calculators into mathematics instruction. The first is convincing parents and administrators that calculators are legitimate tools to use to learn Mathematics: "Convincing people that *computers* enhance instruction does not seem to be a problem, but convincing them that *calculators* can also play this role seems very difficult, possibly, in part, because everyday use of calculators generally involves only routine computation" (p.217). Similarly, Cockroft, (1982), reported a contrast between popular acceptance of the calculator in the workplace and resistance to it in the classroom. However, the report argued strongly in favour of developing calculator use in primary schools, stating that it would help in developing number concepts. The second obstacle identified by Bright *et al*. is the problem of educating teachers in the appropriate uses of calculators in teaching mathematical concepts. Schmidt and Callahan (1992) identified fear as the most common element felt by teachers and principals towards the use of calculators. They reported that teachers feared that calculators would become a crutch, replacing students' mathematics thinking, reasoning, mental computational abilities, and basic skills.

The National Council of Teachers of Mathematics (NCTM, 1989), in addressing concerns in the United States, produced a leaflet entitled *Using Calculators to Improve Your Child's Math Skills.* The leaflet cited research findings that showed that students using calculators develop a better attitude towards mathematics and higher levels of achievement on number skills and facts. It identified three aspects of wider mathematical development promoted by calculator use: concept development, pattern recognition and problem solving. It concluded with the recommendation that use of calculators should be integrated into teaching and assessment materials at all grade levels.

In the UK, the Calculator Aware Number (CAN) project tried to develop public understanding through presentations and workshops, offering opportunities for discussion through open sessions, securing independent and authoritative endorsements and through supporting opportunities for parental involvement (Shuard, 1992). However, nearly 60% of teachers reported rarely or never using calculators in their classrooms. The project team involved in the CAN project reported that most teachers felt insecure and confused about what they should do. However, with growing experience and confidence, the teachers moved to a more developed model of teaching in which children often developed their own ideas from starting points suggested by the teacher. Groves (1993a, b) reports that by the end of their third year of involvement, most teachers claimed to have made substantial changes in their teaching of mathematics, adopting a more open-ended approach.

Evidence of longer-term effects of calculator use amongst pupils has been gathered in the course of the *Calculators in Primary Mathematics* project (Groves, 1993a, 1994a, b; Stacey and Groves, 1994). Performances of samples of primary students who had followed a calculator-aware curriculum for three and a half years were compared with a sample of students, two years senior, in the same schools, who had followed a conventional curriculum. The results showed that project children were familiar with a wider range of numbers, that they were more successful in mental calculation, and that they were better able to tackle real-world problems. They performed better overall on negative numbers, decimals and place-value. Furthermore, Groves (1994b) reports that where primary children have taken part in projects emphasising the development of mental methods of calculation alongside use of the calculator, it has been found that they do not make more use of calculators, but that they do make more appropriate choices of methods of calculation.

Brolin and Bjork (1992) report on a Swedish project (ARK), which provided an analysis of the pocket calculator. It found that students who used the calculator in grades 4-6 (10-12 year olds):

- Gained a better understanding of fundamental concepts
- Gained a better ability to choose the correct operation
- Gained a better proficiency in estimation and mental arithmetic
- Did not lose basic skills in algorithmic calculations

Hembree and Dessart (1992) report the findings of a meta-analysis of the effects of pre-college calculator use. The research analysed results from eighty-eight studies focused on students' achievement and attitude. They concluded that the calculator did not hinder students' acquisition of conceptual knowledge and that calculators help to promote a better attitude towards mathematics and an especially better self-concept in mathematics. Interestingly, Ruthven (1992) reports that, although students in general saw the calculator as, by far, the most efficient and reliable means of carrying out calculation, many expressed a preference for not using it, which could be caused by a general reservation towards calculators or caused by confidence in, and enjoyment of, number. Smith's (1997) meta-analysis of a further twenty-four research studies conducted from 1984 through 1995 supports Hembree and Dessart's conclusions. They state that not only did the use of calculators help to improve mathematical computation, but it also did not hinder the development of pencil-and-paper skills.

Suydam (1979) reported that provided the basics had been developed first in a calculator-free environment, then calculators did not threaten basic skills, and she has suggested that the introduction of calculators as early as preschool does not harm computational ability. Shuard (1992) argues that where the calculator embodies standard mathematical constructs and conventions, such as the place-value system of numeration, this can create valuable teaching opportunities. This highlights the point that how a teacher integrates calculator use in his or her classroom curriculum is a critical factor in student achievement.

Although the research generally favours the use of calculators in classroom instruction, little is known about the impact of calculator use on young children's learning of basic skills. Hembree and Dessart (1992) report that there have been no empirical studies on how to integrate the calculator directly into the learning process, and that developing number sense is one particularly important area in which further research should dwell on the use of calculators. The *Third International Mathematics and Science Study-Repeat* (TIMMS, 1999) states in its executive summary that: "Calculators were used most frequently to check answers, perform routine computations, and solve complex problems" (p.8). In conclusion, it seems that, as yet, there are insufficient in-depth empirical studies conducted into determining how number awareness might benefit significantly from the use of calculators.

Methodology

Subjects

The class involved in this study was a fifth class of thirteen boys and seventeen girls, aged ten and eleven years. The majority of the class came from a middle class background, and three pupils attended learning support classes. The average age of the pupils at the time of testing was 10 years 7 months. Their mean score in the Drumcondra Primary Mathematics Test, administered the previous June was 58.1%, with a standard deviation of 15.2.

The pre-calculator place-value teaching experiment test

Before embarking on the calculator place-value teaching experiment, a written test was administered in order to ascertain pupils' competence at interpreting place-value numbers. The test was teacher designed, did not involve the use of calculators, and was used purely for comparative purposes with a post experiment test. This pre-calculator place-value teaching experiment test involved writing down the value of twelve underlined numbers, two examples of each of the following: tens of thousands, thousands, hundreds, tens, units and tenths. These numbers were presented in a mixed fashion, in order to minimise the risk of getting the correct answer through chance.

Pupils were also given the vocabulary needed in written form, to allow for those who might experience spelling difficulties, although this provided a choice of correct answers for them. As there were two of each example, and as they did not follow each other, it was presumed unlikely that a pupil would get both examples correct, unless he/she had a good understanding of the value of the numbers. Pupils were given fifteen minutes to complete the task.

The calculator place-value teaching experiment

Having established some sense of place-value awareness amongst the pupils, the possibility of using calculators was investigated. All pupils had a calculator available from home, but there seemed to be a resistance towards the idea of using one during Maths class. Comments such as “Wouldn’t that be cheating?” and “My Mammy won’t allow me use a calculator when I am doing Maths” were typical of the general feeling of the pupils towards calculators. In order to alleviate concerns about using calculators, it was explained to the pupils that the calculator was not going to replace their tables book or the learning of tables, but that they would be used for a while each day in the classroom to do extra work in Maths. This time would be called a “special maths session”. Next morning thirty eager pupils appeared in the classroom, armed with a calculator each!

Exploring the function keys

The first activity involved exploring the four main function keys on the calculator i.e. +, -, x, ÷. The = function was explored, as was the operation of the constant function. Pupils were then introduced to the game “Five steps to zero”, mental arithmetic requiring single digit number facts. In this activity, player A enters a three-digit number less than or equal to 900. Player B must reduce the given number to 0 in at most five steps, using any of the four basic operations of arithmetic and a single-digit number at each step. There was a buzz in the classroom during this activity, as pupils tried to beat each other. By the end of this lesson, the pupils seemed familiar enough with the basic operations of the calculator to proceed into the exploration of whole and decimal numbers.

Exploring whole numbers

The next three lessons involved working with whole numbers 1-9999. The lessons involved three distinct stages. Firstly, there was a class discussion and activity period, followed by pupils forming groups of two, challenging each other. Finally, numbers were called out and pupils keyed them into their calculators. Answers were then compared. This gave the researcher an opportunity to assess what progress was being made. Starting with the number 1, another 1 was added to it, and pupils were asked to predict what answer the calculator would show. They then checked the answer. Using the constant function, pupils could move swiftly through the ascending numbers. When they reached 9 a discussion ensued on how the next number would appear: 10. This allowed discussion of the value of the 1 in the 10: one ten. During this lesson numbers from 1-99 were explored i.e. units and tens, in ascending and descending order. Pupils would then spend some time working in pairs challenging each other. Subsequent lessons examined hundreds and thousands, using the same format. The main questions were: "What number will come next?" "What number is one less than?" and "What does the calculator show?"

Exploring the tens-pattern

In order to see the ten-times-larger pattern and to observe that the reverse was also true, a calculator activity called "The Place-Value Mover" was introduced. Students entered a one-digit number e.g. 5, calling it five units. They multiplied it by ten, and by using the constant function, they discussed the value after each operation. The class were asked to predict what would happen if they divided by ten. They predicted and read the answers as they shrank back to 5.

Changing digits back to 0

The game "Wipe out" was introduced as an exercise in reinforcing and consolidating the concepts and notational conventions of place-value numeration. The pupils were challenged to change single digits of a numeral? to 0 by single calculator operations. For instance, pupils might enter 4763 and be asked to wipe out the 7 (which can be done by subtracting 700). This tested their knowledge of the value of numbers being erased.

Exploring estimation strategies in place-value

At this stage, the class was counting on and back in units, tens, hundreds and thousands. Before exploring decimal fractions, the "Range Game" was introduced. Given a number e.g. 26, pupils had to find a number when multiplied by 26 gave an answer in the range 500-600. This game, apart from promoting number sense, embodies computational estimation. Pupils gave their answers in whole numbers only i.e. 20, 21, 22 or 23. When asked how they found these answers, some said that they kept guessing, while others said that they tried to round up the numbers to get a starting point. According to Ruthven (1992b), estimation tends to be adopted "where the user is unable or reluctant to formulate a more direct derivation from the information available" (p. 457). Because proximation strategies offer a relatively lengthy route to a solution, the calculator, because of its speed, drastically lessens the need for a systematic approach, encouraging strategies based on trying out a large number of guesses. Developing proximation strategies can trigger the critical structural insight enabling the user to devise a more direct path.

Exploring decimal numbers

In order to introduce decimal numbers, pupils were presented with a game called "Hit the target". They were asked to find a number, which when multiplied by 26 would give a number close to 580, without using the division function. They answered 22. They were then challenged to find another number. They said that it was impossible! They were encouraged to discuss it amongst themselves for a few minutes. "We have it" came a cry from one group: "22.2". The class looked dazed for a few seconds, until they began to realise that decimal fractions were also numbers. This started an avalanche of responses as pupils tried to get closer to 580. Not only were tenths and hundredths tried, but some of the more able pupils extended the range beyond thousandths! Follow-up lessons involved counting in tenths, by using the constant function again on the calculator. When they reached 0.9 (nine tenths), they were asked, "What will come next?" They said "Ten tenths". They could see from the display that ten tenths was the same as one unit. During the next lesson the researcher and the class discussed how the value of one thousand was the same as ten hundreds, how one hundred was the same as ten tens, how one ten was the same as ten units, and how one unit was the same as ten tenths. They were asked whether or not the tenth could be renamed, leading into the strand of number exchanging. Immediately, the answer came back "ten hundredths". They keyed in 0.01, calling it one hundredth, and using the constant function, counted in hundredths until they reached nine hundredth (0.09). The next question did not have to be asked! They could see that ten hundredth (0.1) was the same as one tenth. Although the work in hundredths had not been planned for, there was no problem in taking place-value a step further. At the end of each lesson, the class was again asked to enter numbers on their calculators as they were called out. It was found to be more beneficial to call out the value of numbers (six thousand, four units and two tenths rather than six thousand and four point nought two). By omitting certain numbers from the range e.g. 6004.02 rather than 6734.52, it allowed progress to be gauged.

The post-calculator place-value teaching experiment test

Having spent approximately fifteen minutes a day working with calculators over a three-week period, the pupils were given a post-calculator place-value teaching experiment test. While this test was identical to the first test in format, the numbers presented were different, but covered place-value numbers ranging from ten thousands to hundredths. Pupils again were given fifteen minutes to complete the test.

Results

Of the thirty pupils participating in the pre-test, only eight pupils (27%) displayed an awareness of the values of all whole and decimal numbers presented. Thirteen pupils (43%) were unsuccessful at recognising the value of any of the twelve numbers presented. Seven pupils (23%) confused the values of the units and the tenths, and two pupils (7%) seemed to recognise the numbers with the value of thousands only. Results of the post-calculator place-value teaching experiment test revealed an improvement in interpreting the value of numbers ranging from tenths to ten thousands. Twenty children (66%) were successful at answering all questions correctly, compared to only eight (27%) who were successful at the pre-calculator test. Four children (13%) failed to score any correctly. This compares with thirteen (43%) who had failed in the pre-test. Of these four, one had missed a week at school during the calculator activities and two had consistently "forgotten" to bring their calculators to school. The fourth pupil suffers from an attention deficit disorder. Six pupils had mixed results compared to nine pupils in the original test. Of these six pupils, the most common mistake was mixing thousands with tens of thousands.

Discussion

Results of the pre-calculator test showed a perceptible weakness in the ability to interpret the place-value of numbers. Of particular concern was the fact that 43% of pupils failed to recognise the value of any whole or decimal numbers presented. According to *Curaclam na Bunscoile* (1999), a pupil entering fifth class should be able to explore and identify place-value in whole numbers, 0-9999. They should also be able to explore and identify place-value in decimal numbers to two places of decimals. The place-value system for recording decimal numbers is an extension of that for whole numbers. The value of a digit depends on the position in which it is placed, and the value of any position is ten times larger than the value of the position to its immediate right. This work begins in junior classes and by the time pupils have reached fifth class, they are expected to have a knowledge and awareness of numbers ranging from hundredths to ten of thousands. Post-calculator test scores reveal a definite improvement in place-value interpretation awareness. Pupils were more successful in all areas of weakness displayed in the pre calculator place-value test. However, the presentation of test scores, alone, limits a fuller evaluation.

The poor pre-calculator test results may seem surprising initially, as children at this stage spend up to five hours of school time at Maths every week. Having discussed the results of this test with teachers in the school concerned, two things became evident. Firstly, children spend quite a lot of time in the junior classes working with manipulatives, such as unifix cubes, Cuisenaire rods and Dienes' blocks, building up place-value awareness and number sense. However, as they progress, the emphasis shifts on to computational skills, and number sense or place-value activities are replaced by paper and pen exercises. Secondly, at third and fourth class levels, a minimum amount of time is spent working with manipulatives, as teachers feel pressurised to teach the computational aspects of numbers and decimal fractions. This pressure would seem to arise from a greater emphasis being placed on textbook work, and an eagerness for pupils to score well in standardised tests annually. Interestingly, these classrooms have very little mathematical equipment available to them compared to junior classes, and calculators are not used in any classroom. The test results can possibly lead to a misapprehension that children who are good at computing have strong place-value awareness. Another possibility for the poor results in the pre calculator place-value test is that pupils were unfamiliar with the presentation or format of the test. An examination of the 5th class textbook reveals that the presentation of questions employed in the test does not appear in it. These results, however, may or may not indicate a lack of place-value awareness. One can only conclude that in this type of questioning, pupils failed to register successfully, which may say more about the test than the actual level of place-value knowledge. The calculator place-value teaching experiment was used as a possible mode of exploration.

Attitudes towards calculators

An examination of the teaching experiment would seem to confirm previous findings by Hembree and Dessart (1992) that calculators help to promote a better attitude towards mathematics and a better self-concept in mathematics. While discussing proximation strategies during one of the calculator place-value lessons, one pupil asked whether he could use the calculator to find estimates when he was doing long division. Long division involves a lot of rote computations, which demands little mathematical thinking. A real comprehension of mathematics comes as a result of understanding what the question is asking, knowing how to set up the problem, deciding which operations are appropriate, and determining whether or not the answer obtained makes sense. A discussion ensued on how pupils felt the calculator could best be used for this activity. One pupil remarked that if calculators were to be used, then there was the chance that people would "cheat" by doing the whole long division on the calculator. Another pointed out that while one would have the correct answer, the calculator would not show how the answer was arrived at! Another pointed out that the calculator could be used for estimation purposes only and for checking whether an answer was correct. There was a general consensus that calculators would be useful for estimation, that it would shorten the amount of time doing basic calculations, that one still had to show how one arrived at an answer, and, therefore, "cheating" was not involved. This "detour" proved helpful in changing pupils' attitudes and in convincing them that the calculator could be a helpful tool in other areas of Mathematics as well.

Effects of calculator usage

The integration of calculator activities into the Mathematics class promoted greater peer co-operation, communication, exchange of ideas, and an increased level of comfort with technology. Much more discussion took place during these activities, especially amongst the pupils themselves. All pupils were able to participate in exercises, which led to increased confidence. Interestingly, Ruthven (1992) reports that although students in general saw the calculator as by far the most efficient and reliable means of carrying out calculation, many expressed a preference for not using it, which could be caused by a general reservation towards calculators or caused by confidence in, and enjoyment of, number. Supporting Groves' (1994b) findings, it was noted that although pupils had access to their calculators during Math periods, pupils used them very much as individual needs arose. Some used them for estimation, some for checking answers and some for computation.

It would seem that the role of the calculator is likely to be particularly influential for pupils with less confidence or less enjoyment of number. During the third stage of a lesson on working with thousands, it was noted that four of the weaker pupils were not keying in correct answers. When four thousand and twelve was called out, they were keying in 400012. It took a while to realise that they were keying in numbers as they heard them: four thousand and twelve. They were asked to key in 4000 and to add 12. When asked what number they would expect the calculator to show, they said: "four thousand and twelve". They pressed the answer, which showed 4012. The calculator had helped to illustrate the correct format for these pupils. "The calculator seemed to provide a means of matching the demands of school work to their capabilities and interests" (Ruthven, p. 453).

The relationship between whole numbers and decimal numbers is a complex area. Traditionally, decimal fraction work has followed fraction work. Many children experience difficulty in seeing the relationship between fractions and decimals e.g. 1/10 and 0.1. With the calculator, the opposite was the case. Pupils saw 0.1 and were encouraged to call it one tenth. Another difficult area that has always arisen with number work is estimation. Pupils find it very time-consuming when it is a paper and pencil activity. With the calculator, time is freed up to concentrate on the problem solving aspect of questions, although this study concentrated mainly on place-value awareness only. The calculator proved a very useful tool for creating enjoyable activities that helped to consolidate the concept of place-value. However, pupils would seem to benefit most when the calculator is used along with, or after children have had experience working with other manipulatives. As calculators cannot exploit the physical properties of numbers, their strength would seem to lie in the consolidation of number awareness, rather than in the early stages of instruction.

Summary

The use of calculators in this study allowed students access to mathematical concepts and experiences from which they were previously limited with only paper and pencil.

The results of the pre-calculator and post-calculator tests have shown a definite improvement in place-value awareness. The study has also shown how calculator use can be incorporated into other areas of the Mathematics curriculum. The research cited has shown that calculator use today allows students and teachers to spend more time developing mathematical understanding, reasoning, number sense and applications, thus developing useful mathematical understanding and mathematical power. The use of calculators allows students, who would ordinarily be turned off by traditional mathematics' tedious computations and algorithms, to experience true mathematics.

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Junior Cycle Second Level Mathematics Education

Junior Cycle Mathematics Examinations and the PISA Mathematics Framework

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In 2003, 41 countries, including Ireland participated in the second cycle of OECD's Programme for International Student Assessment (PISA 2003). This assessed mathematics, reading, science and problem solving. The contrast between the PISA type mathematics questions and those typical of the Irish Junior Certificate examinations has been a subject of discussion. In this paper, the questions from the Junior Certificate mathematics papers for 2003 and the 1974 Intermediate Certificate mathematics papers are mapped onto the 3-dimensional (content, competencies and situations) PISA Mathematics Framework. The results are discussed in relation to PISA mathematics questions, the Irish mathematics curriculum, and other matters of current concern.

Introduction

In 2003, 30 member countries of the OECD, including Ireland, and 11 additional countries participated in the second cycle of OECD's Programme for International Student Assessment (PISA 2003). The major assessment domain in PISA 2003 was mathematics, while reading, science and problem solving were minor domains. The mathematical performance of Irish 15-year-olds placed them at about the international average (21st out of the 39 countries involved). This is in marked contrast to the high performance of Irish students on the reading tests (5th out of 39 countries rated) (Cosgrove et al., 2005). The mathematics performance is of some concern since mathematics is considered a key area of competency in moving towards a more knowledge-based society and economy. In this context, the contrast between the type of questions in the PISA mathematics tests and those typical of the Irish Junior Certificate examinations has been a subject of discussion.

This paper attempts to provide further evidence for addressing the issue. It outlines the framework underpinning PISA mathematics (described in terms of three dimensions: content, competencies and situations) and also that used for the current Junior Certificate mathematics courses. The questions from Junior Certificate mathematics papers for 2003 (the first examination of the recently revised courses) are then mapped onto the PISA mathematics framework. Since the last major root-and-branch reform – as opposed to revision – of the Junior Cycle Mathematics curriculum took place some forty years ago (with the introduction of "modern mathematics" in the 1960s), this mapping is carried out also for a set of Intermediate Certificate papers from that period. The results are compared and contrasted. The paper is intended to promote discussion and reflection on the Irish curriculum prior to the forthcoming review of post-primary mathematics education.

In the paper, the authors draw on their own experiences, the first author as a member of the Mathematics Expert Group for PISA, the second author as a former Education Officer with the National Council for Curriculum and Assessment (NCCA) and a member of the PISA Mathematics Forum. However, where unattributed views are expressed, they are the authors' own and not necessarily those of the groups with which they have worked.

Background

This section describes the PISA mathematics framework and then outlines the framework used in assessing Junior Certificate mathematics in Ireland. In each case, especially the latter, some context is provided in order to explain the rationale for the type of assessment used.

The PISA mathematics framework

The PISA mathematics assessment framework is concerned with assessing how well students can use mathematics in realistic situations (mathematical literacy) (OECD, 2003). It reflects current trends towards emphasising the processes of problem solving and mathematisation, and towards problem-based teaching and learning including team problem solving, and away from mathematics as isolated sets of concepts, principles and procedures taught by exposition and practice. As indicated above, the PISA mathematics framework has three dimensions: *context*, *content*, and *competency*. These are discussed in turn.

Mathematical problem situations and contexts are categorized as (i) *Personal*; (ii) *Social/Occupational*; (iii) *Public*; (iv) *Scientific*, with varying kinds and levels of mathematics arising in these different types of situations and contexts. In the first cycle of assessment for PISA (PISA 2000), a fifth category was used: *Intra-mathematical*, explicitly accommodating situations in which the context for the question was provided by mathematics itself. However, in PISA 2003, this is treated as part of the scientific domain.

The mathematical content is described in terms of four categories that encompass the kinds of problems arising through interaction with everyday phenomena. For PISA purposes, these are called the four "overarching ideas": (i) *Quantity*; (ii) *Space and Shape*; (iii) *Change and Relationships*; (iv) *Uncertainty*.

The competency dimension is divided into three competency classes to reflect the different cognitive demands of mathematical tasks. The classes are: (i) *Reproduction* – performing calculations, solving equations, reproducing memorised facts or "solving" well-known routine problems; (ii) *Connections* – integrating information, making connections within and across mathematical domains, or solving problems using familiar procedures in contexts; (iii) *Reflection* – recognising and extracting the mathematics in problem situations, using that mathematics to solve problems, analysing and developing models and strategies, or making mathematical arguments and generalisations.

This framework underpins the PISA mathematics tests. As with traditional standardised tests, PISA tests consist of a number of items each constituting a *task* for the students. In the 2003 tests, there were 85 such tasks. A task (usually situated in a realistic context) could involve several mathematical topics and several cognitive skills. (Examples are given in OECD, 2003). Sets of tasks (two to six items) are frequently based on the same stimulus material, which may include drawings, diagrams, and photographs as well as text.

The Junior Certificate mathematics assessment framework

Junior Certificate mathematics is offered at three levels: Higher, Ordinary and Foundation. The rationale and aims for each level are described in the *Guidelines for Teachers* (DES / NCCA, 2002) that were issued to support the recently revised syllabus (DES/NCCA, 2000). The account in this paper focuses on the main issues relevant to the discussion here.

The courses are assessed solely by means of a terminal examination. Candidates at Higher and Ordinary level sit two papers, each containing six "long" questions, while Foundation candidates sit one paper containing six such questions; no choice is offered in any of the papers. Individual questions are intended to test cognate content areas and to provide a "gradient of difficulty" spanning the assessment objectives, in particular some or all of the following: (i) *recall*; (ii) *instrumental understanding* (the ability to execute procedures); (iii) *relational understanding* (the ability to understand concepts and the relationships between them); (iv) *application* in familiar contexts. (For fuller accounts, see DES/NCCA, 2002; Cosgrove, Oldham and Close, this volume).

The questions are usually divided into three parts, labelled "(a)," "(b)" and "(c)" respectively. "Part (a)" typically tests recall and/or simple instrumental understanding. "Part (b)" generally focuses mainly on instrumental understanding – testing procedures with which students should be familiar and which they should be able to execute fluently; it may also assess relational understanding. "Part (c)" is intended to address somewhat higher-order objectives; it usually tests the ability to apply knowledge in fairly familiar contexts (often intra-mathematical) and/or to solve comparatively routine problems. "Part (a)" usually carries 20% of the marks for the question, and "parts (b) and (c)" usually carry 40% each.

The rationale behind this structure – and some of its differences from the PISA framework – may be best understood by considering the background to its inception. While many examination questions on both Junior and Leaving Certificate papers over the years had presented some features of the structure, it was explicitly introduced at Leaving Certificate level in 1992 (for the Higher and Ordinary level courses first examined in 1994). Among the issues that the 1992 revision was designed to address (Oldham, 1993), two are of relevance here.

- Questions on the Higher level papers at that time were often of (intra-mathematical) problem-solving type. Even able candidates could not necessarily find a way of addressing the questions in the time available; some good candidates were failing. The level of difficulty appeared to discourage students from attempting the Higher level course; uptake was lower for that of Higher level courses in other subject areas.
- By contrast, the Ordinary level papers were not deemed sufficiently challenging for the more able students presenting for the examinations at that level. Some such candidates, who perhaps could have taken the Higher course, were confident of obtaining a very high grade on the Ordinary course and preferred the less demanding option.

Thus, the "part (a)s" were designed to ease students in to questions, but not to bring their scores close to 40% (the lower bound of a grade D, widely considered as a "pass"). "Part (b)s" were intended to reward diligent learning and hopefully to ensure that suitable candidates who had a sufficiently sound grasp of the main techniques would be able to obtain a grade D or higher. However, only an almost perfect performance on all the "parts (a) and (b)" would allow candidates to achieve the 55% necessary for a grade C without also scoring marks on some of the "part (c)s." For B and especially A grades, students should have to display the higher order behaviours associated with application and problem-solving in comparatively familiar contexts. The solution of problems set in thoroughly unfamiliar contexts, intra-mathematical or otherwise, was felt to be unduly demanding on candidates in a high-stakes examination with a comparatively short time-frame. Nonetheless, "part (c)s" were intended to be to a certain extent unpredictable. For example, it was envisaged that a novel problem, once used as a "part (c)," could not be utilised again because it would no longer be novel; students would have encountered it when working through past examination papers. The problem would have lost a key characteristic of a "part (c)."

Uptake of the revised courses in their first few cycles of implementation, and performance in the corresponding examinations, suggest that the structure went at least some way towards satisfactorily addressing the two issues outlined above. Anecdotal evidence supports this view. Feedback from teachers tended to indicate that they were happy for questions to have a challenging "part (c)" (or, at least, a challenging final stage of a "part (c)") provided that the "part (a)" was genuinely easy and the "part (b)" suitably accessible. Hence, when revised Junior Certificate courses were introduced in 2000 for first examination in 2003, the same basic structure was adopted. It should be noted that the structure was meant to apply for each of the three courses, with due regard for the content and skills expected of the intended cohort of students in each case. A "part (a)" for a Higher course student might well be a "part (c)" for a student at Foundation level.

In what might be regarded as a slight modification of the original concept as described earlier, the *Guidelines* indicate that a "part (c)" would *typically* involve application (DES/NCCA, 2003, p. 99), but that "the ability to apply, *or to execute more difficult examples of familiar exercises, or to demonstrate understanding at the upper level of the syllabus*, is needed for a higher grade [than C]" (DES/NCCA, 2002, p. 96, emphasis added). Thus, elements of competencies that in terms of the PISA framework would be classified as Reproduction might be expected even in a "part (c)." Overall, however – while the mapping between Junior Certificate question "parts" and PISA competencies is not always straightforward – the intended functions of the three parts can perhaps be described in PISA terms as being to test simple Reproduction, more complex Reproduction or perhaps simple Connections, and harder Connections, respectively.

In implementing the structure, some disadvantages have appeared. They tend to reflect areas of difference from the PISA framework. Again, some of the evidence is anecdotal (typically stemming from comments by teachers at meetings of the Irish Mathematics Teachers' Association), but some is at least consonant with research findings as indicated below.

First, the (limited) problem-solving element is sometimes treated as a bolt-on extra for the more able students at the relevant level, rather than as an intrinsic part of the mathematics education experience. The possibility of this happening was foreseen (DES/NCCA, 2002, p. 100), but the merits of the structure were expected to outweigh the disadvantages. A second disadvantage seems to have arisen from attempts by teachers and students to reduce the "part (c)s" from application/problem-solving status to the status of rehearsed procedures. This – if indeed it has happened – has simultaneously lengthened the course in terms of material to be covered, hence rushing the intended concept development, and negated the intentions of the course designers with regard to problem-solving. Such an approach is currently very understandable with regard to the Leaving Certificate: a high-stakes examination, the grade from which contributes to a candidate's "points score" for entrance to third-level courses. (For popular courses with insufficient places for all applicants, the required points can be very high; each few marks in the Leaving Certificate can be seen as crucial steps towards a course and career of choice.) However, especially if implemented also at Junior Certificate level, it may reflect inherently procedural views of mathematics held by at least some students and teachers. Evidence of such views can be found in the work of Beaton et al. (1996), Oldham (2001), Lyons et al. (2003); and Close et al. (2003). Thirdly, marking schemes are now firmly in the public domain; while this is commendably transparent and helpful in many ways, it may also encourage a very mark-focused approach to preparation for examinations and even to initial teaching of topics.

Other issues do not relate so directly to question structure, but rather to other aspects of examination paper design. At Leaving Certificate level, a problem has arisen because the distribution of topics over questions has been less variable than was originally envisaged. It is too soon to say if the same difficulty will occur with Junior Certificate examinations because the current course has been examined only three times, but so far it does appear as if a rather predictable pattern is emerging.[1] Predictable papers can lead to students learning to apply a technique in (say) question 3 rather than in a certain mathematical or realistic context. With regard to the role of non-mathematical contexts, worries have been expressed in the past that the use of realistic contexts would not provide a "level playing field" for candidates. For the current Junior Certificate, however, the role of context is given some acknowledgement, with the possibility of

[1] A late change of plan with regard to the format of the examination papers meant that the *Guidelines* went to press without being able to specify the number of questions per paper and the likely distribution of topics within the papers. The intention was that, while clusters of questions examining major content areas would be located predictably on specified papers (and perhaps in fairly standard positions on these papers), individual techniques would *not* be rigidly associated with specific questions. Perhaps this is not made clear in the *Guidelines*. However, the *Guidelines* do specify that the sample examination papers would provide a *typical* – hence, not a unique – distribution of questions for each topic (DES/NCCA, 2002, p. 92).

context-based questions occurring in "part (b)s" – as well as in "part (c)s" – being explicitly mentioned in the *Guidelines* (DES/NCCA, 2002, p. 97).

The *Guidelines* also discuss the omission of objectives of higher order than application (which figure in the syllabus) from assessment in the current Junior Certificate examinations. They emphasise the difficulty of assessing such objectives under examination conditions, and point out the possibility that, if different forms of assessment were introduced, then the higher order objectives could be addressed. However, it must be said that there has been virtually no demand from the public for such forms of assessment. Equally, the issues of predictability and context freedom have not been controversial; on the contrary, unfamiliar questions and contexts have tended to draw widespread criticisms. It has taken a view from outside the Irish system, in a report to the NCCA, to draw official attention to the issues (Elwood and Carlisle, 2003). Publications around the PISA results may help to do likewise.

Aims of the Study

To complement the recent study in which PISA items were mapped onto the Junior Certificate mathematics content areas (Cosgrove et al., 2005), the present study set out the following aims:

1. To analyse and classify the items (tasks) on the 2003 Higher, Ordinary and Foundation level Junior Certificate examination papers in mathematics using the PISA mathematics framework and to compare them to the classifications of the PISA items;
2. To classify the items on the 1974 Higher and Lower Intermediate Certificate examination papers in mathematics using the PISA mathematics framework and to compare them to the classifications of the 2003 Junior Certificate examination papers and the PISA items;
3. To discuss the results in the light of current international trends and the proposed review of the Irish post-primary mathematics curriculum.

Method

As pointed out earlier, the PISA 2003 mathematics tests contained 85 items. For comparison purposes the sub-tasks of the six questions that students in the 2003 Junior Certificate Mathematics Examination and in the 1974 Intermediate Mathematics Examination were required to answer were classified as separate items. The numbers of items emerging from this analysis are provided in the following table.

Table 1: Numbers of items in the PISA mathematics test and in the 2003 Junior Certificate (JC), and the 1974 Intermediate Certificate (IC), examinations in mathematics

	PISA 2003 Maths Test	*2003 JC Higher Level Maths Exam*	*2003 JC Ordinary Level Maths Exam*	*2003 JC Foundation Level Maths Exam*	*1974 IC Higher Course Maths Exam*	*1974 IC Lower Course Maths Exam*
No. of items	**85**	**77**	**81**	**32**	**79**	**79**

Each item in each question on each examination paper was classified using the three dimensions of the PISA 2003 assessment framework: overarching idea (mathematical content), competency cluster (cognitive process skills), and context (problem situations).

The method of classification used in this study was derived from a method described in the Framework document (OECD, 2003 p.49):

> analyse the demands of the item, then rate each of the eight competencies for that item, according to which of the three clusters provided the most fitting description of item demands in relation to that competency. If any of the competencies were rated as fitting the description for the reflection cluster, then the item would be assigned to the reflection competency cluster. If not, but one or more of the competencies were rated as fitting the description for the connections cluster, then the item would be assigned to that cluster. Otherwise, the item would be assigned to the reproduction cluster, since all competencies would have been rated as fitting the competency descriptions for that cluster.

This method was adapted for the study as follows:

1. A task analysis of the item was carried out by working out answer(s) using the methods considered appropriate for students in the relevant age range. (This can also referred to as the "cognitive walk-through" approach.) The marking schemes provided on the State Examinations Commission (www.examinations.ie) website were useful in this exercise. The results obtained from the analysis were then used to carry out the following classifications.

2. The *overarching idea* of the PISA Framework that appeared to contribute most to the mathematical complexity and solution process of the item was identified and

used to classify the item. If there appeared to be two or more overarching ideas involved then the item was assigned to the more important one.[1]

3. The cognitive process skills involved in the method of working out answers were identified using the list of process skills in the PISA Framework. The process skill that contributed most to the cognitive complexity and solution of the item was isolated and used to classify the item on the basis of the *competency cluster* to which it belonged. If there appeared to be two process skills contributing substantially to the cognitive complexity, and they belonged to different clusters, then the item was assigned to the higher level cluster.

4. Items were classified into five *situation* or context categories, one intra-mathematical (for which the item or problem is purely mathematical), and four extra-mathematical categories: personal, educational/occupational, public, or scientific.

The analysis was carried out by the first author, using his experience of working with PISA items as a member of the PISA Mathematics Expert Group. An example of the application of the method to two questions on paper I of the 2003 Junior Certificate examination in mathematics is provided in the Appendix.

Results

Tables 2, 3 and 4 below provide a summary of the results of the analyses of the 2003 and 1974 examination papers.

[1] It should be noted that this classification was carried out for all items, including those testing topic areas such as sets, deductive geometry and trigonometry that do not appear in the PISA tests; they can nonetheless be assigned to an overarching idea.

Table 2: Percentages of items in the 2003 Junior Certificate and the 1974 Intermediate Certificate mathematics papers corresponding to the four overarching ideas of the PISA mathematics framework

PISA Framework Dimension	*PISA Framework Dimension category*	*PISA 2003 Maths Test (85 Items) % of*	*2003 JC Higher Level Maths (77 Items) % of*	*2003 JC Ordinary Level Maths (81 Items) % of*	*2003 JC Foundation Level Maths (32 item) % of*	*1974 IC Higher Course Maths (79 Items) % of*	*1974 IC Lower Course Maths (79 Items) % of*
Overarching Ideas	Quantity	**27.1**	**16.9**	**30.9**	**53.1**	**13.9**	**23.5**
	Space and Shape	**23.5**	**33.8**	**27.2**	**12.5**	**38.0**	**38.2**
	Change and Relationships	**25.9**	**37.7**	**32.1**	**21.9**	**43.0**	**30.9**
	Uncertainty	**23.5**	**11.7**	**9.9**	**12.5**	**5.0**	**7.4**

Table 3: Percentages of items in the 2003 Junior Certificate and the 1974 Intermediate Certificate mathematics papers corresponding to the three competency clusters of the PISA mathematics framework

PISA Framework Dimension	*Dimension category*	*PISA 2003 Maths Test (85 Items) % of*	*2003 JC Higher Level Maths (77 Items) % of*	*2003 JC Ordinary Level Maths (81 Items) % of*	*2003 JC Foundation Level Maths (32 item) % of*	*1974 IC Higher Course Maths (79 Items) % of*	*1974 IC Lower Course Maths (79 Items) % of*
Competency clusters	Reproduction	**30.6**	**83.1**	**95.1**	**100**	**84.8**	**88.2**
	Connections	**47.1**	**16.9**	**5.0**		**15.2**	**11.8**
	Reflection	**22.4**					

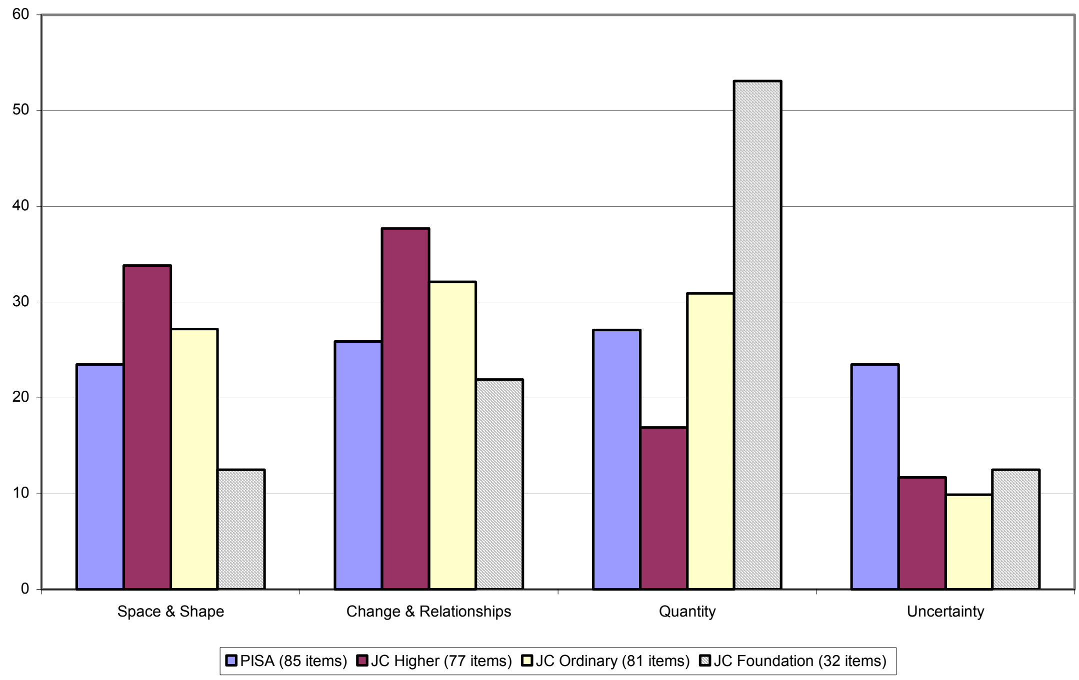
Table 4: Percentages of items in the 2003 Junior Certificate and the 1974 Intermediate Certificate mathematics papers corresponding to the context categories of the PISA mathematics framework

PISA Framework Dimension	*Dimension category*	*PISA 2003 Maths Test (85 Items) % of*	*2003 JC Higher Level Maths (77 Items) % of*	*2003 JC Ordinary Level Maths (81 Items) % of*	*2003 JC Foundation Level Maths (32 item) % of*	*1974 IC Higher Course Maths (79 Items) % of*	*1974 IC Lower Course Maths (79 Items) % of*
Situations	Personal	**21,2**		**12.3**	**21.9**	**1.2**	**13.2**
	Educational Occupational	**24.7**	**6.5**	**23.4**	**6.2**	**7.6**	**7.4**
	Public	**34.1**	**16.9**	**2.5**	**12.5**	**8.8**	**10.3**
	Scientific	**20.0**	**10.4**				
	Intra mathematical		**66.2**	**61.7**	**59.4**	**82.2**	**69.1**

Overarching Ideas

There are considerable differences across specific Irish examination papers in terms of the percentages of items testing each of the four overarching ideas, reflecting the levels of the examination (three levels in 2003 and two levels in 1974). In PISA there are approximately equal percentages of items across the four overarching ideas. In the 2003 Junior Certificate papers, the percentages of items on Quantity range from 16.9% (Higher level) to 53.1% (Foundation level); however, the mean of the figures is approximately 33%, which is near the PISA figure of 27%. A similar situation prevails across two of the other three overarching ideas, where the means across the Junior Certificate papers are not much different from those for PISA (Space and Shape – about 24% versus 23.5%; Change and Relationships – about 30% versus 26%). In the case of Uncertainty there are considerably more items in the PISA tests than in the Junior Certificate papers; the figures are about 11% for the Junior Certificate (and a smaller percentage for the Intermediate Certificate) versus 23.5% for PISA.

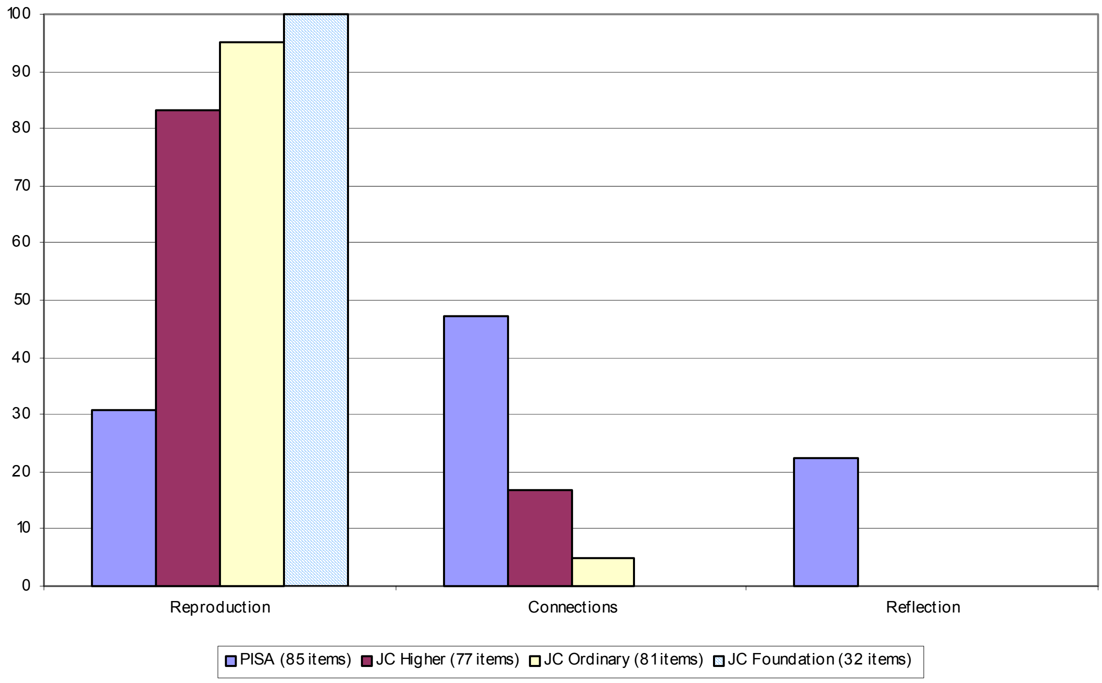
Figure 1: Percentage of items testing the four overarching Ideas distributed across PISA 2003 test and the 2003 Junior Certificate Higher, Ordinary, and Foundation Examination papers



Competency Clusters

In the mathematical process dimension, a major difference can be seen between the PISA tests and the Irish papers. There are considerable discrepancies across specific Irish papers in terms of the percentages of items in two of the three competency clusters (Reproduction and Connections), again reflecting the levels of the examinations. There are no items at all in the Reflections cluster. In PISA there is a reasonable spread of items across the three competency clusters. In the 2003 Junior Certificate papers, the percentages of items on Reproduction range from 83.1% (Higher level) to 100% (Foundation level); the mean of the figures is approximately 93% across the three examination levels compared with the PISA figure of 30.6%. The percentages of items on Connections range from 16.9% (Higher level) to 0% (Foundation level); the mean of the figures is approximately 7% across Higher and Ordinary levels compared with the much larger PISA figure of 47%.

Figure 2: Percentage of items testing the three competency clusters distributed across PISA 2003 test and the 2003 Junior Certificate Higher, Ordinary, and Foundation Examination papers



Situations

In the situations dimension a major difference can also be seen between PISA and the Irish papers. Across the three Junior Certificate papers the mean percentage of intra-mathematical items is about 66% versus 20% for PISA, and that for realistic situations is about 33% versus 80% for PISA.

As can be seen in Table 2 the results for the 1974 Intermediate Certificate examination papers in mathematics are reasonably consistent with the results of the 2003 papers in terms of percentages of items mapping into the four overarching ideas and the three competency clusters. In the case of the situations dimension, there are more intra-mathematical items on the 1974 papers than on the 2003 papers.

The results of the analysis of the 2003 Junior Certificate Examinations in mathematics are consistent with the results of mapping of the 2003 PISA items onto the Junior Certificate syllabus (Cosgrove et al., 2005), where the majority of the PISA items mapped onto the two Junior Certificate content areas of (i) Applied Arithmetic and

Measure and (ii) Statistics, while about three-tenths of the PISA items did not map onto any Junior Certificate content area. That study also found that most of the contexts of the PISA items were considered to be unfamiliar to Irish Junior Certificate students. In fact the correlations among PISA mathematics, science and reading performances (.80 to .85) were significantly higher than the correlation between PISA mathematics performance and performance of the same students on the Junior Certificate examination in mathematics (.70).

Discussion

The analysis highlights many of the mismatches between PISA framework and tests and the Junior Certificate examinations. In this section, some key aspects of this mismatch are discussed; consideration is then given to other issues raised by the comparisons.

Mismatch Between the PISA Tests and the Irish Examinations

As noted above, the results are consistent with the findings of the earlier study by Cosgrove et al. (2005) mapping the PISA items onto the Junior Certificate mathematics curriculum. With regard to mathematical content, the present study has indicated that there were about 50% fewer Uncertainty items in the Junior Certificate papers than in the PISA tests. However, this can be explained by the fact that the PISA category covers statistics and probability, but the latter topic does not occur in the Irish courses at this level (and likewise for a number of other countries participating in PISA). More significantly, the Junior Certificate examinations contained no items on Reflection, very few on Connections and over 90% on Reproduction. This points to a contrast, not only with the PISA framework, but also (albeit to a lesser extent) with the intended balance of the Junior Certificate examinations as described earlier. The imbalance is particularly notable in the case of the Ordinary and especially the Foundation courses. It might be somewhat less noticeable if the classification had been carried out with specific regard to the content of those courses; however, because of its focus on mathematical literacy, PISA explicitly ignores national course content and its variations, and considers item characteristics with regard to the mathematics needed for future life and work. On the other hand, the predictable layout of the Junior Certificate papers perhaps enhances the "Reproduction" aspect, in that students have extraneous clues as to the mathematics that they should use in specific questions. Of course this is not the case for a one-off international test such as PISA.

Classifications for the 1974 Higher and Lower Intermediate Certificate papers in mathematics using the PISA mathematics framework are reasonably consistent with the results of classifying the 2003 papers, suggesting little change relative to that framework. The results also draw attention to the fact that the Intermediate Certificate papers contain few realistic item situations or contexts. However, for the Junior Certificate papers there are rather more items with realistic contexts, pointing to some movement in the direction of greater contextualisation in accordance with the indication in the *Guidelines*.

The results raise a number of issues regarding the present Junior Certificate assessment and its heavy focus on the learning and practising of mathematical procedures. Algorithms or procedures and factual information can be memorised; alternatively, they can programmed into a tool such as a computer and stored. Hence, there is now less need for speed and accuracy by users in implementing these procedures, and more need for a deeper understanding of the conceptual underpinnings of the procedures in order to deal with problems that arise in using the procedures, or in determining when to use them or how best to modify them. The Irish mathematics curriculum, as is the case in many other countries, does not seem to have adapted very much to this development.

Readability

Given the PISA method of testing and consequent constraints of international surveys, it is inevitable that a considerable amount of text is involved in the presentation of items. Therefore the reading demands of items and item units can be considerable for many students, more so than in the Junior Certificate mathematics questions. An analysis of the readability of the English versions of the PISA mathematics tests using a simple measure of readability, the Flesch Reading Ease Formula, indicated that the reading grade level equivalent for the Mathematics tests was 7.7. (For the Science tests it was 8.8, and for the Reading tests it was 9.5.) The corresponding figures for the 2003 mathematics papers were 3.0 for the Higher level, 3.9 for the Ordinary level and 2.9 for the Foundation level. Of course, there have to be some reading demands; but they should not be so great that for some students the assessment constitutes more of a reading test than a mathematics test. Based on these figures it could be argued that the Junior Certificate mathematics papers are more accommodating than the PISA mathematics tests to students in this regard.

Time Considerations

With regard to the conflict between the requirements of large-scale international comparative testing such as PISA and the desire to test a broad range of mathematical competencies in what are often unfamiliar problem settings consonant with the PISA framework, it can be said that time allocations have always been an issue in standardised group testing. However, this is particularly the case in the PISA mathematics tests, where students are required to exercise a broad range of knowledge and skill on each item in a very short time – something like an average of 2.5 minutes per item. This can be compared with the much more generous time allowed in the

Junior Certificate mathematics examinations (in 2003, 5 hours for 77 items on Higher level papers, 4 hours for 61 items on Ordinary level papers and 2 hours for 32 items on the Foundation level paper). The contrast is all the more stark when it is recalled that the Junior Certificate examinations are considered “too short” to allow for much problem-solving activity. Of course there has to be some compromise between competing demands at the end of the day, but one would be particularly concerned about the group of students who could be called “slow thinkers” for want of a better descriptor. These are students who think slowly and carefully, but also deeply, about (mathematical) problems, and given a little extra time, compared with the more typical student, can solve PISA-like problems which they otherwise would not have been able to complete.

High Stakes Assessment

In some countries students take public examinations in mathematics (as well as other subjects) around the third or fourth year of their secondary education.. Examples include the General Certificate of Secondary Education in England and the Junior Certificate Examination in Ireland. Such examinations are considered to be high-stakes tests and there is considerable preparation for them in schools. In other countries, standardised tests may be administered to students and these could also be considered as high-stakes tests. If PISA is administered with the timeframe of preparation for such high-stakes tests, there is a possibility that students will not put in quite the same effort on the PISA tests as they would for their national tests, particularly if, as is the case in some countries (including Ireland), the PISA tests do not particularly resemble those examinations.

Influencing Curricula and Teaching

There is an obvious need for discussion and debate by policy makers, teachers, parents, and the wider public about the PISA mathematics approach. This is particularly relevant in the light of the forthcoming review of post-primary mathematics education, which may provide the first opportunity for some forty years to offer a fundamental critique of the purposes and style of such education, rather than to work basically within the confines of the present courses. In Ireland the current post-primary school mathematics programme is quite formal, abstract, and, it can be argued, not relevant or motivating for many of the students studying it. The enthusiasm and dedication of many teachers who deeply love their subject must be acknowledged; so too must the professionalism and expertise of the inspectors, and now of the members of the State Examinations Commission, who have worked so hard in order to produce examination papers that are as accurate and fair as possible. Nonetheless, despite their efforts, students may be motivated to learn mathematics, not by any intrinsic desire to study the subject, but often by the fear of getting a low grade on the school and public examinations. Mathematics in the classrooms tends to be isolated, by and large, from reality and taught or learnt as a discrete set of skills. Also, teachers may have a narrow view of realistic mathematics – they may see it as simply "shopping" or "house decoration" problems and so consider it as more appropriate for the "remedial" students. Others with a broader view of applications and realistic mathematics see them as being too difficult and time-consuming to cover in the short bursts of class-work in today's busy school timetable or to assess in traditional examinations. Students as well as teachers might find it hard to adjust from a culture emphasising speed and accuracy to a more process-focused approach. There are obvious implications for the difficult issue of classroom management. All these are major obstacles to be surmounted in promoting a more "realistic" approach to mathematics in Irish schools, if that is what we want!

Schools or school systems willing to implement a problem-based approach to mathematics teaching and learning will require training and resources to do so and even then it may take many years before teachers feel comfortable with it. Nonetheless, the authors believe that at least some move in this direction would ultimately be rewarding for students, teachers and eventually the wider public alike. However, that is for other people to decide; let the debate begin!

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Appendix: Classification of Items/Tasks in the 2003 Junior Certificate Higher Level Mathematics Paper I (Questions 1 and 2) Using the Overarching Ideas, Competency Clusters and Situations Dimensions of the PISA Mathematics Framework

1. (a) *Express 45 centimetres as a fraction of 15 metres and write your answer in its simplest form.* Content Area: Quantity (Applied Arithmetic) Competency Cluster: Reproduction Principal Competencies: (1) Recall equivalencies of metric measures. (2) Perform routine computations. Context: Intra-mathematical **1. (b)** *€6000 was invested at compound interest. The rate for the first year was 4% per annum.* **1 (b)(i)** *Calculate the amount of the investment at the end of the first year.* Content Area: Change and Relationships (Applied Arithmetic) Competency Cluster: Reproduction Principal Competencies: (1) Recall concepts of, and formula for calculating, simple and compound interest. (2) Perform routine computations. Context: Public **1. (b)(ii)** *At the end of the second year the investment amounted to €6520.80. Calculate the rate per annum for the second year.* Content Area: Change and Relationships (Applied Arithmetic) Competency Cluster: Connections Principal Competencies: (1) Recall concepts of, and formulae for calculating, interest. (2) Manipulate and reason with elements of formula for calculating compound interest (3) Perform routine computations. Context: Public **1. (c)** *The standard rate of income tax is 20% and the higher rate is 42%. Fiona has tax credits of €1493 for the year and a standard rate cut-off point of €30 000. She has a gross income of €31 650 for the year.* **1.(c)(i)** *After tax is paid, what is Fiona's income for the year?* Content Area: Change and Relationships (Applied Arithmetic) Competency Cluster: Reproduction Principal Competencies: (1) Recall concepts of, and formula for calculating, income tax. (2) Perform routine computations. Context: Public	**1. (c)(ii)** *What would Fiona's gross income for the year need to be in order for her to have an after-tax income of €29 379?* Content Area: Change and Relationships (Applied Arithmetic) Competency Cluster: Connections Principal Competencies: (1) Recall concepts of, and procedures for calculating, income tax. (2) Manipulate and reason with elements of formula and procedures for calculating income tax (3) Perform routine computations. (4) Construct a mathematical model of relationships among variables. (for some students) Context: Public

2. (a)(i) *List the first six multiples of 3 and the first six multiples of 5.* Content Area: Quantity (Number Systems) Competency Cluster: Reproduction Principal Competencies: (1) Recall concept of multiple. (2) Perform routine computations. Context: Intra-mathematical **2. (a)(ii)** *Hence, write down the lowest common multiple of 3 and 5.* Content Area: Quantity (Number Systems) Competency Cluster: Reproduction Principal Competencies: (1) Recall concept of multiple. (2) Perform routine computations. Context: Intra-mathematical **2. (b)(i)** *By rounding to the nearest whole number, estimate the value of* *1/(3.67) + (7.9)^2 x √16.32* Content Area: Quantity (Number Systems) Competency Cluster: Reproduction Principal Competencies: (1) Recall meanings of symbols in exponential notation (2) Recall estimation procedures (3) Perform routine computations. Context: Intra-mathematical *Then, evaluate 1/3.67 + (7.9)^2 x √16.32, correct to two decimal places.* Content Area: Quantity (Number Systems) Competency Cluster: Reproduction Principal Competencies: (1) Recall meanings of symbols in exponential notation (2) Perform routine computations using a calculator. Context: Intra-mathematical **2. (b)(ii)** *Simplify* *(3 √27 x 3)/(9^1/2 x 3^4) into the form 3^n where n* $\in$ ***Z***. Content Area: Quantity (Number Systems) Competency Cluster: Reproduction Principal Competencies: (1) Recall meanings of symbols in exponential notation. (2) Recognise equivalencies exponential notation. (3) Perform routine computations. Context: Intra-mathematical	**2. (c)(i)** *A = {1, 2, 3, 4}, B = {2, 3, 5} and C = {1, 3, 4, 5, 6}.* *List the elements of (A / B) U (C ∩ B)* Content Area: Change and relationships (Sets) Competency Cluster: Reproduction Principal Competencies: (1) Recall meanings of symbols in set notation. (2) Perform routine logic operations with sets Context: Intra-mathematical *List the elements of (A U B) ∩ (C / B)* Content Area: Change and Relationships (Sets) Competency Cluster: Reproduction Principal Competencies: (1) Recall meanings of symbols in set notation. (2) Perform routine logic operations with sets Context: Intra-mathematical **2. (c)(ii)** *U* is the universal set and *P* and *Q* are two subsets of *U*. *#U = 20; #(Q ∩ P) = x; #(P / Q) = 2x;* *#((P U Q)’) = 4; #Q = 2(#P)* *Represent the above information on a Venn diagram* Content Area: Change and Relationships (Sets) Competency Cluster: Connections Principal Competencies: (1) Recall meanings of symbols in set notation. (2) Connect symbolic and diagrammatic representations of set relations Context: Intra-mathematical *Hence find #Q.* Content Area: Change and Relationships (Sets) Competency Cluster: Connections Principal Competencies: (1) Recall meanings of symbols in set notation. (2) Connect symbolic and diagrammatic representations of set relations. (3) Reason with set relations and represent them as an equation Context: Intra-mathematical

Assessment of Mathematics in PISA 2003: Achievements of Irish 15-Year-Old Students in an International Context

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The Programme for International Student Assessment (PISA), an initiative of the OECD, is a large-scale survey of the achievements of 15-year-old students. In the second cycle of PISA, in which 41 countries participated in 2003, mathematics was the major focus. Results of the survey were first published in December 2004. Since there are notable differences between the approach to mathematics taken by PISA and the mathematics curriculum in Ireland, Ireland's performance is explored in more detail in this paper, particularly with reference to the Junior Certificate mathematics syllabus. The paper concludes with a consideration of how the results of PISA 2003 might be used to reflect on the mathematics curriculum in Ireland.

Introduction

The OECD's Programme for International Student Assessment (PISA) is now in its third three-year cycle (PISA 2006), where the focus of the assessment is on science. The focus in this paper, however, is on PISA 2003, where the principal domain of the assessment was mathematics. Students who participated in PISA 2003 also attempted test items pertaining to reading, science and cross-curricular problem-solving, but the majority of testing time was devoted to mathematics.

The objectives and design of PISA represent something of a departure from the most recent international survey in which Ireland has participated, the Third International Mathematics and Science Study (TIMSS) in 1995 (Beaton et al., 1996a, b; Martin et al., 1997; Mullis et al., 1997). First, the sample design is different. TIMSS examined mathematics and science achievements at both primary and post-primary levels. Intact-class sampling was used whereby two classes were sampled from the adjacent grade levels which contained the majority of students aged 13 ('Population 2') and which contained the majority of pupils aged 9 ('Population 1') (Foy, Rust & Schleicher, 1996). PISA defines eligible students on the basis of age, not grade. An age-based sample was chosen since the OECD argues that grade-based samples are not internationally comparable. Age-eligible students are usually spread across several grade levels (ranging in Ireland from second year to fifth year) and study programmes, and countries vary with respect to the modal grade (in Ireland, third year).

The OECD's choice of an age-based sample also reflects a change in the focus of the assessment itself. This is the second major difference between PISA and TIMSS. While all previous international assessments have made some attempt to ensure that the test is representative of the national curricula of participating countries, and/or have published analyses to show how performance relates to students' opportunity to learn the material tested, PISA is an explicit attempt to move away from school curricula and to assess what is deemed relevant for future participation in adult society. The aim of PISA is to assess the achievements of students at or near the end of compulsory schooling. The concept of 'literacy', actively using and building on information and skills learned as a basis for lifelong learning, is key here.

Hence, the OECD makes reference not to mathematics but to *mathematical literacy*, defined as

> … an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (OECD, 2003, p. 24).

The OECD further argues that

> The assessment is forward-looking, focusing on young people's ability to use their knowledge and skills to meet real-life challenges, rather than on the extent to which they have mastered a specific school curriculum. This orientation reflects a change in the goals and objectives of curricula themselves, which are increasingly concerned with what students can do with what they learn at school, not merely with whether they have learned it. (OECD, 2001, p. 14)

Describing the PISA approach as 'broadly oriented' (OECD, 1999, p. 9), the OECD justifies its approach on the basis of broad domain coverage and relevance to current and future needs outside the context of school. However, PISA's approach does not necessarily represent a solution to the question of whether the content of the assessment is equally fair or relevant in the countries participating in the assessment. We take this issue up further in the next section, which reviews in a little more depth the nature of the PISA mathematics assessment, and compares it with the Irish mathematics curriculum at lower post-primary level.

In total, 41 countries participated in PISA 2003, with testing in the spring of 2003 in countries in the Northern hemisphere, and in the autumn of 2003 in the Southern hemisphere (Table 1). Achievement results for all countries with the exception of the UK, which failed to reach the response rate standards (OECD, 2005), were published by the OECD in December 2004 (OECD, 2004a, b). Analyses in the national report for Ireland, published in full and summary form, complement those in the international report (Cosgrove et al., 2004, 2005).

Table 1. Countries Participating in PISA 2003

	OECD Countries		***Non-OECD Countries***
Australia	Hungary	Norway	Argentina
Austria	Iceland	Poland	Brazil
Belgium	Ireland	Portugal	Chile
Canada	Italy	Rep. Of Korea	China
Czech Republic	Japan	Slovakia	Hong Kong
Denmark	Korea	Spain	Indonesia
Finland	Luxembourg	Sweden	Russian Federation
France	Mexico	Switzerland	Slovenia
Germany	New Zealand	Turkey	Thailand
Greece	The Netherlands	United States	Tunisia
		United Kingdom	

PISA and Curriculum

Recently, the manner in which PISA assesses achievement has come under some criticism:

> There is a curious contradiction in the design of PISA. It is intended to be a knowledge base for policy analysis. Yet, it explicitly rejects attempting to assess what pupils have learned in relation to the school curriculum. This puts the onus on PISA to demonstrate that non-curriculum based tests can be used to derive policy conclusions for educational systems. (Smithers, 2004, p. 38)

Smithers (2004) argues that ignoring curriculum does not eliminate it as a factor to be considered, that cross-country differences in the degree of curriculum match is a source of bias in PISA, and the lack of a curriculum match analysis severely limits PISA's explanatory power. Prais (2003) argues that the results are "unlikely to be of specific direct help to schools, or to educational policy-makers" (p. 152). This 'curious contradiction' points to the need for curricular analyses to be carried out as part of the analysis and reporting of the PISA results at national level, and, ideally, at international level. The UK Department for Education and Skills (DfES) recently requested that the representatives on the PISA Governing Board consider participation in a multilateral study, part of which would compare the match between the PISA tests and the content of national curricula. The position of the DfES was that, in and of themselves, the PISA results are "… not sufficient to allow us to draw reliable conclusions about the impact of policies in different countries in recent years. We need a deeper understanding of why the distribution of outcomes which PISA revealed arose" (DfES, 2004, p. 4). However, so far the request has met with little interest and no success.

For those countries who are interested in the issue of curriculum, a lack of comparative information on how PISA matches the aims and content of national curricula is problematic. It appears that only eight countries have analysed the PISA results for their country with respect to the national curriculum.[1] These analyses are not comparable across countries; what might appear unfamiliar in one country might have high familiarity, relatively speaking, in another, and *vice versa*. With these difficulties and limitations in mind, we now review the PISA framework for mathematics and compare it with the Junior Certificate curriculum.

PISA Mathematics and the Irish Curriculum

PISA 2003 Mathematics Framework

The PISA definition of mathematical literacy and the accompanying framework are heavily influenced by the Realistic Mathematics Education movement, which stresses the importance of solving mathematical problems in real-world settings (e.g., Freudenthal, 1973, 1981). Central to this approach is the process of mathematising, i.e., starting with a problem situated in a real-world context, organising the problem according to mathematical concepts, trimming away the reality through such processes as generalising and formalising, solving the problem, and finally making sense of the mathematical solution in terms of the original situation. The framework distinguished between mathematical content and competencies.

There are four content area or 'overarching ideas': Space & Shape, Change & Relationships, Quantity, and Uncertainty. These broadly correspond to more traditional curriculum areas. For example, Uncertainty includes elements of probability and statistics. About one-quarter of the assessment tasks was devoted to each of these four content areas, which formed the basis of the reporting scales in PISA 2003. Results were also reported on an overall (combined) mathematics scale. All scales were scaled to have an OECD average of around 500 and a standard deviation of approximately 100.[2]

[1] This information was gathered via a series of emails sent by the first author to all PISA National Project Managers between June 2004 and July 2005.

[2] The 2003 mathematics scales were scaled with respect to achievement in 2000 (for use in comparing results in 2000 and 2003). Therefore, the OECD means and standard deviations are not exactly 500 and 100 in 2003.

The framework identifies three competency clusters: the Reproduction cluster, the Connections cluster, and the Reflection cluster. These are assumed to form a hierarchy, with problems in the Reproduction cluster requiring recall and application of routine procedures, and problems in the Reflection cluster requiring more advanced reasoning, abstraction, and generalisation to novel contexts. About 30% of items belong to the Reproduction cluster, 47% to the Connections cluster, and 23% to the Reflection cluster. However, the competency clusters do not form the basis of reporting scales.

This is necessarily a very brief overview. A fuller account can be found in OECD (2003; 2004a). Sample tasks with information on the performance of Irish students with reference to the OECD average are located at:
http://www.erc.ie/pisa/P03Sampleitems.pdf.

Junior Certificate Mathematics Syllabus and Examination

The Junior Certificate mathematics syllabus, which is examined at the end of third year at one of three levels (higher, ordinary or foundation), was revised in 2000 and examined for the first time in 2003 (Department of Education and Science/National Council for Curriculum and Assessment, 2000; 2002). Its structure has not changed substantially (Oldham, 2002), although a number of changes have been noted. For examples, there is now no choice on the examination papers, to encourage increased topic coverage; the appropriate use of calculators is recommended, and calculators are now permitted in the examinations; and geometry has undergone some refinements (see Department of Education and Science/National Council for Curriculum and Assessment, 2002, pp. 3-7).

Concepts are organised into topic areas: sets, number systems, applied arithmetic and measure, algebra, statistics, geometry, and functions and graphs. Higher and ordinary level students also study trigonometry. The study of probability (which features in items on the PISA Uncertainty subscale) is reserved for Senior Cycle.

Objectives of the current mathematics syllabus, which apply to all three syllabus levels, may be summarised as follows:

A. Recall of mathematical facts
B. Instrumental understanding
C. Relational understanding
D. Application of mathematical knowledge
E. Analysis of information, including that presented in unfamiliar contexts
F. Ability to create mathematics for oneself (e.g., make informed guesses)
G. Development of psychomotor skills to attain objectives

H. Ability to communicate mathematics

I. Appreciation of mathematics

J. Awareness of the history of mathematics.

Of the ten objectives, six (A, B, C, D, G and H) are assessment objectives, examined through the Junior Certificate mathematics examination, while the remaining four are not.

A comparison of the aims and objectives of the Junior Certificate mathematics curriculum and the PISA mathematics assessment, and of the PISA test items and Junior Certificate examination papers, suggests a divergence in what is learned and assessed. Divergence can be traced first to the fact that the Junior Certificate assessment objectives are likely to take higher priority than objectives which are not assessed. For example, the real-life approach to mathematical problem-solving in PISA implies that the ability to solve problems in novel, authentic contexts is an important prerequisite for most of the items. Indeed, a review of the PISA 2000 mathematics item set suggests that 97% of items had real-life relevance (Nohara, 2001). This skill is not apparent in any of the assessment objectives, although it is mentioned in the Objective E (which is not assessed). In the Junior Certificate, questions are usually presented in a purely mathematical and abstract context, almost always without redundant information. In the PISA assessment, questions are usually embedded in rich real-life contexts (Cosgrove et al., 2005). Second, Junior Certificate mathematics emphasises vertical mathematisation (developing increasingly complex mathematics concepts and skills in abstract contexts) (Oldham, 2002). PISA, in contrast, emphasises horizontal mathematisation (the application of mathematical concepts and skills to organise and solve a problem located in a real-life situation, and the abstraction of concepts and skills from these contexts) (OECD, 2003; Treffers, 1987). Third, it has been suggested that Junior Certificate mathematics tends not to tap processes associated with items in the PISA Reflection cluster (Cosgrove et al., 2005). Fourth, it has also been suggested, from a comparison of the PISA mathematics marking schemes (the majority of which award either zero points or one point per item) with the marking schemes of the Junior Certificate mathematics examination, that the marking of mathematics in the Junior Certificate Examination offers greater scope than the PISA assessment for recognising merit in students' work (Cosgrove et al., 2005).

Irish Student Performance on Pisa 2003 Mathematics

Performance on the Overall (Combined) Scale

Ireland achieved a mean score (503) on the combined mathematics scale that is not significantly different[1] from the overall OECD mean (501), yielding a ranking of 17th out of 29 OECD countries or 20th out of 40 OECD and partner countries.[2] Ireland's mean score is not significantly different from those of eight countries and is significantly lower than those of 10 countries, including Finland, Korea, the Netherlands, Japan, Canada, Australia, and New Zealand. Ten countries (including Poland, Hungary, Spain, and the USA) achieved significantly lower mean scores.

Performance on the Mathematics Subscales

The mean performance of Irish students on Space & Shape (476) is significantly below the OECD mean score of 496. Ireland ranks 27th of 40 countries overall, or 23rd of 29 OECD countries, on Space & Shape. The mean performance of Irish students on the Change & Relationships subscale (506) is significantly above the OECD mean score of 499. Ireland ranks 18th of 40 countries on Change & Relationships, and 15th of 29 OECD countries. Ireland's mean performance on the Quantity subscale (502) is not significantly different from the OECD average (501). Ireland ranks 21st of 40 countries on this subscale and 18th of 29 OECD countries. Ireland's mean score of 517 on the Uncertainty subscale is significantly above the OECD average of 502. Ireland ranks 13th out of 40 countries on Uncertainty and 10th of 29 OECD countries.

Distribution of Performance

Achievements were reported on categorical proficiency scales, which describe student performance at six levels, with students at Levels 5 and 6 demonstrating advanced mathematical skills and reasoning, and students at or below Level 1 described as lacking the skills required to build on for successful participation in adult life (OECD, 2004a). The distribution of the performance of Irish students on these proficiency levels, regardless of which mathematics scale is considered, tells a similar story: Irish performance is characterised by comparatively few low achievers, as well as comparatively few high achievers. For example, on the combined scale, 17% of Irish students compared with 21% of students on average across the OECD are at or below Level 1. Some 11% of Irish students are at Levels 5 or 6, while 15% of students on average across the OECD are at this level.

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[1] Statistical significance was evaluated at the .05 level with adjustments for the number of comparisons being made.

[2] It is important to note that measurement and sampling error introduce some uncertainty around point estimates such as country means, so the rankings are also prone to error. For example, Ireland's ranking on the combined scale with respect to the 29 OECD countries could range (with 95% confidence) from 15th to 18th.

Analyses of PISA Mathematics and the Irish Curriculum

What Approach was Taken in TIMSS?

The approach used in TIMSS 1995 to measure curricular coverage is similar in some respects to that adopted for analyses of the curricula in Ireland (Shiel et al., 2001; Cosgrove et al., 2005) and is described here in order to demonstrate how the analyses of Irish curricula build on the TIMSS approach. In TIMSS 1995, ratings of test items in terms of the intended curriculum, i.e., "instruction and learning goals defined at the system level" (Beaton et al., 1996, pp. A1-A2) were included as part of the international survey design the aim of which was to examine the effect of topic inclusion/exclusion on national curricula on student achievements. TIMSS obtained item-level data by asking mathematics and science curriculum experts in each country to rate each item as to whether or not the topic covered by the item was included in their intended curriculum or not. A dichotomous variable was then associated with each item in each country and it was possible to calculate and compare percent correct of all TIMSS items with percent correct of only those TIMSS items which, according to national curriculum experts, were covered in the intended national curricula.

There are some limitations to the methodology used in TIMSS 1995. First, using a dichotomous variable to indicate topic coverage/no topic coverage is overly simplistic. It does not account for the possibility that students might be familiar with some characteristics of an item (for example, the underlying concept) but not familiar with other characteristics of an item (for example, the context in which the concept is applied). Nor does it allow for the possibility that students may have had the opportunity to learn a topic in a broad, but not a detailed, manner, i.e., that there might be gradations of familiarity. Second, using the same rating for all students at a grade level does not account for differences in curricular coverage which are dependent on the academic track that each student is following. Differences by academic track were not examined in TIMSS until TIMSS 1999 (Martin et al., 2000a; Mullis et al., 2000). In Ireland at least, where Junior Certificate subjects are examined at different syllabus levels, a measure of curricular coverage should not only be multidimensional with respect to the properties of items, but should also be multidimensional with respect to syllabus level or academic track.

The Irish Test-Curriculum Rating Project

The test-curriculum rating project was carried out as part of the reporting plans for the Irish PISA data (Cosgrove et al., 2005). The framework comprises a 3 x 3 matrix whereby the three aspects of the items which are of interest (mathematical concept, context of application, and item format) are cross-classified with the three syllabus levels. Expected familiarity with the items was examined. Ratings range from 1 ('not familiar') to 3 ('very familiar') (Table 2).

Three individuals with extensive knowledge of the mathematics curriculum and/or teaching experience at post-primary level assigned ratings to the items. Initially, items were rated independently, and items on which there was a lack of consensus were flagged. 'Consensus' was defined as the modal rating assigned to a particular scale at a particular syllabus level where there was either perfect agreement across the three raters or, where there was disagreement, the difference did not exceed one scale point. Consensus was then reached on these items through discussion with all the raters. In addition to supplying the familiarity ratings, the concept underlying each PISA mathematics item was located in the mathematical topic area on the Junior Certificate in which it was most likely to be. Results were also aggregated to the student level and associations between expected familiarity with the item set attempted and actual achievement on PISA mathematics examined. The scales and the manner in which ratings were applied are described in more detail in Shiel et al., 2001 (pp. 224-232) and Cosgrove et al. (2005, pp. 269-270).

Table 2. Framework for the Test-Curriculum Rating Project

	Junior Certificate Syllabus Level		
Aspect	*Higher*	*Ordinary*	*Foundation*
Concept: How familiar would you expect the typical third year student to be with the specific mathematical concept(s) underlying this item?	Not/Somewhat/ Very Familiar	Not/Somewhat/ Very Familiar	Not/Somewhat/ Very Familiar
Context/Application: How familiar would you expect the typical third year student to be with the application of the specific mathematical concept(s) underlying this item in the type of context suggested by the item and stimulus text?	Not/Somewhat/ Very Familiar	Not/Somewhat/ Very Familiar	Not/Somewhat/ Very Familiar
Format: How familiar would you expect the typical third year student to be with the application of the specific mathematical concept(s) underlying this item in the type of format suggested by the item and stimulus text?	Not/Somewhat/ Very Familiar	Not/Somewhat/ Very Familiar	Not/Somewhat/ Very Familiar

Results

Table 3 shows the curriculum familiarity ratings for PISA 2003 mathematics for each of the three item aspects. Concepts underlying the majority of items at higher (69%) and ordinary (65%) levels were rated as somewhat or very familiar, while just under half of the items at foundation level (48%) were rated as somewhat or very familiar. In contrast, the contexts in which the mathematics problems were presented were rated as unfamiliar in the majority of items (66% at higher level, 71% at ordinary level, and 80%

at foundation level). Item formats were also largely unfamiliar to Irish students, regardless of syllabus level.

Table 3. PISA 2003 Mathematics Curriculum Familiarity Ratings, by Junior Certificate Syllabus Level

	Not familiar		*Somewhat familiar*		*Very familiar*	
	N	*%*	*N*	*%*	*N*	*%*
Concept						
Higher	26	30.6	21	24.7	38	44.7
Ordinary	30	35.3	25	29.4	30	35.3
Foundation	44	51.8	22	25.9	19	22.4
Context						
Higher	56	65.9	19	22.4	10	11.8
Ordinary	60	70.6	17	20.0	8	9.4
Foundation	68	80.0	14	16.5	3	3.5
Format						
Higher	53	62.4	21	24.7	11	12.9
Ordinary	62	72.9	17	20.0	6	7.1
Foundation	71	83.5	12	14.1	2	2.4

Concept familiarity ratings may also be compared both for the PISA subscale areas and for the three PISA competency clusters (Table 4). At higher and ordinary levels, Irish students were expected to be familiar with the concepts underlying the majority of the items on the Quantity subscale (74%), as well as on the Change & Relationships subscale (77% at higher, 73% at ordinary). Ratings on the Space & Shape items suggest moderate familiarity, while students were expected to be least familiar with items on the Uncertainty subscale. The majority of all Reproduction items are expected to be somewhat or very familiar to students at all syllabus levels, while ratings on the Connections items suggest moderate familiarity. Reflection items are somewhat less familiar to students, particularly at foundation level, where 74% of such items were rated as being unfamiliar.

Table 4. Percent of PISA 2003 Mathematics Items Rated 'Not Familiar' on Concept, by Content Area and Competency Cluster, for Higher, Ordinary and Foundation Level

	Higher	*Ordinary*	*Foundation*
Content Area			
Space and Shape	30.0	35.0	50.0
Change and Relationships	22.7	27.3	50.0
Quantity	26.1	26.1	39.1
Uncertainty	45.0	55.0	70.0
Competency			
Reproduction	19.2	23.1	38.5
Connections	35.0	37.5	50.0
Reflection	36.8	47.4	73.7

Table 5.	*Curriculum Area Ratings for PISA 2003 Mathematics Items Cross-tabulated with Junior Certificate Mathematics Syllabus Level*									
	Junior Certificate mathematics strand area									
	Not in Junior Cert.		*Number systems*		*Applied arith. & measure*		*Algebra*		*Statistics*	
Syllabus Level	N	%	N	%	N	%	N	%	N	%
Higher	26	28.6	8	8.8	30	33.0	5	5.5	18	19.8
Ordinary	30	33.0	9	9.9	29	31.9	4	4.4	16	17.6
Foundation	44	49.4	8	9.0	23	25.8	1	1.1	13	14.6
	Functions and graphs		*Sets*		*Geometry*		*Trigonometry*		*Total*	
Syllabus Level	N	%	N	%	N	%	N	%	N	%
Higher	4	4.4	0	0.0	0	0.0	0	0.0	91	100.0
Ordinary	3	3.3	0	0.0	0	0.0	0	0.0	91	100.0
Foundation	0	0.0	0	0.0	0	0.0	n/a	n/a	89	100.0

Note. Total number of PISA 2003 mathematics items = 85. As evidenced in the totals, 6 items were identified as being located in two Junior Certificate strand areas in the case of higher and ordinary levels, and 4 items in the case of foundation level.

Concepts underlying mathematics items were also classified according to which Junior Certificate mathematics topic area they best fit. This classification indicated that the Junior Certificate mathematics topic areas of sets, geometry, and trigonometry are not assessed at all by the PISA items. There is also little coverage in PISA of algebra, and functions and graphs. The majority of PISA mathematics items whose concepts *are* somewhat familiar to Irish students are located in the Junior Certificate mathematics topic areas of applied arithmetic and measure, and statistics (Table 5). Cosgrove et al. (2005) also compared the ratings in Table 5 for each of the four PISA mathematics content areas. Concepts underpinning the PISA items for each content area were, at times, distributed across several Junior Certificate mathematics topic areas. At higher level, for example, concepts underlying the 18 items associated with PISA Change & Relationships which were found to be on the Junior Certificate syllabus are spread across five Junior Certificate topic areas (number systems, applied arithmetic and measure, algebra, statistics, and functions and graphs). In contrast, almost all PISA Uncertainty items were located in the Junior Certificate topic area of statistics.

Limitations of the Irish Test-Curriculum Rating Project

While the test-curriculum rating project provides a framework to discuss similarities and differences between the PISA approach to assessing achievement and that used in State examinations at the end of lower post-primary school, it does not take into account the likelihood that numerous factors, other than the intended curriculum, affect student achievements. There are likely to be additional characteristics of the PISA assessment which are relevant to this analysis and which have not been included, such as the manner in which students' responses are marked or graded. Second, while it appears that PISA assesses some concepts and skills that are not on the Junior Certificate syllabus, it is also true to say that several aspects of the Irish syllabus are not assessed by PISA. Close and Oldham (2005) have classified Junior Certificate mathematics examination questions on dimensions of the PISA mathematics framework; this provides additional information on the extent of overlap between the two assessments. The results of these analyses should be considered in conjunction with the results presented here since the combined results present a more complete picture of areas of overlap and divergence. Third, the results are interpretable only in a national context so no conclusions may be drawn about the relative level of expected familiarity with the assessments. For example, the low familiarity of Irish students with PISA 2003 Uncertainty items may not be so low, relatively speaking, in an international context. Oldham (2002) has noted that the teaching of concepts relating to probability is reserved for upper secondary level in several other PISA countries. Fourth, student ability and curriculum familiarity may be confounded. Once an adjustment is made for student ability, the relationship between expected familiarity with the concepts underlying the test items and students' achievements is much weaker, and no longer statistically significant.[1] This indicates that, while the data may be useful in *describing* differences and similarities between PISA and the Junior Certificate at the item level, they are of limited use in *explaining* student achievement.

Conclusion

Despite considerable differences in the philosophies underpinning the Junior Certificate and PISA, which are most evident, perhaps, in the manner in which mathematics problems are contextualised in the assessments (and hence, the amount of mathematising required), Irish students performed around the OECD average on PISA mathematics, which is perhaps better than expected. Students in Ireland performed significantly above the OECD average on Uncertainty items. While items assessing probability were unfamiliar to Irish students because the topic does not feature until Senior Cycle, this is also likely to be the case in some other countries. Moreover, skills

[1] A comparison of two ordinary-least-squares regressions which examine the association of familiarity with mathematics concept with student achievement on PISA 2003 mathematics with and without an adjustment for student ability as measured (somewhat inadequately, perhaps) by performance on the Junior Certificate mathematics examination) indicates that the association between curriculum familiarity and performance is much weaker adjusted ($r = .03$; $p = .169$) than when considered alone ($r = .37$; $p < .001$).

underlying other Uncertainty items, which require interpretation of graphed data, may well be learned in other subject areas (e.g., science, geography), especially given the real-life and cross-curricular nature of the PISA assessment (see Nohara, 2001). The comparatively low performance of Irish students on Shape & Space items might be attributable to the fact that the majority of items which were deemed unfamiliar to Irish students in the rating exercise assess spatial visualisation skills, which do not feature on the Irish mathematics syllabus. In any case, the nature of the Space & Shape items merit closer scrutiny, given the particularly weak performance of Irish students.

The low explanatory power of the curriculum rating data (once student ability is controlled for) is perhaps disappointing, but nonetheless consistent with the low explanatory power of curriculum familiarity measures reported elsewhere (Floden, 2002). It suggests that the analyses are better suited to profiling PISA mathematics at the item level rather than explaining performance at the student level. Additionally, other factors which have not been captured in the curriculum familiarity ratings should be considered, such as the marking schemes of the assessments. Other aspects which might be expected to affect item difficulty, such as the extent of multi-step reasoning, may also be worthy of consideration (Nohara, 2001). Moreover, the ratings are global and impressionistic (expected familiarity) rather than having their basis in a more quantifiable measure (e.g., expected number of mathematics lessons devoted to a topic or concept, or proportion of the Junior Certificate mathematics paper devoted to a topic or concept). It is also be the case that a cross-sectional survey design is not optimal for explaining student achievements with respect to what they may have learned. In this respect, longitudinal surveys are better, since they are capable of capturing achievement gains (as opposed to a single, static measure as in PISA) (Goldstein, 1995).

The national symposium for PISA 2003, held in April, 2005, included a lively discussion among participants about the nature of PISA mathematics and how the results might be used to improve mathematics education[1]. It was noted and generally agreed that PISA is not a curriculum model, nor is it a guiding philosophy. One can look to PISA if one is interested in curriculum reform, but there is also a need to look outside PISA. It was also agreed that an effort is required into better understanding and changing classroom and national culture *vis à vis* what mathematics is and why mathematics matters. If, however, it is decided to incorporate elements of Realistic Mathematics Education into the national curriculum, a debate about, as well as an awareness of, the dangers and the benefits is required.

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[1] Presentations from the symposium may be accessed at http://www.erc.ie/pisa/PISA_symposium.html.

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Timeo Danaos: Science Teachers Teaching Mathematics

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In Ireland, mathematics is commonly taught by qualified science teachers due to the shortage of mathematics graduates entering the teaching profession. A further complication is that many of the science graduates may be biologists who traditionally have experienced a specialised emphasis on biometrics rather than the physical sciences. The minimum requirement for teaching mathematics to junior post-primary level is to have taken a module in mathematics to at least first year in university, whereas to senior post-primary level the requirement is at least second year university mathematics. However, this university level mathematics is unlike that taught in school in depth, breadth and rigor and student teachers may approach the teaching of mathematics at junior level constrained by their experiences of senior post-primary and university level maths, teachers and institutions. In short, student teachers who qualify to teach maths feel obliged to imitate their strongest role model whether positive or negative. This work outlines how practical hands-on activities were designed using principles based in science education and applied to mathematics education. Two further aspects discussed are history of mathematics with particular emphasis on Hellenistic and Classical Greek writers and the students designing and making their own apparatus.

Introduction

The Focus of This Study

This research is focused on the relationship between the teaching and learning of mathematics vis-à-vis science teaching and learning and where they are compatible. The subjects of study are scientists teaching mathematics. Broadly speaking, the subject of this paper will be an approach that was taken over the last four years in dealing with the mathematics pedagogy section of the 2nd Year BSc course which in turn was based on my own teaching of science, mathematics and philosophy in Transition Year at Ashbourne Community School, Co. Meath.

The reason that the students have to do a course in mathematics pedagogy is pragmatic. Science teachers in Ireland are often, if not always, called on to teach mathematics at least to junior level in secondary school. Subjects may be taught to the junior level in secondary schools if it has been studied up to second year at university; whereas a subject to be taught at senior level in secondary school must have been studied in the third and fourth year at university.

I am interested in certain questions that this raises. Does a science teacher teach mathematics in a way that is different from trained mathematicians? Science teachers are used to dealing with a good deal of concrete materials and a huge literature has been built up dealing with a constructivist, investigative, hands-on, collaborative, meta-cognitive, or child centred approaches to teaching science. Do science teachers use this experience in teaching mathematics?

Prior experience of mathematics.

One approach to answering part of this question is to examine the prior experience of maths in the student teacher assuming that prior experience frames or contextualises the current state of action or being:

"*We all carry, as teachers and learners, our own bags*
and baggage of knowledge, beliefs and attitudes
with us into any new situation" (Crook and Briggs, 1991).

This may seem a truism, and as such it may be overlooked or felt that such a truth cannot be handled in the context of teacher education. We stress things to know rather than attitudes to inform since the former are thought to be easier to 'acquire' and study for in an exam. Crook and Briggs (1991) describe how in analysing teachers' responses relating to their prior experience while being interviewed, the teacher paradoxically appeared to have had negative experiences in their own mathematical education. They also reported that in one piece of research work undertaken for the 'Cockcroft' Report (1982), as many as 50% of people approached to be interviewed about mathematics in everyday life refused.

The aim of this work is to explore one approach for improving the teaching and learning of mathematics at junior post-primary level. Much work has been done regarding teachers' beliefs regarding the teaching and learning of science and their science content knowledge (Hewson et al., 1995; Smith and Lloyd, 1995; Lloyd et al., 1998; Lenton and Turner, 1999; Smith, 1999; Parker and Heywood, 2000; Levitt, 2001). However, little has been done in Ireland regarding science teacher's beliefs regarding the teaching and learning of mathematics. Crook and Briggs (1991) believe that ones' prior experience (and the subsequent retrospective perception of that experience) of mathematics will contribute to the emerging teachers' views on the teaching and learning of mathematics. Tobias (1978) relates how everyone is exposed to a wide variety of cultural influences, some of which are mythical. Thompson (1984) believes that there is a strong relationship between the conceptions of mathematics held by a teacher and the teaching and learning styles used by that teacher. Because of this, Thompson (1984) is supportive of the idea that the (student) teacher needs to unpack or expose their prior experience so that they can perceive the extent of a particular experience on their own teaching and learning style.

Thompson (1984) relates how most research (in the 1970s) on the relationship between the effectiveness of mathematics teachers and their knowledge focussed on the teachers' knowledge of mathematics. Begle (1972, 1978); Eisenberg (1977); Clark and Peterson (1976); Jackson (1968); Mackay and Marland (1978); Morine-Dershimer and Vallance (1975) (cited in Thompson, 1984) questioned an original assertion that everything a teacher does is rational and reflective: they reported findings that teacher behaviour is mostly instinctive and intuitive. A pessimistic worldview would suggest that we are enslaved or trapped by our past experience since it is this that frames the present. I would suggest that this might be the case until an alternative is experienced (and makes sense) to the student teachers, thus the second element of this paper – the activities – is one alternative way. But a caveat is needed, sometimes we are happy to be in a trap, and we do not want to be set free.

Methodology and Results

Student Teachers' Personal Biographies

Student teachers (n=52) on the Bachelor of Science in Science Education course from three successive years were asked to write a narrative account of their own experience of mathematics in primary, secondary and tertiary levels. The students were directed to describe both positive and negative experiences and to write in a personal reflective style. They were asked to keep the narrative to two sides of A4 single-spaced word-processed text. Finally, they were asked to summarise their narrative as a time-line. The responses were read and categories of significant instances recorded. A tally was taken of the number of times the instance was mentioned. On completion of this categorisation, categories with only one occurrence were included under the most similar except where this was illogical. The tallies are represented in Table 1.

Table 1. Per cent occurrence (n=52) of categories in student teachers' mathematical biographies

Significant feature of maths learning	positive %	negative %
mathematics teachers I had	21	11.5
having to rote learn the times tables	8	13.5
higher achievers were not catered for	0	9.6
level of teacher's preparation	0	3.8
change of teaching method when changing teacher	0	9.6
that there is always a right and wrong	3.8	1.9
any other content besides the times tables	0	9.6

It is clear that the most negative experience of these groups of student teachers was the role of the teacher (see excerpt 1) followed by learning (or failing to learn) the 'times tables' (see excerpt 2). The timelines, because of their nature as linear outlines of events tend not to contain deep reflections but merely evaluations. Consistently, learning mathematics in primary school is dominated by the times tables (Figure 1) as a significant memory even though there was obviously other mathematics taught.

Figure 1. An example of a time line of a student's personal history of learning maths.

"*Overall I believe that the attitude of the teachers and their expectation of me had the greatest influence on my learning of maths. I believe that a good pupil-teacher relationship is necessary to form a basis from which it is easy to relate to each other and in this way to learn from each other. Learning maths is much more enjoyable if a pupil finds the teacher helpful and interesting rather than criticising and condescending.*"
(Student 51639421)
Excerpt 1

"*To this day I can't multiply or divide automatically and have to employ a certain amount of working out, for example to multiply eight by nine, I have to multiply eight by ten and subtract eight*."
(Student 51152801)
Excerpt 2

Crook and Briggs (1991) described as follows the main categories of their research:

- The role of the teacher
- The change from primary to secondary school
- Personal failure
- Gender
- Choice of teaching course
- Turning points

Of these, gender and choice of teaching course did not arise here, the second not being relevant except that many of the students who had had a bad experience of mathematics in secondary school purported a change in their attitude on entering university due to the role of the teacher. I would further assert that problems associated with a change of school, issues regarding personal failure, gender may also be related the role of the teacher. From the results of the survey, *maths teachers I had*, *higher achievers not catered for*, *level of teacher's preparation* relate to the role of the teacher *á la* Crook and Briggs (1991) and in part *change of teaching method* (though this was again in part relevant to the *change from primary to secondary school* or from junior certificate to leaving certificate). *Having to rote learn times tables* as a negative attribute could relate to *personal failure* and interestingly, the positive attribute of *learning tables* intersected with those who saw *that there is always a right and wrong.*

The Activities, an adaptation of a Transition Year module in Mathematics and Science
In response to biographies from previous years, activities were selected (Appendix 1) that could challenge the traditional model of the mathematics teacher in Ireland, that is one who teaches without concrete materials in teacher directed, teacher-centred, drill-based approach. That is not to say that such a model is wholly inappropriate. It is a truism to say that a teacher should have a battery of models of teaching and learning at her/his disposal as a realisation of the idea that not all children learn in the same way all of the time. Separate from this was the issue of the times tables, typically discussion around the times tables became polarised much as the results in Table 1 represented. One session in the B.Sc. course was spent on an introduction to Trachtenburg's algorithmic system of arithmetic (Cutler and McShane, 1978) with the point of

demonstrating that failing to rote learn the times tables may not be a failure in mathematics.

Using the history of a subject in order to teach it is not new and a cursory scan of websites for various third level institutions reveals many course with modules dealing with history e.g. Engineering Department at the University of Patras. In Greece, for example, there is a compulsory component in science on the history of science. However, it does appear to be missing in teacher education courses – student teachers usually have to take modules in the history of education; however many new syllabi have provisions for the 'nature of science'. Dimitriadis *et al.* (2001) proposed a historical approach to the teaching of the nature of science for trainee primary teachers and they found that a complete approach of history, philosophy and methodology of science led not only to the improvement of the teachers' views in relation to the nature of science, but also to a relatively sufficient appreciation of the content. In spite of this, Abd El Khalick *et al.* (1998) have questioned such an approach.

The student teachers on the module tend to believe that mathematics (and science) is fixed and is infallible – there is always a 'right' or 'wrong' answer: in other words, it has no history, if we take history to be a record of its development. One thread, then of challenging students' pre-conceptions of mathematics (and science) is to explore the historical development of it as a discipline because many of the protagonists in the development of laws or norms made 'incorrect' claims also.

It is particularly noticeable that this idea of a fallible maths and science is problematic for the student teacher. They want to know how to teach with an authority that will not be questioned let alone start to question 'mathematics' itself. Thus when Pythagoras' theorem is revised but placed in its historical context of Greek philosophical project, Pythagoras and thus his (mispatronymic) theorem seem less than absolute. Indeed, philosophers such as Aristotle appear to admire mathematics in its early forms but point out that the 'so-called Pythagoreans' thought that the principles of mathematics were the principles of all things and thus they attributed moral, political and cosmological meanings to numbers (Cuomo, 2001 p34-35), thus we should be wary of some (Ancient) Greeks some of the time.

Conclusion

Ultimately the aim of this module and the way it is taught is to produce students who will use their own skills to teach mathematics well but yet incorporate observed best practice. What constitutes best practice is open to question and ensuring it is observed or experienced at a deeper level is problematic also. The student teachers expressed an initial reluctance to break out and teach mathematics in a reconstructed and new way from their experience given the well-documented phenomenon of newly qualified teachers adopting the praxis of an internalised role model. It can take years to develop one's own authentic praxis and some may not choose to. At the end of the course, the students expressed the view that the notion of transferring their skills in teaching science to the teaching of mathematics was something they wished to explore further (as evidenced by their course evaluation sheets). They also thought the idea novel and had not considered even its possibility previously.

Further work is needed in this area. How the student teacher generates their 'role constructs' as a science teacher vis-à-vis mathematics teacher could be examined in more detail using Repertory Grid Analysis. This would also elicit their views on the nature of the two subjects and allow further refinement of the course. Also, how are these role constructs different or similar to those of a fully trained mathematician? How much of the student teacher's reliance on externally generated role constructs is further constrained by the use of text books which for the most part retain a 19th century format in Ireland needs to be examined further also. Textbooks and the preconceived ideas of publishers, teachers, students and their parents, as to what constitutes a 'good' textbook, hold such an authoritative place in education in Ireland that even 'new' textbooks are highly conservative.

Finally the efficacy of the course of activities in the history of mathematics needs to be tested in a quantifiable manner. Changes in motivation, the ability to think transferring ideas between the science and maths domains, and the ability to adapt existing resources to their own context could be starting points for research.

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Appendix 1. Short Course in the History of Mathematics (with science and technology)

1. Number systems: binary, decimal, duodecimal Babylonian, Sumerian, Arabic and Roman

***Practical**: construct an abacus and a sundial*

2. Democrates, Anaxamander, Thales, Pythagoras – what is the world made of; summarising the universe in mathematics, the magic of numbers. The attempt to 'Square the Circle'. Trigonometry Theorems, axioms, propositions and corollaries – what are they?

Practical*: use of the sextant in navigation (Bauer, 1994), construct and use a clinometer and trundle wheel;*

3. Ptolemaic world and cosmological views of the ancients the importance of realising that knowledge is fluid and dynamic, ideas 'wax and wane'

The mathematics of ellipses, Kepler's 3rd Law

Practical: *build an armillary sphere for the Ptolemaic, Brahean, Copernican, and Modern views of the solar system using handicraft/recyclable materials.*

4. The Great Library of Alexandria II: Eratosthenes and the circumference of the earth (as described by Sagan (1991) in the Cosmos television series).

Mathematics: The importance of hands-on experimentation in mathematics, the role of error and estimation, proof and deduction

Practical*: model of Alexandria – Cyene landscape, and using a lamp, model the differences in shadow length of the obelisk and position of the sun over the well: how did Eratosthenes find the circumference of the Earth?*

5. The Great Library of Alexandria I: Aristarchus and the size of the sun and the moon manufacture and use of sundials, latitude and longitude, maps, globes and projections.

Mathematics: scale, proportion, ratio, co-ordinates, angles

Practical: *print-outs of various projections and attempt to fit to papier-mâché spheres*

6. Greek science: alchemy in general, the Leiden Papyrus on mineralogy, dyeing, precious metal augmentation.

Practical*: extraction of silver and lead from Galena.*

8. Greek technology: The Antikithyra Mechanism

(de Solla Price (1974) and Sfetsos (2003) The earliest surviving gear mechanism is the Antikithyra Mechanism, which dates from the 1st century B.C.E. This artefact provides the initial impetus into a discussion of gears and then leads on to experimenting with gear wheels to induce rules concerning ratios of the cogs.

Practical: The activity with gears is taken from the Cognitive Acceleration through Science Education (Adey et al., 1995), project, Activity 6.

Mathematics Teacher Education

An Exploration of the Mathematical Literacy of Irish Students Preparing to be Primary School Teachers

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Concerns about how mathematics is being taught in Ireland have surfaced occasionally (Government of Ireland, 2002a, b) but a discussion of the mathematical skills, which Irish primary teachers bring to the teaching of the mathematics curriculum, has not yet taken place. Nor has the nature of mathematical skills and knowledge actually needed to teach the primary mathematics curriculum been discussed. Like the elephant in the corner, teachers' levels of mathematical literacy are problematic but invisible and ignored. This paper seeks to initiate discussion by shedding some light, albeit in a small way, on a hitherto hidden variable of mathematics teaching and learning in primary schools in Ireland.

Introduction

Ireland is proud of her educational system. A recent policy document boasts of Ireland's "strong record of commitment to education" and advocates building on this "competitive advantage" (O' Driscoll, 2004, p.73). Her economic success in recent years has been attributed in part at least, to a highly educated workforce who has attracted foreign investment (Friedman, 2005). Irish 15-year-olds achieved a mean score of 515.5 on the reading literacy scale in the Programme for International Student Assessment (PISA), (Cosgrave, Shiel, Sofroniou, Zastrutzki and Shortt, 2005) giving Ireland a ranking of 7th out of 40 Organisation for Economic Cooperation and Development (OECD) countries. Yet there are intimations that all is not well on the mathematics education front. PISA 2003 with its concentration on assessment of mathematics places the performance of Irish 15-year -olds (mean score of 502.8) as not significantly different from the OECD country average of 500 points on the combined mathematics scale. This score ranks Ireland about 17th out of 29 OECD countries and does not differ significantly from PISA 2000 results where mathematics was a minor domain (Shiel, Cosgrave, Sofroniou and Kelly, 2001). It is noteworthy that six of the seven countries with mean scores that are not significantly different from Ireland's in reading literacy all have significantly higher mean scores in combined mathematics (Cosgrave *et al.*, 2005, p. 72).

Expressions of concern about mathematics achievement at primary level are low key but consistently critical. Findings of *The 1999 National Assessment of Mathematics*

Achievement (Shiel and Kelly*,* 2001) indicate that the test items in which students in fourth class in primary schools performed least well were those requiring higher-level mathematical processes. Schools inspectors (Government of Ireland, 2002b, p.19) express concern that there is insufficient “application of mathematical skills and concepts to practical situations” evidenced in schools. Three separate studies into the implementation of the new curriculum (Gov. of I., 2005 a, b; National Council for Curriculum and Assessment, 2005) compound fears that the teaching and learning of mathematics is problematic in many Irish schools.

Freudenthal (1991) calls the “wide gap between what is claimed to be taught and accepted as being learned the big lie”. Rowland, Martyn, Barber and Heal (1998), in the context of mathematics teacher subject knowledge spoke of the need to “Mind the gaps”. This very phrase has yet another meaning in the Irish mathematics education context where there are potentially detrimental gaps between the mathematics curriculum as written and the curriculum as it is being implemented in schools (NCCA, 2005). The possible flaws in Irish mathematics teaching have been largely ignored to date, possibly because in a political climate where teacher unions are highly, influential teaching has become “invisible and silenced, the silent discourse of the reform process” (Sugrue, 2004, p.191). The powerful social context identified by Ernest (1988) as creating a barrier between espousal and enactment of a constructivist model of teaching mathematics is evident in Ireland, in all its manifestations (Lyons, Lynch, Close, Sheerin and Boland, 2003, p. 366). Redemption, as “the autonomous mathematics teacher” is possible in Ernest’s eyes through higher level thought which enables the teacher “to reflect on the gap between beliefs and practice, and to narrow it”. Teachers must first acknowledge that such a gap exists. Like the elephant in the corner, pre-service teachers’ levels of mathematical literacy are often problematic but invisible and ignored. The study reported here is part of a larger mixed methods (Burke Johnson and Onwuegbuzie, 2004) pilot study of Irish pre-service primary teachers’ subject knowledge in mathematics.

The Context of the Study

St Patrick’s College in Dublin has an annual intake of 400 Bachelor of Education (B Ed) students embarking on a three-year primary teacher preparation course. Places are awarded on a points system based on leaving certificate examination results aggregated over six subjects. English, Irish and mathematics are compulsory subjects for student teachers. Entry requirements to the B Ed course when the students in the study enrolled in 2002 showed the median was 480 points out of a possible 600. Typically, B Ed students belong within the top quintile of leaving certificate examination results. A further 100 students pursue a Post Graduate Diploma in Education (PG) over an eighteen months course. These have already acquired a primary degree in domains as diverse as Nursing Studies and Agricultural Science. A minimum requirement of grade D3 on the ordinary level mathematics paper in leaving certificate (currently worth five points out of a possible 100 points for mathematics) obtains for all students although almost all students entering teacher education have achieved higher scores in mathematics than is required for entry. Three class groups (N =105 students) were invited to participate in the study according to the day on which their mathematics education workshops fell. A 'convenience sample' of B Ed and PG students (N =71) agreed to participate and students were free to identify themselves or not. There are some differences in education and experience between the two student groups at course intake but the disproportion of second year B Ed (N = 40) to PG (N= 31) students participating makes it difficult to compare samples in a meaningful way. Mathematics education courses in college are similar for both groups of students and no distinction is made between teachers once they have been appointed to schools. As a consequence the two cohorts have been merged for purposes of data analysis and reporting.

The Test Instrument

The innovative philosophy of mathematics underpinning student assessment by PISA is well documented, with its interlinking of skills, content and context (OECD, 1999). The PISA mathematics framework uses four overarching ideas to explore mathematical literacy and at the time of this study only eleven items were released for public use following the PISA 2000 assessment (OECD, 2002). These items relate to two mathematical domains, Growth and Change and Shape and Space. The framework has ranked the items in levels of difficulty from one to six, the higher levels requiring higher order reasoning and thinking skills. (See *Appendix One* for a schedule of items). A twelfth related item, *Continent area 1*, which had been used in an earlier PISA unpublished pilot study was included[1]. In the booklet given to students in this study, each PISA item was preceded by a five point Likert scale where students were invited to assess their confidence in tackling the item and followed by a four point scale to measure its relevance to the primary mathematics curriculum as perceived by the student teachers. An analysis of this aspect of the study will be reported elsewhere. Students participated in the study during the first month of their first mathematics education course so the level of mathematical subject knowledge exhibited could be attributed to their previous mathematical experiences, rather than to their college mathematics education courses. Such a situation might be a fit for the notion of mathematical literacy envisaged by PISA:

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[1] Dr. Seán Close, personal communication, November 13 th 2002.

> An individual's capacity to identify, to understand, and to engage in mathematics and make well-founded judgments about the role that mathematics plays, as needed for an individual's current and future private life, occupational life, social life with peers and relatives, and life as a constructive, concerned and reflective citizen (OECD, 2002, p. 82)

This knowing and using of mathematics “in rich contexts” (Freudenthal, 1991) is germane to the constructivist aspirations of the primary mathematics curriculum, and there is a strong link between the test items used by PISA and the realistic, problem based mathematics envisaged by that curriculum. High levels of mathematical literacy might be expected among student teachers.

Data Analysis

Data generated by the study are analysed in three ways. In general, the scores of the cohort as a whole are discussed for the intimations they give of the strengths and weaknesses of mathematical literacy of the student teachers. Secondly, students’ responses to two items are studied in greater depth. Thirdly, a mathematical literacy profile is developed for two students selected at random from the data set. Student teachers are more than twice as likely to have studied leaving certificate mathematics at ordinary level than at honours level. Grades achieved are much more likely to be A or B at ordinary level (worth between 60 and 35 points) and C or B at honours level (worth between 65 and 80 points). The research investigates how these generally high achieving students after a minimum of thirteen years studying mathematics perform on a small and admittedly limited selection of PISA items.

Table 1: Cross tabulation of number of students achieving each leaving certificate grade against examination levels. Points awarded to each grade are in parentheses.

Leaving Cert Level	*a*	*b*	*c*	*d*	*Grade Undisclosed*	*Totals*
Undisclosed					2	2
Honours	1 (90-100)	6 (85-75)	8 (70-60)	3 (55-45)	1	19
Ordinary	21 (60-50)	20 (45-35)	6 (30-20)	0 (10-5)	3	50

It could be expected that students in teacher education programmes would perform well on any test designed to test the competences of all students towards the end of compulsory schooling. However, an initial analysis of these students’ leaving certificate scores in mathematics vis a vis their scores on the PISA items indicates that higher scores in leaving certificate mathematics do not guarantee high scores on this PISA test.

Achievements of Irish students on PISA 2003 (Cosgrave *et al.*, 2005) were scaled using Item Response Theory (IRT) model, which facilitates efficient summarising of data when a rotated test design has been used. Mean country scores are provided and also proficiency levels for the combined mathematics domain and the sub domains contained therein. For the purposes of this study, only the proficiency levels for the combined subscales of Shape and Space and Growth and Change (renamed Change and Relationships in PISA 2003) are used in the analysis of data provided here. IRT scaling used by PISA 2000 (OECD, 2002, p.88) established the following levels of task difficulty and the eleven test items used in the study were ranked accordingly.

Table 2: The PISA Tasks arranged according to proficiency levels and the percentages of students who offered correct answers to each item

Proficiency Levels	*Mathematical Tasks*	*Descending order of difficulty*	*Student teachers % correct*	*Irish 15 year olds % correct*
Level 6	Modelling of complex problem situations and advanced reasoning	Apples 3 Continent area 2	42.3% 21.1%	9.2% 19.3%
Level 5	Work strategically using broad, well developed thinking and reasoning skills	Racing Car 4 Apples 2	26.8% 67.6%	18.5% 23.3%
Level 4	Select and integrate different representations. Construct explanations based on own interpretations, arguments and actions	Apples1	80.3%	43.2%
Level 3	Execute clearly described procedures, including those that require sequential decisions.	Triangles Farms 2 Farms 1 Racing Car 1	70.4% 80.3% 94.4% 84.5%	52.2% 60.9% 68.6% 66%
Level 2 [1]	Employ basic algorithms and make literal interpretations of the results	Continent Area 1	66.2%	No data available
Level 1	Basic mathematics tasks, in contexts where all relevant information is presented and questions are clearly defined.	Racing Car 3 Racing Car 2	98.6% 98.6%	81.4% 85.9%

Student teachers' responses to items were scored using the PISA scoring rubric provided. Irish student teachers' raw scores were mapped to the six proficiency levels

[14]No item was released at proficiency level two so an 'extra' item, *Continent area 1,* was arbitrarily assigned to this level without IRT scaling.

and the ‘extra’ item already mentioned was included to approximate to a task at proficiency level two for which no item was released. Limited conclusions are drawn about their mathematical literacy for primary teaching from the percentages of students who succeeded in answering a particular question at a particular proficiency level.

The three mathematical competency clusters defined by PISA (Cosgrave *et al.*, 2005) are deemed to be hierarchical in nature, with the ‘reproduction’ cluster one, the easiest. Five items used in this study relate primarily to competency cluster one, six to competency cluster two and one only, *Apples 3* to competency cluster three.

General Findings

Some student teachers do very well on PISA with one student scoring 100% and a further three scoring in excess of 90% on the twelve items. There could be said to be a ceiling at proficiency level 4 for up to 80% of students. There are similarities between the student teachers’ success at task difficulty levels five and six on this PISA test and the achievements of their 15-year-old compatriots in PISA 2003, where Irish students generally were less well represented on items requiring higher order mathematical skills represented by items at levels five and six (Cosgrave *et al.*, 2005).

The student teachers in this study appeared to find the second of the tasks at proficiency level six, *Continent area 2*, which seeks to access estimation skills, more difficult than the generalisation item, *Apples 3*. In this respect they mirror the pattern of scores achieved by Irish 15-year-olds, although in percentage terms, more than four times as many student teachers as potential school leavers scored the full mark[1]. Less than a quarter of the student teachers (N = 15) achieved success on the ‘easier’ of the proficiency level six items and less than half of them (N = 30) achieved success on the ‘more difficult’ of the two items. Just over a quarter of them (N =19) achieved success with the more difficult of items on proficiency level five. This item, *Racing car 4,* was designed to access skills of data analysis from different representations. Considerably less than three quarters of student teachers (N = 48) achieved success on the second proficiency level five item. (See Table 2 for relative difficulty of tasks assigned to each proficiency level).

[1] In making comparisons between the performances of the group in this study and the national cohort, one must be mindful of the relative sizes of the samples. (N= 71) student teachers (N= 500) sample of 15-year-olds.

Table 3: Mathematical tasks with the highest facility among student teachers

Highest Facility		*Student teachers*			*Irish 15-year-olds*		
Task title	*Task content*	*% to score 0*	*% to score 1*	*% to score 2*	*% to score 0*	*% to score 1*	*% to score 2*
Racing Car 2	Read information from a graph	1.4%	98.6%	NA[1]	4.6%	85.9%	NA
Racing Car 3	Read information from a graph	1.4%	98.6%	NA	4.8%	81.4%	NA
Farms 1	Compute area of a square	5.6%	94.4%	NA	27.7%	68.6%	NA
Racing Car 1	Interpret a graph, linking verbal description with particular features	15.5%	84.5%	NA	10.9%	66.0%	NA

A not dissimilar pattern exists between performances of student teachers and Irish 15-year-olds on the four PISA items with highest facility, except for the item, *Racing car 1*, where a greater percentage of student teachers either didn't attempt it or failed to answer correctly. If we exclude performances on items *Triangles* (top of level three) and *Continent area 1* (possibly more complex than a level two task) then more than 80% of the student teachers in the study achieved a mathematical literacy level up to and including proficiency level four. Just fewer than 30% appeared to misread or misunderstand the mathematical language in item *Triangles.* This intra-mathematical item seeks to access the 'communicating and expressing' aspect of mathematical literacy by inviting participants to choose which diagram of a triangle fits the description given in mathematical language. Such an important element of syntactic mathematical knowledge could be deemed essential for primary teaching and is pertinent to the *Primary Mathematics Curriculum*. But the finding that Irish student teachers do not perform well on the Space and Shape domain of PISA, to which this item and items *Continent area 1* and *2* belong is not surprising, since it is on the Shape and Space subscale of PISA 2003 that Irish 15-year-olds deviate most from the OECD mean achievement scores (Cosgrave *et al.*, 2005). Irish mathematics education does not appear to handle certain Shape and Space elements of mathematics very successfully.

[1] These items warrant only one point as maximum score

Table 4: Mathematical tasks with the lowest facility among student teachers

Lowest facility		*Student teachers*			*Irish 15 -year -olds*		
Task title	*Task content*	*% to score 0*	*% to score 1*	*% to score 2*	*% to score 0*	*% to score1*	*% to score 2*
Continent 2	Find a strategy to estimate area of an irregular and unfamiliar shape	32.4%	46.5%	21.1%	24.2%	27.4%	5.5%[1]
Apples 3	Make generalisation of growth of linear and quadratic function	38%	19.7%	42.3%%	58%	9.4%	9.2%
Racing Car 4	Link two different visual representation of speed and distance	73.2%	26.8%	NA[2]	20.9%	18.5%	NA

Taking a Closer Look at Items with Lowest Facility

The primary mathematics curriculum puts special emphasis on the teaching of estimation skills (Gov of I, 1999b, pp. 32-34) and offers four highly procedural strategies, which might be used for estimation. Examination of the student teachers' answers to the estimation item, *Continent area 2*, indicates that higher order estimation skills are weak in more than 75% of the student teachers. Reasonableness of answer does not appear to have influenced almost half of the respondents who offered answers, which were on either side of the 12,000,000 to 18,000,000 square km. accepted by PISA as an accurate estimation. The confusion of perimeter and area has long been recognised as problematical for school children (Foxman, Joffe, Mason, Mitchell, Ruddock and Sexton (1987). It also appears problematical for some student teachers. But the four students whose misconception of the multiplication algorithm involved deleting the zeros at the end of 4500km and 3600km[3] multiplying 45 by 36 and then adding 00 show a much more fundamental lack of facility with numeration and place value. Substantive mathematics for teaching the primary curriculum is also accessed in this item by requiring the use of scale to estimate the area correctly and a good working knowledge of units of measure is required to convert cm squared to km squared.

[1] Two extra categories were coded in the PISA pilot study: 'invalid' and 'omit'. These account for the responses 'missing' from 100%.

[2] These items warrant only one point as maximum score

[3] 4500km and 3600 km are approximate length and width of the shape

Table 5: Pattern of answers to Continent Area 2

Estimation strategy	*Full credit*	*Partial credit*	*Reason*	*No credit*
Drawing a rectangle	5	4 17 4 1 1	Incorrect no. of zeros Estimation too large Estimation too small Incorrect multiplication Incomplete	Changed unit, (x2) without use of scale
Drawing a circle	1	1	Too large	
Adding regular shapes	9	7 3	Too large, Too small	
Perimeter				2
No working shown			Offered l x b formula	4
No attempt				11

Less than 10% of student teachers (N= 7) got both items at level six fully correct. More than 20% (N= 16) got both items totally incorrect. *Apples 3* is rated the most difficult of the twelve items used in this study (OECD, 2002). It is the only item designed to access higher order, competency cluster three skills. 38% of student teachers (N=31) either were incorrect in their attempt or did not address the problem. The (N= 13) students who received partial credit were lacking in an ability to communicate and express their thinking- a mathematics process skill primary teachers are charged with developing in their pupils. A distinction can be made here between operative mathematical understanding and descriptive understanding (Sierpinkska, in Watson and Mason, 1998). Descriptive understanding might be considered an essential skill in teaching mathematics. But it was not *Apples 3,* based on the algebra strand of the mathematics curriculum, which student teachers found most difficult since 42% (N=27) answered this item correctly.

While Irish student teachers are generally found to perform well on number tasks and operations on number (Corcoran, 2005) the second level six item, *Continent area 2* - rated easier for 15-year-olds using IRT scaling by PISA standards (OECD, 2002, p.88) - proved more difficult for the student teachers. A surprising (N= 68) was partially correct or incorrect. The mathematics accessed here involving a combination of estimation skills, number and operations on number, the use of scale and measurement of area, is substantive mathematics which primary teachers report as being well within their repertoire of mathematics teaching (NCCA, 2005, pp.135-136).

Table 6: Pattern of answers to Apples 3 showing numbers of student teachers achieving each mark

Solution strategy used	*Full credit*	*Partial credit*	*No credit*
Algebra	27		
Extended table		4	
Some understanding of relationship between n and 8n		9	
Insufficient or wrong explanations			2 6
Incorrect (Conifers)			19

Shiel and Kelly, (2001) indicate that the test items in which students in fourth class in primary schools performed least well were those requiring higher level mathematical processes such as mathematical reasoning (55%), analysing and solving problems and evaluating solutions (54%) and understanding and making connections between mathematical processes and concepts (53%). Of the six mathematical process skills listed in the *Primary Mathematics Curriculum* (Gov. of I. 1999a), development of applying and problem solving skills, together with communicating and expressing skills linked with formal reasoning, mark this curriculum out as new and different. Their development in children requires considerable depth of mathematical literacy on the teacher's part in order to select problems suitable for teaching a particular mathematics topic to particular children. It also requires a personal confidence in one's own ability to reason mathematically in order to structure lessons which will develop children's reasoning skills. More than 50% of these Irish student teachers show a worryingly low level of these skills as evidenced by their performance in answering the three items enumerated in Table 4

Two individual students' mathematical literacy profiles

Aidan is a second year B Ed student who achieved A in his leaving certificate mathematics examination on an ordinary level paper (50 points). He did not perform particularly well on the PISA items. His raw score was 6 (42.86%) where 9.6 was the mean with a standard deviation of 2.4. Aidan felt relatively confident in his ability to solve all of the problems except the estimation of area item, ***Continent area 2,*** **(proficiency level 6)** which he considered he couldn't solve. But this very item was the only one he considered relevant to the primary curriculum. Others were all deemed "fairly relevant" and in four cases "irrelevant". Aidan's performance on the PISA items places him firmly at proficiency level three where he can operate at a purely procedural level of mathematical literacy. In contrast, Brian also a B Ed student had a raw score of 11 (78.57%) on PISA, which was above the mean of 9.6. He took an honours level mathematics paper at leaving certificate but does not disclose his grade. He is much more confident in his approach to solving the problems and scored partial credit for both items at level six. His only difficulty below level six was in two items at and below level three (***Farms 2*** **and** ***Continent area 1***), both of which require sequential decision-making. Brian deems most of the items to be at least fairly relevant or more so, to the primary mathematics curriculum although in the case of pattern recognition (algebra strand) in items, ***Apples 1*** **and** ***2***, he doesn't offer an opinion. Judged by their responses, these two students represent different levels of the mathematical literacy in the cohort, where the minimum raw score was 4, maximum was 14, median was 10 and mode was 11. They differ most from their peers in being male, where 85% of the participants are female. Findings from PISA 2003 (Cosgrave *et al.*, 2005, p. 99) point to gender as a significant variable in performance in the mathematics domain. Gender differences in performance among student teachers may warrant further investigation.

Discussion

Mathematical skills development is an explicit requirement of the Irish primary mathematics curriculum. Six process skills are specified for development. These teaching objectives resonate strongly with the PISA framework and are not highly developed among many of the student teachers in the study. Nor are the limitations evidenced in mathematical literacy confined to process skills. The substantive mathematics content being accessed by the PISA item *Continent area 2* is to be found in the *Measures strand* for sixth class under the *Area* strand unit At a purely instrumental interpretation, the substantive mathematics accessed by the PISA item *Apples 3* can be located in the *Number strand* for sixth class under the *Number Theory* strand unit (identify and explore square numbers) and in the *Algebra* strand under the *Rules and properties strand unit* (identify relationships and record symbolic rules for number patterns). In other jurisdictions a plethora of studies have been published on mathematics teacher subject knowledge and its links with teaching. Brown and Borko (1992) in a synthesis of the earlier ones established a need for "high literacy" levels in mathematics in order to teach mathematics successfully. They also found that where adequate content knowledge was lacking, student teachers spent scarce planning time learning content they proposed to teach. Rowland *et al.,* (2001) established a significant correlation between high mathematics subject knowledge in trainee teachers and high marks for mathematics teaching during teaching practice.

Cosgrave *et al.,* (2005) cite a model of curriculum as intended, implemented and attained. An analysis of PISA 2000 mathematics items against the three levels of mathematics curriculum at Junior Certificate (JC) level found that less than 30% of item concepts were 'very familiar' to students at JC higher level, while neither context nor item format was rated as 'highly familiar' for any PISA 2000 item, with concept familiarity being the best predictor of achievement on PISA. Yet recognising of mathematics in realistic contexts is hugely important in teaching primary mathematics as outlined by the current curriculum, for two reasons. If teachers subscribe to "the application of mathematics in a wide variety of contexts [which] gives people the ability to explain, predict and record aspects of their physical environments and social interactions" then they must be able to "recognise situations where mathematics can be applied, and use appropriate technology to support such applications" (Gov. of I., 1999b, p.2). Many of the student teachers in this study may have difficulty in fulfilling this expectation. Secondly, constructivist mathematics pedagogy as outlined in the *Primary Mathematics Curriculum* requires that a teacher be able to recognize where a pupil's thinking is leading and while respecting the student's ideas (Ball, 1993), know how and when to head off wrong assumptions by directing the student's thought process along more productive and fruitful lines (von Glaserfeld, 1995).

Conclusions

In Ireland there is no obligation on student teachers to indicate their proficiency in mathematics beyond the entry requirement. The *Mathematics Curriculum* and its accompanying *Teacher Guidelines* (Gov. of I, 1999a, b) are the only official documents, which specify what mathematics primary teachers need to know. The mathematics knowledge, which student teachers bring to teacher education, is largely procedural and their mathematics knowledge base often lacks important syntactic elements such as estimation and generalisation skills. Regrettably, such a foundation may hamper further development of mathematics knowledge during a teaching career (Grossman, Wilson and Shulman, 1989).

There is evidence of contradictory thinking among Irish educational policy makers with regard to mathematics teaching and learning. The *Junior Certificate Mathematics Teacher Guidelines* (2002) optimistically contends that

> …students emerging from the revised [primary mathematics curriculum] should be more likely than their predecessors to look for meaning in their mathematics and less likely to see the subject almost totally in terms of rapid performance of techniques. They may be more used to active learning, in which they have to construct meaning and understanding for themselves (2002, p.5)

Findings from this study are that such an aspiration is unlikely to be realised without considerable changes in classroom culture and in how primary mathematics teaching is conceived and practised by teachers. More pertinent to the findings however, is the report about teacher education, which posits that the relatively poor performance of Irish pupils on mathematics tasks involving problem-solving activities and realistic contexts does not bode well for the success of the more constructivist approach to learning espoused in the revised curriculum, and merits particular attention, both in primary-school classrooms and in teacher education programmes (Gov. of I., 2002a, p.99).

No claims for generalizability can be made because of the size of the sample and the limited number and range of test items. Nor does the interpretative stance of the researcher permit them, but indications from this study are less than favourable. Links have made between mathematical literacy measured by PISA and the mathematical literacy required to teach the primary mathematics curriculum well. Inferences can be made about the limitations in mathematical literacy of many of these student teachers and their preparedness to teach the *Primary Mathematics Curriculum*. Numerous references are to be found in the literature about the difficulty for teachers to teach mathematics in a manner, which is qualitatively different from that in which they themselves learned mathematics (Schifter and Bastable, 1995). This difficulty may well increase for Irish students, working as they do in a culture which values and develops generic teaching skills and competencies while relegating the curricular requirement to develop mathematics process skills to a preference for useful methodologies (NCCA, 2005).

At present, students in St Patrick's College have less than forty contact hours of mathematics education during their time in college. A strong recommendation is made for all student teachers to engage in an extra, realistic, problem-based, PISA-type mathematics course of at least one semester's duration, where they have time and opportunity to develop mathematical thinking, generalisation skills and insight into their own and each others mathematics thus building mathematical literacy in a collaborative environment (Lerman, 1983) before they begin their current mathematics teacher education courses. They and their future pupils would benefit.

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Appendix One: Categorization of items, according to overarching domain, competency cluster, context and item type.

Task title	*Mathematics domain*	*Competency class*	*Situational Context*	*Item type*
Apples 1 [Algebra]	Growth and change	2. Connections & integration for problem solving	Educational	Closed constructed response
Apples 2 [Algebra]	Growth and change	Competency class 2.	Educational	Closed constructed response
Apples 3 [Algebra]	Growth and change	3. Mathematical thinking, generalisation & insight	Educational	Open constructed response' partial credit
Continent area 1 [measures]	Space & shape	1.Reproduction, definitions & computation	Personal	Multiple choice
Continent area 2 [measures]	Space & shape	Competency class 2.	Personal	Open response, partial credit
Racing car 1 [functions]	Growth & change	Competency class 2.	Scientific	Simple multiple Choice
Racing car 2 [functions]	Growth & change	Competency class 1	Scientific	Simple multiple Choice
Racing car 3 [functions]	Growth & change	Competency class 1	Scientific	Simple multiple Choice
Racing car 4 [functions]	Change & relationships	Competency class 2.	Scientific	Complex multiple choice
Triangles [Shape and space]	Space & shape	1.Reproduction, definitions & computation	Scientific	Complex multiple choice
Farms 1 [measures]	Space and shape	Competency class 1	Occupationa l	Short response
Farms 2 [measures]	Space and shape	Competency class 2.	Occupationa l	Short response

Mathematics Professional Development for Primary Teachers: Looking Back and Looking Forward

Seán Delaney, Coláiste Mhuire Marino and University of Michigan

Whether one calls it in-service education, in-career training, lifelong learning, professional learning or professional development, it is now widely acknowledged that teacher education continues beyond the initial pre-service degree or diploma programme. Consequently, professional development is becoming a more significant part of the field of mathematics education in Ireland. The present capacity for professional development in mathematics is a product of decisions that were taken and opportunities that were seized by individuals and by the system in the past. This paper considers the current state of mathematics professional development in Ireland. Journal articles and books on mathematics education in Ireland are used as data. International comparisons are used to highlight aspects of Irish professional development provision in mathematics. It also proposes, for discussion, four aims for the future of mathematics professional development which are achievable within the current infrastructure.

Introduction

Honora Rice has been teaching for ten years. She teaches thirty-one third class pupils in a twelve-teacher school in a large town outside Dublin. Five pupils in the class speak English as a second language and Honora believes that two pupils, who need a lot of individual attention, should be in a special needs class. She works hard and loves her job. Colleagues and parents regard her as a very good teacher.

She particularly enjoys teaching maths and believes that she is good at it. In her opinion, the secret of success, is to "overlearn." Every pupil completes every problem in the textbook and if they get a problem wrong, they must re-write it correctly. Pupils learn tables by rote. Honora has observed some of her colleagues moving away from asking pupils to learn things by rote but she believes that those teachers will eventually revert to the memorizing approach. That method worked for Honora in primary school. Doing every sum in a 'drill and practice' textbook prepared her for secondary school entrance exams. Even if pupils don't understand everything at the time, understanding will follow.

Not seeing the need for it, Honora is unlikely to voluntarily embark on professional development in maths. But she willingly attends mandatory professional development days and is happy to share her beliefs about teaching maths with other teachers. For teachers like Honora, most recent professional development has come in the form of curriculum seminars and school planning days.

Formal professional development begins when a prospective teacher enrolls on a pre-service programme. Some researchers (e.g. Feiman Nemser, 1983, ; Richardson, 1996, , 2003) claim that informal learning to teach begins much earlier and Lortie (1975/2002) describes as a powerful "apprenticeship of observation" the years that a teacher spends as a pupil in school. For the purposes of this paper, professional development refers to how teachers learn once they complete their pre-service teacher education course and they begin to teach. Various terms are used to describe the learning in which teachers engage during their careers. Hyland and Hanafin (1997) refer to incareer development. Sugrue (2002) identifies four widely-used terms: in-service, lifelong learning, professional development and professional learning. Sugrue favours the latter two terms for their precision and their connotations of development. When the term professional development is used in this paper, it incorporates the idea of professional learning.

Honora represents both a hope and a challenge for professional developers in mathematics. She is hardworking, energetic and interested in her work and wants what is best for her pupils. She also has deeply-held convictions about good mathematics teaching which are at variance with aspects of the 1999 curriculum which emphasises understanding, higher-order thinking skills and discussion (Government of Ireland, 1999).

This paper proposes four aims for mathematics professional development of Irish teachers. These aims emerge from analysing data on mathematics teaching and on mathematics professional development in Ireland over the past fifteen years. The data consist mainly of articles and books on the topic, published since 1990 and they are supplemented by primary source data and unpublished research. These data are considered in the light of professional development research in the United States.

The paper begins by describing four aspects of mathematics education in Ireland: professional discourse, assessment of mathematics performance, disadvantage and mathematics and teachers' mathematical knowledge. The first three aspects are highlighted in two recently-published reports by the Department of Education and Science (DES) (Department of Education and Science, 2005a, , 2005b) and the fourth is attracting much attention from international mathematics education scholars (e.g. Ball, Lubienski, & Mewborn, 2001) and has recently been studied in this country (e.g.

Delaney, Zopf, Ball, & Hill, 2005; Wall, 2001). Next, aspects of the professional development infrastructure for mathematics are outlined. These are teachers' professional networks, summer courses, courses for credit, the Primary Curriculum Support Programme and professional reading. In the discussion section, the four aims for future professional development in mathematics are outlined and the paper concludes with a single aim for mathematics education researchers. Beliefs of Honora Rice, a composite portrait of several Irish teachers, are used throughout the paper to illustrate or to counter some of the points that are made.

Themes of Mathematics Education in Ireland

Professional Discourse

In a review of research on contemporary professional development, Wilson and Berne (1999) categorized the research they reviewed under three headings: opportunities to talk about subject matter, opportunities to talk about students and learning and opportunities to talk about teaching. Common to the three categories is the idea of talk. In her book *California Dreaming: Reforming Mathematics Education*, Wilson (2003) contrasts how the disciplines of mathematics and of teaching are practised. Mathematicians present their thinking for public review in articles and at conferences and in seminars and they "engage in sometimes sharp discussions about the validity and quality of one another's work" (p. 198). In contrast, teachers rarely talk to peers about their work or document it. Consequently their ideas about teaching are rarely challenged, or subjected to peer review and then disseminated.

In Honora's school one colleague drafted the school plan for mathematics. Although it was presented to a staff meeting, Honora remembers that the brief discussion centred exclusively on how subtraction should be taught. Irish teachers rarely discuss their teaching with colleagues. According to the 1995 TIMSS report, only seven per cent of Irish fourth grade students sampled, were taught by teachers who reported meeting with colleagues at least once a week to discuss teaching (Mullis et al., 1997, p. 158). This was the lowest meeting rate of the twenty-six participating countries, with only Hong Kong (9%) teachers reporting a similarly low number of meetings.

With designated curriculum planning days, the level of professional discourse and collaborative planning might have been expected to rise. But an evaluation of curriculum implementation conducted this year by the DES found that "whole school planning for mathematics was weak or had scope for development in more than half the schools inspected" (Department of Education and Science, 2005a, p. 26). This study was based on observation, examination of school documents and semi-structured interviews with principals and teachers from 61 classes in 28 schools. The study further claimed that the lack of whole school planning in many schools "impacts negatively on classroom practice" (p. 32). Further research is needed to investigate why professional discourse among primary teachers remains low.

A study of Irish post-primary mathematics teaching, involving video data and interviews with pupils, teachers, principals and parents, identified barriers to professional discourse in Irish post-primary schools. Teachers work in "autonomous units" and this combined with the "speed and intensity" of teaching militates against reflecting on practice. Indeed, of ten teachers interviewed, nine "had not experienced an alternative approach [to teaching mathematics] throughout their teacher training or their teaching career." (Lyons, Lynch, Close, Sheerin, & Boland, 2003, p. 276 and p. 263). Not being accustomed to discussing the teaching of mathematics Honora is not sure how it might change her teaching and she does not know where she and her colleagues would find the time to engage in this kind of professional discourse.

Disadvantage and Mathematics

Professional development could also play an important part in raising mathematics achievement among disadvantaged pupils. The DES regards a child as disadvantaged if "because of economic, cultural or social factors, the competencies that he or she brings to school differ from those valued in schools" (Department of Education and Science, 2005b, p. 14). In a recent study of 1,080 pupils in nine schools designated as serving areas of disadvantage, almost two-thirds of pupils' mathematics scores fell on or below the twentieth percentile and less than 3% scored above the eightieth percentile. The problems were even more pronounced in fifth and sixth classes. The researchers note limitations in generalising these findings: the sample of schools is small and the test scores data had been collected by individual schools prior to the study.

In another part of the study of disadvantaged schools, inspectors interviewed teachers and found that less than 15% of them had attended specific courses on the teaching of numeracy. Many teachers were unaware of courses that they could attend to develop their skills in teaching numeracy and the inspectors conclude that significant improvement is needed in the professional development of teachers to equip them to work with disadvantaged pupils. Honora teaches in a school in a relatively affluent area and it is not designated disadvantaged. Five children in her class, however, are non-native speakers of English and she struggles to support their mathematics learning.

Mathematics and Assessment

Honora administers a mathematics test and a tables test to her students on alternate weeks. In addition, she and her colleagues administer a standardised test annually to all pupils in June. A DES study of literacy and numeracy in disadvantaged schools found evidence of poor practice in relation to the analysis and use of assessment in mathematics in well over half the schools studied (Department of Education and Science, 2005b). Another study by the DES found that this problem is not restricted to schools that serve students who are designated as disadvantaged (Department of Education and Science, 2005a). In this study more than half the 28 schools surveyed had unsatisfactory approaches to assessment in mathematics. One third of the sixty-one teachers assessed pupils using only standardised tests and almost half of the teachers did not use the results of standardised tests appropriately.

The predominance of standardised tests in the DES study seems to contradict a finding of a National Council for Curriculum and Assessment (NCCA) study which found that almost all teachers surveyed reported using teacher observation as a form of assessment at least a few times a week and three-quarters reported using teacher-designed tasks and tests at least a few times a week (National Council for Curriculum and Assessment, 2005). The difference can be explained by the different data that were collected. The NCCA study administered questionnaires to teachers in 170 schools and they interviewed children, parents, principals and teachers in six case-study schools. Observing practice and studying school documents allowed the DES to see shortcomings in teachers' recording of pupils' achievement and discrepancies between documented and actual assessment policies. 10% of teachers in the NCCA study expressed the wish to use more assessment in implementing the mathematics curriculum. The DES study specifically identifies support services (including the School Development Planning Support initiative and the Primary Curriculum Support Programme) which can provide support to teachers in using formative assessment effectively. Most of Honora's assessment is summative rather than formative. Although she believes her weekly tests motivate the pupils, she realises that the results do not inform her teaching. She uses observation as a form of assessment also but finds it somewhat haphazard in nature and is interested in learning how it might become more systematic.

Teachers' Mathematical Knowledge

Growing recognition of the mathematical complexity of teaching mathematics, even to young children, has lead to several researchers studying teachers' mathematical knowledge (Ball et al., 2001). For example, analysing how a pupil got a particular wrong answer or deciding if a pupil's alternative problem solving approach is generalisable requires good knowledge of mathematics. In one study of Irish student teachers, Wall (2001) found that a small number of prospective teachers struggled with the kind of mathematics that sixth class pupils are expected to know. Ball (Ball, Thames, Phelps, & Hill, 2005) and others argue that knowing sixth class mathematics is far from sufficient mathematical knowledge for engaging in the work of teaching mathematics. Honora remains to be convinced that teaching primary school mathematics is mathematically demanding. She studied honours maths in her leaving certificate examination and she is more confident mathematically than several of her colleagues, one of whom refuses to teach senior classes because she knows she would find teaching mathematics challenging.

Professional Development Infrastructure

In the previous section I have explained some of what we know about mathematics education in Irish schools in relation to practising professional discourse, teaching students who are disadvantaged in some way, assessing pupils' mathematical achievement and studying teachers' mathematical knowledge. Using the persona of Honora Rice, I have hinted at the role professional development may play in improving practice in these areas. Professional development is no silver bullet. It cannot be delivered on demand, in a guaranteed timeframe or in equal measures to all teachers. Professional development is an ongoing process which must be credible to teachers and must be seen to enhance teaching and learning. It requires good professional developers and a strong infrastructure that supports and in turn develops the developers. The paper now looks at some components of the infrastructure for professional development in Ireland: professional networks, summer courses and courses for credit, the Primary Curriculum Support Programme (PCSP) and professional reading and writing.

Professional Networks

Between 1979 and 1988 teachers who were interested in sharing ideas and talking about mathematics teaching could join the Primary Teachers Mathematics Group. Dr. Seán Close, a lecturer in mathematics education in St. Patrick's College of Education, established this group in October 1979. Its aims were to (i) provide a forum for primary teachers to exchange ideas and methods about mathematics instruction and to discuss important issues in mathematics in primary schools and (ii) to promote co-operation between teachers both within schools and between primary and post-primary schools. An initial group of twenty members grew to over one hundred and in 1982 the Primary Teachers' Mathematics Group became the primary branch of the Irish Mathematics Teachers' Association (Primary Teachers' Mathematics Group, 1982 est.). For several years the Group organised workshops, talks and seminars and it produced a newsletter *Pegboard*. It also engaged in research activities, and one study resulted in a number of conference and journal papers (e.g. Mulryan & Close, 1982). Eventually, however, the group's membership declined to around 30 members. Funds, which came from membership fees, were low, attendance at the workshops dwindled and the group disbanded in 1988. Seán Close commented that at this time "all the focus [of teachers' professional development interest] was on computers"[1].

In California, Wilson found that participating in mathematics networks offered teachers the experience and the confidence to become teacher leaders and to offer professional development to other teachers (Wilson, 2003). It cannot be claimed that the Irish primary teachers' network made a similar contribution to developing teacher leaders. Nevertheless, three former members of the Irish Mathematics Teachers' Association subsequently became members of the National Council for Curriculum and Assessment's primary mathematics curriculum sub-committee that designed the 1999 mathematics curriculum (Government of Ireland, 1999). Further, in 2001 two former members of the group became members of the PCSP professional development team in mathematics.

More recently, attempts have been made to re-establish a network for teachers interested in the teaching of mathematics, the Primary Teachers' Mathematics Association. This Association was established in 2000 and it organises an annual conference and occasional workshops, delivered by teachers and teacher educators, and it produces two newsletters per year. The Association has a membership of around 100 teachers – less than 0.5% of Irish primary teachers.

Summer Courses

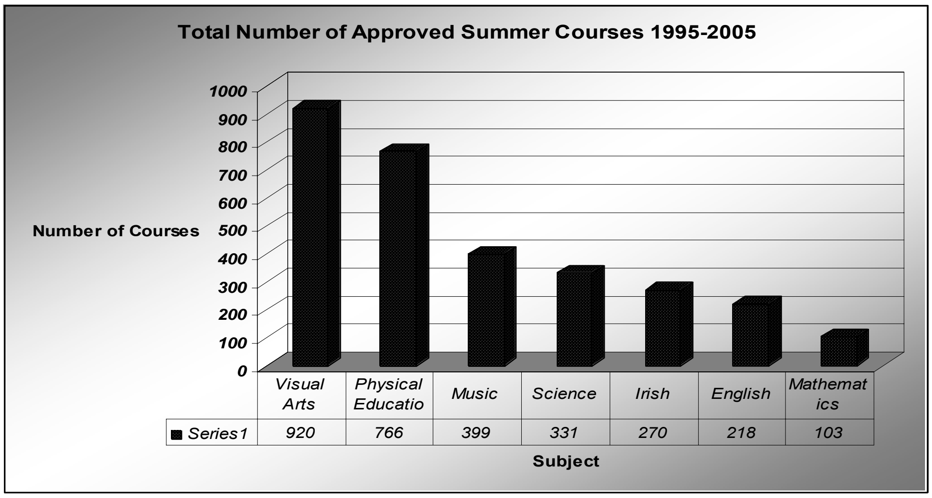
Many primary teachers attend week-long summer courses on curriculum and general education topics and in 2005 teachers could also choose an online course. Little published data exist regarding the quality or impact of these courses on teaching and learning. Figure 1 shows how many courses were approved by the Department of Education and Science (DES) in several curriculum areas between 1995 and 2005. In contrast to other curricular areas, especially visual arts, physical education and music, few mathematics courses were approved. Limited opportunities, therefore, have existed on DES-approved courses for Irish teachers to develop their mathematical knowledge or their pedagogical knowledge in mathematics. This has made it more difficult for teachers to develop the skills and the confidence necessary to become teacher leaders in mathematics.

[1] Dr. Seán Close, personal communication, March 4th 2005.

Courses for Credit

Another route to becoming a teacher-leader is to take a course for certification. Few institutions offered certified postgraduate programs in primary mathematics education prior to 2000. In that year at least one college introduced a part-taught Masters Degree programme in primary mathematics education and at least two institutions are offering Masters Degree programmes in mathematics education beginning in autumn 2005[1].

Figure 1: Summer courses for teachers in curriculum subject areas that were approved by the Irish Department of Education and Science between 1995 and 2005[2]



The Primary Curriculum Support Programme

The Primary Curriculum Support Programme (PCSP) was established to support the implementation of the primary curriculum that was revised in 1999. In-service seminars, school-based planning days, a website and newsletters were the principal professional development instruments used by the PCSP which worked closely with the network of education centres that had been expanded during the 1990s.

Twenty-one mathematics trainers,[3] all primary school teachers, were appointed in 2001 to deliver two one-day seminars to every primary school teacher in the country. One of

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[1] The institutions referred to are St. Patrick's College Drumcondra and Trinity College Dublin. Source: *InTouch*, journal of the Irish National Teachers' Organisation, January/February and March 2005.

[2] Numbers exclude courses on generic topics such as educational disadvantage, learning difficulties, early childhood education, pupil assessment or technology which may or may not have a mathematics component. The numbers relate to courses advertised in the official list of courses and the numbers include some courses which were subsequently cancelled.

[3] Despite its connotations of instruction and drill and practice this term is widely used in relation to members of the PCSP who work directly with teachers.

the criteria for selection was "experience in both attending and delivering courses in primary mathematics."[1] Trainers began their own professional development in June 2001 and this continued in September and throughout the 2001-2002 school year. Specialist input was provided for the newly-appointed trainers by mathematics education lecturers, a teacher, a researcher and representatives of the Department of Education and Science.[2]

A PCSP design team decided that seminars for teachers should emphasise changes in mathematics teaching methodologies. These included more talk and discussion, active learning/guided discovery, problem solving, teaching skills through content, using the environment and collaborative/cooperative learning. Decisions about emphases were made partly on the basis "that the main changes to the maths curriculum were methodological rather than content-based." [3]

In the DES implementation study, teachers attributed their general "good understanding of the structure of the curriculum" to the work of the PCSP (Department of Education and Science, 2005a, p. 31). A full review of the PCSP, commissioned by the DES and the NCCA, has been completed by researchers from Trinity College, Dublin but it is not available for study until it has been considered by the NCCA council on September 22nd.

Although teachers bring credibility to the role of delivering professional development, the California experience prompted Wilson to caution that teachers teaching teachers is not problem-free. She believes that professional development in mathematics should provide participating teachers with deep mathematical knowledge as well as deep pedagogical knowledge. Therefore a teacher leader offering professional development in mathematics needs to have "sound mathematical knowledge" (Wilson, 2003, p. 93). But in her research Wilson observed frequent workshop sessions that were "chock-full of important information about instruction, reform and assessment, and weak on mathematics" (Wilson, 2003, p. 94). Wilson's role as an evaluator of professional development is significant here because without a good understanding of mathematics, she may not have noticed the mathematical shortcomings of the workshop.

Professional Reading

Even if teachers are not participating in professional networks or attending courses, they may be encountering new ideas about teaching and learning through professional

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[1] Valerie O'Dowd, Assistant National Coordinator, PCSP, personal communication, April 10th 2005.

[2] Valerie O'Dowd, personal communication, April 12th 2005.

[3] Valerie O'Dowd, personal communication, April 10th 2005.

reading. Irish teachers, however, have a low level of professional reading. The 1995 TIMSS results showed that pupils at fourth grade level[1] were taught by teachers who reported spending an average of 0.6 hours per week on professional reading and development, the lowest time of any participating country (Mullis et al., 1997, p. 157). This situation may be due to limited access to relevant journals and books rather than a criticism of individual teachers (Martin & Morgan, 1994). The intervening years have seen the publication of an education act, a new curriculum and almost a tripling of expenditure on professional development in the years from 1996 to 2000 alone (Drudy & Coolahan, 2002) and the level of professional reading may have changed.

InTouch, the monthly magazine of the Irish National Teachers' Organisation (INTO), is circulated to almost every primary teacher. A survey of the teacher tips section and the 'Teacher to Teacher' section for 2004 shows that the journal contained no articles on mathematics in that year (See Table 1).

Table 1
Number of articles related to specific curriculum areas in the "Teacher to Teacher" and "Tips" section of the Irish National Teachers' Organisation's journal InTouch *in 2004.*

Visual Arts	Social Environmental and Scientific Education	Music	2 or more subjects integrated.	English	P.E.	Drama	Irish	Maths
13	11	2	2	1	1	1	0	0

The absence of articles in mathematics is significant because *InTouch* is the most widely circulated education magazine in Ireland[2] and the one that primary teachers are most likely to read. On the positive side for mathematics, the first two editions of the magazine in 2005 contained mathematics articles written by members of the PCSP. The topics, mathematics trails and the one-hundredth day of school, were topics that had previously been the subject of workshops organised by the Primary Teachers' Mathematics Association. This is an example of how a network of teachers can potentially disseminate ideas about mathematics teaching to a much wider audience.

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[1] Participating children had an average age of 10.3 years but were not necessarily in fourth class in Ireland (Mullis et al., 1997, p. 24).

[2] It is distributed to its 24,977 members in the Republic of Ireland and to educational institutions.

Discussion

Practising teachers may engage in professional development that is mandatory, voluntary or incidental (Wilson & Berne, 1999). In the United States the various activities combine to form what Wilson and Berne (1999) describe as a “patchwork of opportunities” for teacher learning which are irregular and disconnected. Ball and Cohen (1999) share similar concerns when they bemoan the absence of “anything remotely resembling a comprehensive perspective on professional learning” (p. 4).

In Ireland, with the exception of the PCSP activities, teachers participate in the professional development opportunities described above on a voluntary basis. Many teachers, like Honora, will not become involved in these activities because they have limited time to dedicate to professional development, because they do not perceive the need for professional development in mathematics or because they have experienced unsatisfactory professional development in the past. In the 1999 National Assessment of Mathematics Achievement, only 29% of pupils were taught by a teacher who had attended some mathematics incareer training. The average number of training hours attended by teachers was 9 hours. Of the teachers who attended incareer development 47% of them were either dissatisfied or very dissatisfied with the courses.

Thanks to the PCSP and institutions offering masters degrees in mathematics education, the infrastructure for professional development in mathematics is becoming stronger but more remains to be done. Greater coordination between the different components of professional development is also needed. It is envisaged that the Teaching Council will adopt such a coordinating role ("Teaching Council Act, 2001," 2001) although it will face several challenges around quality, usefulness, opportunities and incentives, balance between mathematics professional development and other subjects, meeting the needs of diverse teachers and evidence of effectiveness.

Despite the challenges, the complexity of teaching mathematics makes professional development highly desirable. Four aims for professional development in mathematics are presented here. Each element of the professional development infrastructure, working in a coordinated way, can play its part in achieving these aims.

1. *To promote professional discourse among teachers*

If teachers at school level engage in sustained and honest professional discourse, ideas about teaching mathematics can be shared and challenged. Discussing mathematics teaching requires language that teachers do not frequently use and this is not something that can be learned in a one-day seminar or even during one week. Professional discourse is a process to be learned and practised throughout the teacher education process. Teacher educators and professional developers can practise it and model it for pre-service and practising teachers who can in turn practise it themselves. As professional developers and teachers practise professional discourse, other barriers to such discourse in schools need to be identified and addressed. Over time, mathematicians and other education partners may join the discourse. Indeed this research conference is an example of such discourse.

Mathematical topics like estimation and data that have been prioritised by the DES could be used to stimulate discussion. Similarly, Japanese lesson study (e.g. Stigler & Hiebert, 1999), where teachers collaboratively plan, observe and discuss a lesson, or video records of teaching practice (Ball & Cohen, 1999) might be used as discourse stimuli.

2. *To develop teachers' mathematical knowledge*

Since Shulman emphasised the importance of subject matter knowledge for teachers (Shulman, 1986), researchers have tried to describe the mathematical knowledge needed to do the work of teaching. Researchers at the University of Michigan (Hill, Rowan, & Ball, in press) have done substantial work at describing the mathematical knowledge that teachers need to teach mathematics and correlating it with student achievement. Developing teachers' mathematical knowledge should be a cornerstone of future professional development.

3. *To raise the mathematical achievement of children who are disadvantaged*

Disadvantaged children are not achieving well in Irish mathematics classes. Teachers feel unprepared to teach these children and they are unaware of relevant and available professional development options. Raising the mathematical achievement level of all children is another priority for future professional development.

4. *To improve the assessment of pupils*

If teachers are to build on pupils' existing mathematical knowledge they need to become skilful in assessing that knowledge and in using the assessment findings to inform their subsequent teaching. Skilful assessment is difficult and many teachers need additional support to assess pupils effectively.

These aims are not exhaustive and do not include professional development in areas such as integration with other subjects, problem solving or technology and mathematics. The aims are not specific to a particular curriculum but rather address the ongoing development of the professional over the course of a teaching career. Although they will not be easy to achieve, given time and determination they are possible within the existing infrastructure and they can contribute to strengthening and developing that infrastructure.

Finally, having outlined four aims for professional development of Irish teachers the paper concludes with an aim for mathematics education researchers. It is that researchers study the capacity and the achievements of Ireland's professional development infrastructure in mathematics. Internationally, the "pedagogy of professional development" (Ball & Cohen, 1999) is in its infancy. By studying professional development we can come to understand how professional development impacts on the practice of teachers like Honora and others and on pupil achievement. Through such research we can learn the extent to which the aims above have been achieved, which additional ones are desirable and what a continuum of mathematics professional development for a primary teacher's career might look like.

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Solving Problems with Student-Teachers: Reactions and Reflections 1995-2005

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Recent revisions to the primary mathematics curriculum are intended to give emphasis to problem solving. This paper describes an attempt by the author to engage pre-service teachers in problem solving in mathematics. It outlines the design and development of a course offered to all students in the Higher Diploma in Education (Primary) programme at one Irish college of education over the past ten years. Using the author's records and reflections and also students' reflective writing, it seeks to identify factors that contribute to successful implementation.

Introduction

Recent revisions to the primary mathematics curriculum are intended to give emphasis to problem solving. However, teachers are unlikely to implement the relevant changes unless they have experienced problem-solving activities. This paper describes an attempt by the author to engage pre-service teachers in problem solving in mathematics. The paper outlines the history of a course in mathematics offered to all students in the Higher Diploma in Education (Primary) programme at one Irish college of education – Froebel College – over the past ten years. Section 1 of the paper examines the literature on problem-solving and similar courses, in particular for teacher education, and relates this to the Irish context. Sections 2 and 3 describe respectively the original design and ongoing implementation of the course taught by the author at Froebel College. Conclusions are drawn in section 4.

1. Mathematical problem solving in teacher education courses

1.1: the international context

Problem solving has long history in mathematics education and is a major concern in many attempts at curricular reform (see for example National Council of Teachers of Mathematics, 2000). An allied but slightly different approach, focusing more on open-ended exploration and less on unique correct solutions, is represented by "investigations" (Ponte, 2001; Ruthven, 2001). Models for investigations, and examples of the type of work that school students can produce when doing them, are found in the seminal work of Banwell, Saunders and Tahta (1972). Both traditional problem-solving and investigational approaches involve aspects of what may be called "mathematical thinking" (see for example Mason et al., 1982). The two traditions are brought together in the Cockcroft Report:

> Mathematics teaching at all levels should include opportunities for…
>
> - problem solving, including the application of mathematics to everyday situations;
> - investigational work (Cockcroft, 1982, §243).

The extensive literature on problem solving and investigations in school mathematics is now mirrored by a corpus of work on the use of problem-solving and/or investigative approaches in teacher education. Among the issues addressed are the rationale for courses; their design; and practicalities involved in implementation. These three themes, which provide a framework for the present paper, are discussed in turn below.

The rationale typically addresses two main perceived needs: for teachers (including student-teachers) to experience the approaches and also to reflect on the experience. With regard to the first aspect, it is assumed that teachers are unlikely to implement problem-solving or investigational approaches effectively unless they have experienced them as learners and come to terms with the type of challenge involved (Even & Lappan, 1994; Lester et al., 1994; Cooney, 2001; Ponte, 2001). In particular, "it is important for students to know that blind alleys and confusion are a normal part of problem-solving activity" (Lester et al., 1994, p. 155). The second aspect stems from the extent to which emphasis on mathematical thinking encompasses a particular view of the nature of mathematics: that it is more about process than about product. This not only supports the use of constructivist approaches to learning, but also means that teachers' beliefs will have to be addressed (Cooney, 2001). Activities that tend to promote reflection are therefore crucial. Lester et al. (1994), in particular, note the importance of teachers engaging in reflective writing.

Course design involves selection of the content and activities to be included as well as the teaching methodology and assessment techniques to be used. Tasks selected, especially those of investigational type, need to have a number of characteristics. They should involve appropriate content and skills, be addressable initially by all students, provide opportunities for varied developments, and allow students to take responsibility for their learning (Ollerton, 1994; Ponte, 2001). With regard to teaching methodology, group and individual work is likely to predominate over whole-class expository teaching. Assessment, to be authentic, requires more than just a statement of findings; the solution process must also be described (Lingefjärd & Holmquist, 2003). Moreover, for classes in which group work is the norm, it makes sense to allow group assessments (Lester et al., 1994).

Implementation involves decision making by the person or people leading the course. Matters for decision include the methods of forming groups within the class, the length of time for which students work on the various tasks, the extent of teacher intervention and the level of rigour demanded (Even & Lappan, 1994; Ponte, 2001). An allied aspect is the provision of appropriate support especially for lower-achieving students, in order to address affective as well as cognitive issues (Pyne et al., n.d.; Wells, n.d.).

1.2: the Irish context

Until recently, problem-solving and allied approaches have not been a major focus for attention in Irish mathematics education. The evidence from national and international studies points to an undue preoccupation with the execution of routine procedures rather than with problem-solving activities: hence, with mathematics as product rather than as process (Beaton et al., 1996; National Council for Curriculum and Assessment, 2005). The OECD PISA study (Cosgrove et al., 2005) has contributed to the discussion. Many PISA mathematics questions involve solution of problems set in realistic contexts; Ireland's rather moderate performance is an indicator of Irish students' unfamiliarity with this approach.

Moves towards increasing the emphasis on problem solving began in the 1990s when work began on the revision of the primary school curriculum. The curriculum was published in 1999 (Department of Education and Science / National Council for Curriculum and Assessment, 1999) and the mathematics element was formally introduced to classrooms in 2002. Since Irish teachers in general have not experienced the approach in their own mathematics education, implementation is bound to present a challenge. As indicated above, teacher education has a major role to play in this respect.

However, teacher education is currently facing another challenge. Concerns have arisen about the mathematical attainment of school leavers, and this has knock-on effects at third level (Morgan et al., 2001; National Council for Curriculum and Assessment, 2005). In particular, the knowledge and skills of some entrants to primary teacher education courses need to be enhanced. Moves have already been made to address the issue. For example, in the Bachelor in Education (B. Ed.) course offered at Froebel College and the other colleges working with Trinity College Dublin, the mathematics course provided for all students has been revised to include more explicit focus on the content of the primary curriculum.

While the B. Ed. remains the most usual route into primary teacher education, it is not the only one. From time to time, courses have been provided to cater for university graduates in subjects other than education who wished to qualify as primary teachers. The current round of such courses began in 1995, with students undertaking an eighteen-month programme leading to the award of Higher Diploma in Education (Primary) (H. Dip.). Challenges in devising an appropriate mathematics course for H. Dip. students are increased by the fact that these graduates are at least three years away (in some cases much more) from their school mathematics experiences. Some students have had no formal contact with mathematics since leaving school, but others have studied the subject at third level – as part of an economics or psychology course, for instance, or even as the main subject of their degree.

The following sections describe the course designed by the author and implemented at Froebel College with eight cohorts of H. Dip. students in the period 1995-2005. In considering the three themes of rationale, design and implementation, two factors should be borne in mind: first, the context as described above; secondly, the fact that the author has also been the lecturer throughout the period and that the analysis is inevitably subjective.

2. Course rationale and design

In designing the course for the initial cohort of H. Dip. students, three main issues had to be addressed: the nature of the course (predominantly content-based or process-based); the extent of focus on topics in the primary curriculum; and the ways in which a course in mathematics could be made to seem relevant in the eyes of intending primary teachers. Decisions with regard to these issues would lead to more detailed specification with regard to aims, content, teaching/learning methods, and modes of assessment.

With regard to the first main issue, the nature of the course, it was decided to focus chiefly on the *process* of doing mathematics. This was done because:

- the approach was deemed to have intrinsic worth
- it would help to prepare students for the forthcoming changes in the primary curriculum
- the constructivist and reflective approaches typically associated with such a course were particularly appropriate for graduate students
- the existing B. Ed. course in Froebel College, aimed at school leavers, was not well suited to the students' varied mathematical backgrounds.

The initial version of the course owed rather more to the "investigational" than to the "problem-solving" philosophy. This was due in part to the author's having personal experience of the former, and in part to the absence of much debate on problem solving in Irish mathematics education at the time. The aims as originally formulated to implement the philosophy included helping students to reflect on their mathematical experience, enhancing their understanding and appreciation of mathematics, and giving them practice in communicating (talking and writing) about mathematics.

The decision with regard to the second main issue – teaching the content of the primary curriculum – was taken at a time at which students' competence in primary mathematics had not been identified as a major concern. Hence, explicit focus was restricted to areas likely to be new to the students. Other topics would be revised naturally as they arose in suitably chosen investigations. The course therefore included three main elements: activities to promote reflection and communication (for example, writing a mathematical autobiography and completing reflective questionnaires); study of unfamiliar material, chiefly probability, in the then forthcoming revised primary curriculum; and experience of mathematical thinking, based largely on doing investigations.

At this point some details can be given about the mathematical thinking element. It would be introduced by the conduct of a classic investigation: the so-called pegboard problem (also known as "frogs"), in which participants attempt to interchange the positions of two sets of pegs on a pegboard subject to certain conditions. Students would work on the problem individually, being encouraged to move to extended versions of the problem when they were ready; results would be compiled and at intervals discussed with the whole group; hopefully, patterns would be identified and conjectures made about a general solution to the problem. Time would be devoted subsequently to examining the rationale for doing investigations and to identifying typical stages that occur when they are being carried out: clarifying the problem, solving it, extending it, and so forth – perhaps ending with a proof of a general solution. Students would then be given a booklet of well-tried investigations covering a variety of content areas (chiefly number, measure and algebra). All these investigations could be addressed by means of mathematics from the primary or lower secondary curriculum, though some could be tackled more thoroughly by use of advanced topics such as calculus. The investigations would be done chiefly through group work, with students forming their own groups, choosing investigations that appealed to them, and addressing them at a level at which they were comfortable – primary, secondary or even tertiary.

The third major issue, that of helping students to find relevance in a course that was not restricted to primary mathematics, was addressed partly by suitable timing and partly by choice of teaching methods. The more obviously relevant content-based section would be offered near the beginning of the course. It would be presented mainly through expository teaching; this would be augmented by activities and discussion that would help to model constructivist approaches and engage mature students in reflection, but the classroom environment would fundamentally be one with which the students were familiar. Hopefully, this would build up students' confidence. The first reflective element, a mathematical autobiography, would be introduced even earlier, in the hope of helping students to find relevance through personal involvement. Students would be asked to record their attitudes to mathematics over the years, accounting for likes and dislikes and for any changes in attitude at different times.

It remains to consider assessment. This would have to match the aims concerned with reflection and communication as well as those dealing with mathematical competence. Hence, students would be required to complete a portfolio, the chief items in which would be their mathematical autobiographies, various reflective questionnaires, and their accounts of selected investigations (describing the process as well as presenting the product). In the case of the accounts of investigations, students would be given the choice as to whether a group made a joint submission or whether group members wrote up their own individual accounts.

3. Course implementation

This section describes and critiques the implementation of the course with eight cohorts of students over the period 1995-2005. The account is based chiefly on two types of source: written records (such as course outlines, the author's lecture-by-lecture "log," and students' results and recorded reflections), and the author's reflections illuminated by hindsight.

Over the ten-year period, the course has evolved in response both to experiences from its implementation and to changing external circumstances. While evolution has been continuous, it is possible to identify three periods (early, middle and recent) in which the course has had more or less distinct characteristics. These periods are discussed in turn in sections 3.1 to 3.3 below. The identification of exact years is intentionally left vague to protect the identity of individual classes and students.

3.1: the early period

This first implementation of the course appears to have been successful (Oldham, 1997). In the section on new content, students participated eagerly in the activities; a "collaborative, appropriately cross-curricular and slightly zany tone was set, in which mathematics appeared as an entertaining activity involving more than 'doing sums'" (Oldham, 1997, p. 92). When investigations were introduced, one student – who had used an investigational approach while teaching in England – shared her experiences with the class; this helped to establish that children, albeit in another country and culture, could cope with such activities. The group investigations were an instant success: "For the first session, in particular, the students appeared deeply absorbed in and (in many cases) fascinated by the problems, and they continued to work hard even after the lecture was supposed to have finished" (Oldham, 1997, p.93). Responses to the final reflective questionnaire, which asked students to comment on a wide range of issues relevant to the course, suggested that almost all members of the group had achieved its objectives. Not all students warmed to the emphasis on open-ended problems; some preferred the clarity of unique right answers; but they were able to give reasons for their views and show that they had critiqued their own beliefs about mathematics.

Class groups over the next few years also produced good work, but were perhaps not as well suited by the course as was that initial cohort. In the content-based section, they did not seem to respond so well to the "slightly zany" tone or to generate the same lively atmosphere. For the unit on mathematical thinking, a lecture log entry about the first group investigation for one cohort reads: "They did hang in there, but it isn't going to be as instant a success as with the first group of grads. – [one student] … is already formulating the view that he likes Maths. for its unambiguity and unique solutions…." This suggests that there was more resistance than within the first cohort to accepting, or at least being prepared to consider, the unfamiliar open-ended investigational approach.

Also, the author may have been less skilful, or less lucky, in her decision making with later cohorts in this period than with the first. Two examples are relevant here. First, during investigations a lecturer needs to decide at what point to intervene if students are spending an undue amount of time in a "blind alley"; too much intervention does not allow students to take responsibility for their learning, but too little can obviate success and undermine students' confidence. With at least one group, the author probably erred on the side of too little intervention, and some students were left without adequate closure for their needs. Secondly, when doing investigations some students may be happy with a partial solution to a problem, and may not wish to develop natural extensions; in particular, they may not want to find a general solution and eventually to devise a proof. With one cohort, several groups chose as their first investigation a problem that leads fairly easily towards development of a formula that can be proved – without use of advanced mathematics – to hold in all cases. Over-emphasis on generalisation and proof at this stage demotivated students who found mathematics difficult and who did not regard proof as intrinsically valuable or beautiful.

The latter episode had an unexpected benefit. One of the less mathematically gifted students was so well able to verbalise her feelings in her reflective accounts that she achieved the highest score for the course. This has been used to encourage students in later cohorts; one does not need to be a good mathematician in order to do well.

3.2: the middle period

The middle period in the development of the course was marked by the fact that, as an unintended outcome of changes in the organisation of the H. Dip. programme, the number of hours allocated was reduced: from around eighteen to about twelve. Some redesign was necessary, and in view of the dominant rationale at the time it was decided to concentrate on mathematical thinking. However, the roles played by the content-based section of the course were missed. The students lost the opportunity to develop confidence about the course in a relatively familiar setting. The author also lost opportunities, in her case with regard to getting to know individual students' abilities and interests. This could exacerbate difficulties with regard to implementation. In particular, it could affect decision making while students were carrying out investigations, as described above, and could also make for problems in advising on the formation of groups.

With regard to group formation, issues arose with regard to both size and composition. Students were advised to work in groups of three or four – big enough to provide diversity and allow group work to continue when a member was absent, but small enough to facilitate the engagement of each member. In practice, over-large groups could be formed when smaller ones coalesced, perhaps because members were missing for a crucial week; a lecture log entry from this period notes that "one girl did say that the 'group of eight' was affected by comings-and-goings / absences." With regard to group composition, the author originally thought that there would be advantages in having mixed-ability (or mixed-attainment) groups, in which members might explicitly teach and learn from each other. However, some of the less mathematical students appeared to be scared rather than encouraged by the performance of able peers, and to have their own already low self-confidence undermined. In recent years, therefore, students have been advised to form similar-ability groups.

Around the end of the period, the number of students in the H. Dip. cohort was increased from around thirty (only twenty-two in the case of the first cohort) to sixty. Timetabling considerations obviated breaking the cohort into two groups, so the monitoring of group work without tutorial assistance became more challenging. Perhaps the greater group size – and correspondingly fewer interactions with individuals – accounts for the difficulty experienced by one such group in coming to terms with the rationale for the course. For example, one person appeared to grasp it only as he filled up his final questionnaire. If that is so, it provides an example of the questionnaire playing its role in encouraging reflection.

Discussion has tended to focus on difficulties, but these do not provide the whole story. Throughout the period, many students produced good and even excellent work. On some occasions, individual investigations were seen to provide exactly the experiences that were hoped for; for example, the lecturer's log for one week contains an entry with regard to "two groups encounter[ing] maths. that they are rusty/muddled about … one group explicitly on area and perimeter, the other… [with regard to] priority of operators." Students also had moments of noteworthy personal success. Log entries include the following: "one student [said that she] saw meaning in an algebraic formula for the first time" and "eventually one girl spotted the pattern – the first success she'd had in maths. (ever??), she said."

In this period, it became clear that the course – perhaps more than any other taught by the author – had become a *high risk* course (compare Oldham, 1997); it worked very well for some students but failed for others, at least in the short term. Comments from the students' anonymous course evaluations and reflective questionnaires ran the whole gamut, archetypal versions ranging from "my attitude to mathematics was changed – from negative to positive – by this course" to "at least I now know how *not* to teach maths."

3.3: the recent period

The third period in the development of the course can be dated from the time at which concerns about the students' content knowledge were becoming more acute. Students' own worries about their competence were seen to be undermining their mental freedom to learn and their openness to approaches that did not appear to relate directly to the classroom: at least, to the mathematics classroom as they had experienced it. They had problems in accepting that children could cope with work that they themselves found difficult. Perhaps also they were worried, rather than liberated, by the amount of freedom they were given to choose the number of investigations they tackled and the time they devoted to each one.

Three developments were instituted. First, more emphasis was placed on the advantages of using primary or at most junior second level mathematics in carrying out the investigations (and correspondingly less heed was paid to promoting opportunities for the better mathematicians to enjoy using the tools at their disposal). In order to provide more explicit links to the primary curriculum, students were asked to list the curricular content and skills used in each investigation submitted for their portfolios. Secondly, more hours were allocated to the course so that the unit on the content of the primary curriculum could be re-introduced. The focus was not only on new content, but also on providing exposition and activities to allow students to develop deeper understanding of key concepts and skills. This represented a fundamental expansion in the course rationale. Thirdly, the aims of the course were recast so as to refer more explicitly to problem solving (again providing stronger linkage to the primary curriculum, now implemented in schools), and to alert students to the fact that although short-term benefits should be obtained from the course, some benefits might be felt only in the longer term.

With the most recent cohort (the intake of 2004 – a group of only thirty again), all the features described here were in place. This group, like the first cohort, seemed particularly well suited by the style of the course. The lecture log for the content-based section contains remarks such as “Lots of lively discussion” and “when it came to playing the [‘guess my rule’] game in groups, I could hardly stop them, so great was their enthusiasm!” Interestingly, the group’s early encounters with investigations were not outstanding; the log entry notes that they “were perhaps under-engaged as a group, compared with other groups in the past”; however, it adds that “some afterwards commented very positively.” The smaller group size and the author’s consequent greater familiarity with group members meant that she could identify and talk with some people who initially struggled – aiming to encourage them and to provide space for them to decide if the investigational approach was to their taste. They subsequently got to grips with the genre and made good progress.

The extended period devoted to primary school content left a comparatively brief window of time for the group investigations. In the short term, benefits may have accrued with regard to imposing stricter deadlines than heretofore, not only for the students (writing up and submitting the first investigation) but also for the author (providing prompt feedback). Moreover, students did not necessarily have to move beyond a “honeymoon period” and engage deeply with issues of generalisation and proof. This may have been very good for their confidence if not so beneficial for their long-term competence. However, in a situation in which attitude is very important, that may be a fault on the right side. It is worth noting that the group contained at least one student who had already worked on problem-solving activities with the children in his class when he was on teaching practice. As with the first cohort (and perhaps some intermediate ones), this may well have helped to convince classmates that such work could be successful in the primary classroom.

The reflective voice speaking so far in this paper is that of the lecturer, albeit informed by students’ reactions and written submissions over the years. It is appropriate to add the voices of the students: in particular, and with their permission, the students in the most recent cohort, as recorded in their final – outstandingly thoughtful – reflective questionnaires. (The prevailing very positive tone of the entries doubtless owes something to the fact that the exercise was used for assessment purposes; however, what is of interest here is the type and depth of analysis.) Issues addressed are concerned both with course rationale and design and with details of implementation; relevant aspects are described in turn.

The questionnaire entries indicated that the students understood the philosophy of the course and had reflected on their beliefs about mathematics. Several spontaneously referred to the process/product issue: insights ranged from "I experienced a recreation of my mathematical understanding…. [Focusing] on process rather than product was an invaluable learning experience" to "I am very much a product person myself." With regard to the elements on primary school content and mathematical thinking, some stated a preference for the former, but more did so for the latter. Typical remarks indicated that the former was useful but the latter was more fun, albeit at times frustrating. One student identified that preference "depends on learning style." Another emphasised the similarities rather than the contrasts, in that activities in the earlier element had paved the way for investigations in the later one. An interesting remark with reference to course design was that "it was definitely a good idea to have the investigative workshops at the end of the course so that we could fully utilise our new-found confidence in our mathematical skills and knowledge!"

With regard to implementation and allied issues, one matter of interest concerns the emphasis that should be placed on extending problems and providing proofs of general results. One student stated that "I get as far as 'enlarged problem formulated and solved', etc and by then I really don't feel inclined to continue towards the development of a general truth." Another said that her group "managed to produce a pattern to our immense satisfaction, but not one of us expressed a desire to prove this pattern." By contrast, for another, "a proof is the missing piece of a puzzle, and without it, the problem is incomplete."

A second matter is the formation and operation of groups. In general, students found group work fun ("social and sociable") and supportive, although occasionally frustrating, with the students' varied backgrounds making for the contribution of different skills. A number of insights were given into the functioning, or malfunctioning, of specific groups. Some were "relaxed" or "balanced and co-operative"; typically, members learnt from each other or clarified their own ideas through discussion. However, others had problems with group dynamics or in accommodating differences in ability or style. One student recommended working in pairs rather than in larger groups.

A final matter, not discussed so far in this section, is the role of writing up investigations: telling the story, describing "blind alleys" – recording the process as well as the product. Typically, students found that it clarified their thinking, but was time-consuming and could interrupt the flow of ideas. With regard to blind alleys, one student wanted to "record only what went right," but for another "it was the story that showed us the blind alley."

4. Conclusion

In reflecting on the development of the course over the ten-year period, several key issues can be identified. They refer to the three themes of rationale, design and implementation.

As regards the rationale, the expanded version – presenting primary school content as well as addressing problem solving and investigational work – appears to have the potential to achieve its aims: to produce appropriate reflection as well as meeting the students' own perceived needs. However, it is crucially dependent on the participants. For the students, the aims are more easily met if class members are not only open to different approaches in theory, but also willing to interact and "play." The most recent group (the 2004 intake) was particularly open and interactive. Also, they had a clear vision, presumably developed earlier in the H. Dip. programme, of the aims of the revised mathematics curriculum. As this becomes more fully implemented, students are more likely to observe classroom activities relevant to the course and hence find it easier than some past cohorts have done to accept the rationale. With regard to the lecturer, much depends on his or her ability to create a climate of openness and confidence: providing enough (but not too much) structure, so that students can feel safe in taking ownership of their learning and critiquing their views of mathematics.

Design issues include the selection of appropriate content (topics and investigations) and assessment techniques. The investigations originally chosen appear to have worked well in general, displaying the characteristics described in section 1. Existing assessment components also seem to have achieved their objectives. However, questions can be asked as to whether some kind of examination of primary school content should be added in order to match the enhanced focus on primary school content – and if so, how this can be done without destroying students' sense of relaxation and enjoyment.

Implementation matters are consistent with the literature discussed in section 1. Where there were mismatches, difficulties tended to emerge. Many involved the lecturer's decision making: for example, how much autonomy to allow students with regard to group formation and task selection, when to intervene during investigations, and how hard to press for students to address generalisation and proof. Perhaps the focus on these latter aspects should come only as a minor element in this short course, at least until the genre is more familiar.

The course remains at least a moderate risk operation, notably as regards short-term outcomes. The tension between productive confusion and frustration on the one hand, and confidence building on the other, is intrinsic to such courses. Some of the benefits are likely to emerge only in the long term (T. Lingefjärd & M. Holmquist, personal communication, July 2003). On a personal note, the author eventually learnt much from courses that initially challenged her beliefs or learning style. She hopes that students can be helped to do likewise.

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Third Level Mathematics Education

What Counts as Service Mathematics Teaching in Irish Universities

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The so called 'mathematics problem' as it is styled in the UK research on mathematics education encompasses issues in the transition from school mathematics to university service mathematics teaching. Similar issues and concerns arise in the Irish context. With such a diverse number of career prospects available to students on exit from Higher Education nowadays and the escalating importance of mathematics in such fields e.g. Business, Engineering, Education or Pure Mathematics, there is an obligation on universities to provide a suitable mathematics education for each student. In this paper the authors attempt to create a meaningful characterisation of Service Mathematics Teaching as it is practiced in Irish Universities today, and to define more clearly the teaching/learning contract that exists between the actors in this sphere of activity.

Introduction

Mathematics has a vital role to play in education and everyday life in general. The very fact that a pass (Grade D) in Leaving Certificate Ordinary Level mathematics is a requirement for entry to most third level courses in Ireland shows how highly the subject is valued (Murphy, 2002). In the U.K. it is usually the only subject that is a prerequisite for engineering courses in university (Croft and Ward, 2000).

Mathematics is an essential tool in Science and Technology and, as a result, is central to the economic success of a country (LMS, 1995; IMA, 1995). In order for engineers to be inventive and original they need to have a high degree of mathematical fluency (IMA, 1995). While there is no doubt that computers are rapidly advancing and have made life easier in the last few decades for engineers, mathematicians, statisticians, scientists etc., one must still have a clear understanding of the underlying mathematical principles involved (IMA, 1995). There are fears in the U.K. that a drop in the level of the mathematical proficiency of undergraduates will lead to them falling behind their peers in other countries and, as a result, the country itself will have to rely on others for inventions and developments (LMS, 1995).

> "To engineers, mathematics is the international language of communication expressing physical phenomena in precise terms. Training engineers who are deficient in mathematics is like having doctors who are deficient in anatomy" (IMA, 1995:13).

It is evident that there *has* been a decline in recent years in mathematical standards at first year undergraduate level in Ireland (Gill and O'Donoghue, 2004) and the U.K. (Hunt and Lawson, 1996). There are concerns that this under-preparedness will have serious short and long-term consequences not only for individual students (i.e. failure and drop out (O'Donoghue, 1999)) but also for the professional reputation of various universities and for the economic progress of a country (LMS, 1995). Many universities in Ireland have responded well to the demands placed upon them, turning to diagnostic testing to help assess the extent of under-preparedness within each of these institutions and establishing Mathematics Learning Centres to help students catch up and fill in the gaps in their knowledge.

The authors believe that many of the problems relating to mathematical under-preparedness in third level service mathematics courses stem from the fact that the nature of service mathematics is not fully understood. In this study, the authors investigate this theme and attempt to develop a comprehensive characterisation of what is Service Mathematics as distinct from Pure or Applied Mathematics.

The purpose of this study is to investigate what constitutes Service Mathematics in Irish universities today and to clarify pedagogical issues related to Service Mathematics teaching e.g. the nature of the didactical contract between students and lecturers. In this paper the authors refer to mathematicians and mathematics students as those pursuing careers/ degree courses in mathematics fields. Service Mathematics refers to degree courses where mathematics plays a part, be it small or large, but is not the main focus of the students' courses.

In this paper, the authors present a preliminary analysis of the data obtained from lecturers and students. After a short discussion on methodology and background issues, the paper is organised to reflect responses to specific questions as solicited by the researchers. The paper concludes with a discussion of pertinent issues.

Methodology

The analysis presented is based on direct observations of classroom practice in each of Ireland's seven universities, and semi-structured interviews with 9 experienced mathematics lecturers involved in service mathematics teaching and follow up interviews with 12 students taking Service Mathematics courses. The qualitative data was analysed using the Constant Comparative Method (Cohn and Manion, 2000) and NVIVO software.

The authors proceeded by first collecting course documentation on each of the service mathematics courses to analyse. The selected lecturers were asked if they would complete a questionnaire (by e-mail), participate in a semi-formal conducted interview and agree to allow the author (OG) to carry out structured observations within a 'typical' mathematics lecture. All agreed.

When this component of the investigation was completed, the authors felt that in order to get a more comprehensive insight into the didactical contract present in third level service mathematics classes, one would have to get the view of the students participating in these courses. Subsequently, this was done with the lecturers' permission.

Background Issues

Service Mathematics

The authors' aim was to investigate with some exactitude what constitutes Service Mathematics as perceived by the lecturers at the chalk face and how they plan, implement, assess and evaluate these courses. The authors were aware, as the literature subsequently confirmed, that there were many interpretations of the nature and meaning of Service Mathematics. For example, Howson et al (1988:1) state that the term service mathematics does not connote a lesser form of mathematics. They refer to it as "…mathematics in its entirety, as a living science, able – as history has ceaselessly shown- to be utilised in, and to stimulate unforeseen applications in varied domains"

While it was once the case that Service Mathematics solely referred to engineering mathematics, this is no longer true. Chevallard (1989: 52) says '… the empire of mathematics is steadily spreading and keeps encroaching on domains which until recently had remained foreign to is influence'. All professions have varying requirements for the knowledge and use of mathematics skills. O'Donoghue (1999) predicted that in the 21st century professionals would require higher levels of mathematical proficiency than ever before.

Kent and Noss (2001) ask the question; what is mathematical knowledge? There is, they claim, different perceptions depending on what domain each professional works in i.e.

science, engineering etc. This occurs, they say, because each person sees a different purpose for mathematics, one that is relevant to their own particular realm (Kent and Noss, 2001).

Didactical Contract

The authors anticipated that an important aspect of the research would hinge on the relationship between students and teachers. Brousseau (1980) generated the theory that an implicit contract exists within every mathematics classroom between all actors in the sphere. Students are presented with mathematical tasks/problems by their teacher/lecturer. The students are required to work on the tasks whilst adhering to various constraints governed by the teacher/lecturer. The expected behaviours of the students from their teacher/lecturer and vice versa determine the didactical contract present in the classroom. Brousseau believed that this contract has a significant impact on the teaching and learning that occurs in the class.

Data Analysis

The authors believe that it is crucial to examine how seriously Service Mathematics is treated within mathematics departments and service departments, how it is planned, implemented, assessed and evaluated. The lecturers were asked explicitly to state what behaviours they expect from their students to fully participate in these service mathematics courses. A number of questions were posed to elicit information from lecturers.

Perceptions of Service Mathematics

Question: ***"What do you understand by Service Mathematics?"***

The data showed that lecturers generally viewed service mathematics as mathematics for students not doing mathematics degrees. This negative perception is interesting because it indicates that lecturers see service mathematics as something that is not chosen for its own sake as a path to some career. Lecturers also distinguish between service mathematics and Applied mathematics. In response to the question: *"Is it Applied Mathematics?"* one lecturer responded: "I wouldn't use the term 'Applied Mathematics' I suppose because it would be confused with Applied maths as a branch of mathematics".

Question: ***"Is Service Mathematics seen as a serious responsibility?"***

Simons (1988) claimed that in many mathematics departments there is a hierarchy of priorities with mathematics research and education of mathematics students at the top of lecturers' lists while service mathematics and mathematics education remain near the bottom. It appears that this is not the case in Irish universities from these lecturers' points of view. The lecturers, the mathematics departments and client departments place great value on service mathematics. All the lecturers stated emphatically that it is important. Some of the reasons it is so important are financial reasons. Many mathematics departments depend on service mathematics for a lot of their income.

It was reassuring to hear though that one of the main reasons it is so important is that these service mathematics students are just as important as any mathematics group. It is important in their eyes to provide an apt mathematics education for these students as they believe mathematics plays an integral part in their other studies and future careers.

In some of the universities there are coordinators for service mathematics courses. The coordinators liaise between the mathematics department and the client department to ensure the smooth running of the courses and deal with any issues that may arise regarding course design and execution e.g. course content, organisation of tutorials, attendance etc.

Course Design

Question: ***"What are your main aims and objectives for the course?"***

All mathematics courses have clear aims and objectives as outlined by the department e.g. to develop the fundamental concepts and basic tools of calculus. The authors wanted to see if the lecturers had any implicit aims and objectives of their own to get a deeper insight into how they view their courses and students. Taking the fear out of mathematics and giving them an interest in the subject were the main ones. It is clear that these lecturers have a love of their subject and wish to transfer that to their students. They want students to realise how important and useful mathematics can be in their courses and careers.

Question: **"** ***Who decides on the course content and subject matter?"***

Collaboration between lecturers from client departments and mathematics departments is essential in constructing a relevant, coherent and realistic syllabus. This is procedure in the universities in Ireland for the service mathematics courses discussed. If any matters arise, they get sorted out through collaboration with both departments involved.

Question: ***"Is there any external input from employers or industry?"***

If students are to enter industry, academia or other areas perhaps future employers should have some input into the type of mathematics students learn (Flanders & Fuller, 1997; Kent and Noss, 2001). Bajpai (1985) states that industry should play a part in the formulation of mathematical syllabi. Apart from the client departments and the external examiners there is generally no other input into the courses in Irish universities.

Question: ***"Do you use relevant real-life examples in your teaching?***

Relevance appears to be a key issue/objective in the teaching of service mathematics. It is felt that if students are aware of how relevant mathematics is to their chosen courses/careers, they will be more motivated to learn mathematics (Simons, 1987). The lecturers interviewed do try to make the content relevant to the students' core areas as much as is possible. Unfortunately, in some subjects this is easier to do than in others.

Question: ***"Do you keep a close eye on the tutorials?"***

The students have one tutorial each week that they are required to attend. These tutorials are run by post-graduate or, sometimes, fourth year mathematics students. The tutorials have generally anything up to thirty students in them though the ones the author observed had no more than 15 present. The lecturers distribute the homework/tutorial sheets and the problems covered in any one tutorial are generally based on material covered in the previous week's lectures. There isn't much interaction between the lecturers and the tutorials other than giving the tutor the solutions. It is left up to the tutor how they teach the tutorials. The lecturers would prefer, however, that the tutorials would be a bit more interactive and student-led.

Teaching Methodologies

Question: ***"How would you describe your teaching?"***

The author asked the lecturers to describe a typical class and describe their own teaching. They would, they say, have different approaches depending on whether they are teaching mathematics groups or service mathematics classes. Class size is a major factor in determining how they would approach a class. Chalk and talk is the general approach to teaching these mathematics classes, again possibly because of the high numbers. Group work and exploratory work are not feasible when lecturing to anything up to 400 students.

Question: ***"What justification or motivation to you give your students for studying mathematics?***

Because these service students do not choose to study mathematics and many would rather not, it is sometimes part of the lecturers' job description to sell the mathematics to the students. The lecturers respond to this by giving students applications of mathematics to their other areas of study which is probably the best motivation tactic. In this way students see how mathematics fits in to their other areas and they can see its usefulness. Informing the students that other faculties require that they do mathematics is probably not sufficient.

Assessment

The modes of assessment vary depending on the university. Some have a 100% final examination at the end of the academic year or end of semester. Some have 75% examinations with coursework to be submitted throughout the year counting towards the rest of the percentage.

Question: ***"Would you say your classes or programmes are exam-orientated?"***

Obviously lecturers have to ensure they cover what students will face on their examination paper and they need to keep the students focused but it is not the focal point of these service mathematics classes. When asked: *"Would you say your classes or programmes are exam-orientated?"* one lecturer responded *"*I don't think that I'm going in to teach question 1 part 1, I definitely don't think that, but there is always in the students' mind a question is this going to be on the exam so I am a little bit focused on the final exam but I don't go out of my way to teach to it".

Murphy (2002) discovered that in Irish secondary schools, the main focus of the senior cycle mathematics classrooms was the Leaving Certificate examination.

> "The Leaving Certificate terminal exam should be the central aim of the mathematics class. This exam should be the core component of each lesson, present as a sole motivation to learn a new topic. The teacher should present work referring to its inclusion/importance in the Leaving Certificate and provide details of the gain/loss of marks at every opportunity" (Murphy, 2002: 159)

It comes as an unwelcome surprise to students to find that the examination at the end of the semester/year is not the sole focus of their new mathematics classroom. It is one of the biggest changes for the students and one they find very difficult to adjust to.

Evaluation

Question: ***"How helpful have you found the student evaluations?"***

A few of the lecturers said that evaluations are carried out on their courses and their teaching where the students are asked to voice their opinions. The lecturers like to get feedback and most take on board the advice and use it to adapt their course, their teaching or even their textbook and notes if necessary.

Expectations

Question: ***"What do you expect from your students?"***

In order to disclose the didactical contract within the classrooms, the authors felt it necessary to ask lecturers what they expect from their students, how much work they expect them to do independently and what they thought the students expected from them. There were no big surprises here. Lecturers expect students to attend class, look over their notes, and do their homework. They also expect certain things like respect, punctuality and attentiveness. As for independent work, the lecturers do not expect a lot from their students but they still expect more than the students interviewed admitted to.

Student Data

The students who agreed to be interviewed were all first year university students who were an average of 18 years old. Mature or foreign students were not considered as the authors wanted to look at issues of transition from Irish second level to third level education. The students were participating in various degree courses which had a mathematics element such as Science, Commerce and Business Studies. Two of the students were doing an Arts degree and had chosen mathematics as one of their first year subjects. Some of the students were interviewed on an individual basis, others were interviewed in pairs or threes. The interview was structured in different sections in an attempt to get an holistic insight into Service Mathematics in Irish universities from the students' perspectives. For the purposes of this paper an overview of the preliminary analysis is presented as an integrated section.

The students' opinions of mathematics at secondary school were, on the whole, very favourable. It was interesting to note that some of the students associated good teachers with liking mathematics and vice versa.

The transition from secondary school to university appears to be a substantial hurdle to cross. It raises many issues for the students which they find difficult to accept and deal with. If something is difficult or different, then automatically they dislike it. They do not welcome the challenge. The pace and difficulty of lectures seem to be what troubles them most. Some felt completely overwhelmed. Some students are dissatisfied if the work is not simplified completely for them. The idea of working it out for themselves seems alien to them as a result of the spoon-feeding they received for so long.

Besides patience and enthusiasm, the most common characteristic of a good mathematics teacher given by the students was clarity of explanation. Students associate good, repetitive examples as the key to good teaching. Drill and practice is what students were used to. Digression from this custom is onerous for many. For the students, third level methods of lecturing are an unwelcome departure from the spoon-feeding and repetitive drill methods they have become accustomed to in second level. Taking responsibility for their own learning is a concept students find difficult to embrace. They acknowledge that it is their responsibility to attend lectures and tutorials but it appears to end here. Being part of a large lecture group, which can be anything up to 400 students, is also undesirable. Students miss the personal touch that they were used to at secondary school. Even if students admit they are not paying attention, they miss having someone telling them to listen or re-focus.

The authors wanted to determine what the students' attitudes to mathematics now are. The negative outlook of the students was quite palpable when they were asked the question: "Would you study mathematics if you were given the choice?" Only one student said they would like to keep it on anyway. Some of the answers to this question gave an interesting insight into how students see the usefulness of mathematics. It seems the only occupation available to mathematics graduates, in their eyes, is mathematics teaching.

It is remarkable, though probably not surprising, that none of the students said that they enjoyed the subject or found it interesting. It is only useful, in their eyes, if you intend to be a mathematics teacher.

What is surprising is that very few of the students stated that they attempted or completed the tutorial problems in advance of the classes. One group said they did as their homework had to be handed up for correction and counted towards their final grade. It appears that if it is not compulsory or contributing to their final grade, then the students do not put in any extra effort outside of their lectures or tutorials.

The authors felt it was important to explicitly ask students what they expected from their lecturers. The question *"What do you expect from your mathematics lecturer?"* seemingly was one that students had not really given much thought to before. Some said they didn't know and others couldn't think of anything outside of what their lecturer was already doing. Some stated that the lecturer should do revision for examinations like they would have been used to at secondary school. The jump from continuous revision and assessment to one final examination at the end of year is difficult for some. Some expected their lecturer to provide sample papers and pick out the 'important' parts of the courses. One could surmise that 'important' means: will it be on the examination paper!

Other students expect their lecturer to just explain everything for them. There seems to be no onus on themselves for working things out. This is interesting because students do not ask questions in their lectures. They seem to feel that lecturers should *just know* when they are having trouble.

Discussion: Issues In Service Mathematics Teaching

Certain issues arise in every mathematics classroom. The authors wanted to explore the different issues that are in some way unique to service mathematics classrooms as opposed to those that arise in mathematics classrooms generally.

One of the first concerns that arise in service mathematics is the attitude of the students. They vary enormously. Mathematics students will view mathematics differently to engineering students who in turn will have different outlooks to business studies students due to the differing roles mathematics plays in their fields, their backgrounds in mathematics, their mathematical abilities etc. Mathematics students study mathematics because they choose to; they like them, they are good at them. When they start off on a mathematics degree they are under no illusions as to how important mathematics is for them and the amount of time that will need to be spent studying mathematics. Service mathematics students study mathematics as part of a course requirement. Many students view mathematics as a nondescript, even expendable subject. They aren't quite sure why they need to do mathematics at all.

Students exhibit negative or neutral attitudes towards mathematics. Many students regard mathematics as an implement needed to get them through the desired degree in Business Studies, Science etc. It is a means to an end for some and many just aim to get the grades needed to barely pass the subject. They just want to get through the module and they often do not care if they do not understand it. As a result motivation levels can be very low (Simons, 1988). Added to this is the feeling that they were never good at mathematics. This is a hurdle in itself for lecturers and students to overcome.

These problems are countered by lecturers in a number of different ways. For example, integrating mathematics with core subjects is one way of demonstrating to students why mathematics is so valuable for them. Lecturers appear to be quite conscious of students who have a mental block and do their best to reassure these students. On the other hand, some students are highly motivated and just get on with it despite the fact that Service Mathematics is viewed as a chore.

Students entering on to service mathematics courses nowadays take with them very diverse mathematical backgrounds and levels of ability. In classes, which may have up to 400 students, lecturers will have all levels, from students with grade A1 in Higher Level Leaving Certificate Mathematics down to those with a grade C3 in Ordinary Level. Added to this, there are mature students, transferees and foreign students. This inevitably causes problems for the lecturers and the students themselves. When asked, *"Is there much variation in levels of ability and knowledge in the service mathematics classes?"* the lecturers said that the variation was huge. Not only do lecturers have to cater for the weaker students, they have concerns about challenging the stronger students too. The biggest difference between the service mathematics groups and the mathematics classes in this respect is that the mathematics groups would be much more homogeneous. It is easier to pitch a class at students of similar mathematical background and ability.

Class size is a universal issue affecting the lecturers on these service mathematics courses. It has various knock-on effects and inhibits teaching. One of the lectures observed was a mathematics group with about 25 students present. In this class, the lecturer used a lot of questioning, introduced some discussion and allowed the students to do some work independently. The lecturer exploited a wide range of resources including the chalkboard, the overhead projector, the book of notes for the course and two sample questionnaires which the students had to study and analyse. Something like this, unfortunately, is not possible in large group situations. Discussion is not feasible with such large groups and students feel intimidated because of the numbers so are not comfortable with speaking out. Some lecturers prefer to teach mathematics classes simply because of large numbers in the service courses. Class size restricts lecturers' teaching styles. While they would like to ask questions, they don't because they do not like to intimidate students. Stopping to ask questions in a larger group makes it more difficult to keep control of the lesson. There are much less opportunities to do group work or have a discussion. Much as the lecturers would like to do this they know it is not feasible. Organisation is all-important when dealing with such large numbers. It seems the students in such large classes are really losing out in many ways.

Overall it is fair to conclude that there is an over-riding feeling of mismatch between lecturers' and students' expectations of their respective roles and responsibilities in the service mathematics classroom.

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A Constructivist Approach to Identifying the Mathematical Knowledge Gaps of Adults Learning Advanced Mathematics

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The changing economic climate in Ireland in recent years (Celtic Tiger etc) has seen a major increase in the number of adults returning to third level education in Ireland. These adults are enrolling in courses across a wide spectrum of disciplines from the social sciences to engineering disciplines. Each of these courses has a mathematical content varying in difficulty depending on the nature of the main discipline. For many adults this mathematical content is intimidating and has a negative effect on their overall confidence and retards their progress. The authors have adapted the APOS Theory framework (Dubinsky et al 1996) in an effort to support the mathematical learning of mature students who are experiencing difficulty. APOS Theory seeks to exploit aspects of Piaget's cognitive development theory, in this case reflective abstraction, to promote the learning of advanced mathematics. Interventions based on a genetic decomposition of the subject matter into "actions, processes, objects and schemas" are central to this approach. This paper describes such a genetic decomposition by the authors of an advanced mathematical topic (vectors).

Introduction

Mathematics education at university level is multifaceted including Mathematics (Pure and Applied), Statistics, and Operations Research etc. It does not comprise a single discipline in the traditional sense nor is it in monolithic enterprise. Mathematics teaching in Irish universities is designed for two types of audience viz. mathematics specialists and users of mathematics. Users of mathematics are students from client departments who study service mathematics. The students who study service mathematics are further discriminated in a number of different ways including gender, age, social background and level of mathematical preparedness. These issues combine to exacerbate the so called "mathematics problem" in Irish Universities and Institutes of Technology. A further complication arises because of the need to make the transition from elementary to advanced mathematics.

To date research on advanced mathematical thinking has focused on normal age students eg. Dubinsky (1991,1996), Schoenfeld (1989a,1989d), Dreyfus (1993), Serpinski(1990, 1999). Issues of numeracy have driven much of the research on adults learning mathematics (O'Donoghue, 1995, 1996, 2000, 2002; Coben and Chanda, 2000; Kaye, 2002; Maguire, Johnson, Yasukawa, 2002). Tall (1991) describes the move from elementary to advanced mathematics as a significant transition from *describing to defining*, from *convincing to proving* in a logical manner based on those definitions and this transition requires a cognitive reconstruction. There is a dearth of research focusing on *adults learning advanced mathematics.*

Cann (1999) speaks of the 'politics of mathematics anxiety' and highlights the misconception of the phrase 'naturally talented' within academic circles. Cann (1999) maintains that unless teachers are aware that the skills to which they owe their own mathematical achievements are not the result of 'natural talent' but can be identified analysed and passed on, students are placed in a position where information, which they require to successfully assimilate material in the style in which it is being presented is being withheld from them. The authors do not dismiss natural talent as Cann does but agree that the skills necessary to become competent at mathematics can be identified, analysed and passed on.

This paper will focus on adults learning advanced mathematics on service mathematics courses at the University of Limerick. We define an adult as a student recognised by the third level institute as a mature student. We define advanced mathematics as any mathematical topic, which is considered too advanced to be included in the ordinary level secondary school mathematics curriculum. As a representation of an advanced mathematical topic we have chosen vectors.

The authors are attempting to adapt the APOS Theory framework (Dubinsky et al, 1996) in an effort to support the mathematical learning of adults who are experiencing difficulty. APOS Theory seeks to exploit aspects of Piaget's cognitive development theory, in this case reflective abstraction (Dubinsky, 1991), to promote the learning of advanced mathematics. Interventions based on a genetic decomposition of the subject matter into "actions, processes, objects and schemas" are central to this approach. This paper describes such a genetic decomposition by the authors of an advanced mathematical topic (vectors). Initially the paper presents the theoretical framework (APOS Theory), followed by a genetic decomposition of vectors (cycles 1 and 2), concluding with a discussion and summary.

This work is part of Golding's doctoral research and is framed by APOS theory and employs interpretative approaches for data collection and analysis to understand how adults learn advanced mathematics (vectors), and includes mathematical life histories, reflective journals, attitudinal scales, and quantitative measures.

The Theoretical Framework

The theoretical framework for our research is an adaptation of APOS theory; a constructivist theory developed by Dubinsky and the Research in Undergraduate Mathematics Education Community (RUMEC) that deals with the way an individual learns mathematics. The purpose of the framework is to facilitate the description of any chosen mathematical concept together with the acquisition of the concept through teaching (Dubinsky, 1991). The framework relates to what Piaget describes as reflective abstraction, a concept he uses for the construction by the individual of logico-mathematical structures. The aim of RUMEC is to build on the cognitive development theories of Piaget and to apply them to more advanced mathematical concepts. The framework for the theory has three individual components: 1) Theoretical Analysis, 2) Instructional Treatment and 3) Observation and Assessment.

In this paper we focus only on the theoretical analysis of the concept.

Theoretical Analysis (APOS Theory)

A theoretical analysis is an attempt to predict how a student mentally constructs the concept during the learning process. Dubinsky et al. (1996) call this a *genetic decomposition* of the concept into its actions, processes and objects. An action is described as something that a student does as a result of some external stimuli. A process can be described as a mindset where the student can actually visualise the whole procedure without having performed any of the action steps. If the student can reverse the process, can reflect on the process and can actually feel comfortable with expanding and refining the process, and can see how actions could be performed on the process as an object, then the student is said to have *encapsulated* the process into an object. An object can be an action at a higher level and a collection of these objects will combine to form a schema. The instructional treatments are developed based on the theoretical analysis and the observations and assessments are used to revise the theoretical analysis for the next cycle.

Theoretical Analysis of Vector Topic (Genetic Decomposition)

An exploratory cycle (year 1 of the project) was undertaken for two purposes, the first, was to learn about the course material and how best to design a theoretical analysis of vectors, which would be suitable to the adults taking this module, the second, was to observe a group of five adults as they studied vectors, to try out various methods of instructional treatment and observation so that a definite approach could be developed for the future research. In the exploratory cycle all five adult students were enrolled in the Science Maths 1 module at the University of Limerick.

The Course Material

Adults taking the "Science mathematics 1" module at the University of Limerick are required to study vectors as part of the module. The course notes given to each student contain two chapters devoted to vectors. These are the first and third chapters in the course notes. Figure 1 describes the content of both chapters and the material for each of the subsections are grouped together in the diagram

Theoretical Analysis of Vectors (exploratory cycle)

The Theoretical Analysis is a genetic decomposition of the vectors into actions, processes and objects. As described in figure 2 vector actions are interiorized into vector processes and these in turn are encapsulated into vector objects. At higher levels the vector objects become vector actions and the process continues in an upward spiral until a complete vector object called a vector schema is attained.

In order to construct a theoretical analysis for the teaching of vectors to adults, the authors have used personal experience as an adult learning vectors (Golding) and the experience of teaching adults in a teaching support role. A major difficulty experienced by the authors is the adult's unwillingness to accept any part of a concept without first fully understanding the reasons why certain approaches are taken. The two key questions, which arise when learning any mathematical concept, are 'how to do it?' and 'why do it?' For an average student 'how to do it' comes first and this is the action stage. The 'why do it' part comes about as a result of accommodations and assimilations, i.e. moving through the stages of process, to objects, and finally to the schema stage where a full understanding is reached.

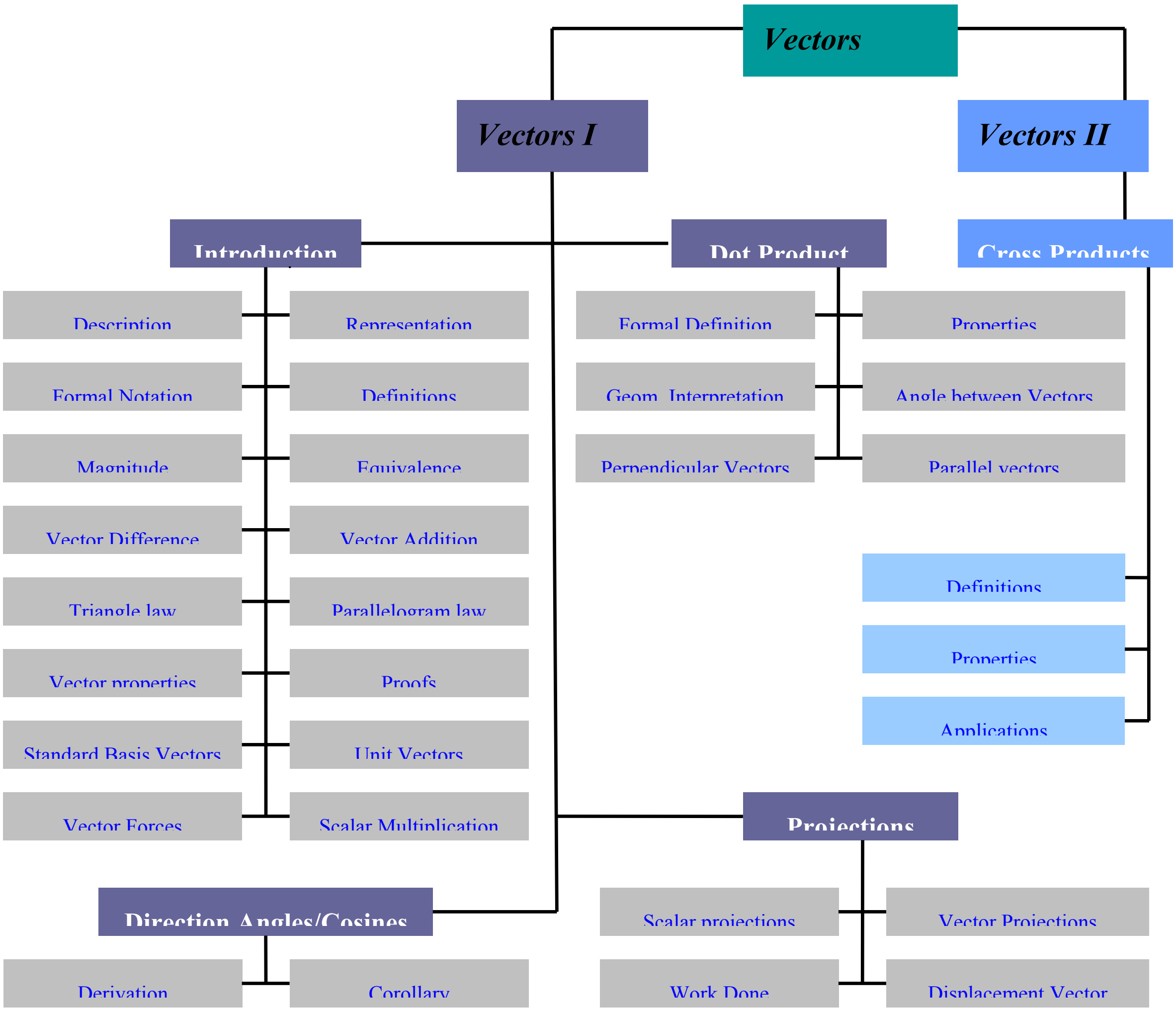

Figure 1: Course Material

From experience the authors found that most adults find it difficult to accept the ‘how to do it’ part without first understanding ‘why’. This is often as a result of some anxiety or fear of getting lost in the topic at a further stage. For this reason it was felt that a complete understanding of the concept of a single vector was necessary before moving on to topics which involved the interaction of two or more vectors. It was also felt that a necessary requirement would be to highlight the prerequisite material needed for a complete understanding of a single vector entity. In the theoretical analysis this prerequisite material is presented as an object/action. An example would be the trigonometric requirements. This can be classified as a trigonometric object, which appears as an action in the single vector object. The adult would need to have fully encapsulated all the trigonometric requirements before attempting to understand the concept of a vector. Coordinate geometry is treated in a similar fashion. Once the adult has developed a complete understanding of a single vector having encapsulated all the prerequisite objects, moving forward will be a smooth transition. In this theoretical analysis five levels of genetic decomposition were identified and each is described below.

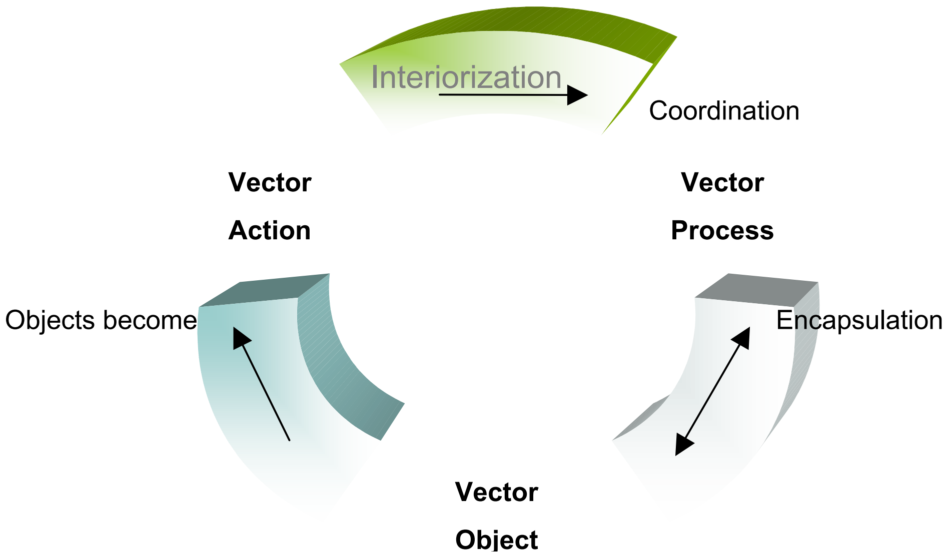

Figure 2: Theoretical Analysis

Genetic Decomposition (Level I)

This level contains a basic Vector Representation Object. As previously stated this theoretical analysis is designed for adults and there are issues with prerequisite material that need to be highlighted. In the course notes it is accepted that a student has a prior knowledge of coordinate geometry of the line and a clear understanding of basic trigonometry. This is not the general case with adults returning to third level education and therefore a way had to be found to include prerequisite material. This was achieved by treating the prerequisite material as objects that are treated as actions at the lowest level. It is at this level that these objects need to be interiorized if the adult is to progress to the process stage. The authors feel that many of the difficulties that the adult student is likely to experience learning vectors can be traced back to a lack of prerequisite material. Figure 3 illustrates the actions and processes that need to be encapsulated to form a basic vector representation object.

Basic Vector Representation Object

This object is the keystone of the theoretical analysis in that it attempts to fill the gaps in the adult's knowledge base that are necessary for a successful move to higher level objects. The first cognitive conflict that the adult is likely to encounter is interiorising both ordered pair actions to develop an ordered pair process. The Cartesian pair representation of a vector as an ordered pair of whole numbers is straightforward and developing this to accommodate rational numbers should also be relatively straightforward.

However, a major conflict arises when the adult is faced with polar coordinates. On the surface these do not appear as two numbers in an ordered pair, they involve trigonometric functions. An adult who has not already encapsulated the required trigonometric object will not progress beyond the action state. This problem will then escalate as the adult attempts to move to higher levels. A successful interiorization of the ordered pair actions into an ordered pair process will mean that the adult can start to assimilate and accommodate each of the other actions at this level into processes, and eventually should encapsulate all three into a basic representation object. This object will then appear as an action at the next level.

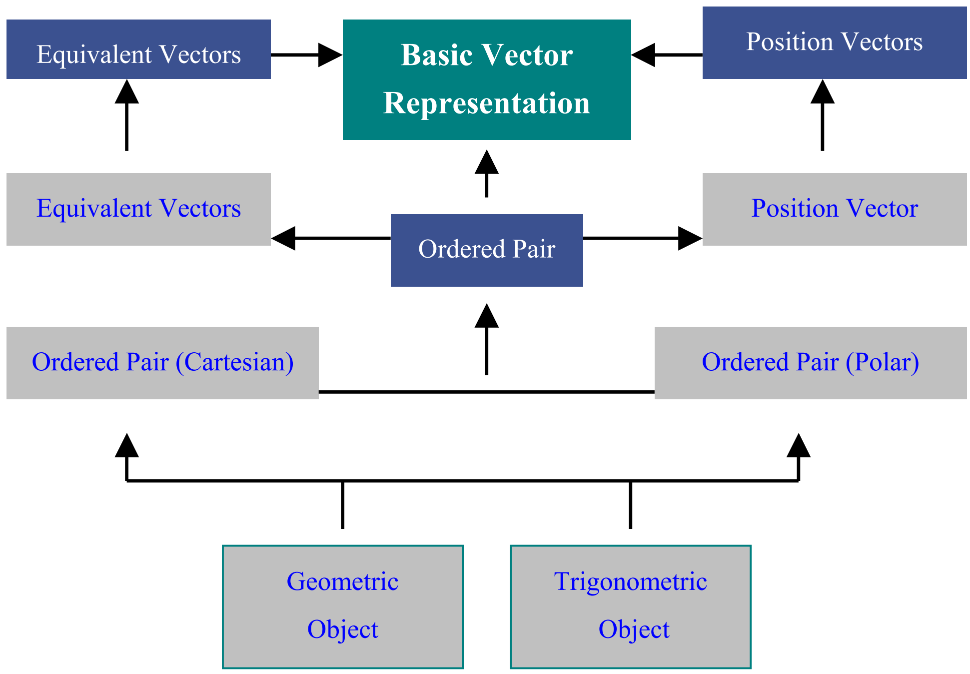

Figure 3: Basic Vector Representation Object

Genetic Decomposition (Level II)

Level II contains a "single vector properties object". Figure 4 describes the genetic decomposition of the single vector object.

Single Vector Properties Object

This object allows the adult to accommodate all the basic properties of a single vector before moving on to operations between one or more vectors. Each process at this level is preceded by an action of the same name.

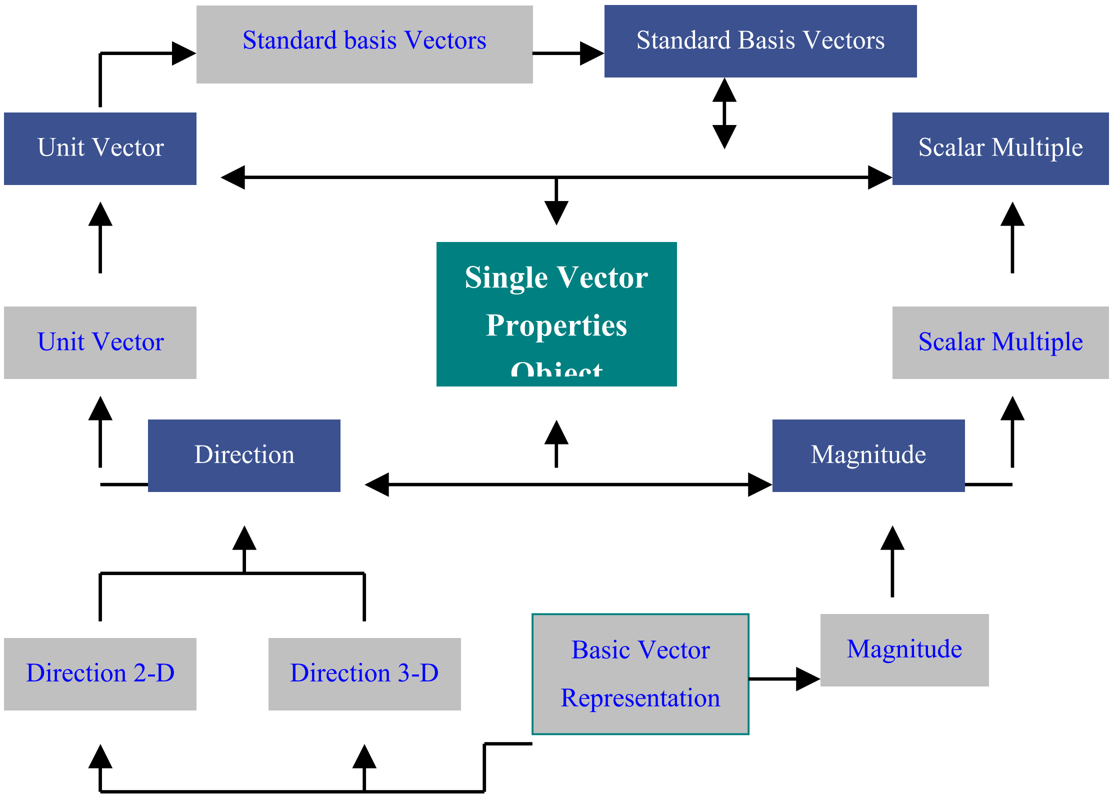

Figure 4: Single Vector Properties Object

Each action is governed by a rule or definition and the adult needs to progress from applying this rule or definition to individual vectors, to a more general case where the adult also incorporates a graphical picture of the property in both the 2-dimensional and 3-dimensional form. As indicated in figure 4, the adult begins with the concepts of direction and magnitude and once they reach the process stage, they should have the necessary knowledge to accommodate scalar multiplication and the unit vector. The standard basis vectors, which require an understanding of the unit vector process, can then be introduced. Interaction between all processes is then necessary to achieve encapsulation of the complete object.

Genetic Decomposition (Level III)

Level III contains two objects, neither relies on the other but both are necessary for progress to the next level.

Sum and Difference Object

Figure 5 describes the actions and process that when encapsulated give the adult a clear understanding both algebraically and geometrically of the *Parallelogram* and *Triangle* laws. As before the object from the previous level is a necessary prerequisite for interiorization of the concepts involved. Many geometric problems and physics applications can be solved using the concepts interiorized here.

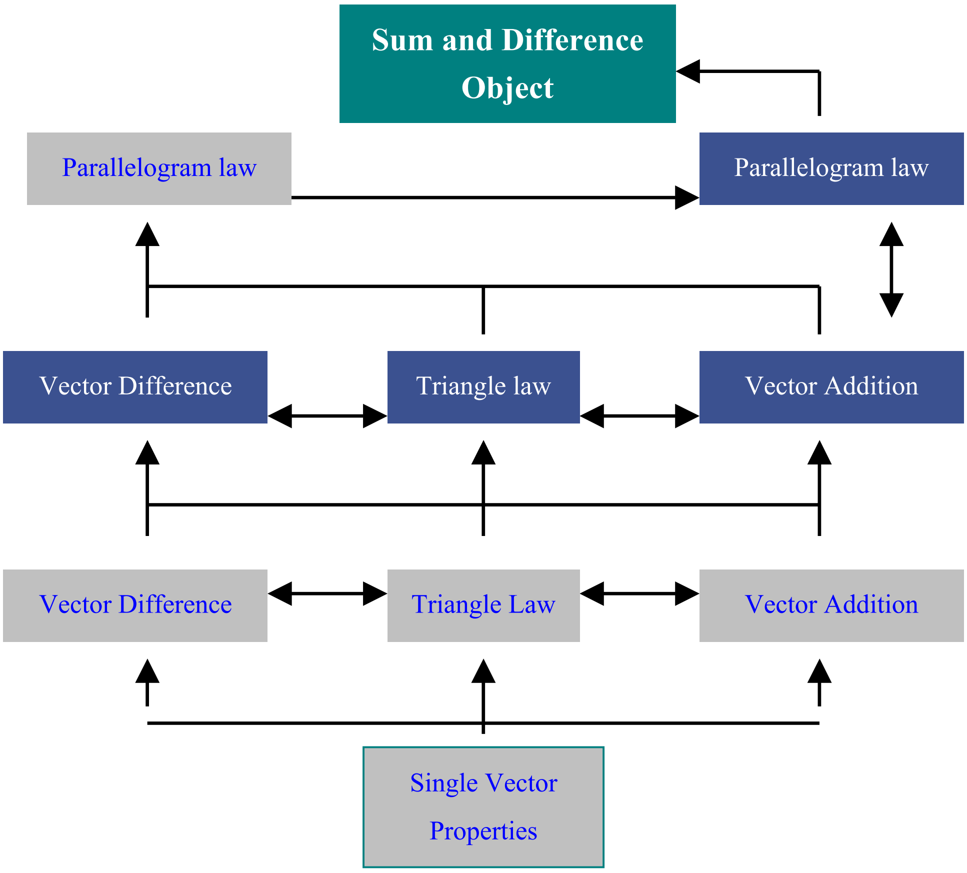

Figure 5: Sum and Difference Object

Dot Product Object

As with previous objects, each process begins as an action and it is only after the adult has mastered all the properties of the action and can apply them in all situations, will he/she move to the process stage. This object plays a key role in many proofs and applications at higher levels.

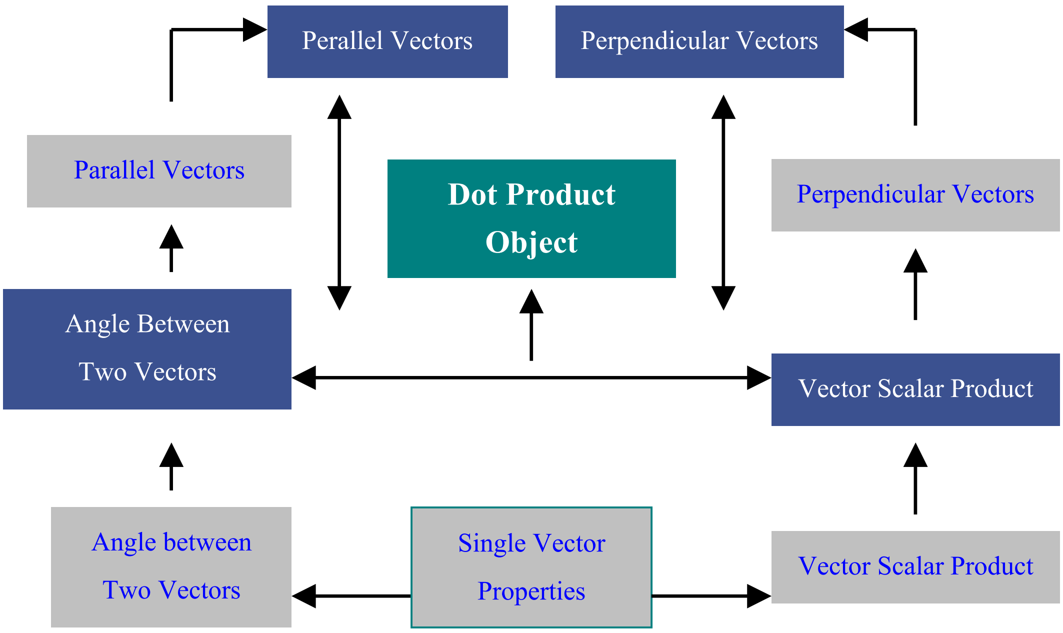

Figure 6: Dot Product Object

Genetic Decomposition (Level IV)

Level IV contains two objects: “the projections object” and the “cross product object”.

Projections Object

Figure 7 describes the actions and processes associated with this object. The vector projection is sometimes known as the orthogonal projection and begins as an action. At this stage the adult can apply a formula to find the vector projection. Similarly the scalar projection also known as the component of one vector along another vector begins with an action. This too is the application of a formula. When the adult has a clear understanding of the meaning of both concepts and can derive each, they will have moved onto the process stage. Encapsulation of the processes into a projections object occurs when the adult can move between both processes to solve geometric and physics applications comfortably.

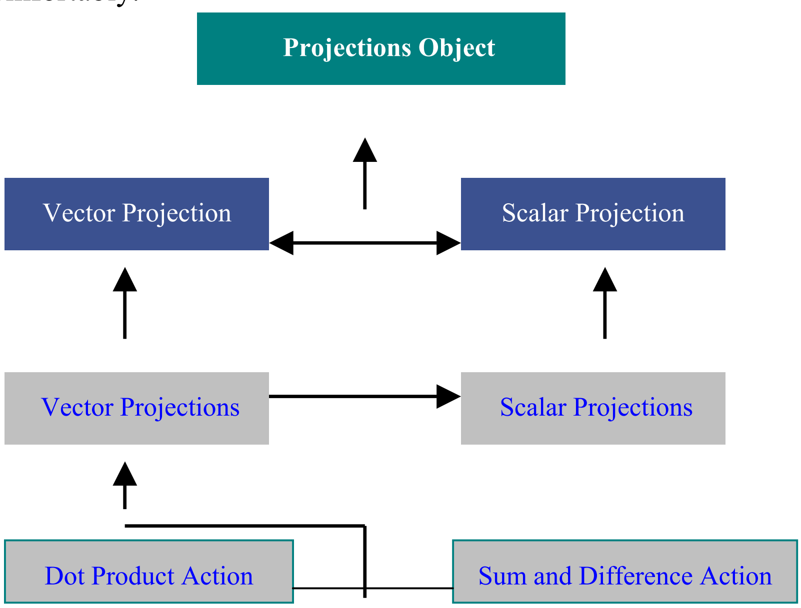

Figure 7: Projections Object

Cross Product Object

This is the final object and is used quite extensively in geometric and physics applications. The action stage comprises of a formula for calculating the cross product based on the determinant method. The student will need previous knowledge of matrices and linear equations so this is included as an action object. Figure 8 shows the breakdown of the object.

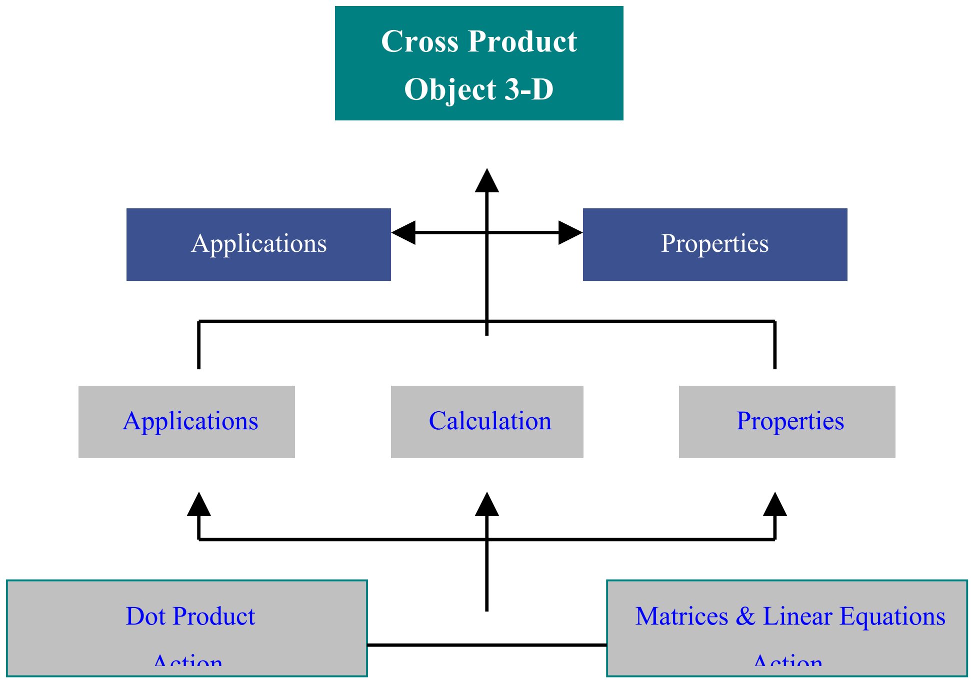

Fig 8: Cross Product Object

As the object is used in many different applications, it is important that the properties are clearly understood. A properties action is included, as the student needs to deal with each property separately at first. The adult reaches the process stage when he or she can use several different properties to prove a result. Encapsulation occurs when interaction takes place between both processes.

Genetic Decomposition (Level V) Vector Schema

This is the highest level and when an adult has encapsulated this object, he or she is said to have a vector schema. This means that they have encapsulated all the knowledge from previous levels and can freely call on this knowledge to solve a wide variety of problems. Figure 9 describes the highest level.

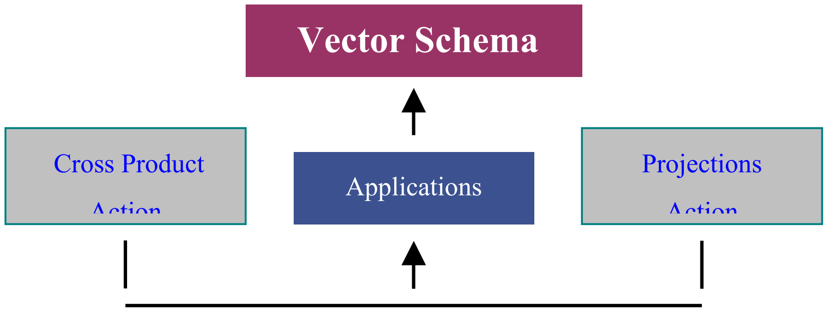

Figure 9: Vector Schema

Theoretical Analysis of vector topic (cycle 2)

The genetic decomposition of vectors given above was developed by the authors from the personal experience of the author (Golding) being an adult learner and from his experience teaching the five adults involved in the first cycle in a support-teaching role. We learned a great deal from this part of the study. At the beginning it was difficult to predict how adults faced with the pressures of advanced mathematics would react when they were asked to take part in a study. We experienced a very positive response from all five adults from the beginning. In casual conversations over coffee, we were able to observe changes in their confidence levels and gain vital insight into their mathematical learning experiences. This theoretical analysis formed the basis for the instructional treatments used in the second cycle. In the second cycle the adults taking part in the study came from both first year undergraduate science maths and engineering maths. A revised theoretical analysis (genetic decomposition) was then developed from the experience and knowledge gained by the authors during the implementation of these instructional treatments in cycle 2 of the study.

The instructional treatments used in cycle 2 were a series of eight workshops designed to specifically follow the theoretical analysis developed from cycle 1. During the implementation of these instructional treatments, in particular the perquisite workshops the author (Golding) experienced the existence of knowledge gaps that we were not previously aware of in cycle 1. Many of these gaps arose from basic algebra and other elementary mathematical concepts. As a result a more indebt granularity was consider appropriate for the revised theoretical analysis. The prerequisite objects were discarded and in their place a more fine-grained approach was taken at the action steps that would highlight the existence of knowledge gaps regardless of their mathematical origin and these issues could then be dealt with on an individual basis if necessary. The granularity of the genetic decompositions was determined by the exigencies of the situation being more or less fine grained as required. The complete revised theoretical analysis is far too detailed to be included in this paper. A more in-depth analysis of this theoretical analysis features in the author's (Golding) research.

An example of a genetic decomposition of an action step is given to demonstrate the revised approach. This new theoretical analysis is an extension to the existing APOS theory approach in that it is specifically developed to suit the learning needs of adults and can be used as a teaching aid to highlight potential gaps in the mathematical knowledge of the adult learner.

To find the angle between two vectors $\vec{a} = \langle 3,4 \rangle$ and $\vec{b} = \langle 2,-3 \rangle$ the adult must apply a formula. The application of this formula requires the adult to perform an action. The genetic decomposition of this action into its elementary mathematical concepts is as follows

Formula $\theta = \cos^{-1}\left(\dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)$

$\vec{a}.\vec{b} = 3 \times 2 + 4 \times (-3) = 6 - 12 = -6$

$|\vec{a}| = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$

1. Greek Symbols,
2. Addition,
3. Negative numbers,
4. Multiplication,
5. Square root properties,
6. Mathematical notation,
7. Inverse trig properties,
8. Indices,

Discussion

The genetic decomposition of this action highlights the number of mathematical concepts that an adult needs to have already accommodated before being able to successfully apply this formula to the given vectors. If this formula were applied in its general terms i.e. with algebraic notation, the granularity would increase even further to subscript notation and general algebra. Developing a theoretical analysis in this format highlights all the areas where potential knowledge gaps might occur. These knowledge gaps can then be addressed at extra support workshops or at individual level if necessary. This type of approach is clearly supportive of adults who arrive at third level education with varying mathematical backgrounds.

Conclusions

Using an approach consistent with the APOS theory methodology we developed our original theoretical analysis based on the experience and data feedback from the exploratory cycle. We then revised and extended the theoretical analysis based on our observations in the second cycle. The authors are satisfied that APOS Theory has potential for framing studies in adult mathematics learning. In particular there is scope for improving our understanding of how adults learn mathematics and using genetic decomposition and instructional treatments based on the genetic decomposition. The original idea of the genetic decomposition (Dubinsky et al, 1996) was an attempt to predict how the concept may be formed in the mind of the learner. In this paper the authors have demonstrated how a genetic decomposition can be used not only to predict how the mathematical concepts might be formed in the mind of the learner, we have taken this process further and used the genetic decomposition as a tool for highlighting potential gaps in the mathematical knowledge of the adult learners.

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Mathematics Assessment:

A Process for Students with Special Needs

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This paper will examine issues affecting assessment and learning for people with disabilities and their teachers. The goal of this research is to develop an assessment process for students with disabilities. Learning support teachers work with students to develop effective personal educational plans. These plans depend on effective assessment and identification of existing skills and knowledge. This presentation will reflect on the principles of assessment that should guide the development of processes suitable for people with disabilities. Such processes will accurately provide the information essential for the development of effective personal educational plans. The process that will be presented is based on a core set of test materials, which when used in the manner suggested will usefully inform the development of individual educational profiles. The central principle behind this assessment process is that of communication between student and teacher.

Introduction

This research was a response to the need to identify the mathematical skills of students with disabilities. The goal was to develop a test package that students with disabilities could use effectively and that would help teachers to identify interventions in the mathematical field. Standardised tests give little or no information about people with low-level skills. Significantly more useful are criterion-referenced tests, which give some of this information about existing skills. Such tests give a snapshot image of a student on a particular day. These tests do not provide diagnostic information, which is essential for an effective educational plan.

Background to Study

Administering various assessment packages to students with special needs over many years caused me to question the format and use of assessment. While being reasonably comfortable with the effectiveness of reading assessments, both formal and informal and the usefulness of the information they provide, I was concerned about techniques and format in the mathematics field. Did the literacy content of many tests prevent some students from 'doing the maths'? Did calculations out of any relevant context suggest deficits where skills were present? Were procedural skills not much less important than analysing, interpreting, responding to and communicating mathematical information? By focusing only on the answer was I ignoring the problem solving strategy that has been used successfully? In an analysis of the intake assessment results and end of course certification levels of National Training and Development Institute (NTDI) students in 1995/96, major discrepancies were noted in the attainment achieved by students with learning difficulties. It was in the light of these questions that assessment of mathematical competence merited scrutiny.

This research began as part of my work with the National Training & Development Institute (NTDI), co-ordinating learning support in 1996. NTDI facilitated and supported the research. John O'Donoghue, University of Limerick, offered to supervise the research and provide much needed expertise in the area of adults learning mathematics.

The research project sought to examine and improve assessment in relation to elementary mathematics. It includes a brief outline of mathematics and mathematical competence. The central focus is the aim of assessment, ways in which assessment could be improved and the intervention itself. The valuable work already done in relation to developing an appropriate assessment system meant that this research had a tremendous resource of data and experience readily available. Utilising the experience and insight of learning support teachers in NTDI helped to focus time and again my thoughts and the thrust of the intervention. Throughout the co-operation and contribution of students was overwhelmingly enthusiastic and practical.

Literature Review

"Classroom practice in mathematics is different from other subjects but this is not necessarily a good thing." (O'Rourke, 1997). My own lack of competence and confidence in the area of mathematics heightened my awareness of the need for training of teachers. The area of adult literacy has received much attention and training, with numeracy treated as a sub-section of literacy or indeed as an ancillary clause in job descriptions. The lack of expertise and skill in teaching this crucially important subject should not be under-estimated. This lack of expertise should not be confused with a lack of interest. In pursuing this research I was fortunate to encounter teachers keen to develop their skills and discuss strategies for teaching and assessment. The government publication, *Learning for Life: White Paper on Adult Education* (2000) has no plans to identify training needs for students and teachers nor indeed a definition of numeracy from which to start.

However the principle that assessment is a process of communication is central, regardless of the subject matter. The teacher is responsible for ensuring that the materials used by every student are appropriate and accessible. This statement is simple yet can be difficult to achieve especially for students with special needs. Students in need of elementary education often have a negative experience of education and poor self-esteem. It is essential that the process of assessment positively reinforces the relationship between the students and teacher. The communication of information to other relevant figures in the family, school or wider community is another phase in this process. The quality of assessment information will significantly affect the individual educational plan of every student.

Assessment can be used as a filtration process and frequently this is in fact the purpose. Equitable instruments need to pay attention to language, design, context, tone and appearance. Language should be straightforward, uncomplicated, stimulate responses that need to be thought out, and use diagrams or models to define vocabulary. A vocabulary list can be checked with students and be made available to them during assessment. Scoring should be based on concepts and not language. At the design stage, the aim should be to ensure that questions are not complex but should stimulate responses that show thought. Context should be appropriate for the student, not the teacher. This is easily overlooked when teachers construct tests. Culture, educational background and socio-economic diversity need to be catered for. The conditions of assessment should be familiar and informal. Context should be realistic and appropriate for a competent student. Ample time should also be given to complete the assessment. By reducing barriers, access to the assessment process is facilitated for more students. "Teacher support is critical" (Belcher, Coates, Franco and Mayfield-Ingram, 1997:1998).

Assessment is most usefully linked to instruction. Teachers can act as guides, observing and discussing what they see in work. Grades are of little use when the same grade may indicate successful scoring on a wide variety of questions. Teachers continually assess in daily instruction, but this is not valued when test results are used externally. However, operating on the assumption that "teachers develop a useful, uniquely detailed sense of what students know and can do through interaction" (Asturias, 1994), integrating this skill into data collection has the potential to be very useful. Teachers can help students learn by understanding their thinking and stimulating or correcting as the need arises. Assessments that allow students to show connections and understanding, by implication, show evidence of sound mathematical thinking. 'Discussion is a powerful key to learning' (Belcher et al., 1997:1998). This powerful phenomenon is often ignored in group-assessments, but it is of particular importance in mathematical assessment as writing skills are often also weak.

Assessment needs lots of organisation so that what is important is available and is clearly recorded. Students are central characters in this organisation and should be treated as partners in the process. This could include student assessment of their own work.

Assessment has the potential to give information about student perceptions of mathematical concepts and processes as well as their ability to function mathematically. The integration of assessment and instruction has educational potential. Making sure that assessment is integral to instruction should mean that the information obtained is really useful for guiding instruction. In short good assessment is good instruction. (Webb & Briar, 1990)

Speaking of the US education system, Kulm refers to two significant features of the impact of testing by saying "Whatever gets tested gets taught" (Kulm, 1990) and the even more depressing "only that which is tested will be taught". Recent commentary on the second level system in Ireland has raised similar concerns. All those involved in the Irish education system, students, parents, teachers and employers have made similar remarks in relation to the Leaving Certificate (*Commission on Points*, 1998). Kulm agrees with the view of Romberg, Zarinna & Collis (1990) that the discrete categories of the content and behaviour model of mathematics fail to reflect the interdependence of mathematical context. In his critique, Kulm reiterates the assertion that this model ignores psychological evidence of mathematical learning, where complex networks of meaning connect new concepts and skills to those already mastered

In 1989 the NCTM recommended ways to improve assessment by paying increased attention to the following:

- Assessing what students know and how they think about mathematics
- Having assessment be an integral part of teaching
- Focusing on a broad range of maths. skills and have holistic view of mathematics
- Developing problem situations that need the application of a number of maths. ideas
- Using multiple assessment techniques – written, oral and demonstration
- The use of calculators, computers and manipulatives
- Evaluation of programmes by considering information on outcomes, curriculum and instruction
- Standardised test should be used as one indicator.

Research indicates a number of concerns about traditional tests. Traditional tests can be used to initiate valuable diagnostic processes. Talking to students will help us understand beliefs that form the basis of test responses. The language used in problems must be chosen carefully because it may cause difficulties for some students. Tests must avoid the possibility of choosing the right answer for the wrong reason. When asked to explain the method they have used, students usually assume that they have made errors. This is further evidence that mathematics is seen as a written subject, which consists of right, or wrong answers.

As part of their everyday function, teachers assess who knows what and what needs to be revised. However, formal assessment systems in many countries including Ireland do not legitimise this. The assessment skills of teachers need to be valued recognised and built upon. Realistic concerns for teachers include doubting their own assessment skills, difficulties in tracking groups at work and offering poor service to students in the judgements they make.

Assessment is the exchange of information but this information is often misinterpreted, misused or mistaken in its meaning. Teachers and students see what happens in a classroom differently. Clarke (1992) identifies two important themes - assessment as a mechanism for the social construction of mathematical competence and the practical implementation of constructive assessment. Outlining the possible distortions involved in existing assessment structures, Clarke recommends the development of alternative methods. When undertaking such development, two factors need to be considered, the necessary exchange of information and that this should be meaningful and productive. These factors will "empower student and teacher". The empowerment of students is particularly important when one considers that "test performance has profound implications for conceptions of themselves as mathematically able and it is from such conceptions that students draw their motivation" (Clarke, 1992:150).

As a teacher and not a mathematician I needed to develop a clearer picture of what it means to be a competent user of mathematics i.e. a numerate person. The following statements are those that I find most useful in developing this understanding. Mathematics is usually defined by using lists of descriptors such as "classifying, abstracting, symbolising, and proving". Stewart & Tall (1977), and Cockcroft (1982) among others note the hierarchical structure of mathematics. Kline (1964) defined it as a highly symbolic language, involving reasoning. Counting is a basic competence for children (Gelman & Gallistel et al., 1978) and is used to solve many problems even when they have difficulty in school (Ginsburg, 1977). Calculation errors often happen due to application of incorrect procedures rather than slips. "Current mathematics education does not adequately engage students interpretative and meaning constructing capacities" Resnick (1987). Resnick also suggests that successful mathematics learners engage in meta-cognitive skills (Brown *et al.*, 1983). These meta-cognitive skills are those which students use to regulate their own learning, by checking understanding, reviewing as necessary and organising both attention and resources to ensure effective learning. The successful learner will check understanding of procedures, monitor for consistency, relate new to prior knowledge and is less likely to practice manipulations without reference to the meaning of symbols (Peterson et al 1984, Resnick 1987). Dweck (1988) adds that strong mathematics' learners work out alternative strategies for attack and solvable sub-problems. The International Life Skills Survey offers the most useful definition of numerate behaviour:

Numerate behaviour involves managing a situation or solving a problem in a real context by responding to mathematical information that is represented in a variety of ways and requires the activation of a range of enabling processes and behaviours. Given the wide range of definitions acceptable in all of the above literature it is obvious that the assessment techniques used to identify these competencies will be many and varied.

Cycles of Intervention

This intervention involved the development of elementary mathematics materials, which could be used by learning support teachers as part of a process, to identify the learning strengths and weaknesses of students with disabilities, on vocational training courses. The critical evaluation of materials by students and teachers took place at each phase of this research. The significant improvements that were made on the basis of these evaluations were the simplification of the literacy demands, the use of colour and graphics and the inclusion of material relevant to catering such as recipes and menus. The development of styles of presentation suited to various disability groups was a result of professional development on the Computers and Assistive Technology Course in U.C.D.

As this research took place within the confines of a vocational training centre for people with special needs, the research was of a clinical nature. The students availing of vocational training within NTDI have a wide range of special needs. Broadly these needs may be defined as learning impairment, physical disability, sensory disability and mental health difficulty. The data from each group also helped facilitate reflection and maintain objectivity. This objectivity is particularly difficult to achieve in action research, as the researcher is part of the context in which the study takes place. The benefits of active involvement, situational knowledge and the ability to respond to the concerns of students and teachers far outweigh this difficulty. This cycle has been repeated four times during this research. As mentioned above, it should continue to be repeated at regular intervals to ensure that the process remains current and relevant.

The first phase actions took place in late 1996:

- Met learning support teachers to evaluate existing assessment

- Collected list of assessment tools advocated by learning support teachers
- Initial test booklet produced
- Test administered to catering students, who had already completed in house assessment
- Inclusion of test as class-work, to be completed over time
- Evaluation of initial booklet by catering students in Dundalk
- Evaluation by interested learning support teachers
- Review of data from student and teachers.

Comparison of former assessment material and initial booklet indicated that little had in fact been changed. The questions were still largely computational and the presentation followed traditional lines. Scoring was still based on answers, though methodology was noted. Catering students expressed the view that this had been easier to follow. Learning support teachers commented on the absence of a standardised score or mathematical age. This comment was quite interesting as learning support teachers often question the value of both of these figures, particularly in relation to adults. These issues indicate the training needs that arise when new assessment procedures are implemented.

In a separate development within NTDI, assessment of mathematical skills was divided into four vocational areas, service, business, horticulture and industrial. This attempt to put elementary mathematical skills in context was a significant step forward. Unfortunately the answer was still the measure of competence.

Phase 2 in1997 saw the following developments:

- Second booklet developed
- Introduction of colour
- Some graphical material included
- Language simplified in problems
- Problems with a catering context, recipes and menus
- Provision of working out space
- Test administered to catering students who had already experienced in house assessment
- Evaluation by catering students
- Evaluation by learning support teachers
- Review of data from students and learning support teachers.

This phase elicited a more positive response from students and learning support teachers. The anxiety of assessment diminished for students, as the work was now included in weekly instruction. The time constraint for teachers of providing useful information about the students' learning needs for the programme-planning meeting (6 weeks after training started) was eased.

Phase 3 in 1998 saw the following developments:

- Third workbook developed
- Increased use of colour
- Increased amount of tasks included
- Test administered to catering students who had already experienced in house assessment
- Evaluation by catering students
- Evaluation by learning support teachers
- Review of data from students and learning support teachers
- Author reviews "Measuring Up" software package.

Students' responses were again favourable. Teachers also expressed positive reactions. The possibility of including more complex tasks from catering training situations was raised.

The final cycle in 1999 saw the following developments

- Development of fourth booklet divided into two parts
- Part one had more advanced problems, and was discussed with students first
- Part two had a wide variety of less complex problems, and was administered when students presented with difficulties in relation to part one
- Inclusion of pages with blue background and yellow text for students with visually impairment, format to be adapted as necessary
- Dissemination of this workbook to expert panel for use in their particular situation and comment
- Test administered to students from catering and also Vocational Skills Foundation (a pre-vocational course)
- Evaluation by students
- Evaluation by expert panel, including learning support teachers
- Review of data from students and expert panel

The final review was positive both on the part of students and the expert panel. A psychologist expressed some reticence about the presentation of the more complex tasks first, concerned about the possibility of damaging the self-esteem of students. Learning support teachers however felt that the inclusion of the materials as part of instruction negated this criticism.

Discussion and Recommendations

During this intervention the involvement of students and learning support teachers was particularly successful. The integration of assessment into teaching successfully lessened the anxiety of students. Learning support teachers had the opportunity to focus on individual student's strategies and competencies in elementary mathematics. This qualitative assessment has much in common with adult literacy programmes.

The final process which this intervention supports involves a change in attitude about assessment, a selection of resources which learning support teachers can adapt for students and guidelines as to the way to summarise and communicate this information to the student and others. The research and development of the teaching of mathematics is progressing apace, and will continue to initiate changes in curriculum and assessment.

This intervention had as its goal the design of a system of assessment of elementary mathematics appropriate for people with disabilities. The process has taught me a great deal about assessment, learning theory, elementary mathematics, the needs of students with disabilities, the skills and learning needs of teachers and the complexities of organisational change. Communicating regularly with students and learning support teachers was a vital part of this process. It ensured that I was aware of the needs of each group and fuelled the creation and development of a dynamic and supportive process in which change is possible and indeed encouraged.

The intervention has clarified the following principles of assessment:

- Assessment is part of a process of communication,
- Assessment should positively reinforce the student teacher relationship,
- We need a clear understanding of the purpose of assessment,
- Assessment results should be clearly understood by student and teacher,
- Assessment is integral to good quality instruction,
- Assessment should initiate action,
- The quality of assessment information will significantly affect the educational plan of each student,
- Assessment should involve student evaluation of personal work,
- Student evaluation should affect instruction,
- Teacher training in assessment methodologies is essential,
- Validation of existing teacher skill in assessment is needed,
- Any system will need to be regularly evaluated by all users.

The implementation of these principles of assessment would improve the quality of learning support provision, in any context. Future developments of this process will be less time consuming.

Following the Equal Status Act 2000, all educational organisations will need to increase their awareness of access difficulties and implement proactive strategies to meet the needs of all students. In the light of the professional evaluation and issues of access, further development of a computer-based interactive format has potential application to many educational settings. The development of this process as a model of elementary mathematics assessment has continued in St. Brigid's school, where I currently teach.

The implications of this research for assessment are clear and positive. Assessment needs to be an integral part of quality instruction. The form of assessment will also be governed by the purpose of the test. In order to achieve this, the purpose and impact of the various techniques need to be closely examined by anyone using assessment. Teachers need training in the various methodologies. The validation of teacher skill in ongoing assessment is essential, as it is an integral component of best practice. The best international developments are both creative and practical.

Conclusion

In summary then, assessment for people with disabilities must be constructed with reference to the changing perspectives on disability, inclusion, equality, and education. The responsibility for promoting awareness of disability and resolving issues of access for people with disabilities rests with society as a whole. Assessment processes and instruments will be essential to support individual educational profiling and planning. The design of such instruments for students with disabilities needs to be approached hand in hand with the professional development of teachers.

The most significant recommendation I could make would be for the development of quality in-service training for teachers of mathematics. This training need has been reflected in discussion with teachers. Training is needed in the subject area of mathematics, as many learning support teachers do not have an adequate background in the subject itself. Assessment, which has a crucial role in education, requires particular training. In the light of such training the confidence and competence of teachers would improve. The provision of quality programmes of education requires that the teachers are competent in the subject area and trained to teach the material. Innovations and research in mathematics education need to be passed on to teachers, in a structured and supportive system.

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Mathematics Learning Support in Irish Primary Schools: A Study

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Much work is currently being undertaken in relation to the provision of literacy learning support to children with learning difficulties. However, despite the aims of The White Paper: Charting Our Education Future (1995) and the Learning Support Guidelines (2000) it is clear that mathematics learning support does not receive the same attention in our mainstream, non-disadvantaged primary schools. The purposes of my study were firstly, to investigate the literature on mathematics learning support in order to provide a structure and a basis for my research; and secondly, through a research study, to establish what exactly learning support teachers are doing in relation to the provision of mathematics learning support.

Introduction

The research was concerned with the following questions. These are:

- What provision is being made in Irish primary schools for mathematics learning support? If this provision is being made what form does it take? If there is no supplementary mathematics teaching in particular schools, why is this so?
- What are the aims of mathematics learning support teachers? How do they view their role in the primary school?
- What teaching methods do mathematics learning support teachers employ when teaching children with mathematics learning difficulties? How does one approach the teaching of specific concepts to these children? How does one overcome the issue of the complex language of mathematics?
- Do schools have a policy in place that deals specifically with the area of mathematics learning support? Do mathematics learning support teachers consult regularly with their colleagues and their student's parents? Are standardised mathematics assessments and Individual Pupil Learning Programmes part of the school's mathematics learning support policy?

The Aims of Learning Support

The aims of education are the same for all children whether or not they have learning difficulties. These general aims are to enlarge the child's knowledge, experience, imaginative understanding, moral values and capacity for understanding, and also to enable the child to enter the world after formal education as an active, responsible participant in society, with as much independence as possible. It is expected that at the same time the child will attain a certain level of literacy and numeracy while in school. Learning support allows children with learning difficulties the opportunity to achieve these aims (The Warnock Report, 1978; Learning Support Guidelines, 2000). A child has a learning difficulty if he finds it more difficult to learn than the rest of the children in his class. Learning support allows a child to progress through an individualised programme of work with the opportunity of talking and working with a few of his classmates which is absolutely essential for the low attainer. It allows children with learning difficulties to be connected with and included in mainstream education because mainstream education cannot provide for the diversity of learners that are in schools. It provides for inclusive education. It allows each child the opportunity to develop to his full potential (Duncan, 1978; Report of the Special Education Review Committee, 1993; Gross, 2002).

The Role of the Mathematics Learning Support Teacher

Traditionally, the role of the learning support teacher has been to attempt to remediate the problems of children who are perceived to be performing at a level which is significantly below that of their peers. The learning support teacher supplements the work of the class teacher by providing more intense tuition to pupils whose attainments in basic literacy and numeracy are very significantly below average (The Report of the Special Education Review Committee, 1993).

It is necessary for the learning support teacher to examine what is required to help children learn mathematics in a meaningful and structured way. This only occurs to the extent that learning occurs. Effective teaching rests heavily on the consideration of how children learn.

> The process of building bridges from the concrete to the symbolic and helping children cross them is at the heart of good teaching and is a constant challenge. (Reys et al., 2001, pp. 13)

The role of the learning support teacher is to link, connect or establish meaningful bridges from the context to the mathematics.

The importance of teacher empathy and awareness for the difficulties experienced by children with learning difficulties is vital in the view of many researchers (Johnson and Myklebust, 1967; Duncan, 1978; Bley and Thornton, 1991; Williams, 1995; Stakes and Hornby, 1996; Reys et al., 2001).Teachers need to be aware of their students needs and provide for them individually. The mathematics learning support teacher needs to consider the child carefully when planning a programme of work. The timing of the introduction of a mathematical concept and the pace at which the pupil develops skills are vital to a slow learner's pace of work. The learning support teacher must bear in mind that the remediation programme must be based upon the nature of the deficit, not on the importance of quantitative thinking. The role of the mathematics learning support teacher is to redefine the work for the pupil who is incurring difficulties, to consider alternative solutions and to concentrate on the mathematics skills in which the child has some proficiency. A positive teacher attitude towards the teaching of mathematics is important. Learning support teachers who enjoy teaching mathematics and share their interest and enthusiasm for the subject tend to produce students who are positively disposed towards mathematics also.

In teaching mathematics to pupils who are low attainers, the children need to be encouraged to generalise and abstract from particular situations and experiences so that they can move from the concrete to the abstract and back again. This relates to the essential nature of mathematics as the discovery and application of numerical and spatial relationships. The process of abstraction involves generalising, which is necessary in the transfer of mathematical ideas from one context to another. Learning support teachers need to acknowledge the importance of procedural and conceptual knowledge in mathematics learning. Research informs us that understanding should come before proficiency in skills.

The children in learning support need lots of practice to achieve mastery, rather than the teacher moving on prematurely. They need to be given much more time on task which research has shown to be particularly low in children with mathematics learning difficulties. This will contribute towards building the child's confidence and self-esteem, which again features prominently in research on the characteristics of children who do not learn well, and which teachers are uniquely able to influence as a route to enhancing achievement (Gross, 2002).

> Learning how to break down skills and concepts into small steps takes practice. It is however an essential competency for special needs work. It is also an immediate confidence builder for both teachers and pupils, shifting as it does from what the child can't do to what, in the easier smaller steps, they can do and what the next learning steps might be. (Gross, 2002, p. 39)

This type of teaching promotes children's "reflective awareness" of mathematics (Maclellan, 1997). If children are aware of what they are doing in mathematics they will be able to transfer their knowledge to help them in real-life situations. The teacher needs to question, discuss, hint, suggest and instruct children as to what they must do in order that they have any chance of abstracting the mathematical concept that the teacher intends to teach.

Mathematics Learning Support Teaching Strategies

Mathematics is a special type of knowledge and calls for special teaching methods. The teacher must find ways of teaching concepts other than the traditional "show and tell" method in order for the children to remember the ideas (Copeland, 1984). One must also examine how the children have learnt. Children have fears about mathematics that must be dealt with.

> There may be gaps in the knowledge hierarchy and errors in it as well as general delay developed over time. (Montgomery, 1998, pp. 154)

The teacher needs to check out these origins and processes (Wong, 1991). This can aid the teacher in developing the most appropriate method of teaching and form of intervention. If a teacher goes back to a child's previous work, areas of success and weakness are quickly detected and this gives the teacher an idea of where to begin. The teacher needs to observe how the children approach certain tasks. The teacher can then evaluate student demeanour, student attention to task and the extent to which the student can read and follow instructions. The teacher should examine the student's work to identify specific error patterns.

Pacing is the key to teaching children with mathematics learning difficulties. These children require more practice to achieve mastery, more examples to learn concepts, more experiences of transfer, and more careful checking for readiness for the next stage of learning (Smith and Rivera, 1991; Norwich and Lewis, 2001).

> Teaching methods should focus upon concept development and attainment and the development of schemata or protocols for further learning in each subject area. These will have transfer possibilities and make learning more efficient and effective. Much of the learning of these pupils should take account of the fact that they are developmentally younger and thus methods traditionally used are more appropriate even well into secondary education (Montgomery, 1998, pp 37).

When teaching mathematics to children with learning difficulties it is vitally important that it is done in a sequential and hierarchical manner. Children need to understand the basic concepts fully before they can build on them and are ready to move on to the next concept. Mathematics is a hierarchical skill in which one step is dependent upon a complete grasp of earlier steps and that it is not possible to skip over numbers and words to understand a problem. In the opinion of Anghileri (1995, pp. 7)

> Children will address all new experiences and problems by relating back to past experiences that will help them understand new phenomena and structure new relationships within their personal framework of knowledge.

Many children with learning difficulties often have trouble with reading too. Shuard et al. (1984) carried out a study, the results of which concur with a similar study carried out by Raiker (2000). This study investigated the relationship between reading and mathematics. It was found that readability could significantly affect success in mathematics and that a problem presented verbally may not be able to be solved, whereas if it was written in numbers it could be easily solved. Mathematical language is extremely complex. It uses a technical vocabulary that overlaps with the vocabulary of everyday language. This can cause confusion and the development of flawed understanding. Teachers of children with learning difficulties need to incorporate the teaching of mathematics vocabulary into their daily work.

> This is based on the hypothesis that the understanding of mathematical words is fundamental to the development of sound concepts and mathematical thinking.(Raiker, 2002, pp. 50)

If children do not understand the mathematics language and vocabulary being used in class, objectives cannot be achieved, knowledge is incomplete and misconceptions are developed.

One of the challenges for the teacher of children with mathematics learning difficulties is to develop mathematics that has meaning and purpose for the children. The objective should be to find purposeful activities in meaningful contexts. Low attainers will surprise their teachers with their competence and their determination to succeed if the mathematics is presented in a meaningful context (Haylock, 1992). Wherever possible the child should work on real numbers based on real life situations in their immediate environment (Gross, 2002).

> Children need to acquire the ability to read numbers and count, to tell the time, to pay for purchases; to weigh and measure ; to understand timetables, graphs and charts; to carry out any necessary calculations; to make sensible estimations and approximations. (The Cockcroft Report, 1982, pp. 10, paragraph 32)

School Policy

The Learning Support Guidelines (2000) outline the necessity of having in place in each school a policy on learning support. It is the responsibility if the principal to ensure that provision is made for all the children in the school. Primary schools need to develop a policy which makes it clear that it is the responsibility of all the teaching staff to provide for all the children in their care. A school's policy on learning support should identify the experiences that children with learning difficulties have in relation to their problem and achievements. The policy, according to the Learning Support Guidelines (2000) should ensure that the instruction and learning that these children receive is sufficiently differentiated to meet the needs of all pupils. The policy should allow the learning support teacher to plan strategically by allowing the teacher to assess the needs. and what the next learning steps might be.

This type of teaching promotes children's "reflective awareness" of mathematics (Maclellan, 1997). If children are aware of what they are doing in mathematics they will be able to transfer their knowledge to help them in real-life situations. The teacher needs to question, discuss, hint, suggest and instruct children as to what they must do in order that they have any chance of abstracting the mathematical concept which the teacher intended to teach.

Difficulties and successes that the school has experienced in relation to learning support.

When developing a school policy in relation to mathematics learning support, one needs to consider that school factors can be much more influential than home background factors with regard to the amount of progress made by pupils. It is asserted that the impact of the school is four times more important for literacy and ten times more important for mathematics (Dean, 1995; Morris and Parker, 1997; Gross, 2002).

Research Methodology

The research methodology employed in this study was a survey. This strategy was deemed appropriate for the purposes of achieving the objectives of this research, as surveys, while providing wide and inclusive coverage gather data at a particular point in time from a specific population in a standardised format with the intention of describing the nature of existing conditions (Denscombe, 1998; Cohen, Manion and Morrison, 2000).

This particular study examines and evaluates the teaching of mathematics by learning support teachers in mainstream non-disadvantaged primary schools. Factual information was collected describing what is currently happening. Comparisons in practice between schools can be determined. How other learning support teachers are grappling with this issue can be ascertained.

The participants were selected because they were learning support teachers living in an area which was geographically convenient for the researcher. The main criterion was that the schools in which the participating learning support teachers taught did not have a disadvantaged classification. The focus of this study was on how learning support teachers in mainstream primary schools were dealing with the teaching of mathematics in view of the large numbers attending for literacy support and their interpretation of the

Learning Support Guidelines (2000) given that it is expected that they are teaching mathematics to all children who are scoring at or below the tenth percentile on standardised tests. The group selected consisted of thirty learning support teachers working in an area of North Leinster.

A semi-structured type of questionnaire was deemed to be the most suitable for this research, as the structured aspect would allow for the quicker processing of data, while open-ended questions would simultaneously provide an opportunity for respondents to express their views and opinions, to explain and qualify their responses, and to avoid the limitations of pre-set categories of response.

Summary of Findings

Many of the schools which do provide mathematics learning support do so on a very small basis. Teachers allocate a few classes per week to the teaching of mathematics. For many teachers time is an issue as they are already under pressure providing for all the literacy learning support needs in their schools.

The research findings indicate that learning support teachers favour providing their students with sufficient time to work with concrete materials and manipulatives. Learning support teachers state that it is important that their students are able to shop, tell the time and perform basic measurement procedures. They should be able to estimate. The emphasis needs to be on the practical and concrete elements of the primary mathematics curriculum. Mathematics games should be incorporated into the work of the mathematics learning support teacher and all suitable resources and textbooks should be utilized as these are specifically designed for children with learning difficulties.

The research established that most learning support teachers are using at least one method of assessment when determining which children should receive learning support. Some teachers use more than one method which is advisable. These teachers are better able to put together a complete representation of their students' needs, learning abilities and styles and therefore design a programme of work to cater specifically to cater for these individual needs.

Mathematics learning support teachers see their role as one which will enable students, at the very least, to be able to carry out basic mathematics transactions, which will allow them to live full and independent lives. But the research also draws attention to the fact that many learning support teachers do not see this as being feasible. The number of mathematics classes conducted is very small, the numbers in need of supplementary teaching is increasing all the time and few teachers have done any special education diploma courses.

Learning support teachers realise the importance of providing the children they teach with meaningful experiences in mathematics which will help them make sense of mathematics. In this way they allow the children develop a sense of confidence regarding mathematics which is essential for progress and success. Learning support teachers hope to provide their students with enough competency in mathematics to be able eventually to return to their classes and work alongside their peers.

The respondents stated that when teaching children with learning difficulties one has to deal with children who are forgetful and who find abstract thinking difficult. These children have poor concentration, lack in motivation and have poor retention. They also tend to have low self-esteem. Parents also find it difficult to help their children. Occasional supplementary mathematics classes will not address these issues.

Learning support teachers believe in developing their student's self-confidence. This is done by starting off with relatively easy work and then progressing to more difficult tasks. This helps to keep the children focused and motivated and helps develop their thinking skills. The learning support teacher needs to modify the work their student's peers are doing. Children who attend mathematics learning support need this. The class textbooks are too abstract and too vague and the language is often too difficult for them to read and understand. These children need work which has been adapted to suit their individual learning needs and that which is real and applicable to daily life.

The respondents believe in the value of having an agreed policy on mathematics language and teaching methods in their schools. Children with learning difficulties are easily confused and if they are taught a particular concept one year in a particular way using certain language, and then the following year it is all changed because the class move on to a new teacher, problems ensue. To minimise such confusion it is necessary that all staff members are as one when it comes to the language used and the methods employed when teaching certain elements of the mathematics curriculum.

A school policy in relation to mathematics learning support needs to be in place in each primary school. The research findings demonstrated that many schools do not have such a policy in place as of yet.

The class teacher and the learning support teacher meet regularly to discuss their students. Progress, or the lack of it is discussed, and suitable work is organised for the underachieving student. The research also establishes that meetings between parents and learning support teachers are all too few. The role of parents in a child's education is extremely important and particularly so when that child has learning difficulties. Parents need to assist their children in their mathematics learning just as so many do with literacy.

The research findings demonstrated how learning support teachers feel about constructing Individual Pupil Learning Profiles given that they are now obliged to do so. However, many learning support teachers dislike constructing them and using them, finding them time consuming and preferring their own methods of record keeping and curriculum construction. Those who found them useful felt that they were valuable in providing a structure for planning and evaluating programmes of work for children with learning difficulties.

Recommendations

These recommendations follow from a consideration of the outcomes of the current study and the findings of the literature reviewed on the subject.

1. All primary schools should endeavour to provide mathematics learning support on a full-time basis to all children who score at or under the tenth percentile on standardised mathematics tests. It is incumbent on the Department of Education and Science to provide schools with enough personnel so that children with mathematics learning difficulties will be given an opportunity to learn mathematics to some basic level that will take them through to adulthood.
2. The mathematics work conducted by the learning support teacher should not involve complex, abstract algorithms. The work should be within the capabilities of the child at all times.
3. Mathematics learning support teachers should incorporate fun activities and games into their work to eliminate drudgery and encourage the enjoyment of mathematics. This in turn will develop confidence and self-esteem.
4. All mathematics work needs to be modified to suit the needs of each individual child's learning difficulties. It is necessary that the mathematics is real, practical and meaningful to each child. The mathematics should be taught in a hierarchical manner, allowing each concept to be full mastered before progressing to the next level.
5. Using learning support as a quick fix solution for mathematics learning difficulties should be avoided. These children need time to learn mathematics. Many of them need to go back to the basic and start again doing lots of concrete work and using manipulatives. Practice takes time. Each step needs to be properly assimilated to ensure retention.

6. A mathematics learning support policy needs to be devised in each school. It should identify the experiences and problems that children with mathematics learning difficulties have. It should ensure that the instruction that these children receive is directly suited to their individual needs and requirements. The role of the mathematics learning support teacher should be clearly outlined.
7. All learning support teachers should implement at least two forms of assessment. Screening and diagnostic tests readily provide the kind of information that is necessary when putting together a complete representation of one's students' needs.
8. All mathematics learning support teachers should be facilitated to participate in completing special education diploma courses as soon as they are appointed. Because this area of education is so specialised, teachers need this facility so that they can give their students the best support possible.
9. Regular consultation with colleagues and parents is necessary. Class teachers need to be facilitated by their principals so that they are free to meet their learning support colleagues on a regular basis. Parents should be encouraged to meet the learning support teacher at least once a term to discuss their children and seek advice on how best to help them.
10. The mathematics learning support teacher and the mathematics post holder should take responsibility for the distribution of information to new teachers and students regarding mathematics language development, teaching methods and procedures. This will ensure the smooth implementation of the school's mathematics policy.
11. Learning support need to be delivered on a regular basis. None of the problems or issues raised will be solved if supplementary teaching is only provided on an ad hoc basis.
12. All mathematics learning support teachers should develop Individual Pupil Learning Profiles. They provide consistency in learning support and add professionalism to the teaching provided.

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How Can School Mathematics Education Benefit From the Research on Adult Learning Mathematics?

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Mathematics has a central role in our evolving technological and knowledge based society but it is often available to learners in a way that leaves them with a view of mathematics as a collection of abstract skills and procedures unrelated to everyday life (Bowl and Burton, 2001). The failure of this approach is confirmed by the negative experiences reported by many adult learners of mathematics. Effective mathematics teaching can change adults' relationship with mathematics. What is important for effective adult mathematics provision is that adults are not faced with the same diet of abstract mathematics that is decontextualised and delivered through transmission mode that has failed them in the past. Methods of teaching that focus on technique and lack any overview, and which promote manifestations of competitiveness rather than cooperation are discouraging to adult learners (Tobias, 1990). Adults learn best when mathematics teaching builds on positive attitudes and is interactive and co-operative, practical and relevant, set in a social, historical and cultural context and enjoyable and fun.

The field of adult mathematics education is an exciting area of research that has emerged and developed since the 1990's. The research domain spans the sub-fields of sociology, adult education and mathematics education" Wedege (1998). Within this research field there has been significant research that has led to the development of pedagogies that meet the needs of adult learners.

This paper will discuss how current research on adults learning mathematics, including, pedagogy and professional development, can inform current developments in the school mathematics education.

Introduction

Adults are not like school students, but they too are mathematics learners. So we can expect to find from research on adult learners, data and thoughts, which will inform and extend our constructs and concepts of mathematics learning in general *(Bishop, 1998, p.3)*'.

The field of adult mathematics education is an exciting area of research that has emerged and developed since the 1990's. The research domain spans the sub-fields of sociology, adult education and mathematics education" Wedege (1998). Although the area is still considered to be under-researched, 'under theorised and generally under developed (Coben 2005), a substantial body of international research has accumulated since the 1990's. What is becoming clear is that adult mathematics education can inform and be informed by mathematic s education. This paper will give a brief outline of research in adults mathematics education and highlight how this research provides a new lens to look at school mathematics; new conceptual tools for considering school mathematics; new teaching and learning methodologies; innovative approaches to professional development and a rich research resource which details the prior mathematical experiences of adult learners, the past pupils of school mathematics.

Adults Learning Mathematics - a Research Domain

The nature of adult mathematics education as a research domain is the subject of much debate. The domain has been called a 'moorland', an analogy that evokes images of a field with informal boundaries, subject to public ownership and open access visualised as a series of concentric circles, with adult learning maths at the centre surrounded by related disciplines. Among these related disciplines adult education, mathematics education and mathematics are displayed as being most significant (Benn, 1997). However the problematic nature of the relationship between adult mathematics education and related disciplines, has also been highlighted by researchers who reject the concept that adult mathematics education is the sum total of contributions from related disciplines. (Coben, 2000 and Wedege, 1998). More recently the domain has been described as a '*specialist area concerned with adults learning mathematics in what ever circumstances they do so'*. FitzSimons et al (2003). Using Wittmann's (1998) portrayal of mathematics as a design science, FitzSimons et al (2003) proposed that adults learning mathematics could be viewed as a sort of 'core', the goal being to develop a coherent picture of mathematics learning and teaching when these activities involve adults. Thus a field specific framework(s) for adult mathematics education that integrates all contributions from the core and elsewhere is needed. The core is described by the authors as being likely to include questions on:

- *The nature of mathematics and the relationship between various forms of mathematics.*
- *The measurement of adults' mathematical ability and performance.*
- *Research into adult numeracy and workplace mathematics.*
- *Attitudinal and affective factors in adult mathematics learning.*
- *Issues of teacher training for adult mathematics education.*

(FitzSimons et al, 2003)

The development of adults learning mathematics as a research domain has been supported by the establishment of Adult Learning Mathematics – A Research Forum (ALM) in 1994. ALM is a registered charity and international forum that brings together researchers and practitioners in adult mathematics teaching and learning (http://www.alm-online.org/).

Research in Adults' Mathematics Education

Research in adult mathematics education has drawn on mathematics education, adult's mathematics education (through ALM and others since 1993) and other disciplines e.g. education, psychology etc. A comprehensive review of the research on adults learning mathematics has been completed by FitzSimons and Godden, (2000) and by Coben, (2003). The research had included population surveys e.g. IALS and ALLS; mathematics specific projects and projects where adult mathematics is linked with literacy, ESOL and ICT in a range of settings including the workplace (Coben 2005). The domain also includes research on; teaching and learning methodologies; workplace mathematics; bridging mathematics; adults learning advanced mathematics; adult mathematics histories and ethnomathematics.

Adult Learners, Mathematical Knowledge and Understanding

A key research theme within the domain has looked at adult learners' mathematical knowledge and understanding using a number of different research approaches (FitzSimons et al 1996). The first approach is centred on knowledge and understanding of school mathematics, these are investigated because of their relevance to courses in Further/Higher Education mathematics programmes. Examples of research work in this field include O'Donoghue's (1999) work on intervention,. The second approach is research based on subjective reading and interpretation of tasks by adults. Research work in this field includes for example Colleran's (2000) work on problem solving,. The third approach is research, where adults are asked about their use of mathematics (interpreted from their own frame of reference - usually school mathematics). Examples of research work in this field includes, Cockcroft, (1982). Other research has also focused on the mathematical knowledge and understanding of adults in non-educational situations. In this kind of research, interest is shown in how adults manage their affairs, not in their application of school mathematics. Examples of research work in this field includes for example Colwell (2000); Carraher et al (1985). Research has also focused on how adults learn advanced mathematics for example the work completed by Golding and O'Donoghue (2005). Some researchers have examined existing approaches of child centred mathematics education to determine what can be learnt from traditional mathematics education research for application to adults' learning mathematics for example Schmitt (forthcoming).

Knowledge Transfer

The transfer of learning means the use of ideas and learning from one context to another. In the traditional view, the transfer of learning from school to everyday situations should be relatively straight forward, especially for those who have understood the mathematics properly. However, transfer is often not accomplished by a particular learner in a particular situation. Adult learners have diverse numeracy practices which are embedded in the contexts in which they occur (Coben, 2005). Lave advocated a particularly strong form of situated cognition in her earlier work, (Lave, 1988) and suggested that there is a disjunction between doing mathematics in school and doing mathematics in everyday life. Although in her more recent work (Chaiklin and Lave, 1993) she does acknowledge that no practice could be completely closed. Evans (2000) describes teaching practices which, he argues will support transfer. He argues that in teaching and learning, bridges between practices can be built by analysing similarities and differences between discourses (e.g. school versus everyday mathematics) in order to identify points of interrelation between school mathematics and everyday contexts (Schliemann, 2000). Evans (2000) also argues that mathematical activity is always cognitive and emotional. He suggests that the feeling triggered may encourage or impede the transfer of mathematical skills knowledge and understanding.

Hoyles and Noss, (1998) found that individuals e.g. nurses and pilots use what they describe as contextual anchors when solving problems. As soon as the context is removed and the same mathematical problems are asked in an abstract way, difficulties are experienced in solving the problems. They suggest that in these contexts nurses and pilots use 'situated abstraction' in which they use some kind of local mathematical models, which are only partly valid in a different context, as each context gives rise to its own modification of the method. Learners encapsulate entities in ways that are different from how similar concepts are taught in schools. This practice has implications for school mathematics that need to be explored further.

Ethnomathematics

The dominant view of mathematics considered it to be both value free and culture free (Clarkson et al, 1997; Harris, 2000), but research on enculturation (Bishop, 1988), and the emergence of the field of ethnomathematics, especially through the work of D'Ambrosio (1985), have brought the issue of values into greater focus. Ethnomathematics is described as the 'mathematics, which is practiced among identifiable cultural groups' (p. 45). The quality of mathematics learning would be improved if more were understood about this phenomenon (Clarkson et al, 1997). D'Ambrosio holds the view that the mathematics curriculum in school should incorporate ethnomathematics. Ethnomathematics incorporates 'folk mathematics' (Mellin-Olsen, 1987), oral mathematics (the mathematics transmitted orally from one generation to the next) (Carraher et al, 1985), hidden or frozen mathematics (the mathematics people have lost) (Gerdes, 1996), workplace mathematics (Harris, 1991; Wedege, 2002).

The adult learner

Knowles (1990) argued that there are four definitions of the term adult: biological, legal, social and psychological, and that these can differ from country to country. The last definition occurs at a point where self-direction comes into operation and is, he claimed, the most crucial from the viewpoint of learning. Significantly, in mathematics education the definition revolves around the student and not the level of mathematics being studied, (FitzSimons and Godden, 2000). Knowles identified six main distinctions between adult learners and traditional school-based learners (need to know: self concept; role of experience; readiness to learn; orientation to learning; and motivation).

Adult learners of mathematics are the past pupils of school mathematics. Throughout their prior experiences of formal education adult learners have been influenced by their own mathematical experience, in terms of how it was delivered and with what resources (Ingleton and O'Regan, 2002). These experiences are not neutral or isolated but reflect larger economic, cultural and political considerations (FitzSimons and Goddard, 2000).

School mathematics is the primary source of quantitative literacy for most adults (Steen, 1997). However, the goals of school mathematics are not necessarily those of adult mathematics and recommendations in this field often represent the viewpoint of different interest groups (employers, curriculum developers etc.). This point is reinforced by Willis (1990), who observes that school mathematics 'cannot be expected to simply transfer to adult life, and be expected to fulfil a 'lifetimes requirements, they may lay a foundation but clearly they will not encompass nor anticipate all future numeracy requirements'. There is clear evidence that school mathematics in Ireland and other countries is failing its people. This failure has manifested itself in a number of ways e.g. a general unwillingness of adult learners to engage further with mathematics. diminishing numbers of students enrolling in Higher mathematics in upper secondary school, widespread under preparedness for Higher education, in numerate disciplines, and significant numeracy problems in the adult (IALS, 1997) and 15-16year old population (PISA, 2005)

This failure is of more concern in today's knowledge-based technological society in which the need for adult mathematical skills, including the interpretation and communication of data and problem-solving, is increasing in the workplace, encouraged by challenging business goals and the use of ICT (NRDC, 2003). Poor mathematics skills constitute a disadvantage in relation to entering the labour market and subsequent success within it (Bynner and Parsons, 2000). Individuals with the lowest level of participation in the labour market were individuals with poor mathematical skills rather than poor literacy skills.

Studies of adults 'maths life histories' have identified four central influences on how adults view their mathematics.'

- The 'brick wall' – the point (usually in childhood) at which mathematics stopped making sense.
- The 'significant other' – someone perceived as a major influence on the person's maths life history. The influence might be positive or negative, past or present,
- The 'door' marked 'mathematics', locked or unlocked, which people have to go through to enter or get on in a chosen line of work or study.
- 'Invisible maths' the mathematics someone can do, but which they may not think of as maths at all, 'just common sense'. (Coben, 2005, pg. 20).

Research into adult maths histories can inform future developments of school mathematics. The data collected can help develop a range of strategies to ensure that current pupils within the school system can, if there is a need, be supported in overcoming 'the brick wall'; will only encounter positive and encouraging influence from 'the significant other'; will be confident in their mathematical abilities to access the door marked mathematics and will have an understanding of the importance of mathematics in everyday life.

Mathematics and Numeracy

One of the observations of the ICME (2000) working group on adult mathematics education was that a conceptual shift from mathematics to 'numeracy' might afford the opportunity to broaden the school curriculum, and look at it through a new lens (FitzSimons et al, 2001).

> The construction of this new thing, numeracy, gave us a chance to be creative, to claim for it the essentials we had for so long foregone [in maths]. Numeracy involved mathematics of course - how could it not? - but it was to be a mathematics in conversation with the world, where matters of life and death, survival and destruction, were not irrelevant matters but core concerns. It was to be a mathematics used by people, meaningfully, appropriately, purposefully, justly - and enjoyably. (p.286)

Johnston and Yasukawa, 2001

Since the Cockcroft report (DoES, 1982) the academic debate around the relationship between mathematics and numeracy has been wide reaching (Gal et al, 2000). In the

IALS survey 'numeracy' was conceptualized through the lens of quantitative literacy and defined as *'the knowledge and skills required to apply arithmetic operations either alone or sequentially to numbers embedded in printed material'* (Morgan et al, 1997, p.vii). In essence it depended on the stimuli that were text based and required considerable literacy skills (Manley et al, 2000). The Adult Literacy and Life skills (ALL) Survey that is currently taking place internationally, explicitly surveys the numeracy abilities of adults (this survey also includes Document and Prose Literacy and problem solving, and the survey further incorporates an assessment of other variables via a background questionnaire). The view of numeracy incorporated in the ALL project is broad and based on the premise that numeracy is the bridge that links mathematical knowledge, however learnt, with the demands of the real world. Numeracy is defined as: *the knowledge and skills required to effectively manage the mathematical demands of diverse situations* (p. 78). In the ALL context numerate behaviour is seen as involving five facets each with several components e.g. managing a situation or solving a problem in a real context, responding to information about mathematical ideas that are represented in a range of ways and which require activation of knowledge, behaviours and processes. This broad conceptualisation of numeracy has been adopted by the National Adult Literacy Agency (NALA) who have defined numeracy as:

> ‘Numeracy is a lifeskill that involves the competent use of mathematical language, knowledge and skills. Numerate adults have the confidence to manage the mathematical demands of real-life situations such as everyday living, work-related settings and in further education, so that effective choices are made in our evolving technological and knowledge based society.’ NALA, 2004.

This conceptualisation of numeracy now underpins future numeracy development in adult basic education in Ireland. (NALA, 2004).

A recurring theme in the discourse is the idea that numeracy has the ability to 'bridge' or integrate. Numeracy has been described as *'the critical awareness that builds bridges between mathematics and the real world in all its diversity'.* (Johnston, 1994, p. 34). In this context the conceptualisation of numeracy described through five strands of making meaning in mathematics has particular relevance (Johnston, 1994). This view incorporates a vision of numeracy that is about using and understanding 'all of mathematics', not just number skills to make sense of the real world:

- *<u>Meaning through ritual</u> – a minimal strand where meaning is acquired through rote learning or atomised content.*
- *<u>Meaning through conceptual engagement</u>- where mathematical meaning is constructive through problem solving process and cognitive dissonance.*

- *Meaning through use – where meaning is developed through use in every day context.*
- *Meaning through historical and cultural understanding - where meaning is enhanced by understanding of the genesis and cultural use of specific mathematics.*
- *Meaning through critical engagement - where meaning is generated by asking "In whose interest" type questions and also questions about the appropriateness and the limits of the math's model in the real situation.* (Johnston, 1994. p.32)

Traditional school mathematics is concerned with strand 1 primarily and less so with strand 2 and beyond. Adult numeracy, even when narrowly defined, because of its emphasis on everyday contexts goes further than school mathematics and also captures strand 3. Johnston argues that numeracy is a social activity and should incorporate a critical aspect of using mathematics. She puts forward the view that a fully developed concept of numeracy should incorporate all five strands. Mathematics education that incorporates all five strands of meaning making could together produce a multifaceted understanding of mathematics.

Adult Mathematics Curricula

Effective curricula for adults incorporate a view of mathematics as a social practice and allows for the diversity of adult experiences and the pluralistic nature of mathematics (Coben, 2001). This perspective allows curriculum developers to draw on research carried out, for example, on mathematising in the workplace (Hoyles et al, 1999), studies in ethnomathematics (D'Ambrosio, 1988) and studies in social justice such as Knijnik's work with the Landless Movement (Knijnik, 1993, 1997, 2002). Curricula that focus upon atomised skills and techniques may not allow the adult learner to engage in situations they can explore meaningfully (Johnson, 1999).

Coben (2001) described a way of evaluating numeracy curricula based on a concept she borrowed from Kell (2001). She suggested that the curriculum should be considered in relation to the nature of the mathematical demands of adult lives. The purpose of teaching and learning in the wider social, political and economic context and the processes and practices of mathematics learning and teaching with adults also need to be considered. Coben goes on to distinguish between Domain 1 and Domain 2 numeracy. Domain 1 'is characterised by formalisation and standardization of the curriculum' (p. 143), usually associated with a formal standardised curriculum, often competence based with an accredited outcome focus. Domain 1 usually has explicit equivalence with education levels in schools. This domain has what she describes as 'little use value' but is valued by adults and governments and so has 'high exchange value'. In contrast, Domain 2 numeracy education is about mathematics practices and processes in adults' *lives which may be informal and non-standard*'. This domain is far removed from the mathematical processes and procedures taught in schools. This domain has what she describes as 'high use value but no exchange value' (Coben and Thumpston, 1996). Domain 2 numerate practices are linked to an individual’s 'common sense', their invisible mathematics (Coben, 2000) and thus are not always obvious to the individual.

Maguire and O’Donoghue, (2003) developed a three phase organising framework for mapping the increasing sophistication in the conceptualisation of numeracy. This framework (Figure 1) sees the development of the concept of numeracy as a continuum with three phases: Formative, Mathematical, and Integrative. The phases represent an ever-increasing sophistication in the conceptualisation of numeracy. Starting from a very limited concept of formative numeracy considered as basic arithmetic skills (Formative Phase) moving through to a concept of numeracy as being 'mathematics in context’, which recognises the value of making explicit the importance of mathematics in daily life (Mathematical Phase). The postulated continuum culminates in a conceptualisation which views numeracy as a complex, multifaceted sophisticated construct incorporating, the mathematics, communication, cultural, social, emotional and personal aspects of each individual in a particular context (Integrative Phase).

Adult Numeracy Concept Sophistication

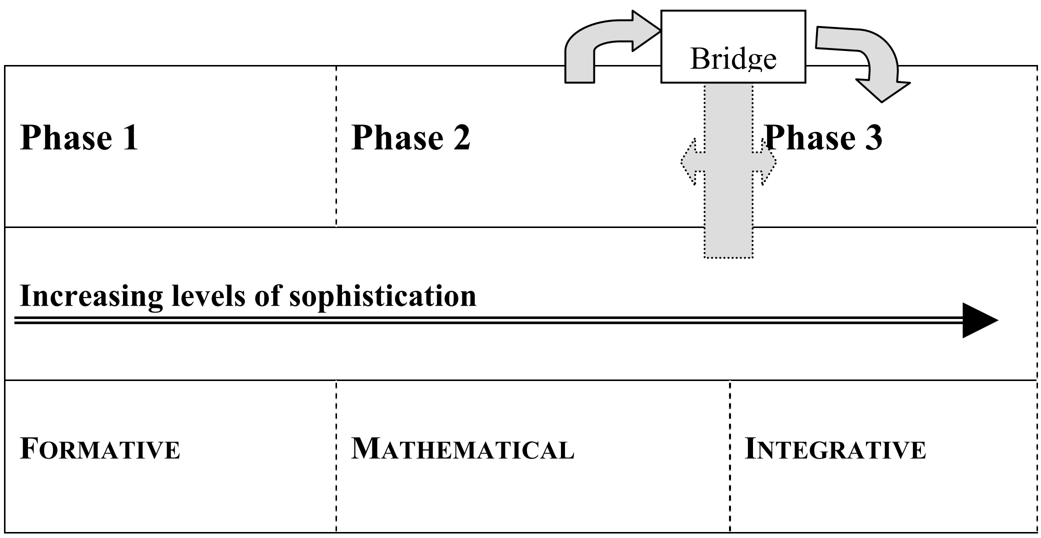

Figure 1 **A continuum of development of the concept of numeracy showing increased level of sophistication from left to right.**

More recently Coben (2004) has combined the view of Domain 1 and 2 numeracy with the framework described by Maguire and O'Donoghue (Figure 2). Using the matrix developed she suggests that it is possible to position conceptions of adult numeracy along the horizontal axis in terms of their degree of sophistication and along the vertical axis in terms of whether they operate in Domain 1 or Domain 2. She suggests a positioning for the Adult Numeracy Core Curriculum and the ALLS concept of numeracy*:* the English Adult Numeracy Core Curriculum (ANCC) is in the top left and the ALL survey is located in the bottom right hand quadrant.

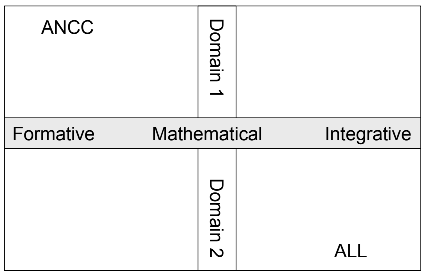

Figure 2 **Combining the Concept framework with Domain 1 and Domain 2 numeracy**.

These conceptual tools, developed through research in adult mathematics education may have significant value in facilitating meaningful communication and discussion on mathematics curricula, provision and practice of mathematics education both in a national and international context.

Teaching and Learning

Methods of delivery which focus on technique and lack any overview, and promote manifestations of competitiveness rather than cooperation are discouraging to adult learners (Tobias, 1990). Adults learn best when mathematics teaching builds on positive attitudes and is interactive and co-operative, practical and relevant, set in a social, historical and cultural context and enjoyable and fun. The danger of regarding mathematics, in an absolutist way, is that it may lead to an absolutist pedagogy, which reinforces negative attitudes in adult learners and ensures that mathematics remains the same collection of rules and facts to be remembered (Benn, 1997). Principles of teaching and learning that are embedded in good practice and reflect the dynamic nature of learning environments and have been synthesized as follows by Bickmore-Brand (2001, p. 252):

Context: creating a meaningful and relevant context

Modelling: providing opportunities to see the knowledge, skills and or values in operation by a 'significant 'person.

Metacognition: making explicit the learning processes

Community: creating a supportive learning environment

Interest: starting from where the learner is at.

Scaffolding: challenging learners to go beyond their current thinking,

Responsibility: getting learners to accept responsibility for their learning.

The pivotal role tutors play in engendering numeracy in adult mathematics education and the requirement for appropriate professional development to meet their training needs, has been clearly revealed (Maguire, 2003). A model of professional development (SENAME - successfully engendering numeracy in Adults Mathematics Education) to help to address this urgent need in Ireland and else where has been developed and is currently being piloted in ABE . SENAME has as its objective the aim of developing an adult numeracy tutor who will be a professional, autonomous, confident lifelong learner; who will have a good understanding of themselves as tutors of adult mathematics/numeracy, a appreciation of the relationship of mathematics and numeracy, and who will have a full understanding of the particular needs of adults as learners and who will be conscious of the beliefs and values inherent in their classroom. At the same time the model integrates and recognises the expertise that a tutor may bring to the domain of adult mathematics education through alternative career paths.

The model incorporates a view that professional development is not a once-off activity but something that allows for the development of a wide range of skills and knowledge, increasing complexity and specificity in particular areas, in the context of a tutors own lifelong learning. The model is dynamic and capable of responding to the developing needs of tutors to accommodate change in an integrated way. Teachers of mathematics and tutors of numeracy share common challenges; consequently this model of professional development may have relevance in guiding the provision of professional development for all those involved in delivering mathematics education.

Concluding Remarks

The domain of adult mathematics education was unrecognised as recently as 1990 but because of many intersecting interests of mathematics educators, education policy makers and governments promoting economic development the area is blossoming as a serious domain of endeavour for researchers and practitioners. This paper has highlighted ways in which this research can inform school mathematics and how it may benefit by being considered through the lens of meaning making in mathematics or through the lens of numeracy. In addition, the paper described two communication and conceptual tools that may have a significant value for facilitating meaningful discussion and information exchange between those working in school mathematics internationally. Finally, the paper briefly outlined teaching and learning approaches that are used in the adult mathematics classroom and described a model of professional development that may provide a first step to ensuring meaningful mathematics are taught and learnt by all learners of mathematics.

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