Proceedings
of

THIRD NATIONAL CONFERENCE ON RESEARCH IN MATHEMATICS EDUCATION MEI 3

24th and 25th September, 2009

St. Patrick's College, Drumcondra, Dublin 9

Theme: "Mathematics for All" - Extending Mathematical Capacity

Editors: Dolores Corcoran, Thérèse Dooley, Sean Close and Ronan Ward

St. Patrick's College, Drumcondra, Dublin 9.

AN ROINN OIDEACHAIS AGUS EOLAÍOCHTA | DEPARTMENT OF EDUCATION AND SCIENCE

ST PATRICK'S COLLEGE · COLÁISTE PHÁDRAIG · INTELLECTUM DA MIHI DNE

NDP National Development Plan 2007 - 2013

TABLE OF CONTENTS

POST-PRIMARY MATHEMATICS EDUCATION

THIRD-LEVEL MATHEMATICS EDUCATION

ACKNOWLEDGEMENTS

We wish to thank the keynote speakers, presenters of research reports, and all participants in the third MEI conference on Research in Mathematics Education in Ireland. Our main aim – that of searching for ways to achieve more equitable outcomes for all students of mathematics – is reflected in our conference theme, 'Mathematics for All: Extending Mathematical Capacity'. In order to achieve this, we seek to bring together those who have an interest in mathematics education, to consider future directions for mathematics education research in Ireland in the light of recent developments and international trends, and to consider ways of improving linkages with mathematics education communities within Ireland and in other countries. We hope that we succeed in this and that the conference provides participants with an interesting and challenging programme of presentations, and with opportunities for discussion and making contacts and renewing friendships. We are happy that Irish mathematics education is flourishing and we welcome links with the wider scientific research community.

We also express our sincere gratitude to all those who supported the conference including: Dr. Pauric Travers, President of St. Patrick's College; Professor Kathleen Lynch, UCD, and our sponsors - St. Patrick's College Research Committee, the Department of Education and Science, The Teachers' Council, Irish National Teachers' Organization and the Centre for the Advancement of Science Teaching and Learning (DCU).

Thérèse Dooley (Chairperson, MEI3 Organising Committee)

ORGANISING COMMITTEE

Sinead Breen

Seán Close

Dolores Corcoran

Thérèse Dooley

Elizabeth Dunphy

Maurice O'Reilly

Joe Travers

Ronan Ward

EQUITY AND HIGH ACHIEVEMENT: THE CASE OF RAILSIDE SCHOOL

Jo Boaler

University of Sussex

The low and inequitable mathematics performance of students in urban schools has been identified as a critical issue for our times. This paper will report the results of a four-year study of approximately 700 students as they progressed through three different Californian high schools. One of the findings was the important success of "Railside" high school where the teachers taught mixed ability classes and used a particular pedagogical approach called 'complex instruction'. The students at Railside learned more mathematics, enjoyed mathematics more and progressed to higher levels. In this paper I will detail the teachers' methods and the impact they had upon the students.

INTRODUCTION

In order to consider the impact of different teaching approaches upon students' understanding of mathematics, a team of graduate students and I conducted a longitudinal four-year study of three high schools. The study involved monitoring approximately 700 students as they progressed through four years of three schools that were chosen because they offered different mathematics teaching approaches. The school that is the focus of this paper was given the pseudonym of 'Railside', as it was located next to the railway tracks in an urban setting in California. The students were from diverse ethnic and cultural groups and were largely from low-income homes. The other two schools that we called 'Greendale' and 'Hilltop' were in more suburban settings with less ethnic diversity. Greendale school had little ethnic diversity and almost all students were white; at Hilltop school most students were white or Hispanic. Table 1 gives details of the three schools and their populations of students. In order to monitor and analyze the teaching approaches in the three schools we observed over 600 hundred hours of lessons, many of which were videotaped. These lessons were analyzed in different ways as set out below. In addition, we interviewed students in every year of the study to consider their reported experiences and interpretations of mathematics class. Students were typically interviewed in same sex pairs and we interviewed approximately sixty students each year, sampling high and low achievers from each approach in every school, taking care to interview students from different cultural and ethnic groups. In addition to interviews, we also administered questionnaires to all the students in the focus cohorts in years 1, 2 and 3 of the study (when most students were required to take mathematics). The questionnaires combined closed, Likert response questions, with more open questions that we analyzed and coded. The questionnaires asked students about their experiences in class, their enjoyment of mathematics, their perceptions about the nature of mathematics, learning, and students.

Table 1: Schools & Students

School:	Railside	Hilltop	Greendale
Approx school population	1500	1900	1200
Study Demographics	38% Latino/a, 23% African American, 20% White, 16% Asian or Pacific Islanders. 3% other groups.	57% White, 38% Latino/a, 5% other ethnicities.	90% White, 5% Latino/a, 5% other ethnicities.
ELL students	25%	24%	0%
Free/reduced lunch	31%	23%	9%
Parent education, % college graduates	23%	33%	37%

The observations, interviews and questionnaires combined to give us information on the teaching and learning practices in the different approaches and students' responses to them. Teachers from each approach were also interviewed at various points in the study although the teachers' perspectives on their teaching were not a major part of our analyses. In addition to monitoring the students' experiences of the mathematics teaching and learning, we also assessed their understanding in a range of different ways, administering tests to all students in the different school approaches as well as longer applied assessments that were given to focus groups and videotaped (Fiori & Boaler, 2004).

DATA ANALYSIS

Data from our classroom observations were analyzed in three different ways. First, we drew upon our observations from class visits and videotapes to produce 'thick descriptions' (Geertz, 2000) of the teaching and learning in the different classes. We collectively watched over 600 hours of lessons and different observers of the classes and the videotapes discussed and highlighted the most salient features of each approach and the differences between them. Second, we conducted a quantitative analysis of time spent in classes. This involved spending a year observing videotapes and deciding upon the different, mutually exclusive ways in which students spent time in class. These included such categories as 'teacher talking', 'teacher questioning with whole class', 'students working alone', and 'students working in groups'. When agreement was reached on the categories that should be used, three researchers coded lessons until over 85% agreement was reached on the coding. We then completed the coding of over 55 hours of lessons, coding every 30-second period of time. This yielded 6,800 coded segments. We also recorded the amount of time that was spent on each mathematics problem in class. This coding exercise was only performed on year one classes as it was extremely time intensive and we lacked the resources to perform the same analysis every year but qualitative observations of lessons suggested that similar differences between the different approaches pertained to each year of the study. In addition to these qualitative and quantitative analyses of lessons, we performed a detailed analysis of the questions teachers asked students. This level of analysis fell between the qualitative and quantitative methods we had used and was designed in response to our awareness that the teachers' questions were an important indicator of the mathematics on which students and teachers worked (see Boaler & Brodie, 2004). Our coding of teacher questions was more detailed and interpretive than our coding of instructional time but it was sufficiently quantitative to enable comparisons across classes.

In order to analyze the detailed student interviews they were first read by teams of researchers and then coded (Glaser & Strauss, 1967; Miles & Huberman, 1994). Codes were first identified by different researchers using a process of open coding; the list of agreed upon codes was then used to re-code all interviews. Questionnaires were analyzed quantitatively with both individual questions and scales of questions being subject to exploratory and confirmatory statistics. The assessments in our study ranged from tests that were scored, blind, and statistically analyzed, to problem solving sessions that were videotaped and assessed. For the videotaped assessments individual student work was graded and rubrics were developed and used to assess the interactions of groups as they worked (Fiori & Boaler, 2004). In addition to the individual analysis of each data source (lesson observations, interviews, videos, questionnaires, assessments) the findings from these multiple sources were then analyzed and understood in relation to one another, thus illuminating trends and themes across sources and affording the opportunity to triangulate the data. As data were analyzed by a team of researchers themes were discussed and agreed upon by groups of people, which served to increase confidence in our analyses and findings (Eisenhart, 2002). In addition, constant comparison across cases (Glaser & Strauss, 1967, 1991) was used to illuminate critical defining features and practices of each school. This allowed us to capture subtle aspects of each learning environment that may have otherwise been overlooked. The analyses were shared with the teachers as a form of *member check* (Glesne & Peshkin, 1992), further enhancing the validity of the findings. The most important teaching and learning interactions identified in this paper emerged from the observations of classes, the student interviews, and the questionnaires and they will be the main data sources reported in this paper.

TIME SPENT AND ACHIEVEMENT

The majority of students in our study experienced one of two teaching approaches. The two suburban schools Greendale and Hilltop offered a choice between a ‘traditional’ sequence of courses taught traditionally and a ‘reform’ sequence taught using more open problems and groupwork, but few students chose the reform courses and we had insufficient numbers to include in our analyses. Most of the students at Greendale and Hilltop therefore experienced a traditional approach, as named by the schools. The teachers lectured and the students practiced methods, working their way through short questions. Our coding of videotapes allowed us to categorise the ways in which students spent time in their classes. This showed that approximately 21% of the time in ‘traditional’ algebra classes at Greendale and Hilltop was spent with teachers lecturing, usually demonstrating methods. Approximately 15% of the time teachers questioned students in a whole class format. Approximately 48% of the time students were practising methods in their books, working individually; approximately 11% of the time they worked in groups, and students presented work for approximately 0.2% of the time. The average time spent on each mathematics question was 2.5 minutes. The second major approach in our study was the one offered at Railside school in which the teachers posed longer, conceptual problems; students worked in groups and they often presented their work while teachers questioned presenters and other students. Our coding of videos showed that teachers lectured to classes for approximately 4% of the time. Approximately 9% of the time teachers questioned students in a whole class format. Approximately 72% of the time the students worked in groups while teachers circulated the room teaching methods and asking the students questions of their work, and students presented work for approximately 9% of the time. The average time spent on each mathematics problem at Railside was 5.7 minutes. An additional important difference between the schools was that Greendale and Hilltop employed ability grouping and students were placed into one of three different levels of classes at the beginning of high school. At Railside all students were placed into heterogeneous classes.

The students at Railside started high school at significantly lower mathematics levels than the students in the more suburban schools ($t = -9.141$, $p < 0.001$, $n = 658$), but within two years they were out-performing the other students scoring at significantly higher levels on mathematics tests ($t = -8.304$, $p < 0.001$, $n = 512$). In addition to their high achievement, which they showed on a range of tests, the students at Railside also pursued with mathematics for longer. By year 4, 41% of Railside seniors were in advanced classes of pre-calculus or calculus (similar to one year of A-level mathematics) compared to approximately 27% of seniors in the other two schools. In addition questionnaire and interview results each year showed that the Railside students were enjoying mathematics more than students at the other schools. Railside teachers were also extremely successful at reducing the achievement differences between groups of students belonging to different ethnic groups at the school. At the beginning of high school Asian students (predominantly east Asian), Filipino, and White students were each outperforming Hispanic and Black students. At the end of Year 1, only one year after the students started at Railside, there were no longer significant differences between the achievement of White and Hispanic students, nor Filipino students and Hispanic and Black students. In subsequent years the only consistent difference that remained was the high performance of (East) Asian students who continued to significantly outperform Black and Hispanic students, but differences between White, Black and Hispanic students disappeared. Achievement differences between students of different ethnicities at Hilltop, where approximately half of the students were White and half Hispanic, remained with the White students outperforming Hispanic students, reflecting inequities that are fairly typical for urban schools in the US (Haberman, 1991; Kozol, 1992). At no time in the study were there any achievement differences by gender in any of the schools and girls and boys were represented equally in the different classes.

ANALYZING THE SOURCES OF SUCCESS

The Department, Curriculum and Timetable

Railside school has an unusual mathematics department. Twelve of the thirteen teachers work collaboratively, spending vast amounts of time designing curriculum, discussing teaching decisions and actions, and generally improving their practice through the sharing of ideas. A study conducted by Horn on the ways in which the department collaborate, found that the teachers spent around 650 minutes a week planning, individually and collectively (their paid work week provides 450 minutes of preparation time) (Horn, 2002). Unusually for the United States, the mathematics department strongly influences the recruitment and hiring of teachers, enabling the department to maintain a core of teachers with shared philosophies and goals. The teachers share a strong commitment to the advancement of equity. The department has spent many years working out a coherent curriculum and teaching approach, and they strive to ensure that good teaching practices are shared. One way in which this is achieved is through a practice that the department calls "following." The co-chairs structure teaching schedules so that a new teacher can stay a day or two behind a more experienced teacher, allowing the new teacher to observe lessons and activities during their daily preparation period before they try to adapt it for their classrooms (Horn, 2002; 2005). In addition, teachers share teaching practices and moves in weekly meetings in order that students experience a consistent approach.

The mathematics department has focused in particular upon the introductory algebra curriculum that all students take when they start the school. The algebra course is designed around key concepts with questions from various published curriculum such as CPM, IMP and a textbook of activities that use algebra Lab Gear (Picciotto & Wah, 1994). A theme of the algebra and subsequent courses is multiple representations, and students are frequently asked to represent their ideas in different ways, using words, graphs, tables and symbols. In addition, connections between algebra and geometry are emphasized even though the two areas are taught in separate courses. Railside follows a practice of 'block scheduling' and lessons are 90 minutes long, with courses taking place over half a school year, rather than a full academic year. In addition, the introductory algebra curriculum that is generally taught in one course in US high schools, is taught in the equivalent of two courses at Railside. The teachers have spread the introductory content over a longer period of time partly to ensure that the foundational mathematical ideas are taught carefully with depth and partly to ensure that particular norms – both social and socio-mathematical (Yackel & Cobb, 1996) – are carefully established. The fact that mathematics courses are only half a year long at Railside may appear unimportant but in fact this organizational decision has a profound impact upon the students' opportunities to take higher-level mathematics courses. In most North American high schools mathematics classes are one year long and they begin with algebra. This means that students cannot take calculus unless they are advanced, as the typical sequence of courses is algebra, geometry, advanced algebra then pre-calculus. If a student fails a course at any time they are knocked out of that sequence and have to retake the course, further limiting the level of content they will reach. At Railside the students could take two mathematics classes each year. This meant that students could fail classes, start at lower levels, and/or choose not to take mathematics in a particular term and still reach calculus. This relatively simple scheduling decision is part of the reason that significantly more students at Railside took advanced levels classes at school than students in the other two schools.

Another important difference between the classes in the three schools we studied was the heterogeneous nature of Railside classes. Whereas incoming students in Greendale and Hilltop could enter geometry or could be placed in a remedial class, such as 'math A' or 'business math', all students at Railside entered algebra classes. The department is deeply committed to the practice of mixed ability teaching and to giving all students equal opportunities for advancement.

Recent years have pointed to the importance of school and district contexts in the support of teaching reforms. Such support is undoubtedly important but Railside is not a case of a district or school that initiated or mandated reforms. The reforms put in place by the mathematics department were in line with other school reforms but they were driven by the passion and commitment of the mathematics teachers in the department. The school, in many ways, provided a demanding context for the reforms, not least because they had been managed by five different principals in the six years we were there, and they had been labelled an 'under-performing' school by the state because of low state test scores. Neither of these factors is atypical in poorly resourced school districts in the US. The department fought to maintain their practices at various times and worked hard to garner the support of the district and school, and while the teachers felt well supported at the end of our study Railside does not represent a case of a reforming district encouraging a department to engage in new practices. Rather, Railside is a case of an unusual, committed and hard working department that continues to grow in strength through its teacher collaborations and work.

Groupwork and Complex Instruction

Some mathematics departments in the US employ group work with limited success, particularly because groups do not always function well, with some students doing more of the work than others, and some students being excluded or choosing to opt out. At Railside the teachers employed additional strategies to make group work successful. They adopted an approach called ‘complex instruction’ designed, by Liz Cohen and Rachel Lotan in the US (Cohen, 1994; Cohen & Lotan, 1997) for use in all subject areas. The system is designed to counter social and academic status differences in classrooms, starting from the premise that status differences do not emerge because of particular students but because of group *interactions*. The approach includes a number of recommended practices that the school employs, as I shall now describe:

Roles

When students are placed into groups they are also given a particular role to play, such as ‘facilitator’, ‘team captain’, ’recorder/reporter’ or ‘resource manager’ (Cohen & Lotan, 1997). The premise behind this approach is that all students have important work to do in groups, without which the group cannot function. At Railside the teachers emphasize the different roles at frequent intervals, stopping, for example, at the start of class to remind ‘facilitators’ to help people check answers or show their work or to ask the group: ‘what did you get for number 1?’. Students change roles at the end of each unit of work. The teachers reinforce the status of the different roles and the important part they play in the mathematical work that is being undertaken. Although I will not write more about the roles in this paper, they contribute to the complex interconnected system that operates in each classroom; a system in which everyone has something important to do and all students learn to rely upon each other. Readers may consult the literature on complex instruction (or visit www.complexinstruction.org) for more information on the roles.

Multidimensionality

In many mathematics classrooms there is one practice that is valued above all others – that of executing procedures (correctly and quickly). The narrowness by which success is judged means that some students rise to the top of classes, gaining good grades and teacher praise, whilst others sink to the bottom with most students knowing where they are in the hierarchy created. Such classrooms are uni-dimensional – the dimensions along which success is presented are singular. A central tenet of the complex instruction approach is what the authors refer to as ‘multiple ability treatment’. This ‘treatment’ is based upon the idea that expectations of success and failure can be modified by the provision of a more open set of task requirements that value many different ‘abilities’. Teachers should explain to students that ‘no one student will be “good on all these abilities" and that each student will be "good on at least one"' (Cohen & Lotan, 1977, p. 78).

At Railside the teachers create multidimensional classes by valuing many dimensions of mathematical work. This is achieved – in part – by having more open problems that students can solve in different ways. The teachers value different methods and solution paths and this enables more students to contribute ideas and feel valued. But multiple solution paths are not the only contributions that are valued by teachers. When we interviewed the students and asked them "what does it take to be successful in mathematics class?" they offered many different practices such as: asking good questions, rephrasing problems, explaining well, being logical, justifying work, considering answers, and using manipulatives. When we asked students in 'traditional' classes what they needed to do in order to be successful they talked in much more narrow ways, with 97% of the students naming the same practice of "paying careful attention".

The multidimensional nature of the classes at Railside was an extremely important part of the increased success of students. Put simply, when there are many ways to be successful, many more students are successful. Students are aware of the different practices that are valued and they feel successful because they are able to excel at some of them. Teachers at other schools may not encourage practices outside of procedure execution because they are not needed in state tests, but the fact that teachers at Railside valued a range of practices and more students could be successful in class made students feel more confident and positive about mathematics. This probably enhanced their success on tests even when tests assessed a more narrow range of mathematical work.

The following comments given by students in interviews give a clear indication of the multidimensionality of classes -

Janet:	Back in middle school the only thing you worked on was your math skills. But here you work socially and you also try to learn to help people and get help. Like you improve on your social skills, math skills and logic skills. (Year 1)

And:

Jasmine:	With math you have to interact with everybody and talk to them and answer their questions. You can't be just like "oh here's the book, look at the numbers and figure it out".
Interviewer:	Why is that different for math?
Jasmine:	It's not just one way to do it (…) It's more interpretive. It's not just one answer. There's more than one way to get it. And then it's like: "why does it work"? (Year 1)

These students recognize that helping, interpreting and justifying are critically valued practices. The following student describes another valued dimension:

Jorge: A math person is a person who knows like, how to do the work and then explain it. Like explaining everything to everyone so they could get it. Or they could explain it the hard way, the easy way or just, like average – so we could all get it. That's like a math person I think. (Year 1)

Jorge shows that he appreciates something sophisticated – that explanations can vary and that a "math person" can explain in different ways. Given the frequent explanations these students give and hear, it may be unsurprising – but nevertheless important – that he appreciates the distinguishing qualities of different explanations.

One of the practices that I have come to regard as being particularly important in the promotion of equity, is justification. At Railside, students are required to justify their answers at almost all times. There are many good reasons for this – justification is an intrinsically mathematical practice (RAND, 2002; Martino & Maher, 1999), but this practice also serves an interesting and particular role in the promotion of equity. Many teachers struggle to deal with the wide range of students who attend classes, particularly in introductory classes such as high school algebra, which include students who are motivated with a wealth of prior knowledge as well as those who are less motivated and /or lack basic mathematical knowledge. Teachers want to help all of the students but the gap between the knowledge of lower attaining and higher attaining students can be very difficult to address. At Railside, school classes have as wide a gap as any I have seen but the teachers embrace the diversity they encounter and one practice that helps them support the learning of all students is justification. The following two students give some indication of the role of justification in helping different students:

Int: What happens when someone says an answer?

Ana: We'll ask how they got it

Latisha: Yeah because we do that a lot in class. (…) Some of the students – it'll be the students that don't do their work, that'd be the ones, they'll be the ones to ask step by step. But a lot of people would probably ask how to approach it. And then if they did something else they would show how they did it. And then you just have a little session! (Year 3)

It is noteworthy that these two students do not talk about students who are slow, dumb or stupid, as other students in our study do; they talk only about students 'that don't do their work', a point to which I shall return later in the paper.

The following boy was achieving at lower levels than other students and it is interesting to hear him talk about the ways he was supported by the practices of explanation and justification:

Juan: Most of them, they just like know what to do and everything. First you're like "why you put this?" and then like if I do my work and compare it to theirs. Theirs is like super different 'cos they know, like what to do. I will be like – let me copy, I will be like "why you did this?

> And then I'd be like: "I don't get it why you got that." And then like, sometimes the answer's just like, they be like "yeah, he's right and you're wrong" But like – why?" (Year 2)

Juan also differentiates between high and low achievers without referring to such adjectives as 'smart' or 'fast', instead saying that some students 'know what to do'. He also makes it very clear that he is helped by the practice of justification and that he feels comfortable pushing other students to go beyond answers and explain 'why' their answers are given. At Railside the teachers have carefully prioritized the message that each student has two important responsibilities – both to help someone who asks for help, but also to ask if they need help. Both are important in the pursuit of equity, and justification has emerged as a helpful practice in the learning of a wide range of students, particularly because the act of justification makes mathematical methods more transparent and learnable for different students.

Assigning Competence

An interesting and subtle approach that is recommended within the complex instruction literature is that of 'assigning competence'. This is a practice that involves teachers raising the status of students that may be of a lower status in a group, by, for example, praising something they have said or done that has intellectual value, and bringing it to the group's attention; asking a student to present an idea; or publicly praising a student's work in a whole class setting. This practice was one that I could not fully imagine until I saw it enacted. My first awareness of it came about when a quiet Eastern European boy muttered something in a group that was dominated by two happy and excited Latina girls. The teacher who was visiting the table immediately picked up on it saying 'Good Ivan, that is important'. Later when the girls offered a response to one of the teacher's questions he said, 'Oh that is like Ivan's idea, you're building on that'. He raised the status of Ivan's contribution, which would almost certainly have been lost without such an intervention. Ivan visibly straightened up and leaned forward as the teacher reminded the girls of his idea. Cohen (1994) recommends that if student feedback is to address status issues, it must be public, intellectual, specific and relevant to the group task (p. 132). The public dimension is important as other students learn about the broad dimensions that are valued; the intellectual dimension ensures that the feedback is an aspect of mathematical work, and the specific dimension means that students know exactly what the teacher is praising.

The practice of 'assigning competence' is subtle and requires highly sensitive teachers. It is a practice that many good teachers employ, to a greater or lesser extent, without necessarily being aware of it or having a name for it. This practice is also an important part of the teachers' commitment to equity and to the principle of showing what different students can do in a multifaceted mathematical context.

Teaching Students to be Responsible for their Own Learning

A major part of the equitable results attained at Railside is the serious way in which teachers expect students to be responsible for each other's learning. Many schools employ group work which, by its nature, brings with it an element of responsibility, but Railside teachers go well beyond this to ensure that students take the responsibility very seriously.

One way in which teachers nurture a feeling of responsibility is through the assessment system. Teachers grade the work of a group by, for example, rating the quality of the conversations groups have. The teachers also occasionally give group tests, which take several formats. In one version students work through a test together, but the teachers grade only one of the individual papers and that grade stands as the grade for all the students in the group. A third way in which responsibility is encouraged is through a practice of asking one student in a group to answer a follow up question after a group has worked on something. If the student cannot answer the question the teacher will leave the group to have more discussions and return to ask the same student again. In the intervening time it is the group's responsibility to help the student learn the mathematics they need to answer the question. The teacher move of asking one member of a group to give an answer and an explanation, without help from their group-mates, is a subtle practice that has major implications for the classroom environment. This practice means that students are responsible to everyone in their group. In the following interview extract the students talk about this particular practice and the implications it holds:

Int:	Is learning math an individual or a social thing?
Gisella:	It's like both, because if you get it, then you have to explain it to everyone else. And then sometimes you just might have a group problem and we all have to get it. So I guess both.
Bianca:	I think both - because individually you have to know the stuff yourself so that you can help others in your group work and stuff like that. You have to know it so you can explain it to them. Because you never know which one of the four people she's going to pick. And it depends on that one person that she picks to get the right answer. (Year 2)

The students in the extract above make the explicit link between teachers asking any group member to answer a question, and being responsible for their group members. They also communicate an interesting social orientation that becomes instantiated through the mathematics approach, saying that the purpose in knowing individually is not to be better than others but so "you can help others in your group". The four practices I have described so far – those of taking group roles, multidimensionality, assigning competence and encouraging responsibility are all part of the complex instruction approach. I will now review three other practices in which the teachers engage that are also critical to the promotion of equity. These relate to the challenge and expectations provided by the teachers.

Challenge and Expectations

High Cognitive Demand

The teachers at Railside provide a huge amount of support to students, making themselves available to students after school and in the evenings. In interviews the students talk at length about the support of the teachers. But the teachers also expect a lot from the students and give them complex, mathematically challenging work. Importantly the support that teachers give to students does not serve to reduce the cognitive demand of the work (Stein, Smith, Henningsen & Silver, 2000). The cognitive demand that is expected of all students is higher than other schools partly because the classes are heterogeneous and no students are precluded from meeting high level content, but teachers also enact a high level of challenge in their interactions with groups and their questioning.

A very important feature of the curriculum that teachers use, that would not be seen in the curriculum materials, is the act of asking follow up questions. The curriculum example analyzed in the longer version of this paper (Boaler, 2004) involves students finding a perimeter of a complex shape made up of algebra tiles. The perimeter is 10x +10 and the teacher asks the follow up question: ‘where is the 10?’. This is complex because the 10 does not exist in a particular place in the diagram and its detection involves an understanding of algebraic variables. The students are aware that the teachers keep levels of mathematical work high and they appreciate it. When we interviewed students and asked them what it takes to be a good teacher, many of them mentioned the high demand placed upon them, for example:

Ana: She has a different way of doing things. I don’t know, like she won’t even really tell you how to do it. She’ll be like, ‘think of it this way’. There’s a lot of times when she’s just like – ‘well think about it’ – and then she’ll walk off and that kills me. That really kills me. But it’s cool. I mean it’s like, it’s alright, you know. I’ll solve it myself. I’ll get some help from somebody else. It’s cool. (Year 3)

The following students, in talking about the support teachers provide, also refer to their push for understanding:

Int: What makes a good teacher?

John: Patience. Because sometimes teachers they just zoom right through things. And other times they take the time to actually make sure you understand it, and make sure that you actually pay attention. Because there’s some teachers out there who say: ‘you understand this?’ and you’ll be like “yes”. But you really don’t mean “yes” you mean “no”.

> And they'll be like "OK" And they move on. And there's some teachers that be like – they know that you don't understand it. And they know that you're just saying "yes" so that you can move on. And so they actually take the time out to go over it again and make sure that you actually got it, that you actually understand this time. (Year 2)

The students' appreciation of the teachers' demand was also demonstrated in our questionnaires. One of the questions started with the stem: 'When I get stuck on a math problem, it is most helpful when my teacher …' This was followed by answers such as 'tells me the answer' 'leads me through the problem step by step' and 'helps me without giving away the answer'. Students could respond to each on a four-point scale (SA, A, D, SD). Almost half of the Railside students (47%) *strongly* agreed with the response: "Helps me *without* giving away the answer," compared with 27% of students in the 'traditional' classes at the other two schools ($n= 450$, $t = -4.257$; $df = 221.418$; $p<0.001$).

Effort Over Ability

In addition to the actions in which teachers engage, challenging through difficult questions that maintain a high cognitive demand, the teachers also give frequent and strong messages to students about the nature of high achievement in mathematics, continually emphasizing that it is a product of hard work and not of innate ability. I have already described the multidimensionality of classrooms and the fact that teachers take every opportunity to value something students can do, but they also keep reassuring students that they can achieve anything if they put in the effort. This message is heard by students and they communicate it to us in interviews, with absolute sincerity, for example:

> Sara: To be successful in math you really have to just like, put your mind to it and keep on trying – because math is all about trying. It's kind of a hard subject because it involves many things. (…) but as long as you keep on trying and don't give up then you know that you can do it. (Year 1)

In the year 3 questionnaires we offered the statement "Anyone can be really good at math if they try", 84% of Railside students agreed with this, compared with 52% of students in the traditional classes ($n= 473$, $t = -8.272$; $df = 451$; $p<0.001$). But the students do not only come to believe that they can be successful, they develop an important practice that supports them in that – the act of persistence. It could be argued that persistence is one of the most important practices to learn in school – one that is strongly tied to success in school as well as in work and life. We have many indications in our data that the Railside students developed considerably more persistence than the other students. For example, as part of our assessment data we give students long, difficult problems to work on for 90 minutes in class, which we videotaped. The Railside students were more successful on these problems, partly because they would not give up on them and they continued to try to find methods and approaches even when they had exhausted many.

When we asked in questionnaires: 'How long (in minutes) will you typically work on one math problem before giving up and deciding you can't do it?' the Railside students gave responses that averaged 19.4 minutes, compared to the 9.9 minutes averaged by students in traditional classes ($n=438$, $t = -5.641$; $df = 142.110$; $p< 0.001$). This response is not unexpected given that the Railside students worked on longer problems in class but it also gives some indication of the persistence students are learning through the longer problems they experience.

In the following interview extract the student links this persistence to the question asking and justification highlighted earlier:

Ana: Because I know if someone does something and I don't get it I'll ask questions. I'm not just going to keep going and not know how to do something.

Latisha: And then if somebody challenges what I do then I'll ask back and I'll try to solve it. And then I'll ask them: "Well how d'you do it?" (Year 3)

Clear Expectations and Learning Practices

The final aspect of the teachers' practice that I will highlight also relates to the expectations they offer the students. In addition to stressing the importance of effort the teachers are very clear about the particular ways of working in which students need to engage. Cohen & Ball (2001) describe ways of working that are needed for learning as 'learning practices'. The teachers are very clear about helpful learning practices and they frequently stop the students as they are working to point out valuable ways in which they are working (an example is given in the longer version of this paper). The teachers also spend time before projects begin setting out the valued ways of working, encouraging students to, for example, pick 'tricky' examples when writing a book (that is one of the projects they complete) as they will "show off" the mathematics that they know. The teachers communicate very clearly to students which learning practices will help them achieve. This was also true of the teachers in the school in England that I studied (Boaler, 1997, 2002) who also brought about more equitable outcomes.

Relational Equity

Many researchers are concerned to understand and promote equitable schooling practices. In studying equity most researchers look for reductions in achievement differences for students of different ethnic and cultural groups and genders, as well as equitable treatment in schools. Post (2004) refers to these two versions of equity as equal outcomes and equal access. The version of equity that I would like to consider in addition to these two important aspects of equity is closer to Elizabeth Anderson's conception (1999) that she refers to as 'democratic equity'. Anderson is a philosopher whose concern is not with test scores or other measures of educational achievement but with an individual's standing in society. 'Democratic equity is identified by an individual "standing as an equal over the course of an entire life" (Anderson, p. 319)' (from Post, 2004, p. 11). This conception shares some aspects of the version of equity I would like to propose. In observing Railside classes over many years I have come to appreciate what I will term as *relational equity* (Boaler, 2008). This conception of equity also goes further than students' test results in schools and considers the ways students are acting in school and the ways they learn to regard and relate to one another. Unlike Anderson's conception of equity, relational equity is about students' relationships with other students and with subjects and with the intersection of the two, as I shall outline below. In proposing this conception of equity I am asking the question – what would it mean to teach students to act in equitable ways in classrooms? This question was raised, for me, through observations of the Railside classrooms and the behaviour of the teachers and students. Indeed it would be hard to spend years in the classrooms at Railside without noticing that the students were learning to treat each other in more respectful ways than is typically seen in schools and that ethnic cliques were less evident in the mathematics classrooms than they are in most schools. Further such behaviour did not just *happen* to take place in a mathematics classroom, it was fundamentally related to the students' conceptions of and work within mathematics. I propose the term *relational* equity because it is about students' relationships with each other developed through mathematical work as well as students' relationships with the subject of mathematics. Such relationships led to more equitable achievement results at the school as well, I contend, to more equitable ways of working that students would take into their lives.

Richard Schweder, a cultural anthropologist and cultural psychologist talks about the importance of considering different perspectives on issues in the working of a democratic society:

> It is often advantageous to have more than one discourse for interpreting a situation or solving a problem. Not only alternate solutions but multidimensional ones addressing "several orders of reality" or "orders of experience" may be more practical for solving complex human problems. (Schweder, 2003, p. 100)

The Railside students learn through their mathematical work that alternate and multidimensional solutions are important which leads them to value the contributions of the people offering such ideas. This is particularly important at Railside as the classrooms are multicultural and multilingual. It is commonly believed that students will learn respect for different people and cultures if they have discussions about such issues or read diverse forms of literature in English or Social studies classes. I propose that all subjects have something to contribute in the promotion of equity and that mathematics, often regarded as the most abstract subject removed from responsibilities of cultural or social awareness, has an important contribution to make. For the equitable relationships that Railside students developed and that I have discussed more fully in Boaler (2004) are only made possible by a conception of mathematics that values the contribution of different insights, methods and perspectives in the collective solving of a particular problem with a particular solution. The relational version of mathematics that these students learn in which they come to value different contributions to a problem and different relationships between mathematical methods enables the respectful working relationships I have set out. It seems to me that it is important to consider whether students are learning to respect students from different ethnic and cultural groups and genders in our schools today yet such concerns are not captured in notions of equity that are measured by test scores. Relational equity builds from the relations within subjects. As students learn to appreciate and make connections between different mathematical methods and ideas and to value the contributions of different perspectives on problems, given by students who think in quite different ways, they will also learn something extremely valuable about people, relations and ideas.

CONCLUSION

Railside School is not a perfect place - the teachers would like to achieve more in terms of student achievement and the elimination of inequities, and they rarely feel satisfied with the achievements they have made to date, despite the vast amounts of time they spend planning and working. But research on urban schools and the experiences of mathematics students in particular tells us that the achievements at Railside are extremely unusual. In this paper I have attempted to convey the work of the teachers in bringing about the reduction in inequalities as well as general high achievement that they achieve. In doing so I hope also to have given a sense of the complexity of the relational and equitable system that they have in place (see also Boaler, 2009). People who have heard about the achievements of Railside have asked for their curriculum so that they may use it, but whilst the curriculum plays a part in what is achieved at the school it is only one part of a complex, interconnected system. At the heart of this system is the work of the teachers, and the many different equitable practices in which they engage.

REFERENCES

Anderson, E. S. (1999). What is the Point of Equality? *Ethics, 109* (January 1999), 287-337.

Boaler, J. (1997). *Experiencing School Mathematics: Teaching Styles, Sex and Setting*. Buckingham: Open University Press.

Boaler, J. (2002a). The Development of Disciplinary Relationships: Knowledge, Practice and Identity in Mathematics Classrooms. *For the Learning of Mathematics*, 22(1), 42-47.

Boaler, J. (2002b). *Experiencing School Mathematics: Traditional and Reform Approaches to Teaching and Their Impact on Student Learning*. (Revised and Expanded Edition). Mahwah, NJ: Lawrence Erlbaum Association.

Boaler, J. (2003). When Learning No Longer Matters: Standardized Testing and the Creation of Inequality. *Phi Delta Kappan*, 84(7), 502-506.

Boaler, J. (2008). Promoting 'relational equity' and high mathematics achievement through an innovative mixed ability approach. *British Educational Research Journal.* 34 (2), 167-194

Boaler, J. (2009). *The Elephant in the Classroom: Helping Children Learn and Love Maths*. London: Souvenir Press.

Boaler, J., & Brodie, K. (2004). *The Importance of Depth and Breadth in the Analysis of Teaching: A Framework for Analysing Teacher Questions.* Paper presented at the Psychology of Mathematics Education NA. Toronto, Ontario.

Cohen, E. (1994). *Designing Groupwork*. New York: Teachers College Press.

Cohen, E., & Lotan, R. (Eds.). (1997). *Working for Equity in Heterogeneous Classrooms: Sociological Theory in Practice*. New York: Teachers College Press.

Fendel, D., Resek, D., Alper, L., & Fraser, S. (2003). *Interactive Mathematics Program: Integrated High School Mathematics*. Berkeley, CA: Key Curriculum

Press.

Fiori, N. & Boaler, J. (2004). *What Discussions Teach us about Mathematical Understanding: Exploring and Assessing Students' Mathematical Work in Classrooms.* Paper presented at the Psychology of Mathematics Education NA. Toronto, Ontario.

Horn, I.S. (2002). Learning on the job: Mathematics teachers' professional development in the contexts of high school reform. Unpublished Ph.D. Dissertation, University of California, Berkeley.

Horn, I.S. (2005). Learning on the job: A situated account of teacher learning in two high school mathematics departments. Cognition & Instruction, 23(2).

Lee, C. D. (2001). Is October Brown Chinese? A Cultural Modeling Activity System for Underachieving Students. American Educational Research Journal, 38(1), 97-141.

Maher, C., & Martino, A. M. (1996). The Development of the Idea of Mathematical Proof: A 5-Year Case Study. *Journal for Research in Mathematic Education*, 27(2), 194-214.

Picciotto, H. & Wah, A. (1994). *Algebra: Themes, concepts, and tools*. Aslip, IL: Creative Publications.

Post, L. (2004). Working Toward Equity: Teacher Communities Changing Practices. Unpublished PhD Dissertation, Stanford University: Stanford, CA.

RAND, M. S. P. (2002, October). *Mathematical Proficiency for All Students: Toward a Strategic Research and Development Program in Mathematics Education* (Dru-2773-Oeri). Arlington, VA: RAND Education & Science and Technology Policy Institute.

Schweder, R.A. (2003). *Why Do Men Barbeque? Recipes for cultural psychology*. Cambridge, MA: Harvard University Press, 74-133.

Stein, M. K., Smith, M., Henningsen, M., & Silver, E. (2000). *Implementing Standards Based Mathematics Instruction: A Case Book for Professional Development*. New York: Teachers College Press.

Yackel, E., & Cobb, P. (1996). Sociomathematical Norms, Argumentation, and Autonomy in Mathematics. *Journal for Research in Mathematics Education*, 27(4), 458-477.

TEACHERS ASSESSING YOUNG CHILDREN'S MATHEMATICAL DEVELOPMENT: HOW CONFIDENT ARE THEY?

Janette Bobis

University of Sydney, Australia

Research has indicated that teachers' knowledge of children's mathematical thinking can be particularly influential in determining their instructional strategies and can potentially increase their abilities to cater for various levels of children's mathematical understanding (Swafford, Jones and Thornton, 2000). This study sought to explore just how confident teachers are in their abilities to assess students' mathematical development and the extent to which they perceived such knowledge impacted on their instructional decision-making. Twenty-two primary school teachers, who had participated in a professional development program that specifically focused on understanding and applying a theoretical framework of children's cognitive development in number, were asked to rate their confidence in identifying students' levels of development according to the learning framework. Follow-up interviews elicited explicit information about each teacher's ability to utilise the framework to assess individual student's mathematical development and to plan appropriate instruction as a consequence. Teachers' self-ratings of their abilities to utilise a theoretical framework to assess student progress and undertake subsequent planning for instruction were generally lower than what they were able to demonstrate in reality. Interviews revealed that the majority of teachers held extensive knowledge about students' mathematical development and could clearly articulate a trajectory of learning appropriate for each child's level of development.

INTRODUCTION

It has only been since Shulman delivered his seminal paper on teacher knowledge in 1986, in which he delineated three categories of knowledge—subject matter content knowledge, pedagogical content knowledge, and curricular knowledge, that the study of teacher knowledge has really become a central focus of educational researchers and policy makers. Since then, many educators and researchers have advanced their own frameworks and interpretations of various components of teacher knowledge (e.g., Hill, Rowan and Ball, 2005). Importantly, attention has changed from looking solely at what content knowledge teachers possess to the quality of their understanding of how children learn that content (including major growth points in their development of understanding) and of the specific pedagogy needed to teach it (Hill, Ball and Schilling, 2008).

When studying teachers' knowledge of mathematics it is essential to also consider teachers' perceptions of that knowledge (Li and Kulm, 2008). While distinct from knowledge (Thompson, 1992), teachers' perceptions of, and confidence in, their knowledge seems to particularly impact on their classroom instruction (Cai, Perry and

Wong, 2007). Without confidence in their knowledge to determine students' mathematical development, teachers' may question their abilities to plan appropriate instruction.

'Confidence' is a dimension of attitude that has been studied quite extensively in relation to teachers' mathematical content knowledge, particularly in primary and middle school teachers (Beswick, Watson and Brown, 2005), but there is comparatively little research that explores teachers' confidence to assess students' mathematical development and to plan appropriate instruction as a consequence. The explicit assessment of teachers' confidence to apply their knowledge of how children learn mathematics was an important component of a larger study designed to explore the impact of a professional development program on teachers' knowledge and classroom practices. Before presenting the specific research questions of the study relating to teacher confidence, background to the professional development program is required to fully appreciate the context for the study and research questions.

BACKGROUND TO THE STUDY

The Count Me In Too numeracy program is an on-going professional development initiative of the Department of Education and Training in New South Wales (NSWDET, 2007), Australia. Its aims are to help teachers understand children's mathematical development and to improve children's achievement in mathematics. The Count Me In Too program began in 1996 as a pilot program involving 13 schools and gradually expanded to involve nearly 1700 primary schools over a ten-year period across the state. Key aspects of the program include the *Learning Framework In Number* (LFIN) and a diagnostic interview or *Schedule in Early Number Assessment* (Wright, Martland and Stafford, 2006). The LFIN is used by teachers to not only identify the level of development each child has attained but provides instructional guidance as to what each student needs to work towards. A stimulus for the current study, was the need to know what teachers understand about the LFIN and how it impacts on their teaching and assessment practices. Importantly, the LFIN has had a major impact on the development of syllabus and curricula documents throughout Australia, New Zealand and a growing number of other regions in the world. Hence, the implications of this study have potentially quite substantial application.

The Learning Framework In Number

Learning frameworks, also known as progress maps or learning trajectories provide a description of skills, understandings and knowledge in a sequence in which they typically occur, thus giving a virtual picture of what it means to progress through an area of learning. Thus they provide a pathway or map for monitoring individual development over time. A student's location on a framework can be utilised as a guide to determining the types of learning experiences that will be most useful in meeting the student's individual needs at that particular stage in their learning. A number of professional development programs now exist that utilize such theoretical frameworks with the aim of increasing teachers' understanding of children's mathematical thinking (e.g., Bobis, Clarke, Clarke, Thomas, Wright, Young-Loveridge and Gould, 2005; Van den Heuvel-Panhuizen, 2001).

The Count Me In Too *Learning Framework In Number* was initially developed by Wright (1994) and has since undergone further development through the impact of a wide range of research in early number (e.g., Gravemeijer, 1994; Mulligan and Mitchelmore, 1997). In brief, the LFIN consists of five key and interrelated components:

1. Building addition and subtraction through counting by ones
2. Building addition and subtraction through grouping
3. Building place value through grouping
4. Building multiplication and division through equal counting and grouping
5. Building fractions through sharing and partitioning

The LFIN provides a description of the knowledge and skills characterising major stages of development in each of these components. Teachers use these stage descriptions to profile their students' knowledge in each key component. Such information then provides instructional guidance as to what each student needs to progress. An important step in a teacher's ability to utilise the framework in their instructional decision-making is their understanding of how all components are interrelated. A visual representation of the *Framework* is presented in Figure 1 and a more detailed description of the LFIN is available in Wright, Martland and Stafford (2006) and at the NSW Department of Education and Training website (
www.curriculumsupport.education.nsw.gov.au/countmein/).

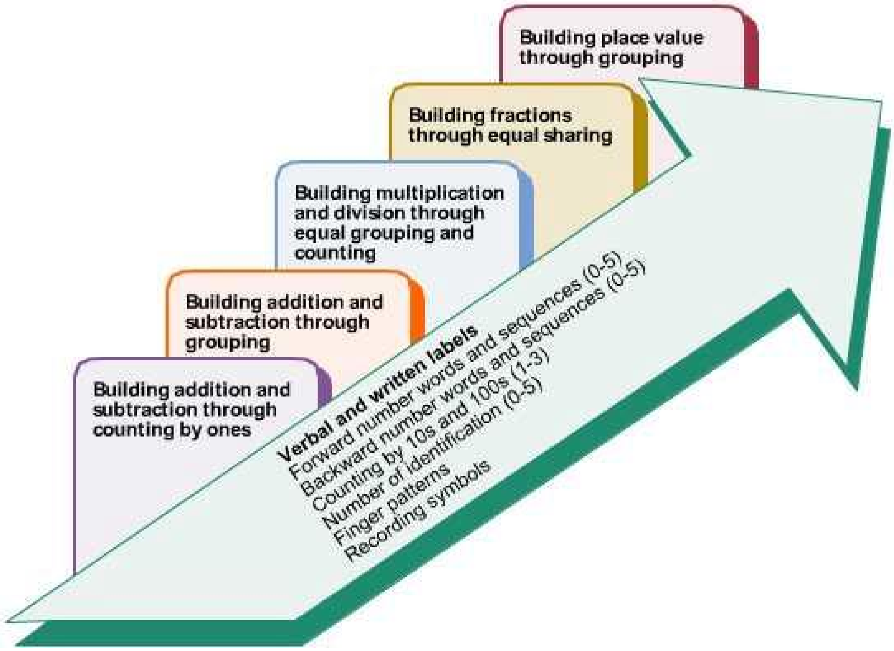


Figure 1. *Learning Framework In Number* model showing five major knowledge components and important prerequisite number skills.

THE STUDY

The overall aim of the larger study was to explore teacher knowledge of the *Learning Framework in Number* from the Count Me In Too numeracy program and the impact this knowledge has on their instructional decision making for teaching mathematics. This paper focuses on a component of the larger study, namely, teachers' perceptions/confidence of their knowledge of the *Framework* and the impact these perceptions have on their ability to assess learning and plan for instruction. In particular, the following research questions were addressed:

1. What are teachers' perceptions about the extent of their knowledge of the *Learning Framework In Number* [LFIN]?
2. How confident do teachers feel about identifying children's levels of mathematical development on the LFIN?
3. Can teachers use their knowledge about children's mathematical development as indicated on the LFIN to plan appropriate instruction?

Information for the larger study was gathered from three main sources—survey, interviews and teaching documents. During the survey component, teachers were asked to rate themselves on a number of aspects. Three aspects related to their understanding of the LFIN, their confidence using it to assess children's mathematical development and the extent to which they perceived this knowledge impacted on their ability to plan appropriate instruction. Following the survey, teachers whose responses were considered representative of the sample, were invited to be interviewed. Each of these teachers had been initially invited to participate in the study due to their on-going involvement in the Count Me In Too numeracy professional development program that was operating in their schools. Only data from the survey and interview component (interviews and teaching documents) related specifically to the issue of teacher confidence for these teachers are reported here.

Participants and Procedure

Twenty-two teachers (21 female and 1 male) drawn from three different schools agreed to participate in the interview component—8 teachers from School A and 7 teachers from both School B and School C. Each interview took approximately 45 – 60 minutes and was conducted in a private office within the school grounds during school hours. Relief teaching was provided for the duration of the interview so that teachers were not inconvenienced by time taken for the interview.

The interviews established background biographical details for each interviewee before seeking information specifically related to professional learning and the implementation of the numeracy program and the LFIN. Teachers were then asked about their confidence concerning the identification of individual students' abilities using the LFIN and the subsequent planning for student instruction. The interviews were digitally recorded and later transcribed for the purpose of analysis. It was requested that teachers bring documents such as school management plans and individual or collaborative teacher programs to the interview to help support their oral explanations of planning and teaching practices.

Analysis

Background data on each survey respondent were collated and descriptive statistics were used to analyse the items requiring teachers to provide personal ratings for certain aspects of their knowledge and utilisation of the LFIN. Interview data were transcribed and read for emerging themes that specifically related to teacher confidence surrounding the LFIN and instruction.

Documents in the form of school management plans and individual teacher programs were analysed in conjunction with interview data since all documents provided by teachers were intended to support, elaborate and verify their responses to interview questions. Teacher programs and lesson plans, when available, were analysed to determine the type and level of impact the LFIN had on planning for student learning at the school and classroom levels.

RESULTS AND DISCUSSION

Background information for each interviewee, along with their self-ratings on the survey

Interviewee	Grade	Approx. Months/Yr using CMIT	Self-rating for perceived understanding of LFIN (1 to 4)	Self-rating for confidence to use LFIN to assess students' (1 to 4)	Self-rating for extent LFIN perceived to impact on instruction (1 to 4)
1. Jane	5 & 6	5 months	2	3	3
2. Kathy	K	8 years	3	3	3
3. Roberta	3 & 4	5 months	2	2	1
4. Lilly	1	1 year	3	3	3
5. Lana	3 & 4	1 year	2	3	2
6. Ann	K	2 years	3	3	4
7. Kate	3 & 4	1 year	1	1	2
8. Mark	3 & 4	1 year	2	3	3
9. Alyson	3	6 years	3	3	2
10. Naomi	1	5 months	2	1	2
11. Tania	1 & 2	8 years	3	3	3
12. Mandy	2	4 years	3	3	3
13. Maxine	4 & 5	6 years	1	3	3
14. Katie	5 & 6	5 years	3	4	3
15. Narelle	1	1 year	1	3	2
16. Robyn	2	2 years	3	3	3
17. Vera	K	2 years	3	3	3
18. Violet	K	2 years	2	2	3
19. Kristen	1 & 2	2 years	1	1	1
20. Erica	1	5 months	2	3	2
21. Natalie	K & 1	5 months	3	4	3
22. Christine	3	10 years	3	3	4

Table 1. Summary of background information and teachers' self-ratings for understanding and confidence using the LFIN to assess and plan for instruction from survey results

component for their understanding and confidence using the LFIN are presented in Table 1.

As can be seen from the table, teachers' experience implementing the numeracy program (and using the LFIN) ranged from just 5 months to 10 years. The mean number of years experience of the entire sample was slightly less than 3 years (2 years 9 months). While the program had been implemented in each school for at least 3 years, teachers' experience differed for various reasons; some had been exposed to the program in a previous school, others had begun employment after its initial introduction at their current school and some schools had chosen to 'roll-out' the implementation starting with more junior classes in the first year and gradually extending the program to the higher grades in the second and third years of the program. In research terms, it would have been more ideal if there was greater consistency in teachers' experience with the program, but that was not an initial aim of the study and nor is it realistic in a school context.

Table 2 summaries the number and percentage of interviewees selecting each level rating on the three aspects of teacher confidence. It can be seen from the table, that no interviewees rated their understanding of the LFIN at the highest level (Level 4) and only 2 teachers considered their ability to assess students' mathematical ability or to plan appropriate instruction using the LFIN to be Level 4. The majority of respondents rated themselves at Level 3 for each aspect dealing with teacher confidence on the survey.

Rating	Understanding of LFIN	Confidence using LFIN to assess students	Extent LFIN impacts on instruction
1	3 (14%)	2 (9%)	2 (9%)
2	7 (32%)	2 (9%)	6 (27%)
3	11 (50%)	15 (68%)	12 (55%)
4	0 (0%)	2 (9%)	2 (9%)

Table 2. Number (and percentage) of interviewees selecting each level rating on the three aspects of confidence

An initial purpose of the interviews was to validate and elaborate on information obtained from the survey data collected. In particular, interviews were a critical means by which teachers' reasons for their personal ratings about the LFIN and its perceived impact on their knowledge and instructional decision-making could be verified. Individual and grade-team teaching programs were presented by interviewees to help explain and justify descriptions of their planning and teaching practices. Reference to these documents is integrated into the discussion of interview data.

Overall, interviewees expressed reluctance to give themselves an 'Excellent' or Level 4 rating for any of the aspects they had been asked to rate. Two Year 3/4 teachers (Roberta and Kate) commented that with less than a years experience working with the *Framework*, they "still had a lot to learn". Mark thought that his understanding was currently a Level 2, but expected "given time, I will be able to delve into it more".

The majority of teachers rated their confidence using the LFIN to identify children's stages of development as a Level 3. Generally, teachers agreed with Jane that "I haven't done it enough to be sure" and Mark that "while I'm at least a Level 3 for confidence, I'm heading for a Level 4—I just need another year to consolidate what I've learnt". Kate also thought that another year implementing the program would ensure an increase in her confidence. Narelle, a Year 1 teacher who had been implementing the numeracy program for approximately one year, explained her confidence in using the LFIN had grown enormously since she had used it to assess a second group of students. Hence, she rated her confidence using the *Framework* to assess her students as a Level 3. She only rated her understanding of the LFIN a Level 1 because she felt "there is so much more to learn about it—and I'm learning new things all the time". As was the case for many interviewees, the perception that there is still much more to learn about the *Framework* and how to use it to guide instruction seemed to weigh heavily on teachers' minds—resulting in conservative self-assessments of their knowledge about the LFIN and their interpretation of its use.

More than half the interviewees indicated that "more time" working with the LFIN as part of the numeracy program would allow them to become more "confident with my understanding of the *Framework* and how to implement it in the classroom". The initial survey data provides some support for this belief; there was a clear trend linking the length of time a teacher had been utilising the Count Me In Too program to their self-rated confidence level. That is, the more time teachers indicated that they had been implementing the program, the more confident they felt about their understanding of the LFIN and their ability to use it to guide their assessment and instructional decision-making. However, the fact that so many teachers with one year or less experience implementing it also rated themselves at Level 3, indicates that 'time' alone is not the only factor impacting on teachers' confidence levels. 'Time' by itself is not considered the determining factor for a teacher's improved confidence level. Rather, the professional support at the school and, in particular, the stage/grade level, was considered more influential. This was evident when interviewees explained their self-ratings for the extent to which the LFIN was perceived to impact upon their instruction.

The degree to which the LFIN was perceived to impact on classroom instruction was most evident when teachers discussed their program and lesson planning strategies. The majority of K-4 teachers considered "[LFIN] activities are embedded into the number programs—they are not seen as something separate". Evidence of this integration was found in the teachers' programs. Kindergarten teachers, Violet and Vera presented their written programs to exemplify the explicit links contained in it to the LFIN. Violet explained that a programming "proforma" was used throughout the infant classes to provide some consistency within stages and that the *Framework* was used to "gear activities to specific groups of students". In addition to syllabus outcomes, specified content and processes, explicit reference to aspects of the *Framework* were listed for the focus of each lesson and accompanied by lists of activities intended for varying student ability levels as indicated on the LFIN.

Not all programs for each teacher explicitly referred to strategy development named on the LFIN, but the learning experiences and activities followed a sequential development akin to the knowledge, skill and strategy development outlined on the LFIN. Additionally, named activities and resources in the programs made direct reference to curriculum resources that have the LFIN embedded in their structure (NSW DET, 1999; 2003). Mandy commented:

> The *Framework* is the reason for the choice of activities. We use the results of the [assessment] to determine how we group the children initially and what activities go into the program—that's why I have the results of the [assessment] and my observations at the front of my program.

The extent to which instruction was planned and implemented in accordance with the LFIN was less obvious in teaching programs for teachers of upper primary classes. Jane and Katie explained that collaboration amongst teachers of the higher grade levels in their respective schools was not as strong as it was for teachers of the younger grades and that this was an obstacle to the degree to which the LFIN was utilised. For instance, Jane was aware that a particular Kindergarten teacher at her school was considered by other teachers to have in-depth understanding of the LFIN and supported the implementation of it in other in K-2 classrooms through collaborative programming and lesson planning.

A quarter of interviewees commented that while they were quite confident with their knowledge about the LFIN and their ability to use it for the grade they were currently teaching, they were not so confident for students in other grades. Naomi was reluctant to rate herself more highly than Level 2 for her understanding of the LFIN and a Level 1 for her confidence using it to assess students because "I still have to come to this document (the *Framework*) to find out how to move them". This low self-rating was in contradiction to the fact that she was easily able to suggest modifications to activities to suit varying abilities of students without having to refer to support documents later in the interview. A possible reason for this apparent contradiction is that Naomi rated herself in terms of 'future' use of the LFIN and was therefore unsure of where a new group of students would take her. However, having used the LFIN in her planning, she was very comfortable relating the instructional decisions she had already made.

Use of the Framework to assess and inform planning for specific students

The final component of the interview aimed to elicit explicit information about the LFIN to determine how confident teachers are to use their knowledge of it to assess student mathematical development and to plan appropriate instruction as a consequence. It did this by asking interviewees to provide authentic examples (their own students) of how they utilise the LFIN to assess students' mathematical development and plan for appropriate instruction. During the interview, teachers were asked to discuss specific students' stages of development in relation to mathematics content currently or recently studied in class. They were also asked to elaborate on the types of teaching and learning experiences they provided for these students to help them progress to the next stage of strategy development according to the LFIN. Interviewees were informed that they could refer to support materials such as programming documentation or the numeracy program-related resources to assist their response, but all indicated that this was not necessary because it related to programming and teaching that was familiar to them. To assist the presentation of these data and their analyses, a rubric was established to assist with analysis of teachers' responses. Each interviewee's ability to articulate particular students' strategy development and elaborate on appropriate learning experiences for them were categorised according to the rubric. Table 3 presents the results of the categorisation process for each interviewee and provides sample responses for each category on the rubric to assist with validation of the categorisation process. Importantly, an allocation to a particular level on the rubric does not indicate that one teacher is considered a better teacher than any other. Rather, they are considered to have a different level of understanding of the LFIN as could be interpreted from their responses during the interview, or they are perhaps less or more likely to utilise the LFIN to guide their instructional decision-making than another teacher.

No interviewee was considered to have provided the lowest level response (Level 1) despite the fact that 3 teachers had previously rated their understanding of the LFIN as a Level 1 (see Table 2). Overall, interviewees provided quite lengthy and very detailed descriptions about the strategy development of individual and small groups of children from their classes. Only two interviewees' responses were categorised as Level 2, meaning that these teachers were quite comfortable describing the strategies particular students displayed but generally could not articulate a clear or appropriate direction that strategy development should take to help the child advance. Given that the majority of interviewees had only about two years (or less) experience implementing the numeracy program, it is encouraging that 20 out of the 22 interviewees provided responses that were considered to be at the top two levels on the rubric to this set of interview questions.

Level	Number (percentage) of interviewees responding at each level	Description of response level	Sample responses for each category
1	0 (0%)	Description of behaviour provided or unreasonable/ inappropriate response indicating little or no understanding of task or unable to make sense of response.	N/A
2	2 (9%)	Strategy development described or LFIN referred to but inappropriate stage selected or rationale for selection unclear. No follow-up suggested or some understanding evident of follow-up activities but may not be the most appropriate given stage selected.	These kids can add and subtract 2-digit by 2-digit numbers mentally without the number line. They use splitting and combining strategies. Some take a lot of time but do it mentally. I don't think they are just visualising the algorithm.
3	8 (36.4%)	Appropriate strategy described or LFIN referred to. Follow-up learning experiences mostly appropriate.	She is low on the LFIN. When I covered the counters she couldn't do the tasks. She's very visual. So I give her addition tasks with 'dot' dice. I'm trying to get her to count-on from the number without counting all the dots.
4	10 (45.5%)	Comprehensive understanding of strategy development LFIN and/or uses LFIN to justify choice of appropriate follow-up learning experiences.	My weakest are still Emergent. They can't recognise all the numbers or even count to 10 yet. So we are working on number recognition and counting forwards and backwards to 10. I'm encouraging them to add two groups of counters together. A few are just starting to work at the Perceptual stage.

Table 3. Categorisation of interviewees' abilities to articulate use of the LFIN to assess students' strategy use and plan appropriate instruction

Ten of the interviewees' responses were considered to be at the most sophisticated level (Level 4). This group of teachers demonstrated a rich understanding of strategy development in early number work and were able to clearly articulate appropriate instructional decisions they had made to enhance the strategy development of particular students. A comment by Maxine, a Year 4/5 teacher with 6 years experience implementing the program, provides a logical explanation for the ability of teachers to provide such comprehensive responses during the interview; namely, the familiarity they have with their own students: "These are my students I'm talking about—it shouldn't be hard for me to know what they can and can't do".

Given the overall success of interviewees responding to this component of the interview, evident by the degree of detail provided and the amount of coherence between teachers' interpretations of students' strategy development and their recommended follow-up for instruction, it seemed that interviewees had rated their own abilities to use the LFIN to assess and plan for instruction quite conservatively. When specifically questioned as to reasons for their conservative self-ratings despite the obvious extent of knowledge demonstrated by more than three-quarters of the interviewees, many responded that they still needed to “turn to the *Framework”* or support documents to confirm their judgements about individual children’s stage of strategy development and the types of activities needed to help them progress. Thus, most teachers perceived that their knowledge was insufficient or lacking because they felt the need to consult other sources to verify their assessments or plans for instruction.

CONCLUSION AND IMPLICATIONS

As opposed to studies that report alarming findings about limited teacher knowledge of either mathematical content or of how students learn mathematics (Li and Kulm, 2008; Sullivan, Clarke and Clarke, 2009), this study raises a concern of a slightly different nature. The findings reveal two seemingly contradictory sides of teacher knowledge. From a positive perspective, teachers’ knowledge of early number content, of how children’s mathematical thinking in this area progresses and of the pedagogy required to teach it were rich and extensive. Conversely, teachers’ perceptions of that knowledge were quite conservative; often expressing a lack of confidence in their knowledge needed to inform their instructional decision-making. Hence, even when teachers possess extensive knowledge of students’ learning, they may still lack confidence in that knowledge and this may have a detrimental impact on their abilities to make instructional decisions based on that knowledge.

To help improve the effectiveness of teacher professional development it is essential that we understand more fully the complexity of teacher knowledge and how it is utilised to inform teacher instructional decision-making. While research surrounding teacher knowledge—particularly knowledge of how children’s mathematical thinking develops—has shown that such knowledge can be influential in determining classroom instruction (Swafford et al., 1997), the research reported here has highlighted the need to further explore the potential impact teachers’ own perceptions of that knowledge may have. From a theoretical perspective, it is feasible and important to extend current models of teacher knowledge to include affective aspects such as teacher confidence. The practical implications of this research emphasise the necessity that providers of teacher professional development explore ways to support the development of teachers’ confidence and expertise in teaching mathematics. Such strategies may include an emphasis on teacher professional conversations designed to assist within- and between-school consistency and validity of teacher judgements and decision-making about assessment and instruction.

REFERENCES

Beswick, K., Watson, J., & Brown, J. (2005). Teachers' confidence and beliefs and their students' attitudes to mathematics. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Melbourne). Sydney: MERGA.

Bobis, J., Clarke, B., Clarke, D., Thomas, G., Wright, R., Young-Loveridge, J. & Gould, P. (2005). Supporting teachers in the development of young children's mathematical thinking: Three large scale cases. *Mathematics Education Research Journal*, 16 (3), 27-57.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L. & Empson, S. B. (1999). *Children's Mathematics: Cognitively Guided Instruction*. Portsmouth: Heinemann and National Council of Teachers of Mathematics.

Cobb, Cobb, P., Yackel, E., & Wood, T. (1992). Interaction and learning in mathematics classroom situations. *Educational Studies in Mathematics*, 23, 99-122.

Gravemeijer, K. (1994). *Developing realistic mathematics education*. Utrecht, The Netherlands: CD-B Press.

Heuvel-Panhuizen, M. van den (2001). *Children learn mathematics: A learning-teaching trajectory with intermediate attainment targets for calculation with whole numbers in primary school.* Utrecht: Freudenthal Institute, Utrecht University/SLO.

Hill, H., Ball, D., & Schilling, S. (2008). Unpacking pedagogical content knowledge: Conceptualising and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39 (4). 372-400.

Hill, H., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, 42 (2), 371-406.

Li, Y. & Kulm, G. (2008). Knowledge and confidence of pre-service mathematics teachers: The case of fraction division. *ZDM Mathematics Education*, 40, 833–843.

Mulligan, J. T. & Mitchelmore, M. C. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education,* 28, 309-330.

Sullivan, P., Clarke, D. & Clarke, B. (2009). Connecting mathematics tasks to learning opportunities, *Mathematics Education Research Journal*, 21 (1), 85 -105.

Swafford, J. O., Jones, G. A., & Thornton, C. A. (1997). Increased knowledge in geometry and instructional practice. *Journal for Research in Mathematics Education*, 28, 467-483.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. B. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127–146). New York: Macmillan.

Wright, R. J. (1994). Working with teachers to advance the arithmetical knowledge of low-attaining 6- and 7-year-olds: First year results. In J. Ponte and J. Matos (Eds.), *Proc. 18th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 377-384). Lisbon, Portugal: PME.

Wright, R. J., Martland, J., & Stafford, A. (2000). *Early numeracy: Assessment for teaching and intervention.* London: Paul Chapman Publications /Sage.

MATHEMATICS TEACHING AS A SHARED ENTERPRISE: LEARNING BY 'DOING'

Dolores Corcoran

St Patrick's College, Drumcondra

Japanese lesson study is a form of teacher professional development which is both 'contextualised' in terms of particular students' mathematical learning and 'situated' in the act of teaching, thereby presenting possibilities for the substantial and sustained changes in mathematics teaching which can be achieved when teachers focus on children's learning of mathematics (Franke, Carpenter, Levi, Fennema,2001). This paper describes a teacher preparation experiment that used lesson study to develop participants' mathematical knowledge in teaching along four dimensions –Foundation, Transformation, Connection and Contingency (Rowland, 2007). A group of prospective teachers participated in a year long education elective course- Learning to Teach Mathematics Using Lesson Study. Drawing on the work of Wenger the lesson study group was conceived as a community of practice and notions of belonging and engagement with the enterprise of the community informed analysis of mathematics teaching as an "economy of meaning" (Wenger, 1998, p. 197). Accountability to the lesson study enterprise and negotiation of a shared repertoire resulted in alignment of the practice with 'reform' interpretations of the primary mathematics curriculum (1999a, b).

INTRODUCTION

Mathematics education has flourished as an academic discipline over the past fifty years. In the early stages, the learning and teaching of mathematics was viewed mainly in cognitive terms and mathematical knowledge was conceived primarily in hierarchical systems of root and branch. The acquisition metaphor for learning of mathematics in terms of and rules and procedures often positioned the teacher as 'expert' delivering mathematics, largely by drill and practice to large groups of individual learners with the 'ability' to 'grasp' the material presented (Povey, 2002). In such a context, the mathematical knowledge of teachers became an object of research, leading to an interest in identifying and measuring the mathematics needed in, and for, teaching (Ball, 1988). This paper seeks to present a different theory of teacher learning, by examining the enterprise of teaching mathematics in which the students in my study engaged (Corcoran, 2008). 1988 was a pivotal year in mathematics education research marked by a decidedly "social turn" (Lerman, 2000, p. 23) with the publication of research by anthropologist, Jean Lave (1988) – who proposed a socio-cultural theory of mathematical problem-solving in adults. The work of Lave and other scholars influenced the emergence among the mathematics education community of theories that construe "meaning" and "reasoning" as products of situated and social activity pertinent to the teaching of mathematics. Thus, moving "out of trees of knowledge and into fields of action," practice came to be defined as "a dialectical relationship between persons acting and the settings of their activity" (Lave, 1988, p. 145). Mathematics teaching can be conceived as such a 'field of action' where multiple practices co-exist, overlap and influence each other, so that the meanings of mathematics and of teaching have come to be problematised and contested. The result is a proliferation in research themes and topics drawing on an increasingly wide range of disciplines to contribute to an intensely textured fabric of research findings about the origins and meanings of mathematics and the multiplicity of factors which contribute to the optimal teaching of mathematics in different school settings. Reconciling all these findings and factors is a difficult task for individual teachers and Hiebert, Gallimore and Stigler observe that "archived research knowledge has had little effect on the improvement of practice in the average classroom" (2002, p. 3).

Research into mathematics teaching in Irish post-primary schools has suggested that self-styled, results-affirmed, "good" and "successful" teachers of mathematics equated improved learning with the memorisation of formulae and procedures (Lyons, Lynch, Close, Sheerin and Boland, 2003, p. 366). More recent research findings of mathematical knowledge in teaching (MKT) among Irish primary teachers were "discrepant" and "equivocal," leading to questions about reasons for the "only fair to good" degree of fit between the teachers' MKT scores and the work of mathematics teaching as investigated in the study (Delaney, 2008). Other scholars have sought to improve mathematical achievement in children by focusing on the interactions between teachers, children and mathematics (Kilpatrick, Swafford and Findell, 2001). There is recognition of a 'reform agenda' where a particular ideology embraces how children learn mathematics, what mathematics they should learn and how that mathematics should be taught. Ireland adopted the reform agenda in primary mathematics with the introduction of the primary mathematics curriculum based on constructivist principles (Government of Ireland, 1999a, b). Within this arena, the setting for my research is a college of education for primary, generalist teachers, where a mathematics teacher development intervention using lesson study was implemented.

OVERVIEW OF THE LESSON STUDY ELECTIVE COURSE

The research was conducted in the context of an elective module in education - Learning to Teach Mathematics Using Lesson Study, which spanned the academic year 2006-07. Six third year Bachelor of Education student teachers participated, with the author acting as tutor. The lesson study protocols of collaborative lesson preparation and post-lesson reflection were adopted to further the course goal of learning to teach the primary mathematics curriculum for maximum pupil benefit. Each member of the elective group was involved in planning, observing, analysing and revising 'research' mathematics lessons intended to promote children's mathematical reasoning. The prticipants taught one research lesson each.

Table 1: Lessons taught during lesson study cycles 1, 2 and 3

Lesson study cycle	School	Class / ages	Topic	Student teacher pseudonyms
Cycle 1	St Peter's NS	4 th /9-10 yrs	Weight	Treasa
	St Paul's * NS	4 th/ 9-10 yrs	Weight	Finola
Cycle 2	St Peter's * NS	5 th/ 10-11 yrs	Fractions	Bríd
	St Paul's NS	3 rd/ 8-9 yrs	Fractions	Ethna
Cycle 3	St Peter's NS	3 rd/ 8-9 yrs	Division	Róisín
	St Paul's * NS	5 th/ 10-11 yrs	Fractions	Nóirín

*Researcher present

The elective course revolved around these research lessons and extended over three cycles of lesson study. Research lessons were taught at two different school sites (see table 1 for details). Because the student teachers came to the schools to teach the research lessons only, these were called 'dive-in' lessons. As such, they lacked some of the rich potential for learning about their pupils –contextual aspects –available to class teachers working on lesson study within their own schools. However, the act of teaching and observing the research lessons for different age groups of children in different school settings constituted a valid lesson study experience for the student participants. A mathematical theme was chosen by the group for the research lessons in each cycle and student teachers volunteered to teach particular lessons. On the research lesson days, the group divided into two with some members accompanying each 'teacher'. St Peter's NS is a large suburban school in an affluent neighbourhood. St Paul's NS is a large, urban, Band 1, DEIS school (Gov. of I., 2005).

THEORETICAL FRAMEWORK

The research project draws on theory at three levels; 'grand' theories, 'intermediate' theories and 'entry level' theories. The grand theory of constructivism as espoused by the primary mathematics curriculum underpins the research and this is viewed using a "situated cognition lens" (Boaler, 2002). In this paper, I describe two intermediate theories which were integral to the design and implementation of the research and most useful in the analysis of data.

The Lesson Study Process as a Practice

Lesson study is said to be premised on the Confucian saying that "seeing something once is better than hearing about it one hundred times" (Yoshida, 2005). Its ultimate purpose is to gain new ideas about teaching and learning based on a better understanding of children's thinking. Thus, the observation of actual research lessons is at the core of the lesson study process. Yet the lesson study cycle encompasses much more than studying children's responses while observing a lesson. It requires time dedicated to an intensive process in which teachers collaboratively investigate all aspects of the content to be taught and instructional materials available – and to the detailed post-lesson review session. While the practice of lesson study is embedded in Japanese school culture for over a century (Isoda, 2007), and its contribution to the success of mathematics teaching has come to be recognised in many diverse school systems (Asia-Pacific Economic Cooperation Education Network (APEC), 2008), discussion of lesson study has focussed more on the methodological aspects –the 'how to' of lesson study –than on the development of theory as to how it works and why (Lewis, Perry, and Murata, 2006). In one study where Japanese and US teachers worked together on lesson study, three essential lenses were found to characterise the participation of Japanese teachers. These were classified as the "researcher lens", the "curriculum developer lens" and the "student lens" (Fernandez, Cannon, and Chokshi, 2003). These perspectives, of researching what is actually happening during a lesson, of thinking about the mathematics to be taught in terms of where it fits in the curriculum -what went before and is to come after -and the focus on pupils' responses to the lesson were habitual in the practice of the Japanese teachers and notably absent in the US teachers. However, the three lenses, while useful, were not sufficiently defined as an analytic framework to suit my purpose in attempting to theorise teacher learning during a lesson study cycle. So I looked to a social theory of learning for my first intermediate theory.

First, I drew on the work of Lave and Wenger (1991) and of Wenger (1998) to conceive of the lesson study elective group, tutor included, as *community of practice* implementing lesson study to learn to teach primary mathematics well. The community element was foremost as we worked, my students and I, to define the enterprise on which we had embarked. Choosing the lesson study elective was an act of identity work on the part of the participants. The potential of lesson study as a means of bridging the gap between planning and pupil outcomes was discussed at the first meeting and the possibilities for developing mathematics teaching, inherent in observing pupils' responses were explored. As an example of lesson study in practice, we looked at the video "Can You Lift a 100kg?" (Mills College Lesson Study Group, 2000) and observed Japanese teachers preparing for and discussing the lesson. In response, one student, Treasa in her first journal entry proposed "presenting children with practical problems instead of immediately demonstrating how to do maths seems like the best place to start." This premise is an example of an entry level theory as to what constitutes 'good' mathematics teaching which came into play as we engaged in the practice of lesson study.

The Knowledge Quartet (KQ) as an Analytic Tool

The second intermediate theory which proved useful for preparing and reviewing research lessons was the Knowledge Quartet (KQ) (Rowland, Huckstep and Thwaites, 2005). The KQ is a practice-based framework for mathematics lesson observation and analysis, developed inductively from analysis of videotaped lessons taught by novice teachers. The four dimensions of the KQ are termed *foundation, transformation, connection* and *contingency.* Foundation includes teachers' knowledge, beliefs, and understanding of mathematics and mathematics pedagogy, acquired before and during teacher preparation; this dimension is seen as underpinning the other three. Transformation encompasses the ways in which the teacher's own knowledge is transformed to make it accessible to the learner, especially through the use of representations and examples. Connection pertains to knowledge displayed when teachers make connections between and among mathematical ideas; it includes issues of sequencing and judgements about conceptual complexity. Finally, contingency is manifested in the ways that a teacher responds to unanticipated events as they emerge during instruction. This could be described as 'thinking on your feet'. The eighteen contributory codes of the KQ suited my research purpose and they seemed to encompass the lesson study lenses already mentioned.Their use with the group was intended to build community by becoming part of the "*shared repertoire* of ways of doing things" (Wenger, 1998, pp. 82-84). By using the KQ the group would establish common ways of thinking and talking about teaching, essential to a shared enterprise.

DATA ANALYSIS

Three distinct aspects of the lesson study elective course emerged, and these were used to frame analyses. First, students participated in the course by *engaging* with the group in preparing, teaching and reflecting on lessons, i.e. by 'doing lesson study'. Secondly, participants also engaged with the elective course by 'doing mathematics' together regularly. This aspect of engagement with interesting mathematics was for the students themselves and independent of mathematics to be taught in lessons. Thirdly, students participated in the elective by 'being a lesson study elective student', where *belonging* meant pursuing activities related to the elective enterprise but not essential to lesson study, for example, watching DVDs about lesson study and writing reflective journals.

In lesson study cycle one, the lesson preparation and post-lesson reflection meetings were audio-recorded. In cycle two these sessions were video-recorded. In cycle three, I was not present at the preparation meeting, although two post-lesson reflection meetings were audio- recorded and all records were transcribed. Student participants also wrote a reflective journal for eight of the lesson study sessions, and these were also important data. I made personal observation notes for the research lessons at which I was present. For the purpose of coding, at first, I drew on concepts of *participation* and *identity* borrowed from Wenger (1998), but gradually the data analysis became more inductive, as various fresh indicators of mathematics teacher development were generated from the data. While analysis of video-tape has potential for learning by teachers about how children learn mathematics (Hall*,* 2000), and each group was glad to share in reflecting on the research lesson DVD of the other, I became convinced of the value of first hand observation of children by fellow teachers. There are at least two reasons for this: the scope of a single video lens is limited by the 'eye' of the person holding it, and the collation of multiple accounts of observers noticing different elements of the teaching learning/episode helps build a more complex, possibly more challenging and certainly more useful picture of the learning ecosystem, which is the mathematics classroom.

'Doing' Lesson Study

Researching people who are mutually engaged in a *shared enterprise* presents a rich tapestry of interactions and interpretations of how that learning occurs. Rogoff, Matusov and White contend, "learning involves transformation of participation in collaborative endeavour" (1996, p. 388) and I sought evidence of this in my analyses of the lesson study *community of practice*. An agreed goal of each lesson study cycle was to establish what mathematical ideas or concepts the student teacher wanted pupils to engage with as a result of the particular lesson being planned, and to study children's responses to the mathematical task(s) during the lesson with a view to assessing the kind of mathematical thinking in evidence in the class. This attempt to link focused mathematical teaching objectives with close observation of children responses became the *shared enterprise* of the lesson study *community of practice*. During the initial planning session and in later sessions students worked together *to align* content objectives from the curriculum with suitable contexts in which to base problems and activities designed to promote children's reasoning about the mathematical ideas underpinning each teaching objective chosen.

'Being' a Member of the Lesson Study Elective Community of Practice

The goal of this research was developmental while contributing to practice and building community, by *negotiating meaning* and thus doing *identity work*. Towards this end, I proposed to do three things: to engage the students in doing some mathematics in each session, to invite exploration of people's relationships with mathematics (Kaasila, Hannula, Laine and Pehkonen, 2006) and to engage personally with all group members as 'knowledgeable other' (Watanabe and Wang-Iverson, 2005) so that the protocols of lesson study and the KQ framework might enhance learning about teaching for the whole group. The focus on 'affect' in relation to learning mathematics was welcomed by the group as an opportunity to strengthen the *identity of participation* of the members (Wenger, 1998, p. 215). One student welcomed the opportunity to participate in the elective because:

> The idea of planning lessons with others appeals very much to me as well. At times, I find it a bit daunting and isolating planning on my own. I can feel quite uncertain whether what I'm doing is right or not, whether my ideas are valid and good enough. I think this elective will help me to be less anxious and more relaxed about planning in Teaching Practise (sic) and also for my future teaching (Journal 1_Nóirín).

Another person also specifically mentioned the group ambience as favourable also along lines of a shared identity:

> I enjoyed this session. The atmosphere within the group is good and I believe that we will work well together and our understanding of maths will be enhanced. It was good to know that others in the group have a 'fear' of maths and are somewhat uncomfortable teaching it (Journal 1_Bríd).

'Doing' Mathematics

Each of the lesson study sessions included some element of exploring mathematical ideas by the participants themselves. The mathematical tasks were intended as a gentle introduction to thinking about primary school mathematics in other than the traditional algorithmic terms. Students' reflections on their responses to the problems ranged from comments on emotions like "fear ... panicked and confused" (Journal 1_Ethna), through perceived personal deficiencies "I always doubt my ability to do it" (Journal 1_Bríd) to the more measured "very interesting" (Journal 1_Nóirín) and realisation of "how indoctrinated we are" (Journal 1_Treasa). Doing mathematics as a group became an essential element of the lesson study elective and while not explicitly part of the lesson study protocol could be subsumed under the "purposeful learning" of the goal-driven pre-lesson planning phase of each cycle (Fernandez and Chokshi, 2005, p. 73). The students accepted that to pose realistic problems and to focus on children's responses were aspects of good mathematics teaching practices that were challenging for them because of their own fragile relationships with mathematics. Therefore, it was agreed that if we as a *community of practice* were to direct our research gaze on how children respond to mathematical tasks then we had to direct our research gaze on our own doing of mathematics also.

Preparing the Lessons

Part of data analysis for lesson study cycle one focused on collaborative planning for the two research lessons on weight and the observing and discussing of these lessons (see table 1). When the group turned to planning the lesson, a tension emerged for some group members. Students were happy to devote three hours on Thursday mornings to their elective, on the expectation that individual responsibility for lesson planning should now belong to the group and Bríd expressed frustration with this:

> While I found a lot of what we did today very helpful, I feel we strayed too easily from one of the major objectives we had for the session; to plan a lesson … I understand that today was the first experience we had of this approach [group planning] but we need to make sure that we give enough time to planning the lessons we will be teaching. If not we need to change our approach from a group approach to an individual one. I personally like the group approach but we have to be more disciplined about it (Journal 2_Bríd).

Bríd's frustration may be indicative of an insecurity felt by other members of the group. "I think we should have given more time to this as we hadn't finalised what we were going to do by the end of the session" (Journal 2_Treasa). What some students perceived as straying from the objective - “at times we could wander from that and begin including less relevant things” (Journal 2_Nóirín) - was from my perspective, a process of exploration of the teaching resource materials available, discussion of the meanings of ‘mass’ and ‘weight’, leading to agreement to focus on the attribute of ‘weight’. Finola who opted to teach one of the lessons declared:

> After the workshop I had a lot of ideas about weight, how to start a lesson with a problem, to have a particular goal in mind and to give them a ‘problem’ to figure out, that is not an ‘activity’. The lesson [workshop] helped me to really think about types of problems and finding the right problem to suit your goal (Journal 2_Finola).

As can be expected of student teachers, there was very little experience among them of ten-year-old children’s current state of mathematics learning, or what they might be expected to know. As a consequence, the effort expended in deciding which specific learning outcomes were being planning for, extended the students considerably. There is some evidence of “collective orientation” (Gellert, 2008) in the brainstorming manner in which students sought to adopt a suitable starting point for the lessons. Every member of the community contributed to this endeavour, where the initial question was “when we begin a lesson, where should we focus?”

3	Voices:	On a problem ... Context.
4	Dolores:	Right…so we get a context, so ...
5	Ethna:	Like a problem.
6	Dolores:	A problem?
7	Bríd, Ethna:	(*Simultaneously*) Yeah. Yeah.
8	Treasa:	Why would you weigh yourself?
9	Dolores:	How would we weigh ourselves?
10	Nóirín:	What is weight?
11	Dolores:	So you’re going to talk about it?
12	Finola:	But, like, that’s too vague isn’t it?
13	Bríd:	How would we weigh…?
14	Treasa:	… a dog.
15	Finola:	You’d need like shopping or something ...
16	Ethna:	If you had five things…
17	Voices:	*(Indistinct)*.....biscuits…
18	Finola:	Something
19	Bríd:	How would we weigh one biscuit in a packet of biscuits? Is that what you mean?

21 Treasa: No, like five boxes of biscuits together weigh twelve kilogrammes, find the weight of each box (cycle1_preparation meeting 2).

Here the entire group was engaged in working out collectively a good starting point for a lesson on weight. The group engagement represents a genuine grappling with *meaning making*, which was by no means trivial in relation to how to teach primary mathematics well. By line 8, the context and a problem were connected in the proposal to weigh oneself. The question posed in line 10 may indicate discomfort with the problem-based context as it suggests a reversion to the more common practice among student teachers of introducing a mathematics topic by 'talking' about it. Finola's question (line 12) may either be in response to the earlier suggestion about weighing oneself (line 8) or to Nóirín's question (line 10). In either case, a "collective orientation" is in evidence since these students are interacting with each other in mutual engagement to devise a context for the lesson. Treasa finishes Bríd's sentence (lines 13 and 14) while Finola persists in seeking a viable realistic context in which to teach weight. In line 21, Treasa suggests a word problem, indicative of reliance on a traditional context for teaching the topic. During this planning session and in later sessions students worked together to *align* content objectives from the curriculum with suitable contexts in which to base problems and activities designed to promote children's reasoning about the mathematical ideas underpinning the chosen objective. Much of the work of the *community of practice* over the following weeks was to critically align their own experiences and beliefs about mathematics teaching with the skeletal list of teaching objectives in the primary curriculum and to flesh out the resultant lessons from available resources in order to maximise the children's learning. In the light of the first research lessons, it appeared that suggesting contexts and providing resource materials is not sufficient support for student teachers. *Alignment with the aims of the enterprise* requires development of a deeper understanding of the mathematics to be taught.

Learning from the Lessons Observed

The format agreed for the post-lesson discussion was that the two people who taught the lessons would give their own perspectives on the lessons first. Others were invited to bring to the discussion, four or five points they would like to raise in connection with pupils' thinking during the lesson they had observed. Treasa reported on her lesson in the absence of a video record and her report included the observation:

> I never did that kind of lesson where they had to do the problem first. That gave me satisfaction. Afterwards when I was doing the reflection, I realised that I hadn't really observed them in 'a maths way'. The others heard much more than I did (cycle 1_reflection meeting 1).

This comment raises issues along the Foundation dimension of the KQ about students' generic teaching skills and assumptions about the pedagogy associated with 'a maths way' of doing things. Later in the same session, Finola, speaking in general terms of her own lesson, said "I should have taken the focus off my lesson plan and put it on the children; I was kind of nervous where it would bring me". There was an acknowledgement here of a Contingency dimension to the lesson, to which Treasa responded, "But that's the way we were trained. All the focus is on us". This notion of a struggle to forge a new practice for teaching mathematics as different from what they perceive as the performative nature of their previous experiences of mathematics lessons recurs often in the data.

Just how challenging a learning trajectory these students had set for themselves was apparent in Treasa's account of her own lesson. It seemed that she was assessing herself by the broad parameters for Teaching Practice (TP) dictated by the college rather than on how the children interacted with the mathematics.

> I made sure to explain each activity to the children and to check for understanding using questions. When each explanation was finished I asked if there were any questions, of which there were only a couple. I think this helped the activities to go smoothly as the children knew exactly what they had to do next (Journal 3_Treasa).

Her colleague observer, Bríd declared that:

> Treasa had good presence in the classroom and maintained good discipline throughout. Her questioning skills were good and she prompted them very well in order to draw out more responses from them. She was well resourced and the use of a variety of food items weighing 1kg helped the children to engage better with the lesson and allowed them to see that 1kg can be made in many ways. Using their schoolbags and books also made the lesson more meaningful for the pupils and allowed them to relate better to the topic (Journal 3_Bríd).

Such a report could well have been written by a TP supervisor and would be a credit to both author and subject. But this kind of teaching was not the "good mathematics, well taught" we were aspiring to. Instead, there were saliencies with the phenomenon recorded by Schoenfeld (1988) where he observed 'good' teaching having a detrimental effect on students' learning of mathematics. Sharing of journals, discussion of and reflection on the enactment and outcomes of the first two research lessons in their different school contexts resulted in two considerably more differentiated, sophisticated, child-focused, problem-based research lessons in cycle two (Corcoran, 2009).

'Doing' Lesson Study: Cycle Three

Student participants conducted the planning for research lessons in cycle three in the absence of a 'knowledgeable other'. Using the KQ framework, there is strong evidence of learning along each of the dimensions in the data here, prompted by efforts to interpret the primary mathematics curriculum. Transformation issues were explored by Bríd, who reported:

> We also spent a lot of time debating whether or not to supply counters for the children to work with. Would they hinder or help them in their problem solving and would the distribution of them take time from the maths? Through our discussion we felt it best not to use them as they might distract the children from the actual problem (Journal 6_Bríd).

The focus had shifted to actual mathematical details of the lesson. Suitability of problem context, choice of example (3 as a divisor), whether or not to use counters, *et cetera*, had become more central in the planning. All six student teachers' journal entries corroborate this engagement with the details of planning a successful lesson on division.

Recognising the need to know more

In cycle three, the group also engaged in planning a lesson on percentages where the difficulties encountered by the group resulted in their abandoning the task until they could get 'expert' help. After outlining suggestions made to teach a lesson on percentages, which linked with fractions but was not aligned with what the community by now considered good practice, the embryonic lesson plan was shelved, in favour of a variation on Bríd's lesson, taught in the previous cycle. The group was learning its own limitations and how to counteract them. Nóirín balked at teaching the proposed lesson on percentages, because she didn't know how it would relate to the current understandings of fractions held by the class in St Paul's. The student teachers' connection of the two mathematical topics raised is indicative of the presence of the curriculum developer lens (Fernandez et al, 2003). These students were becoming aware of complexity in the mathematical connections teachers are required to make, in teaching the curriculum well.

Learning to be Sensitive to Context

By cycle three, the difference in school context between St Peter's and St Paul's had become a factor which students acknowledged in their planning and sought to research in the delivery and observation of lessons. Equity issues became part of their *shared repertoire* and a motivational factor in wanting to improve teaching. There is evidence of considerable sensitivity to the particular classroom context in this journal entry from Treasa, the same person who, two lesson study cycles earlier, had evaluated her own lesson in St Peter's as 'performance'.

> I think the children in St Paul's would have benefited from having the circles drawn on the worksheet. The lesson did give a good idea of where the children's weaknesses lie i.e. in terms of knowing what the fractions are called, how to divide equally, equivalence etc. However, I think it is important to note that when I prompted the children, some of them were able to come up with relevant answers so just going on their worksheets and what they did on the board is not a totally accurate representation of their actual knowledge. It was a question of activating their prior knowledge. I think if Nóirín had probed a bit more when going through the first problem she may have found that they could figure out how to do it (journal 7_ Treasa).

IDENTITY IN TERMS OF LEARNING TO TEACH MATHEMATICS

By identity, I mean the learning that occurs while individuals are mutually engaged in a *worthwhile enterprise* (Lave, 1996). By identity-work, I also mean the narratives people share while participating in a *community of practice*. Identities are formed through participation and identification with the goals of the enterprise and as such are socially formed. These students' "knowing and knowledgeability" (Roth and Lee, 2006) of and for good mathematics teaching was exhibited through their belonging to a *community of practice* dedicated to developing this. Participation in the enterprise of studying mathematics teaching by engaging in actual teaching, and then reflecting critically on it as a group of individuals who are all similarly engaged contributes to the identity of an individual engaged with seeking to teach mathematics well. It also contributes to the community of practitioners intent on building knowledge of and through the enterprise of lesson study.

The Case of Treasa

By putting the spotlight on the practice of the lesson study community as a whole, I wish to illustrate how one participant re-positioned herself, through identity work as a pre-service primary teacher learning to teach mathematics well. Treasa was a mature student who elected to take the lesson study course, without previous exposure to the research except by word of mouth from her friends. At the outset, Treasa was both confident and competent in the classroom but not - to use her own words -"in a maths way". However, over the course of the first semester and into the following one, Treasa worked hard to align her thinking about teaching mathematics with that of the *community of practice* and further, to align the *community of practice* with wider constellations of practices within the college and beyond.

The Work of Alignment

Alignment as a form of belonging is a powerful aspect of *communities of practice*. It requires expenditure of personal energy and power and Treasa worked to build the lesson study group through alignment. An indication of alignment work on Treasa's part surfaced in reflecting on Ethna's lesson in lesson study cycle two. The researcher lens was evident in her thinking from then onwards.

> If we are going to do a third and a fifth [class] it would be nice to do one in groups and one in pairs and we could compare them (cycle 2_preparation meeting1).

Her researcher lens developed further into an 'improvisational' approach to mathematics teaching and she queried whether the degree to which Ethna had scaffolded the children's thinking about sharing cookies during her research lesson cycle two had in fact constrained them. Treasa's espousal of a problem-based approach to teaching mathematics appeared to align her participation in the enterprise with all three perspectives identified as critical to maximising the potential of lesson study (Fernandez, *et al*, 2003). She reflected on her own lesson:

> I didn't really take notice of how children were communicating or what they were saying which would be an important part of assessment ... This was because I hadn't actually planned what I was going to look for while circulating (journal 3_Treasa).

Her fidelity to noting details of children's actions and responses was remarkable among the group and her openness in commenting on lessons and questioning others' ideas added value to the enterprise. But Treasa's work of alignment was not without tension. In reflecting on the preparation session in cycle three she noted:

> I felt that I acted as devil's advocate a lot of the time, asking "what if?" questions e.g. "What if the children say X?" or "What if X happens?" This may have bothered the group who were trying to get the lesson completed. I had to explain to them that I was trying to see it from all possible angles and I think they understood then. I hope this was helpful and the right thing to do, but it did often lead to more lengthy discussion or backtracking (journal 7_Treasa).

There is evidence of identity-work here as Treasa seeks to reconcile attention to the details of lesson planning she believes the lesson study enterprise requires – possibly adopting a leadership role within the community - with a need to be liked, and to blend in with her colleagues.

Treasa's Mathematics Learning Trajectory

Treasa was intent from the outset on alignment of the lesson study enterprise with broader perspectives on mathematics teaching. Her first reflective journal articulates:

> A thought that comes into my head is that exams are focused on getting correct answers, so this is something that would obviously need to be changed if we are to take the certainty out of Maths (journal 1_Treasa).

Subsequent journals dwell on her increasing engagement with mathematics at a personal level and with a desire to understand how children reason about mathematics. Her reflections always have a personal application with a growing focus on the mathematical ideas underlying the lessons and conjectures about how the lesson might be improved to better teach the topic. There is evidence of strong imagination and alignment work in Treasa's identification with the Empson (2002) article:

> I found it fascinating that Kolan found that she never needed to show a child how to solve a problem, that the answers came through discussing the children's different strategies for problem solving and through careful questioning. I also found it interesting how she suggests that you can get children to solve addition and subtraction problems on fractions without specific instruction, once you have done lots of equal-sharing problems and that the children understand equivalence. I wish I was teaching this lesson now. Can't believe I'm getting excited about this way of teaching Maths! (journal 2_Treasa).

As well as engaging in deep personal reflection while refining her researcher perspective Treasa also appears to have engaged in constructive feedback of others' lessons and used the KQ framework to do so:

> There was no clarification about how many people were actually at the party [1][...] I think I would have asked the class if anyone had gotten an answer without any pieces left over [...] Nóirín decided to go on to the second problem without tackling the difficulties brought up by the first problem. There didn't seem to be any contingency there. Nóirín didn't "show willingness to deviate from her own agenda" (Knowledge Quartet). It was her first lesson and I know it is difficult to think on your feet. You just want to get through the lesson as planned. At the time, I immediately thought she should not go on to the second problem, so I feel that I would have been able to handle it in that situation. I would have started over with the problem, clarifying with the children how many were actually at the party (journal 8_Treasa).

The identity-work of colleagues contributed inextricably to the negotiation of the meaning of good mathematics teaching by all members. This process was not entirely smooth, and accommodating to tension - as individual subjectivity and intersubjectivity emerged simultaneously (Roth and Lee, 2006) - was important, even essential, but mostly subterranean or covert work for the whole group.

MATHEMATICS TEACHING AS A SHARED ENTERPRISE

In relation to teaching mathematics, the lesson study community of practice became a vehicle for testing alternative ways of be(com)ing teachers and facilitated identity work by affording affirmation for effort in a spirit of *research as ongoing communal practice*. But recognising mathematics teaching as a shared enterprise has further potential than that gleaned from the dive-in lessons of this study. Among the types of knowledge development occurring in teachers who participated in lesson study in a Japanese elementary school Lewis (2009) enumerates situated, contextualised, pedagogical knowledge of the type invaluable to teachers - e.g., how teachers present a particular problem dramatically influences students' solution strategies; that several very quiet children spoke up when the lesson design required all students to physically participate

---

[1] A boy and his seven friends was construed (incorrectly) by children in St Paul's as seven people in total.

and record their findings; students' (non)-use of previously learned material. This "low stakes, high yield" assessment knowledge of particular students (Shulman, 2007) is highly contextualised to specific teaching situations and lends itself particularly to challenging settings. The design and implementation of a new curriculum, for example, Project Maths is such a setting, where the potential of realising mathematics teaching as a *shared enterprise* is enormous (NCCA, 2009). The work of 'our' lesson study enterprise included: learning to distinguish between 'routine' mathematics teaching and reform practices; learning to translate teaching as 'performance' into pedagogy for understanding. Not surprisingly, given the two sites for lesson study, it also included interrogating established mathematics teaching practices in search of more equitable outcomes (Dooley and Corcoran, 2007). Lesson study appears to be a potent recipe available to teachers to address the challenges associated with teaching mathematics for equity in 'designated disadvantaged' schools. This was particularly noticeable in relation to lessons in St Paul's NS where detailed attention to children's responses could greatly influence the flow of a lesson and impact either positively or negatively on learning outcomes for pupils. In the past year, one such school staff has begun to engage in the teaching of mathematics as *a shared enterprise*. If this practice continues it will be possible to examine the effects of sustained lesson study on pupils' mathematical outcomes and compare them with the outcomes of existing practice. Making mathematics teaching *a shared enterprise* allows teachers to augment their personal mathematical and pedagogical skills, by refining their goals and focusing on what and how students learn mathematics as a result of their combined practices. Enhanced outcomes do not result from individual effort but from *participation in practice*.

Mathematics Teaching and Matters of Interpretation

Evidence from the lessons taught by the student teachers in this study indicates that they often experience difficulty in interpreting what is meant by contested terms like 'problem-based teaching' or 'realistic mathematics.' Mathematical process skills, such as 'communicating and expressing mathematical ideas', are widely interpreted to mean the more generic notion of [teacher] 'talk and discussion'. Curricular guidelines on mathematics pedagogy, for example, the optimal use of materials or mathematical representations, are filtered in the light of past experiences. A *community of practice* by definition functions on an *economy of meaning*, which suggests that some meanings do achieve superior status (Wenger, 1998, p. 198). The role of a 'knowledgeable other' is crucial in this economy (Watanabe and Wang-Iverson, 2005). The lesson study *community of practice* became an important site where meanings of mathematical practices and mathematics teaching were negotiated through *engagement* and *alignment*. Alignment with a reform interpretation of the mathematics curriculum, with good teaching practices, with recent research findings was critical to the lesson study enterprise. *Accountability to the enterprise* begets *negotiation of meaning* in a highly reflexive manner and participation in the lesson study *community of practice* involved negotiating and renegotiating meanings for an increasing number of mathematical ideas and practices.

The shared enterprise in which these student teachers engaged was to focus on children's responses to the mathematics lessons the group had planned and one member had taught. As a result, they all grew in self-confidence, a self-confidence that recognises personal agency and thrives on communal support. If mathematics teaching were to become a shared enterprise such as the lesson study I have described, with as its focus on learners' mathematical thinking, then it could be expected that more teachers and ultimately more school staffs would become increasingly more adept at fastening "the link between whole-school planning, classroom practice and improved outcomes for pupils" (Gov. of I., 2006, p. 82). The outcomes would surely mean a transformation in practice and alignment with the theme of this conference – Mathematics for all: Extending mathematical capacity.

REFERENCES

(APEC) Asia-Pacific Economic Cooperation Education Network (2008). Available online at (http://www.criced.tsukuba.ac.jp/math/apec/apec2008/index_en.php

Ball, D. (1988) *Knowledge and Reasoning in Mathematical Pedagogy: Examining What Prospective Teachers Bring to Teacher Education*, PhD Thesis (Michigan State University, East Lansing).

Boaler, J. (2002) *Experiencing School Mathematics: Traditional and Reform Approaches to Teaching and their Impact on Student Learning* (London, Lawrence Erlbaum).

Conway, P. and Sloane, F. (2005) *International Trends in Post-Primary Mathematics Education* (Dublin, NCCA).

Corcoran, D. (2008). Developing mathematical knowledge for teaching: a three-tiered study of Irish pre-service primary teachers. PhD Thesis (University of Cambridge).

Corcoran, D. (2009) Developing Mathematical Knowledge in Teaching: A Study of Irish Pre-service Primary Teachers. Paper delivered at the Educational Studies Association of Ireland Conference, April, Kilkenny.

Delaney, S. (2008) *Adapting and using U.S. measures to study Irish teachers' mathematical knowledge for teaching*. PhD Thesis (University of Michigan State).

Dooley, T. and Corcoran, D. (2007) Mathematics: A Subject of Rights and Wrongs? In P. Downes and A. L. Gilligan (Eds.) *Beyond Educational Disadvantage* (Dublin: Institute of Public Administration) 216-228.

Empson, S. (2002) Using Sharing Situations to Help Children Learn Fractions, in D. Chambers (Ed) *Putting Research into Practice in the Elementary Grades: Reading from Journals of the NCTM* (Reston, VA, NCTM) 122-127.

Fernandez, C., Cannon, J. and Chokshi, S. (2003) A US—Japan lesson study collaboration reveals critical lenses for examining practice, *Teaching and Teacher Education, 19*, 171-185.

Fernandez, C. and Chokshi, S. (2005) Incidental versus Purposeful Learning, in P. Wang-Iverson and M. Yoshida (Eds) *Building Our Understanding of Lesson Study* (Philadelphia, PA: Research for Better Schools) 73-76.

Franke, M., Carpenter, T., Levi, L. and Fennema, E. (2001) Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal.* 38 (3) 653-89.

Gellert, U. (2008) Routines and collective orientations in mathematics teachers' professional development, *Educational Studies in Mathematics*, 67 (2), 93-110.

Government of Ireland (1999a) *Primary School Mathematics Curriculum* (Dublin, Stationery Office).

Government of Ireland (1999b) *Primary School Mathematics Teacher Guidelines* (Dublin, Stationery Office).

Government of Ireland (2005) *DEIS (Delivering Equality of Opportunity in Schools): An Action Plan for Educational Inclusion.* Available on-line at http://www.education.ie/servlet/blobservlet/deis_index.html

Government of Ireland (2006a) *An Evaluation of Planning in Thirty Primary Schools* (Dublin, Stationery Office).

Hall, R. (2000) Video recording as Theory, in A. Kelly and R. Lesh (Eds) *Handbook of Research Design in Mathematics and Science Education* (New Jersey: Lawrence Erlbaum) 647-664.

Hiebert, J., Gallimore, R. and Stigler J. (2002) A knowledge base for the teaching profession: What would it look like and how can we get one? Educational Researcher 31, (5) 3-15.

Isoda, M. (2007) Where did Lesson study Begin, and How Far Did it Come? In M. Isoda, M. Stephens, Y. Ohara and T. Miyakawa (Eds) *Japanese Lesson Study in Mathematics: Its Impact, Diversity and Potential for Educational Improvement* (Singapore, World Scientific) 8-15.

Kaasila, R., Hannula, M., Laine, A. and Pehkonen E. (2006) Faciliators for change of elementary teacher students' view of mathematics, in J. Novotna, H. Moraova, M. Kratka and N. Stehilikova (Eds) *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*, Prague, PME, 3, 385-392.

Kilpatrick, J. Swafford, J. & Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics.* National Research Council. Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences & Education, Washington, DC: National Academy Press.

Lave J. (1988) *Cognition in Practice: Mind, mathematics and culture in everyday life* (UK, Cambridge University Press).

Lave, J. (1996) Teaching, as Learning in Practice, *Mind, Culture, and Activity*, 3 (3), 149-164.

Lave, J. & Wenger, E. (1991). *Situated Learning: Legitimate peripheral participation.* (UK: Cambridge University Press).

Lerman S. (2000). The Social Turn in Mathematics Education Research, in J. Boaler (Ed.) *Multiple perspectives on mathematics teaching and learning* (Westport, CT, Ablex)

Lewis, C. (2009). What is the nature of knowledge development in lesson study? *Educational Action Research.* 17 (1) 95-110.

Lewis, C. Perry, R. and Murata, A. (2006) How Should Research Contribute to Instructional Improvement? The Case of Lesson Study, *Educational Researcher*, 35 (3), 3-14.

Lyons, M., Lynch, K., Close. S., Sheerin, E. and Boland. P. (2003) *Inside Classrooms: The Teaching and Learning of Mathematics in Social Context* (Dublin, Institute of Public Administration).

Mills College Lesson Study Group (2000) Can You Lift 100kg? DVD

Povey, H. (2002) Promoting Social Justice In and Through the Mathematics Curriculum: Exploring the Connections with Epistemologies of Mathematics, *Mathematics Education Research Journal*, 14 (3), 190-201.

NCCA (2008) Project Maths. Available on line at: http://www.projectmaths.ie/about_projectmaths

Rogoff, B., Matusov, E. and White, C. (1996) Models of teaching and Learning: Participation in a community of learners, in D. Olson and N. Torrance (Eds) *The Handbook of Education and Human Development* (Oxford, Blackwell) 388-414.

Roth, W–M. and Lee, Y-L. (2006) Contradictions in Theorizing and Implementing Communities in Education, *Educational Research Review 1*, 27-40. Available at: http://www.sciencedirect.com.

Rowland, T., Huckstep, P. and Thwaites, A. (2005) Elementary Teachers' Mathematics Subject Knowledge: The Knowledge Quartet and the Case of Naomi, *Journal of Mathematics Teacher Education*, 8 (3), 255-281.

Rowland T. (2007) Developing Knowledge for Mathematics Teaching: A Theoretical Loop, in S. Close, D. Corcoran and T. Dooley (Eds) *Proceedings of the Second National Conference on Research in Mathematics Education* (Dublin, St. Patrick's College) 13-26.

Schoenfeld, A. (1988) When good teaching leads to bad results: The disasters of a "well-taught" mathematics course, *Educational Psychologist*, 23, 145–166.

Shulman, L. (2007) Counting and Recounting: Assessment and the Quest for Accountability. *Change: The Magazine of Higher Learning*. 39 (1), 20 – 25.

Watanabe, T. and Wang-Iverson, P. (2005) The Role of Knowledgeable Others, in (Eds) P. Wang-Iverson and M. Yoshida, *Building Our Understanding of Lesson Study* (Philadelphia, PA, Research for Better Schools) 85-92.

Wenger, E. (1998) *Communities of Practice* (UK, Cambridge University Press).

Yoshida, M (2005). An overview of lesson study, in (Eds) P. Wang-Iverson and M. Yoshida, *Building Our Understanding of Lesson Study* (Philadelphia, PA, Research for Better Schools) 3-14.

MATHS FOR AL

Jan van Maanen

Freudenthal Institute, Utrecht University (NL)

"Al" reminds us of our great examples Al-Kwārizmī, Ibn al-Haytham and of our present day Muslim students, especially my former student Al Jupri ("Al" for his teacher), and of course of the word algebra.

Some 20 years ago, I taught an in-service history course for maths teachers. One of the students designed and tried a worksheet for his 15-year old pupils about solving the quadratic equation according to Al-Kwārizmī (9th century), and he inserted fragments of the original Arabic text. He reported enthusiastically about one of his Muslim pupils. The boy, who could read Arabic, was much more confident in the classroom now that he knew about the medieval Islamic contribution and its ongoing importance.

A second case concerns the history course that I taught to Al Jupri and his fellow students. I made it an 'empty reader' course. It started with a pile of white paper, the course reader, empty at the start, but (to be) completed with their own lectures by the end of the course. After some struggle the group liked the idea and, although they found it difficult to figure out their own data and views about an historical subject, in the end a refreshing series of lectures by the students emerged. I would never have made such an impact on their independent learning had I used a more conventional approach.

SHARING EXPERIENCE

What do we need if we want to make mathematical instruction available to as many human beings as is possible, in any case to those who long for it and to those for whom it would be beneficial?

It is challenging to work on a complete list of what is needed, but that is more the goal for an extensive and intensive international cooperation. My aim for this paper is much more modest, that is to present two cases in which the history of mathematics had a positive contribution towards the 'maths for all' ideal. The two cases, both of which discuss experiences with students in an academic history of mathematics class, are anecdotal. However, in the final section, I shall put them in a broader societal perspective and in relation to a not so often discussed aspect of history in mathematics teaching.

If the two cases bring inspiration, raise discussion and provoke new ideas that will be a step towards 'maths for all', a small step in the right direction.

AL-KWĀRIZMĪ IN A SECONDARY SCHOOL CLASS

As an undergraduate student of mathematics I was able to take history of mathematics and history of science courses, as well as a pre-service teacher training course. When I had completed my masters, I took a teaching job in a Dutch grammar school (*gymnasium*). But since there were only a limited number of hours available, the job was only part time. Somewhere in my second year as a teacher I felt well at home in my classes, and I discussed with my thesis supervisor, the well-known Utrecht historian of mathematics Henk Bos, the possibility of doing some historical research in my spare time. This worked out well, and eight years later I completed my PhD about the Dutch mathematical audience of René Descartes.

My historical research had a considerable effect on my teaching, which I continued to do on a part-time basis. Often enough I hit upon historical sources that were closely related to a subject I was also teaching in my school. At a certain moment I decided to combine the two, and to make a small project for one of my classes, based on a medieval text that I was studying. Instead of working through 'dry' textbook exercises about angle bisectors and perpendicular bisectors, the pupils now had to encounter and use these constructions to settle a 14th century quarrel about land property (Van Maanen, 1992). From that time on, I regularly presented to my classes a subject within its historical framework. When, in another class, we were working on quadratic equations, one of the girls, Janneke, asked whether she might use the 17th century method for solving the quadratic (which consisted of a rhetoric description of what steps to take in order to construct the solution). This led to an interesting classroom discussion, in which we checked whether the two methods were really equivalent. They were, and Janneke, who was uneasy at working with the coefficients in the solution formula of the general quadratic equation, used its rhetorical equivalent (Van Maanen, 1997).

This experience on the overlap of teaching mathematics in a secondary school and researching the history of mathematics led to the invitation of Utrecht Polytechnic (Faculty of Education) to teach a course in the history of mathematics to teachers who wanted to improve their qualification. The course had three main components:

1. to acquire knowledge about the history of mathematics, and to work actively with this knowledge
2. to learn to work with primary historical sources
3. to learn how to integrate the history of mathematics in the classroom (make a classroom resource, try it with a class and report the experience to fellow students).

In the year 1990/1 a student, Edwin, reported about his work in the classroom (component 3). He had decided to make use of a primary source that we had read and discussed in one of the sessions about the history of algebra (components 1 and 2). In the text al-Kwārizmī discusses the equation

> *Roots and Squares are equal to Numbers*; for instance, "one square, and ten roots of the same, amount to thirty-nine dirhems"; that is to say, what must be the square which, when increased by ten of its own roots, amounts to thirty-nine?

or, as we would write it nowadays:

$$x^2 + 10x = 39$$

Our course material contained the Arabic text of this problem (see Figure 1, taken from Rosen's 1831 edition with English translation, which is now also available on-line and in a reprint, Kwārizmī 1997) and a Dutch translation.

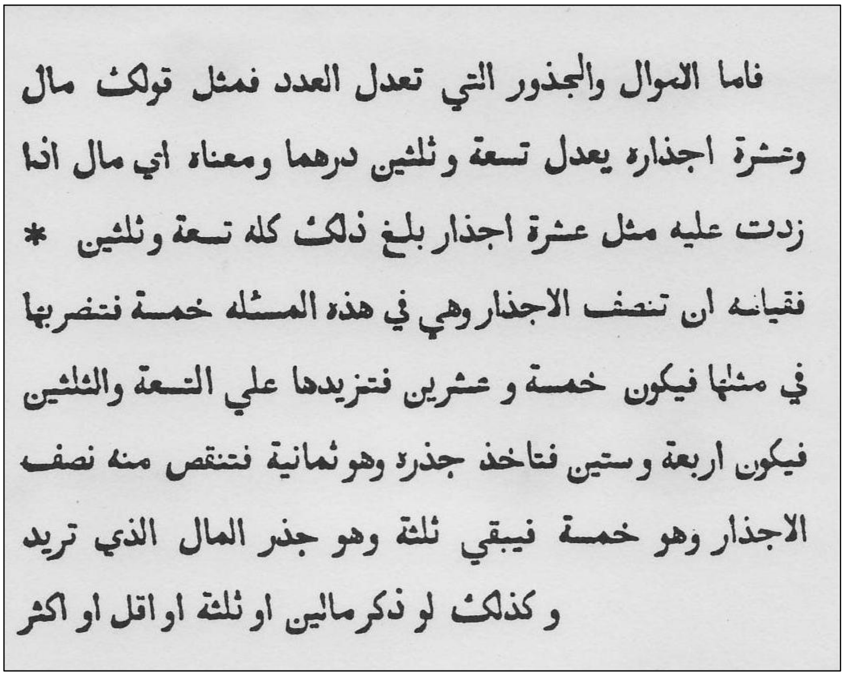

فاما الاموال والجذور التي تعدل العدد فمثل قولك مال
وعشرة اجذاره يعدل تسعة وثلثين درهما ومعناه اي مال اذا
زدت عليه مثل عشرة اجذاره بلغ ذلك كله تسعة وثلثين *
فقياسه ان تنصف الاجذار وهي في هذه المسئلة خمسة فتضربها
في مثلها فيكون خمسة وعشرين فتزيدها على التسعة والثلثين
فيكون اربعة وستين فتاخذ جذرها وهو ثمانية فتنقص منه نصف
الاجذار وهو خمسة فيبقي ثلثة وهو جذر المال الذي تريد
وكذلك لو ذكرمالين او ثلثة او اقل او اكثر

Figure 1: $x^2 + 10x = 39$ in the wording of al-Kwārizmī (edition 1831, p. 8)

Al-Kwārizmī solves the problem as follows (the solution in Arabic is also in Figure 1).

> The solution is this: you halve the number of the roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now take the root of this, which is eight, and subtract from it half the number of the roots, which is five; the remainder is three. This is the root of the square which you sought for; the square itself is nine.

Some pages later (pages 13-16) he gives two geometrical proofs of the procedure. In both

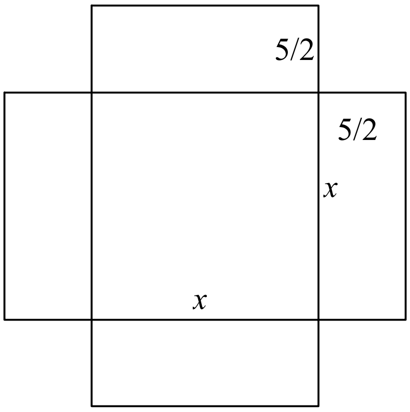


Figure 2: Al-Kwārizmī's first geometrical 'proof' of the solution,

cases he starts drawing the unknown x and also the square on it (x^2). The first proof equally divides the ten roots ($10x$) along the four sides, in attaching four rectangles of $\frac{5}{2}x$ to the square. The cross-figure therefore represents $x^2 + 10x$, which was known from the equation to be 39. The figure can be completed as a perfect square by adding four squares in the corners,

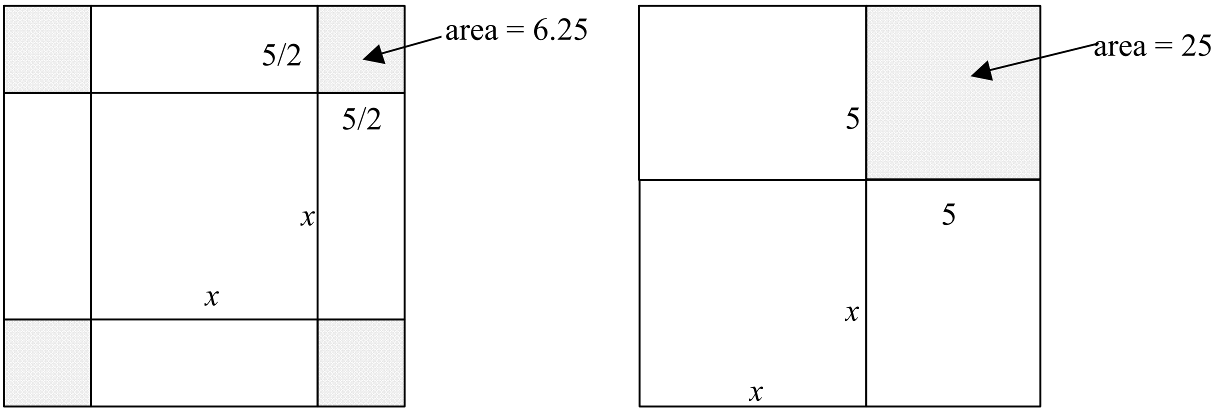


Figure 3: two diagrams by which al-Kwārizmī 'proves' his algorithm

each 2.5×2.5, that is by adding $4 \times 6.25 = 25$ to 39. The area of the completed square therefore is 64, and therefore its side is 8. If you subtract $2 \times 2.5 = 5$ from 8, you will find x, and therefore $x = 3$. In the second proof, two rectangles of area $5x$ are attached to the square, and the gnomon that results has area $x^2 + 10x$, as in the previous case. Now a square of size 5×5 completes the diagram to make a perfect square, which has again area 64. The conclusion, of course, is the same as in the first case.

With this knowledge of the historical source, Edwin designed a worksheet that he presented to a class in the lower vocational stream of his secondary school. He retained the Arabic text (Figure 1), discussed the equation and its solution, and told about the historical context around it, in which he also explained the origin of the words *algebra* and *algorithm* (the first being a Western version of the Arabic *al-jabr*, which was part of the title of Al-Kwārizmī's book about equations and which originally meant restoration or the setting of a bone, and *algorithm* stems directly from the Latin version Algorismus of the name Al-Kwārizmī).

In his report to his fellow students (24 October 1990) Edwin described the lesson, and he focussed especially on the reaction of one boy. The boy, with whom Edwin never had much contact, now responded enthusiastically. He could read the Arabic text, since he was learning Arabic in a Koranic school, and he was happy to know about the crucial role that Arabic speaking scientists had played in the transmission of ancient scientific knowledge to 'our' Western world and the creation of new mathematics such as algebra. The boy also was more confident in the discussion in the classroom now that he knew about the medieval Islamic contribution and its importance for the world as a whole. Edwin concluded with an evaluation of using history in the mathematics classroom. It supports the work in an intercultural classroom, because mathematics was constructed all over the world, just as the classes now have pupils from all over the world. In another part of his report, in which he discussed the approximations of π, he drew the conclusion that history is a good source of inspiration for doing interesting mathematics; in this case it presents interesting and relevant numerical problems.

After his Utrecht degree Edwin did a London MA in Mathematics Education. He is currently working as a teacher educator at the same Polytechnic in Utrecht where he did his degree. When I asked him about his recollection of the episode, he answered with a real letter, from which several passages are relevant here. Edwin, after 19 years, still preserved his notes and also my written feedback. He was pleased to receive my text for comment, and continued "My compliments. Our dossiers run nicely in parallel." Yet Edwin thinks that I gave more colour to his presentation than it had in his perception. He had spoken about the contribution of the medieval Muslim scientists, and this impressed all pupils in class. The indigenous Dutch pupils were used to the story about the 'dark ages' and had not expected that these major developments had taken place in the near East and around the Mediterranean. And the Moroccan pupils were especially excited indeed, more than the Turkish, since Arabic was their mother tongue. Moreover, says Edwin, the mere experience that a Dutch mathematics teacher knew something about the Arabic culture was very important for these children. They were, in a sense, hanging between their own community and Dutch society. Many of them had the feeling of belonging nowhere. Edwin's stories helped them to build their cultural identity, which strengthened their engagement. Edwin adds: it also improved my relationship with these pupils. Once in a while, he gave them copies of Arabic texts, to take home, but whether the boy who could completely read the fragment from Al-Kwārizmī himself really existed, says Edwin, “I do not remember whether there was such a pupil.” And he continues: “As I see these things now, that does not matter too much.” As an author I agree, but the proviso that has to be made is clear: at this point maybe I told a story rather than factual history.

AL JUPRI AND HIS CLASS TAKING A COURSE IN THE HISTORY OF MATHEMATICS

In the academic year 2007-2008, I taught a course in the history of mathematics through English to an international group of students. The course was offered by the Dutch national masters programme in mathematics, *MasterMath*, and students from Amsterdam and Utrecht attended. One of them was Al Jupri, an Indonesian student, who was based at the Freudenthal Institute, with six other students from Indonesia. Five of them were Muslim, and sometimes special arrangements in the course timetable were necessary, such as on Fridays to accommodate their attendance at the afternoon prayer. The Amsterdam group had two Greek students, one from Bulgaria, one from Nepal and one from Sudan, also a Muslim, who treated us marvellously on the occasion of the *Eid ul-Fitr* to celebrate the end of *Ramadan*. For practical reasons one Dutch student also took this course, although in another group it was also taught in Dutch.

A description of the course would make a long story, but one can also taste it by reading the presentation of the course in the *survey of courses*. This was the text that made the students decide to take the course. The “credit points” fit to the European Credit Transfer System, i.e. one point is awarded for 28 hours of student work when successfully completed.

Name of the course: History of Mathematics

Credits: 8 cp credit points

Instructor: van Maanen, Jan (Utrecht University, Freudenthal Institute)

E-mail: maanen@fi.uu.nl

Aim

Students learn to construct a survey of the historical development of mathematics. They learn to read, digest and evaluate primary and secondary sources, and to research the general historical, biographical and bibliographical backgrounds of an area of their special interest.

Description

The course is a combination of short lectures, in which the teacher presents a model of how to deal with a historical research question. Primary sources will be studied in order to answer the question, and the outcomes will be evaluated against existing secondary views on the subject.
Starting with minor questions and gradually moving on to broader topics students work towards their final, specialized topic. The research about the special topic results in an essay, which is also presented in a lecture.
Reviewing the work of fellow students belongs to the tasks to be completed,
as does a research task in the University Library.

Organization

Lectures intertwined with exercises and group work. The lectures in the final weeks are given by the students, and together build a survey of the history of mathematics.

Examination

Discussion of studied literature	(20%)
Exercises on reading primary sources	(20%)
Review task	(10%)
Essay about the special subject	(25%)
Lecture about the special subject	(25%)

Literature

V.J. Katz, A History of Mathematics. An Introduction. (2nd Edition, Reading MA: Addison-Wesley 1998)

Prerequisites

The course can be taken by students who are admitted to a master programme in (applied) mathematics or in mathematics education.

This is the so-called ‘empty reader’ course (so-called by me, and “reader” refers to the course-book, not to the teacher, nor to a person who is reading), and in the first session I deliberately emphasize that we start we a pile of, say, 80 empty pages of A4-paper, with empty red covers. “From this course-book you will learn the history of mathematics,” I then tell the students. As the description of the course indicates, I hold myself responsible for discussing with them how to deal which a historical research question, not for answering these questions for them (which would, moreover, be my own questions, not theirs).

It was not for the first time that I taught history from an empty reader. Some of the earlier groups strongly objected. As a teacher, why would you be satisfied hearing bad stories from students, instead of the beautiful and factually correct stories that you could tell yourself (and indeed that I taught in several courses before I introduced my first empty reader). In Al Jupri's group the opposition was more under the surface. I think that the students suppressed their uneasiness. Indeed, for Indonesian students, the majority in the class, it would be unthinkable not to do what the professor says. If the professor presents an empty reader, their academic culture imposes on them that they work with that empty reader, even if they prefer a written text. By the way, the course description does include Victor Katz's great *History of mathematics*, but clearly not intended as the textbook, rather as the central reference work for the group. Also, the fact that the reader is empty does not exclude that the teacher is full of ideas and initiative; and indeed, I was.

My first and main initiative was to provoke from the students questions about the history of mathematics, things that they did not know yet but that they would like to find out by their own research. I explained my criterion for sound historical knowledge: as much as possible it should be based on direct, primary sources. And if these are not available, or not accessible e.g. for linguistic reasons, then a modern edition or a translation could be used… but only after a precise search for the sources that can inform us about our question. If we learn our history in the textbook-only mode, we only copy the views of others, often without even knowing how often the stories in the book were already retold before they are told to us. This seems to contradict the praise of Katz's book in the preceding paragraph, but that is only seemingly, since Katz founded major parts of his text on source reading, and he is also precise and complete in his references, so that his reader can check the story in the book by consulting the sources.

A second initiative was to show to the students how I myself deal with such open questions and also with reading primary sources. I presented a number of sources, chosen from a long time span, asked them to find out what the meaning of the document was and whether or not we could learn anything from it for our global understanding of mathematics history. The students submitted their work in writing, and, in addition to their grade, they received written feedback. When the answers differed considerably, I made this the topic for a discussion in class, and we tried to build a common view on the topic. A typical example of such a task is displayed below.

Exercise (14 September 2007)

General description of the points to do when reading texts:

- If you have a picture of the original text at hand, try to trace the structure of the text (particularly the numbers) in the original.
- Read the text and try to understand its contents.
- Summarize the contents, using your own phrasing.
- If, in your opinion, the text is special in one way or another, then make a statement about this.

Apply this programme to the following document:

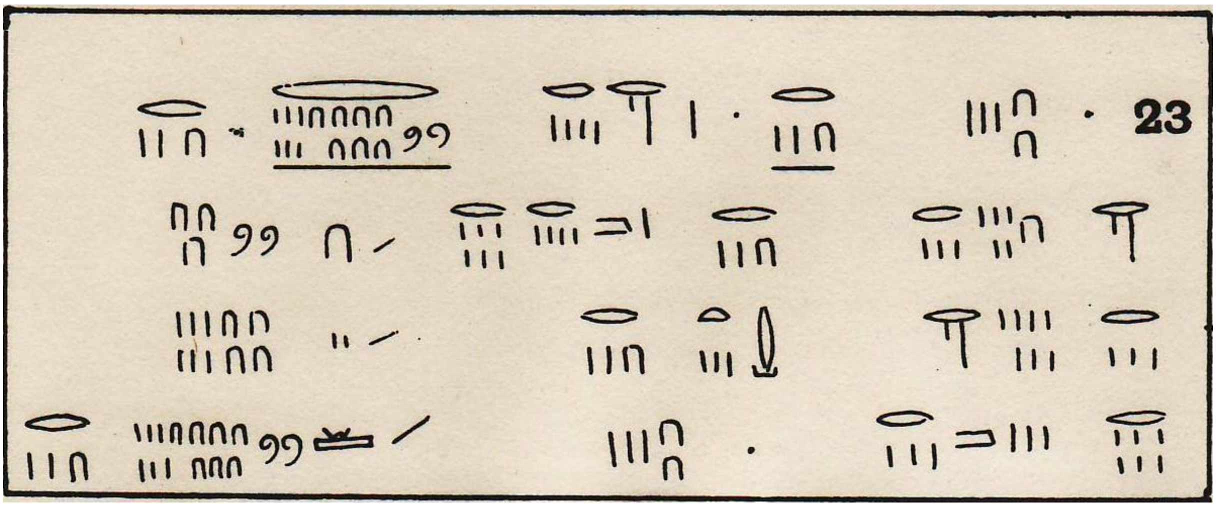


Figure 4: expressing 2:23 in unit fractions, from the Rhind-papyrus (Peet 1923, plate A)

One should realize that the students did not have "**expressing 2:23 in unit fractions**" in the caption. Their task was precisely to find that out, and to read why this particular calculation was important in ancient Egyptian arithmetic (the papyrus dates between 1788 and 1580 BCE). Other tasks asked them to compare developments in different cultures, using Katz's book as a reference, to study the meaning and development of a concept (quadrature and area, for example) or to systematically find research articles about a certain topic. For this purpose we spent a session on the use of electronic databases like MathSciNet and MathEduc.

Homework for this Friday [9 November 2007]:

- select three old books from the catalogue of Utrecht University library (old is: published before 1800).
- give their reference in the format that we discussed last Friday
- also give their Catalogue number (you will find this under the heading "Availability")
- send this information to me by mail, this Thursday at the latest.

Homework was devoted to finding out whether Utrecht University Library owned books that were of special importance for the individual students, and a session was located in the library reading room to work with these books. For most students this was the first time that they had a mathematical text in their hands that was more than say 25 years old.

During the semester they gradually built up a repertoire of knowledge and skills that equipped them to do a small research project of their own (the 'special topic' in the course description). Work on the special topic had been introduced in the course overview in the first session, and it formally started two weeks later, when it was presented in a homework task, as follows:

> There will be no lecture on 28 September [2007]. Use this time to start preparing the work on your 'special subject'. In the final three sessions of the course every student discusses a historical subject of his or her own choice. The result will be a short paper (c. 10 pages) and a talk. The paper describes the historical question, sketches the historical and mathematical background of the question. Then it presents at least one relevant primary source, it gives a survey of the existing literature and finally it answers the question.
>
> The result to be handed in on 5 October will be a one page description of your search. Propose two possible topics for this 'special subject' task. For each topic, describe what you have read about it, and state at least one question that you would like to answer.

In all, the course mimicked the scientific research process. It also involved students mutually reviewing each other's abstract and first draft, and writing a final version of their paper after I had given my comments on a first version.

The lecture topics that came out reflected the interests and backgrounds of the students. The student from Sudan, for example, was intrigued by the discussion about the oldest mathematical artefacts, especially the Ishango bones, which were discovered in Africa near Lake Edward by the Belgian geologist De Heinzelin, and which are now displayed in the Brussels Museum of Natural Sciences. Next to a precise understanding of the meaning of the notches on the bones, he wanted to know to what extent mathematics has its origin in Africa. More students had this wish to study the role of mathematics in their own culture, but this was not always the case, as can be concluded from the lecture titles (included in the schedule for the final three sessions):

> **Session 7 December 2007: The use and notation of numbers**
>
> Ancient mathematical instruments, esp. the Ishango bone (Sudan)
> Infinity in Euclid's *Elements* (Indonesia)
> Proclos' expansion on the *Elements* (Greece)
> Zero (Nepal)
> The transfer of the Hindu-Arabic numerals to Europe (Indonesia)

Session 14 December 2007: Geometry and applications

Ptolemy's theorem and trigonometry (Bulgaria)
Elementary geometry in the Middle East (Indonesia)
Mathematical astronomy (Ptolemy, Copernicus) (Indonesia)
The spiritual meaning of numbers (Netherlands)

21 December 2007: Algebra and friends

Al-Khwarizmi's work: Algebra (equations) (Indonesia)
The *Géométrie* of Descartes (Jan van Maanen, Netherlands)
The *spira mirabilis* or logarithmic spiral (Indonesia)
The theory of matrices (Indonesia)
Felix Klein, his role in mathematical education (Greece)

The account so far gives rise to a variety of consequential questions. What was the quality of the lectures and of the corresponding papers? To what extent do these lectures survey the history of mathematics, or in different words: am I satisfied with this table of contents? How did the students evaluate the course? These are legitimate and important questions, but I shall only deal with them in passing. The reason is that I want to conclude with a reflection on the global topic of the paper, indicated in the title "Maths for Al". The questions will thus turn into trying to determine what the benefits of this course were for Al Jupri, and what it can bring to all?

Happily and unsolicited Al Jupri wrote to me about the course. He had always submitted his homework by email, but a not purely administrative email exchange developed when he had submitted the first version of his lecture (the 21 December lecture about Al-Khwarizmi and algebra). I had sent my comments by email, and in my long series of remarks I had mistakenly corrected Al Jupri by stating that Al-Khwarizmi, whose name is more complete Muhammed *ibn Musa* (= son of Musa) Al-Khwarizmi, was a brother of the so called *Banu Musa* (sons of Musa). This led Al Jupri to write to me (10 January 2008):

> `Notes: In the last email, you said that:`
>
> `"Al-Khwarizmi is himslef called "Ibn Musa", so he was one of the sons of Musa. The sons were called "the Banu Musa", so they were not only colleagues but also brothers."`
>
> `I have tried to read several literatures, but I did not find that Al-Khwarizmi was one of the Banu Musa. According to`
>
> `http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Banu_Musa.html,`

the Banu Musa consist of three, they were Jafar Muhammad ibn Musa ibn Shakir, Ahmad ibn Musa ibn Shakir and al-Hasan ibn Musa ibn Shakir. Although Al-Khwarizmi is also called Ibn Musa, but he does not have the last name "Ibn Shakir".

Therefore, I still write in my paper that The Banu Musa are Al-Khwarizmi's colleagues.

Of course, Al Jupri was completely right: not all John-sons are brothers. And so I wrote to him (6 February 2008), first quoting the above passage:

You are right; I made a mistake!

This prompted an immediate and very informative, reaction with five points (7 February 2008), which gives insight in the impact of the course (and its teacher). With Al Jupri's consent I shall quote it in full and give some comments in between.

Dear Prof. Jan,

Thank you for this information, I am happy to receive it.

There are several "things" I want to say (write):

1. From this course I learned many things. I learned how to be a historian. The special thing is: to be a historian, it would not be sufficient if we only read "story" and memorize it without learning from primary sources. This is really "something new" for me.

Al Jupri's first point covers my central view on sound historical work: "as much as possible it should be based on direct, primary sources."

2. From your lectures, I learned how to give good and interesting lectures (frequently you give jokes, word "puzzles", and tell "mathematics aspects" carefully). But, I think I need many experiences...

3. I also admire your carefulness in using words, sentences, (I learned from your correction to my homeworks and paper). I try to be careful, but frequently I made mistakes.

Points 2 and 3 go beyond friendliness. The teacher is an example, and the student makes explicit whatever actions and behaviour of the teacher he recognized and welcomed.

4. Previously, when I was sending my "critical thinking" to your correction, I was rather doubt and afraid. But, now when you say it is all right, I am happy. I am very appreciate to your objectiveness. This is something that I have to follow (to be an objective teacher).

Here is a crucial point. Critical thinking (in this case about the Banu Musa) is permitted. Although Al Jupri does not state it, I think that this was also "something new" for him.

```
5. Once more, thank you for your permission since I wrote about you
and attached your picture in my web blog:

http://mathematicse.wordpress.com/2007/11/03/bila-guru-matematika-
keliru/
```

Find out and see. Especially for those who read Bahasa Indonesia.

```
Someday, If I write about you (maybe something related to the "our
wonderful" History Mathematics course), I will inform you previously.

I think I have written too much. So, I should stop now.

Thank you Prof. Jan,

Regards,

Al Jupri
```

Behold this uncommon type of symmetry: the student writing about the teacher, and the teacher writing about the student and with great respect for one another.

THE BROADER PERSPECTIVE

The first thing to observe now is that we have just encountered two interesting cases (I hope at least that you found them interesting), that these both deal with students in small domains (teacher students and M.Sc. students), and also that the results strongly depend on the personal capacities of the teacher (who happened to be a teacher with historical research experience). So, can anything in the above story be generalized into the perspective that 'maths for all' has in say Ireland or the Netherlands? The Dutch context, to which I shall limit myself here, is best described by the term 'black' school, in which 'black' is not so much a colour, it rather tells that the school is intercultural, that many pupils (often the majority) are non-Dutch speaking. Sometimes these schools are located in old, poorer quarters of town, and in that case the building may be old as well. These schools, primary and secondary, will not think too much about the history of arithmetic and mathematics. Their main and indeed major problems are in a different area: how to teach abstract concepts in a multi-language classroom? How to involve parents who do not always support the pedagogical climate of the school, and who regularly need their child as an interpreter? How to keep the children motivated if there is no work for them after school; and often no further education either, since sometimes parents do not support their children? These are some of the visible parts of the huge iceberg, and further reading in policy statements, further thinking and discussion will doubtlessly produce many more of these "Hows".

I am not even embarking on answering these questions. A lot of work has been done and is being done in this field. At the macroscopic and political level this is indispensable. Still, for their classes also a 'black' school needs a humane and objective teacher, a teacher with a broad view and with knowledge about the cultural background of her or his pupils. A teacher like Edwin or Al Jupri.

My thanks are to them, without them and without their many fellow students, I could not have developed these thoughts.

REFERENCES

Kwārizmī, Muhammed b. Mūsā al- (1997). *The algebra of Muhammed b. Mūsā al-Kwārizmī*. Frankfurt am Main: Institute for the History of Arabic-Islamic Science (reprinted from the 1831 edition and translation from the Arabic by Frederic Rosen; the 1831 edition is also available on-line, e.g. http://www.archive.org/details/algebraofmohamme00khuwuoft, retrieved 23 August 2009)

Maanen, J. van (1992). Teaching geometry to 11 year old "medieval lawyers". *Mathematical Gazette*, 76 No. 475, 37-45

Maanen, J. van (1997). New maths may profit from old methods. *for the learning of mathematics*, 17 (2), 39-46

Peet, T.E. (Ed., transl.). (1923). *The Rhind mathematical papyrus: British Museum 10057 and 10058*. Liverpool: Liverpool University Press

REALISTIC MATHEMATICS EDUCATION IN AN IRISH PRIMARY CLASSROOM

Patricia Cassidy

Gardiner Street Primary School, Dublin

In this research study, the implementation of a short experimental programme based on the principles of Realistic Mathematics Education (RME) was investigated. The research was carried out with a group of sixth class pupils in a school designated as disadvantaged. They were presented with a series of problems, based on real-life situations, on which they worked in small groups, followed by whole-class discussion of their solutions. The focus of the study was on the children's response to their experience of the RME approach and on its impact on teaching and learning in the classroom. The findings indicated that pupils of all levels of ability displayed a positive attitude towards the problem-solving activities. While high and average-achieving pupils became more independent over the duration of the programme, lower-achieving pupils continued to require a high level of teacher support. In this paper, the rationale for the study and how it related to the principles of RME is outlined. There is also a description of some excerpts from lessons which illustrate how some of the pupils engaged with the activities.

INTRODUCTION

The revised Irish primary school mathematics curriculum (Government of Ireland, 1999a) advocates a constructivist approach to mathematical learning, one which involves children as active participants in the learning process as they construct their own internal structures. An implication of this approach is that mathematics must be taught through problem-solving, usually undertaken by pupils in pairs or small groups. It is recommended that the problem-solving tasks in which children are engaged should be based on their own real-life experiences. This problem-based approach to the teaching of mathematics has been implemented in the Netherlands for over thirty years, where it is known as "Realistic Mathematics Education" (RME). The focus of this study was the implementation of an experimental programme based on the principles of RME in a primary school classroom.

BACKGROUND

Assessments of Irish primary school pupils' achievement in mathematics carried out over a number of years indicate that their problem-solving ability is a cause for concern. In the latest study, the 2004 National Assessment of Mathematics Achievement (NAMA 2004) (Surgenor, Shiel, Close & Millar, 2006), it was found that "pupils achieved the highest scores on items assessing basic mathematics skills … and the lowest on items assessing higher-order skills, including Applying and Problem-Solving" (p. 10). The authors note that a similar finding was made in the previous assessment (NAMA 1999) and that studies carried out in 1977, 1980 and 1985 also indicated that "pupils were strongest in dealing with operations with whole numbers and weakest in the area of problems" (p. 1). It seems that despite the introduction of the revised Primary School Curriculum in 1999, with its increased emphasis on problem-solving, pupils' achievement in this area is not improving. NAMA 2004 also found that the attainment levels of pupils attending schools designated as disadvantaged were significantly lower than those of pupils in non-designated schools, with 35% of pupils in disadvantaged schools achieving scores that would indicate they have only minimal mathematical skills.

Among the recommendations made by the authors of the report, two relate to the development of mathematics process skills. The first is that a stronger emphasis should be placed on the teaching of higher-order mathematics skills "by implementing in a systematic way the constructivist, discussion-based approaches outlined in the Guidelines accompanying the 1999 Primary School Curriculum: Mathematics" (p. 37). The second suggests that "the Department of Education and Science should support the implementation and evaluation of pilot projects linked to problem-based approaches to teaching mathematics such as Realistic Mathematics Education (RME)" (p. 37).

Two studies carried out by the Inspectorate of the Department of Education and Science (DES) also suggest the need for greater emphasis to be placed on the use of constructivist methodologies in mathematics teaching so that pupils are active in their own learning. In an evaluation of the implementation of the revised curriculum in mathematics conducted by the Inspectorate of the Department of Education and Science (DES, 2005a), it was found that, in a significant number of classrooms, there was an over-emphasis on didactic methodologies, teacher talk and the use of a single textbook. The recommendations made by the authors of the report on Literacy and Numeracy in Disadvantaged Schools (LANDS) (DES, 2005b) also focus mainly on teachers' classroom practice, including the provision of more problem-solving tasks based on pupils' own experiences and less reliance on textbooks in the teaching of mathematics.

In investigating how the mathematical development of pupils in my own class might be enhanced through engaging them in problem-solving tasks involving real-life situations, Realistic Mathematics Education emerged as a model on which an experimental programme could usefully be based.

THEORETICAL FRAMEWORK

Realistic Mathematics Education (RME) is an approach to teaching and learning mathematics first introduced in the Netherlands in the early 1970s. The history and philosophy of RME are outlined by Treffers and Beishuizen (1999) and Van den Heuvel-Panhuizen (2000). It was developed in response to demands that mathematics education should move away from the traditional 'mechanistic' approach which focused mainly on the teaching of procedures. RME emphasizes the use of problem situations from the real world, not only as examples of the application of mathematics, but also as the starting point or source for learning mathematics (Treffers, 1993). In RME, pupils are active learners in a process through which they develop mathematical tools and insights. They are also given opportunities to share their experiences with others. The RME approach, therefore, has much in common with the socio-constructivist view of mathematics education, described by Wood and Sellers (1997) and Simon (1995), who contend that the nature of the interaction that occurs among pupils in a classroom influences their mathematical learning.

Through their exploration of problem situations, children are encouraged to discern mathematical structures and procedures. This process is known as 'progressive mathematisation', and can occur at different levels of understanding. Mathematisation is seen as a constructive, interactive and reflective activity. The teacher's role in RME, as outlined by Van den Heuvel-Panhuizen (2000), is not to pass on a fixed body of knowledge, but to design and implement classroom activities that will stimulate pupils to engage in mathematical thinking and discussion. She states that the kind of learning environments provided for pupils must create opportunities for the construction of mathematical knowledge and the possibility of coming to higher levels of comprehension. Scenarios need to be developed that have the potential to elicit growth in pupils' understanding. Gravemeijer (1999) also refers to the importance of mapping out a route along which the students can find the intended mathematics for themselves. He emphasizes that RME theory is primarily a theory about knowledge construction: "the idea is not to motivate students with everyday-life contexts, but to look for contexts that are experientially real for the students and can be used as starting points for progressive mathematisation" (p.158). In the RME classroom, Elbers (2003) contends, pupils are engaged in mathematical discussion rather than in applying algorithms and textbook rules. They generally work in small groups, and this activity is alternated with class discussions about pupils' ideas.

Five fundamental learning and teaching principles that characterize RME have been identified by Treffers (1987, cited in Treffers and Beishuizen, 1999). They are:

> Learning as a constructive activity and the use of context problems.
> Learning as the development through higher levels of abstraction and the role of models.
> Learning through reflection, in particular on children's own constructions.
> Learning as a social activity through interactive teaching.
> Intertwining of the various learning strands within mathematics teaching.
> (pp. 34-36).

The principles that underlie RME are very similar to those on which the Primary School Curriculum: Mathematics (Government of Ireland, 1999a) is based. Throughout the curriculum documents there is a strong emphasis on children being actively involved in their own learning, on connecting mathematics to children's real-life experiences, on collaborative learning in small groups and on the importance of interaction among pupils and between pupil and teacher. High priority is given to the development of pupils' problem-solving abilities. It is suggested that children need to develop their own learning strategies, rules and procedures, although the Teacher Guidelines (Government of Ireland, 1999b) also state that "direct instruction is very important in mathematics" (p.4). Despite some differences in approach, particularly in relation to the teaching of place value, it would seem that many of the methodologies developed in RME could be used to support the implementation of the primary school mathematics curriculum.

METHODOLOGY

The aim of this study was to examine the implementation, in one primary school class, of an experimental learning programme based on the principles of RME. The methodology employed for the study was action research, an approach which uses the 'teacher as researcher' model (Bell, 1999) in which teachers research their own practice in their own classrooms. Cohen, Manion and Morrison (2000) describe action research as "a small intervention in the functioning of the real world and a close examination of the effects of such an intervention" (p. 227). Action research highlights the link between research and practice, and is, they contend, "a powerful tool for change and improvement at the local level" (p. 226).

The programme entailed firstly devising a series of instructional activities which would enable children to construct new mathematical ideas and concepts. Over a five-week period during February and March 2007, these activities were implemented through a combination of collaborative group work and interactive whole-class teaching. The impact of this approach on pupils' engagement in mathematical learning was monitored throughout the project and the activities were modified as required. At the end of the programme, the overall effectiveness of the RME approach in this particular classroom situation was evaluated. Data for the study were collected through the videotaping of group work sessions, examining samples of children's work and filling in classroom observation sheets which formed the basis of more detailed accounts of lessons recorded in a teacher's journal.

The participants

The group consisted of seventeen girls, aged eleven and twelve years, in sixth class in an inner-city primary school. The school is designated as disadvantaged and, in common with many schools in this category, pupils' levels of mathematics achievement (as measured by the Drumcondra Primary Mathematics Test administered at the end of the previous school year) would be regarded as generally very low. During the study, the class was divided into five groups of three or four pupils. Children of similar ability were grouped together so that tasks could be differentiated where appropriate. Pupils in two of the groups, A and B, were described as being of average ability, while in the remaining three groups, C, D and E, the pupils' levels of attainment were considered to be well below average. Pseudonyms are used to protect the anonymity of the pupils.

The programme

The programme consisted of a series of five activities, each of which was based on a situation that was realistic for the pupils. Each activity presented pupils with a number of problems which aimed to encourage them to move on to higher levels of mathematical thinking and towards the development of new concepts. The activities were based on the length and area strand units of the mathematics curriculum for sixth class. These topics were chosen because I had observed that many pupils had a poor understanding of the basic units of length and that no pupil seemed to have developed the concept of area. The programme was covered during eighteen lessons, each one lasting approximately fifty minutes.

Aims of the study

The research questions which the study set out to investigate were:

- To what extent does the implementation of this programme reflect the characteristic principles of RME?

- How does the RME approach used in my class affect pupils' mathematical development in the content areas addressed in the programme?
- What are the implications of this approach for my own mathematics teaching and for the implementation of the primary school mathematics curriculum in my school?

"THE NEW PLAYGROUND"

In this paper, I will focus on how one group (Group B) engaged with one of the activities during two consecutive lessons. This was Activity 2 – "The New Playground" – on which the class worked for four lessons during the second week of the programme. In this activity, pupils were given the task of designing a playground for a town that had never had one. They were asked to draw a plan of the playground and to write a letter to the children of the town explaining the plan. Two questions within the activity introduced the concept of area. The first involved the placing of a 60 metre perimeter fence so as to maximize the play space, while the second required them to calculate the number of rubber tiles required to cover the surface of the playground.

Lesson 1

Each group was given a worksheet outlining the activity and, after a short class discussion to ensure that everyone understood the main details, it was agreed that the first task was to decide how the fence should be put up so as to make the largest possible play area. The groups were reminded that they should discuss the problem first and then show their ideas on the centimetre-squared paper supplied. As they worked, the groups were recorded on video and asked to explain their thinking. Initially, no group considered the actual problem posed in the task – that of looking at the space enclosed by the fence and how they could make this as large as possible. They all focused on drawing the required perimeter and were anxious then to move on to designing the playground. As each group completed their first drawing, I asked them if they could have drawn it differently. When they had produced a second drawing, they were encouraged to look at the space enclosed in each and to consider whether one was bigger than the other.

Group B had drawn two plans for the playground – one as a 15m square and the other as a 16m by 14m rectangle. When asked about the space inside each figure, they initially took a visual approach. Some confusion about the difference between the length of the perimeter and the amount of space enclosed by it was evident in this group. Eventually one group member, Beth, recalled the formula for finding the area of a rectangle which she had learned in a previous class, but demonstrated that she did not yet have a conceptual understanding of this measurement, as the following transcript illustrates.

Beth:	The rectangle is bigger.
Becky:	No, they're both the same.
Teacher:	Can you explain why?
Becky:	*(Pointing to the rectangle)* This looks longer, but if you take this row here off and move it over here *(indicates the side of the rectangle)* you'd have a square.

Teacher: Can you think of a way to work out how much space is inside each one?

Beth: If you add the four sides of this *(square)* you get 60, and the four sides of this *(rectangle)* you get 60, so the space is the same.

Teacher: Does that tell you how much space is in here? *(indicating the space inside the square)*

Beth: Oh, I know! You have to multiply the length by the width.

Teacher: Can you explain why you would do that? *(Beth looks confused)*

Barbara: There's 15 boxes in each row and there's 15 rows, so you multiply 15 by 15.

Teacher: So, to find out how much space is inside each shape, what do you need to do?

Barbara: Multiply.

Teacher: And when you multiply, you find out what?

Barbara: How many boxes there are.

Teacher: So, what you've decided is, you're going to find out how many boxes are in each shape, and then you can see whether one is bigger than the other or if they're both the same. Is that right?

The pupils in this group were very reluctant to explain their thinking. I tried to model the kind of questions which I hoped they would eventually use with each other. Now that they had decided on a procedure to follow, they quickly discovered that the area of the square was larger than that of the rectangle. There was time for only a brief class discussion at the end of this lesson. Three different representations of the playground that had emerged from the group work were drawn on the blackboard. Pupils were encouraged to focus again on the question they had been trying to answer during the activity, namely, 'which playground plan would have the most space?'. I acknowledged that some groups had come up with ideas for working this out, but suggested that we would wait until the next lesson to talk about these.

Lesson 2

Before moving on to the next part of the 'Playground' activity, I felt that pupils needed more time to explore the concept of area. In this lesson, therefore, the focus was on determining the area of rectangles and discovering that rectangles with the same perimeter do not necessarily have the same area. In a brief class discussion, pupils talked about the playground plans they had drawn the previous day. Today's task was to see if they could find any other ways of drawing the plan and then to work out how much space was inside each one so that they could choose the largest. Group B took a systematic approach to drawing new rectangles, and were therefore able to make an interesting discovery. I asked them about the first two shapes (a 15cm square and a 16cm x 14cm rectangle) and encouraged them to use the word 'area' to describe the space they had measured.

Teacher: Tell me about the shapes you are drawing.

Beth: See, we started with a square and then we took one box off here *(the width)* and put it down here *(the length)*.

Becky:	So we kept doing that and they keep getting skinnier.
Teacher:	What did you find out about the square and this rectangle?
Becky:	The square is one square bigger than the rectangle. The square is 225 and the rectangle is 224.
Teacher:	What will you try next?
Beth:	17 by 13 *(pointing to the next rectangle).*
Teacher:	What do you think the area will be?
Becky:	It'll be one square less, it'll be 223. *(Uses a calculator to work this out.)* Oh, it's 221, it's 3 less.
Teacher:	Could you try another one. What do you think you'll find?
Becky:	We could try 18 by 12. I think it'll be about 4 less. *(Uses calculator.)* 216. It's 5 less. Oh, now I get it! It's a pattern, look! First it goes down by 1, then by 3, then by 5. All the odd numbers.
Teacher:	Very good. You've discovered something very interesting. Now, could you see how many more rectangles you can make and draw them on the pages? See how the pattern continues and be ready to tell the class about it later.
Becky:	I know the next one! I bet it'll be…eh, it'll go down 7, so it'll be eh, 209. *(Uses calculator).* See, I was right!

Led by Becky, Group B set about this task enthusiastically. They could now be seen to move from the level of 'horizontal mathematisation' where they had been involved in applying mathematics to solve a problem within a certain context, to 'vertical mathematisation' as they discovered patterns and worked on more formal mathematics (Treffers and Beishuizen, 1999). They continued to draw all the rectangles it was possible to make by decreasing the width and increasing the length by a unit each time, and they became increasingly excited as they found that the pattern continued as predicted. During the class discussion which followed, they explained their findings to the rest of the class. Pupils were encouraged to use the terms 'area' and 'perimeter' and by now most of them seemed to have made progress in understanding the difference between the two. At the end of this lesson, I asked the class if they thought the pattern Group B had discovered would work with any perimeter. Pupils were asked to think about this question and we returned to it briefly at the beginning of the next day's lesson.

DISCUSSION

The literature relating to RME suggests that when children work collaboratively in small groups to solve realistic problems, they can begin to discern mathematical structures and procedures. Their initial informal strategies are gradually developed towards higher levels of understanding, giving them access to more formal mathematics. Learning is seen as a social activity and interactive whole-class discussion encourages reflection and the exchange of ideas. The teacher steers the learning process by providing a "guided opportunity to re-invent mathematics" (Van den Heuvel-Panhuizen, 2000, p. 9).

This study examined the implementation of an experimental programme, based on the principles of RME, with sixth class pupils who had not previously experienced this problem-based approach to mathematical learning. The aim was to examine how pupils responded to this approach and to look at its implications for classroom teaching. The scale of the study was too small to allow any definitive conclusions to be drawn from it about the effectiveness of the RME approach in the classroom. However, a number of positive aspects of the implementation of the programme emerged from the data analysis, suggesting that this is potentially an effective way of enhancing pupils' mathematical learning. These were:

- The use of familiar real-life situations resulted in pupils being task-orientated and becoming very involved in trying to solve problems.
- Pupils began to engage, to a small extent, in mathematical discussion while working in small groups.
- Most pupils made some progress in understanding new mathematical concepts.
- Group work provided opportunities for the teacher to engage in discussion with the pupils, to elicit explanations and to encourage them to clarify their thinking.
- Most pupils participated well in class discussion and were able to explain and justify their solutions.
- The activities allowed for differentiation to suit the varying abilities of the groups, but all pupils had opportunities to contribute to class discussion.
- Pupils displayed enthusiasm for the problem-based learning activities and maintained a positive attitude throughout the programme.
- The activities facilitated formative assessment as they revealed pupils' strengths and weaknesses in mathematics.

The implementation of the programme, however, was not without difficulties. Significant gaps in the basic mathematical knowledge of three groups of low-achieving pupils were obvious and these groups required a high level of teacher support in order to complete the tasks. It was difficult for one teacher to devote sufficient time to each group and pupils sometimes had to await attention when they were unable to proceed independently. As the programme progressed, these difficulties were minimised by adapting the tasks to suit the pupils' ability level. However, in general it was found that the lowest-achieving pupils required guidance from the teacher in order to make any progress in mathematical understanding. One possible solution would be for a learning support or resource teacher to work in the classroom with the class teacher to support pupils as they engage in problem-solving activities in groups. The adaptation of the RME approach for low-achieving pupils is a matter for debate in the Netherlands (Baxter *et al*, 2001; Kroesbergen *et al*, 2004). This study indicated that participation in RME-based learning activities was beneficial in a number of ways for these pupils. The tasks facilitated differentiation so that all pupils could experience success and feel they were part of the classroom community. The value of giving less able pupils opportunities to work at an informal level was highlighted by Dekker (2007). She describes formal mathematics as the 'tip of the iceberg', while beneath the surface is the large 'floating capacity' of informal and pre-formal understanding which enables pupils to reach that formal understanding. Unfortunately, she contends, the response of many schools to pupils who are experiencing difficulties with mathematics is to provide them with more practice at the same formal level they did not understand to start with (pp. 10 -11).

The effective implementation of RME requires that pupils are able to work together in groups. In the initial stages of the programme, a number of difficulties emerged in relation to this.

Certain pupils tended to dominate some groups while others remained uninvolved. Pupils tended to argue rather than listen to each other and discuss their various opinions. The continued use of this methodology over a number of weeks facilitated the development of skills which led to some improvements being observed during the later activities. This points to the need for pupils, from an early age, to have opportunities to develop discussion skills as outlined in the Teacher Guidelines that accompany the Primary School Curriculum: Mathematics (Government of Ireland, 1999b, p. 30).

A third area of concern arising from the study is the length of time required to implement RME-based activities. During this programme, a considerable proportion of classroom time was devoted to one strand unit of the mathematics curriculum. It must be acknowledged that much of this time was needed because pupils were unused to the methodologies employed and if this approach were used in the future, when they had become more accustomed to this way of working, the time would be utilised more efficiently. It should be noted that while the focus was on one particular topic, pupils were also developing some of the knowledge and skills they had previously acquired in other areas. Nevertheless, it seems clear from the study that implementing a problem-based learning approach to mathematics is time-consuming. Pupils need to be given sufficient time to develop and experiment with strategies, to discuss these and to reach higher levels of understanding. The implications of this for the implementation of the overall mathematics curriculum in this way are beyond the scope of this study but they merit further investigation.

This study indicated that a problem-based approach to mathematical learning can be implemented successfully in a primary school classroom. The literature on RME helped to illustrate the kind of activities and teaching methodologies that can contribute to this. The constructivist approach to mathematics which is central to the revised curriculum sees all children as active participants in the learning process. RME presents an example of how this can become a reality in the classroom.

REFERENCES

Baxter, J.A., Woodward, J. and Olson, D. (2001) Effects of Reform-Based Mathematics Instruction on Low Achievers in Five Third-Grade Classrooms, *The Elementary School Journal,* Vol. 101, No. 5, pp. 529-547.

Bell, J. (1999) *Doing Your Research Project: A guide for first-time researchers in education and social science.* Maidenhead: Open University Press.

Cohen, L., Manion, L. and Morrison, K. (2000) *Research Methods in Education*. London: Routledge Falmer.

Dekker, T. (2007) The Dutch Experience, Threat or Treat? In Close, S., Corcoran, D., and Dooley, T. (Eds), *Proceedings of the Second National Conference on Research in Mathematics Education* (pp. 6 – 12). Dublin: St. Patrick's College.

Department of Education and Science (2005a) *An Evaluation of Curriculum Implementation in Primary Schools*. Dublin: The Stationery Office.

Department of Education and Science (2005b) *Literacy and Numeracy in Disadvantaged Schools: Challenges for Teachers and Learners*. Dublin: The Stationery Office.

Elbers, E. (2003) Classroom Interaction as Reflection: Learning and Teaching Mathematics in a Community of Enquiry, *Educational Studies in Mathematics*, Vol. 54, No. 1, pp. 77-99.

Government of Ireland (1999a) *Primary School Curriculum: Mathematics*. Dublin: The Stationery Office.

Government of Ireland (1999b) *Primary School Curriculum: Mathematics. Teacher Guidelines.* Dublin: The Stationery Office.

Gravemeijer, K. (1999) How emergent models may foster the constitution of formal mathematics, *Mathematical Thinking and Learning*, Vol. 1, No. 2, pp. 155-177.

Kroesbergen, E.H., Van Luit, J.E.H., and Maas, C.J.M. (2004) Effectiveness of Explicit and Constructivist Mathematics Instruction for Low-Achieving Students in the Netherlands, *The Elementary School Journal,* Vol. 104, No. 3, pp. 233-251.

Shiel, G. and Kelly, D. (2001) *The 1999 National Assessment of Mathematics Achievement*. Dublin: Educational Research Centre.

Simon, M.A. (1995) Reconstructing Mathematics Pedagogy from a Constructivist Perspective, *Journal for Research in Mathematics Education*, Vol. 26, No. 2, pp. 114-145.

Surgenor, P., Shiel, G., Close, S., and Millar, D. (2006) *Counting on Success: Mathematics Achievement in Irish Primary Schools.* Dublin: The Stationery Office.

Treffers, A. (1993) Wiskobas and Freudenthal Realistic Mathematics Education, *Educational Studies in Mathematics*, Vol. 25, No. 1/2, pp. 89-108.

Treffers, A. and Beishuizen, M. (1999) Realistic Mathematics Education in the Netherlands, in Thompson, I. (ed.) *Issues in Teaching Numeracy in Primary Schools*. Buckingham: Open University Press, pp. 27-38.

Van den Heuvel-Panhuizen, M. (2000) Mathematics Education in the Netherlands: A guided tour, *Freudenthal Institute Cd-rom for ICME9*. Utrecht: Utrecht University.

Wood, T. (1993) Creating an Environment for Learning Mathematics: Social Interaction Perspective, *Journal for Research in Mathematics Education. Monograph*, Vol. 6, Rethinking Elementary School Mathematics: Insights and Issues, pp. 15-20.

Wood, T. and Sellers, P. (1997) Deepening the Analysis: Longitudinal Assessment of a Problem-Centered Mathematics Program, *Journal for Research in Mathematics Education*, Vol. 28, No. 2, pp. 163-186.

CHARACTERISTICS AND CURRICULUM VALIDITY OF TEST ITEMS FOR THE NATIONAL ASSESSMENT OF MATHEMATICS ACHIEVEMENT IN 6TH CLASS

Seán Close, David Millar, and Gerry Shiel, Educational Research Centre, St Patrick's College, Dublin 9.

The 2009 National Assessment of Mathematics Achievement (NAMA 2009) involved administering a test of mathematics to a representative national sample of pupils in 6th Class. In preparation for the assessment, a pilot test was developed and administered in 2008. Test items, designed to be compatible with the content and process skills outlined in the Primary School Mathematics Curriculum (PSMC), were devised and discussed by a mathematics expert group. These items were clustered into booklets and administered to pupils in 31 schools. The items were then analysed to ascertain their suitability for inclusion in the main test, drawing on (i) difficulty level; (ii) ability to discriminate between high achievers and low achievers; and (iii) the functioning of alternate responses in the multi-choice items. This paper discusses the selection of items for inclusion in the final tests for 6th Class, and examines in detail the characteristics of the items that were either dropped from the 6th Class item pool or substantially revised, mainly because pupils did very poorly on them. Yet, many of these items were judged to represent important aspects of the 6th Class PSMC. Findings are discussed with reference to the philosophy and content of the PSMC for 6th Class and its treatment in current text books.

INTRODUCTION AND BACKGROUND

Part of the NAMA 2009 survey involved the assessment of pupils' mathematics achievement in Sixth class. In preparation for the main survey in May 2009, a pilot study was carried out in May 2008 involving the administration of test booklets containing a total of 175 trial items to approximately 1000 pupils in 32 schools. The items from the 6th Class pilot study were then analysed to examine their psychometric characteristics, with a view to eliminating poorly performing items, revising some items and then selecting the items to be included in the main survey test, while maintaining a balance across content strands and process skills in line with the test framework and item specifications (ERC, 2008). In engaging in this process, issues arose with regard to the curricular validity of items being deleted or substantially revised. This paper analyses and discusses a selection of the problematic items in the light of the philosophy, aims, and teaching objectives of the Primary School Mathematics Curriculum (DES/NCCA, 1999a), and the treatment of the teaching objectives involved in the three most popular mathematics textbooks in use in schools.

THE NAMA PILOT STUDY FOR 6TH CLASS

The NAMA pilot study for 6th Class consisted of four main stages:-

- Design and construction of the pilot study test
- Administration of the test to pupils in the schools in May, 2008
- Analysis of results and identification of problematic items
- Revision of the test for administration in the main study in May 2009

Pilot Study Test Design and Construction

175 items were written by members of the Mathematics Expert Group[2] for try-out in the pilot study, based on analysis of the 6th Class content and objectives in the PSMC handbook, the most commonly used 6th Class mathematics textbooks, and the Drumcondra Primary Mathematics Test (DPMT) for 6th Class. The distribution of the 175 pilot study items across the content strands (Table 1) and process skills (Table 2) of the curriculum was designed to reflect the distribution recommended in the draft NAMA framework (ERC, 2008) and also to reflect the distribution of objectives in the Primary School Mathematics Curriculum (DES/NCCA 1999a).

Table 1: *Classification of items by Content Strand – 6th Class, NAMA Pilot Study*

Content Strand	Number and Algebra	Shape & Space	Measure	Data	Total
Number of Items (Pilot Study)	75	40	40	20	175
Percent of items (Pilot Study)	43	23	23	11	100
Percent of Objectives in PSMC – 6th Class	43	21	24	12	100

The distribution of the pilot study items across content strands in Table 1 is very close to the distribution of objectives across content strands in the PSMC for 6th Class and as recommended in the NAMA framework document (ERC, 2008).

Table 2: *Classification of Items by Process Skill – 6th Class, NAMA Pilot Study*

Process Skill	Understand & Recall	Implement	Integrate & Connect	Reason	Apply & Problem Solve	Total
Number of Items (Pilot Study)	18	31	9	55	62	175
Percent of items (Pilot Study)	10.3	17.1	5.1	31.4	34.9	100
Percent of Items - NAMA Framework*	10.7	14.7	10.7	38.7	25.3	100

* *PSMC objectives were not categorised by process skill. This categorisation was carried out by a curriculum specialist*

When categorised by PSMC process skill the distribution of pilot study items across process skills is quite close to the distribution indicated in the NAMA framework for 2009 (ERC, 2008), though the category - Integrate and Connect - is somewhat under-represented, and problem-solving is somewhat over-represented (Table 2).

[2] The Centre is assisted in its work on NAMA 2009 by a Mathematics Expert Group. This group consists of members of the Inspectorate, teachers, lecturers in mathematics education, and ERC staff members.

Sixty-four (37%) of the items were multiple choice (MC) and 111 (63%) were constructed response (CR), which is near enough to the NAMA framework target ratio of two constructed response items to one multiple choice item.

About half of the pilot items involved tasks embedded in a practical context of some sort (e.g. shopping, home or social activities), while the other half involved tasks of a purely mathematical nature.

Administration of the Pilot Test

The pilot test of 175 mathematics items was divided into 7 blocks of 25 items each (A, B, C, D, E, F, and G). For two of these blocks, A and B, calculators were not permitted and for the other five, C, D, E, F, and G, calculators were permitted. The seven blocks were distributed across 10 test booklets so that each pupil booklet included a calculator block (A or B) and any two of the other 5 blocks, i.e. a total of 75 items per pupil (Table 3). In this way, of the approx. 1000 pupils in the pilot study, approx. 500 took each of the two non-calculator blocks and approx. 400 took each of the other five blocks. The open response items were hand-scored using a scoring rubric. It should be noted that although the pilot study involved a representative sample, the resulting pupil data were not weighted as the sample was too small to make statistically supported inferences about the population as a whole.

Table 3: *Structure of Test Booklets -6th Class, NAMA Pilot Study*

Booklet	First Section (Non-Calculator)	Second Section (Calculator)	Third Section (Calculator)
1	A	C	E
2	B	C	F
3	A	D	G
4	B	D	C
5	A	E	D
6	B	E	F
7	A	F	G
8	B	F	D
9	A	G	C
10	B	G	E

Analysis of Results of the Pilot Study

The mean percent correct scores on each of the seven blocks of items piloted are given in Table 4 below. Although the mean percent score across all the blocks was approx. 51.3%, which is a little below the intended figure of around 55%, there was considerable variation among blocks, with blocks D and G being the most difficult at 46% and block A being the easiest at 61% - a difference of 15%. The two non-calculator blocks were about the same level of difficulty. This was also the case with the ten 3 x 25 item pupil booklets where mean percent scores ranged from 52% to 54% - a difference of just 2% (Table 5).

These results suggested that revisions to the test in preparation for the main study should, in general, aim to reduce somewhat the difficulty level of the harder blocks while to a lesser extent increasing the difficulty level of the easier blocks. At the same time there was a need to maintain, as near as possible, the distribution of items across content strands and process skills as per the NAMA mathematics framework

Table 4: *Mean Percent Correct on the 7 Blocks of Items-6th Class NAMA Pilot Study*

Block	*A**	*B**	*C*	*D*	*E*	*F*	*G*
Mean Percent Correct	61	60	50	46	49	47	46

*Non-calculator blocks

Table 5: *Mean Percent Correct on the 10 Booklets - 6th Class NAMA Pilot Study*

Booklet	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Blocks	*ACE*	*BCF*	*ADG*	*BDC*	*AED*	*BEF*	*AFG*	*BFD*	*AGC*	*BGE*
Mean Percent Correct	54	52	52	53	53	53	52	52	53	52

Identification of Problem Items

Analysis of the performance of the pilot study pupils on each of the 175 items in the test yielded measures of the quality of the items, including (i) *difficulty level* which is measured in terms of the proportion of pupils obtaining the correct response, (ii) *point-biserial* index of discrimination which measures the degree to which pupils who do well on the test do well on the item, and, (iii) the functioning of alternate responses or *distracters* in the multi-choice items. These measures were then used to identify problematic items in the test that needed to be modified or replaced. IRT analysis of the data was used to look for differential item functioning on the basis of gender but no effect of this kind was found. Feedback from teachers in the pilot schools was also considered where specific items were mentioned.

Overall, 27 out of a total of 175 items were deleted. 13 of these were surplus to requirements i.e. sufficient items to cover the framework components they represent had already been selected. The other 14 were considered to be problematic items based on item statistics. 9 were very difficult (with 20% or less obtaining a correct response), 12 had pupil scores that correlated poorly with their performance on the test as a whole (i.e. point-biserial index less than 0.3), and 5 of them (multiple choice items) had poor distracters (e.g. positive point bi-serials for incorrect options). Two other problematic items, not deleted, were substantially revised, making a total of 16 problematic items in all.

Analysis and Discussion of the Problematic Items

The problematic items from the pilot study raise some interesting issues from curriculum and teaching points of view, and it is worthwhile to consider in more detail possible reasons for the failure of these items. Following is a selection of 13 items of this type (Table 6). Each item is discussed in relation to four aspects:

- What competencies it requires of the pupil
- Plausible response processes
- Reasons for deletion or revision of the item
- Curriculum and teaching considerations.

With regard to the discussion of curriculum and teaching aspects, these are based on analyses of the PSMC curriculum handbook (DES/NCCA, 1999a) and teacher guidelines (DES/NCCA, 1999b) and of the three most commonly used mathematics textbooks for 6th Class[3].

Shape and Space

Six of the selected problematic items are in the Shape and Space strand and 3 of these are in the strand unit Lines and Angles.

Table 6: *NAMA Pilot Study Items Deleted or Substantially Revised*

Item	Format	Strand Unit	Item Difficulty	Item Point-biserial	Action taken
Shape and Space					
C9	MC	2-D Shapes (Tesselation)	.22	.27	Deleted
C10	MC	2-D Shapes (Symmetry)	.50	.29	Deleted
D6	MC	2-D Shapes (Properties)	.72	.39	Revised
D12	MC	Lines and Angles (Types)	.41	.20	Deleted
G13	CR	Lines and Angles (Measures)	.16	.27	Deleted
C7	CR	Lines and Angles (Properties)	.51	.25	Deleted
Measure					
G17	CR	Area (Scale)	.02	.23	Deleted
F16	MC	Area (Units)	.06	.12	Deleted
C17	CR	Area (3-D Shapes)	.13	.47	Revised
E21	CR	Money (Exchange Rates)	.14	.34	Deleted
G20	CR	Money (Percentages)	.12	.49	Deleted
Data					
E24	CR	Data (Interpret)	.05	.27	Deleted
F22	CR	Chance	.15	.12	Deleted

[3] Action Maths: Folens (2002); Maths Matters: Educational Company of Ireland (2003); Mathemagic: Fallon (2003).

PSMC Objective: Tessellate combinations of 2-D shapes

Item C9

Which of these shapes cannot be used to tile (tessellate) a flat surface?

A) Hexagon B) Parallelogram C) Kite D) Pentagon

Difficulty: .22 Point biserial: .27

Item G8 (Difficulty: .27 Point Biserial .20) tested same objective and was deleted for similar reasons

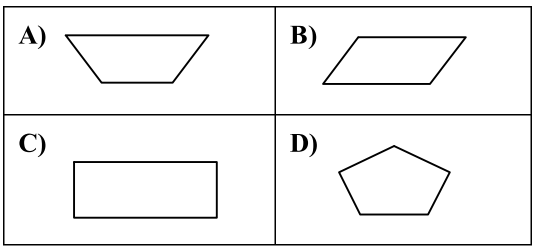
Item C9 requires pupils to know that pentagons are the only shapes among the four shapes listed that cannot tile a flat surface without leaving space in between the shapes. Responses could result from reasoning based on drawing and examining tilings of the four shapes or on previous experience in tiling shapes by manipulation. This item proved to be very difficult for the pilot study sample with only 22% of pupils choosing the correct response from the four options. The poor Point-biserial level (point biserial 0.27) suggested that many of those who obtained the correct answer guessed it rather than responding correctly on the basis of knowledge of tessellations. Hence the item was eliminated from the test as it was seen to be contributing very little to the value of the test as a measure of the mathematics achievement of 6th Class pupils.

Since the objective on tessellations is on both the 3rd/4th Class and 5th/6th Class programmes of the PSMC the elimination of this item raises the question of whether or not it is receiving adequate instructional emphasis in the classroom. An indirect way of obtaining information on this question is to examine the treatment of the topic in the textbooks used by teachers in teaching the objective. Analysis of the treatment of tessellations in the 6th Class books of the three most widely used textbook series revealed that, in all three series, less than half a page was devoted to the topic of tessellation out of almost 200 pages in each of the 6th Class textbooks. Even then the emphasis was on lower order cognitive tasks such as naming shapes in given tessellations and identifying shapes that tesselate. There was no evidence of engagement of pupils leading to clarification of the concept or to generalisations about shapes that tessellate or not, and why.

PSMC Objective: Classify 2-D shapes by their lines of symmetry

Item C10

Which of these shapes has just two lines of symmetry?



Difficulty: .50 Point-biserial: .29

This item, C10, required pupils to know that shape C (a rectangle) has just two lines of symmetry whereas A has one, B has none, and D has five. Responses could involve reasoning based on mental images of lines of symmetry in the shapes or on physically drawn lines of symmetry on the shapes. Though only moderately difficult (0.50), the marginal point-biserial of 0.29 indicated that pupils who answered the item correctly tended not perform well on the overall test and some responses may have involved guessing. The item was eliminated from the test. These item characteristics are surprising in that Symmetry is a strand unit of the PSMC from 2nd to 6th Classes and is allocated significant coverage in the various textbooks for these class levels and so would be expected to be relatively easy and to discriminate well among pupils. A possibility is that pupil misconceptions about symmetry of shapes arising in earlier grades were carried through to 6th Class e.g. that a parallelogram has lines of symmetry or that a rectangle has more than two lines of symmetry. Some pupils may be confusing rotational symmetry and line symmetry.

PSMC Objective: Make informal deductions about 2-D shapes and their properties

Item D6

If a quadrilateral has 4 right angles then it must be:

A) Rectangle B) Kite C) Parallelogram D) Rhombus

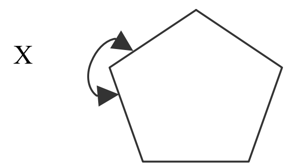
Difficulty: .72 Point-biserial: .39

This item requires pupils to deduce that if a quadrilateral has 4 right angles then it must be a rectangle and not a kite which has to have adjacent sides equal, nor a rhombus which has to have four sides equal, and is more than just a parallelogram which has to have opposite sides equal. The item statistics for this item were satisfactory - in fact it was a relatively easy item. However, feedback from test class teachers indicated that there was some concerns about this item. The item expects a pupil to reason that a rectangle also has the properties of a parallelogram and that there are also parallelograms which do not have four right-angles whereas rectangles always have. This would be unfair to pupils who are taught that a rectangle is not a parallelogram. The item was substantially reworded though still testing the same objective. The item aims to assess pupils' ability to make informal deductions about 2-D shapes and their properties – this kind of reasoning corresponds to Level 3 on the Van Hiele scale of Geometric Thought – Informal Deduction about shape and space (Level I being Recognition and Visualisation of shapes and Level 2 being Analysis of shapes), which is considered to be within the capabilities of senior primary school pupils (Crowley, 1987).

PSMC Objective: Recognise, classify and describe angles and relate angles to shape

Item D12

What is the angle X called?



A) Acute angle B) Obtuse angle C) Reflex angle

D) Right-angle

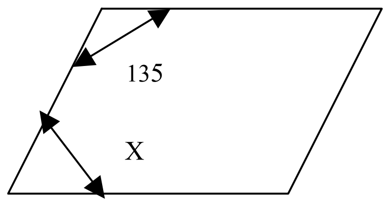
Difficulty: .41 Point-biserial: .20

This item simply asks the pupil to select the type of angle marked on a pentagon. It requires recall of definition of a reflex angle as an angle greater that 180 degrees with perhaps mental visualisation of a 180 degree and/or 90 degree angles in the process. It proved to be surprisingly difficult and to have a low point biserial for the correct response. In fact, more pupils (45%), including many more able pupils, selected the third option (C) rather than the correct one (B) (41%), suggesting considerable confusion about the difference between the two types of angles. For these reasons it was deleted. When we look at the treatment of the topic Lines and Angles in the 6th Class textbooks angle types are dealt with either abstractly as entities formed by different intersections of two half lines, or as the internal angles of 2-D shapes. No reference was found in any of the series to the external angles of 2-D shapes despite the fact that they are not excluded in the PSMC handbook where the relevant objective states: "recognise, classify and describe angles and relate angles to shape", and both internal and external angles are dealt with in the mathematics curriculum in the first year of post-primary schooling.

PSMC Objective: Explore the sum of the angles in quadrilaterals

Item G13

Look at the parallelogram. How many degrees is angle X?



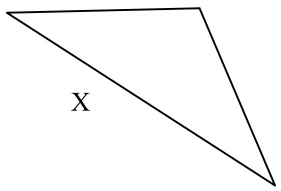
Difficulty: .16 Point-biserial: .27

This item requires pupils to use the rule that the sum of adjacent angles in a parallelogram is 180 degrees, or that the sum of the four angles is 360 degrees and the property that opposite angles in a parallelogram are equal. Since the item proved to be very difficult and performance on the item did not correlate well with performance on the test as a whole it was deleted. This objective is given particularly significant coverage in the textbook series for 6th Class which contrasts with the fact that only 16% of pupils obtained the correct answer to an item which closely parallels tasks on this objective in the textbooks. Some pupils who did not have knowledge of these two rules appear to have measured angle X with their protractors and thus obtained an incorrect answer that way since the angle marked 135 degrees in the diagram measures 120 degrees and the angle X measures 60 degrees and not 45 degrees, which is the correct answer obtained by applying one of the rules mentioned above. This may also have created uncertainty in the minds of pupils who applied the rules correctly and also measured the angle with their protractors. In retrospect, diagrams in Shape and Space tasks should accurately reflect given angles and proportions, and indeed this was the case for similar items in the main study. Another possible contributing factor is that pupils made a computational error when subtracting 135 from 180 degrees.

PSMC Objective: Use angle and line properties to classify and describe triangles and quadrilaterals

Item C7

Look at this triangle. Draw a line perpendicular to side X



Difficulty: .51 Point-biserial: .25

Item C7 requires pupils to use their ruler to draw a line at right angles (perpendicular) to the side X of the triangle. Since parallel and perpendicular lines are introduced in the curriculum for 3rd and 4th Classes and revised in 5th and 6th Classes, it should have been an easy item but in fact was moderately difficult and had a poor discrimination index, and hence was deleted from the test. Inspection of the treatment of the topic in textbooks for 4th, 5th, and 6th Classes shows that the main emphasis is on identification of perpendicular lines, there is very little work on actually constructing one line perpendicular to another. This imbalance should be noted by teachers and textbook writers.

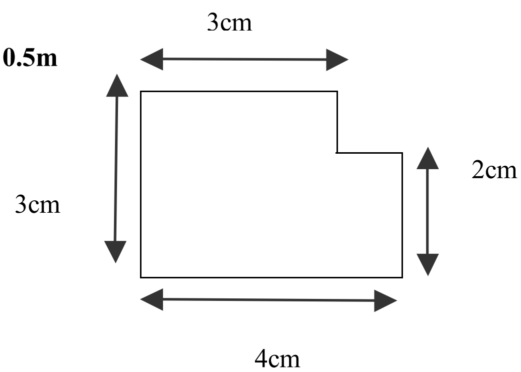
Measures

Five of the problematic items were in the Measure strand with 3 of these in the strand unit Area.

PSMC Objective: Find the area of a room from a scale plan

Item G17

This is a plan of bathroom. The scale is 1cm : 0.5m



What is the area of the bathroom?

Difficulty: .02 Point-biserial .23

This item requires pupils to calculate the area of a room from a plan giving the dimensions in cm and a scale for converting cm to metres. To solve the problem pupils could convert the cm dimensions to metres using the scale (the scale 1cm:50cm could be used instead) and then calculate the area in square metres, e.g. (1.5m x 2m) – (0.5m x 0.5m) = 2.75 sq m or 2 ¾ sq m. This task was so difficult it seemed to be beyond the capabilities of even the most able pupils and so was deleted. The textbooks for 6th Class give significant coverage to area, scale and the relationship between sq cm and sq m and include plans of bedrooms and classrooms in their examples. The scales were usually 1:10, 1:50, 1:100 etc and none were 1cm:0.5m though it is the same as 1:50 in cm (a fact which pupils obviously missed in the test item).

The treatment of the topic in the textbooks does not seem to lead to the deeper understanding of scale needed for this item

PSMC Objective: Identify the relationship between square metres and square centimetres

Item F16

How many square centimetres are there in 5 square metres?

A. 50 000 B. 25000 C. 5000 D. 500

Difficulty: .06 Point-biserial .12

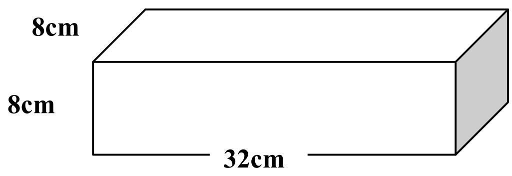
Item F17 (Difficulty: .04 Point-biserial .22) tested same objective and deleted for similar reasons

This item required pupils to convert sq cm to sq m. To obtain the correct answer for F16 pupils needed to know that there are 10 000 sq cm in a sq m and to use this fact to convert 5 sq m to sq cm by multiplication. It proved to be too difficult for the vast majority of pupils, being answered correctly by only 6% of them, despite calculator access, and was deleted, as was another item, F17, which tested the same objective with only 4% getting it correct. Inspection of the coverage of the topic Area in the 6th Class textbooks revealed that even though the relationship between sq cm and sq m is clearly explained, there are no actual tasks requiring pupils to convert a specified number of sq cm to sq m or vice versa, either in isolation or as part of a practical problem, although the PSMC indicates that pupils are expected to be able to make such conversions.

PSMC Objective: Measure the surface area of specified 3-D shapes

Item C17

What is the total surface area of this cuboid?



Difficulty: .13 Point biserial: .46

This item requires pupils to visualise in some way the 6 faces of a cuboid and then to calculate the total surface area of the 6 faces using the dimensions of 4 side faces as 32cm x 8cm each, and 2 end faces as 8cm x 8cm each. The item had a good point biserial index but was very difficult, with just 13% of pupils answering it correctly. Treatment of surface area of 3-D shapes in the main 6th Class textbooks is quite variable, with one series giving it minimal mention with just half a page on the topic and another two treating it in a more substantial way. The revised item made use of a wire-frame type drawing showing the inside of cuboid, instead of the above drawing, which might help pupils to visualise the 6 different faces.

PSMC Objective: convert foreign currencies to euros and vice versa

Item E21

On a particular day a £1sterling is worth €1.25 and €1 is worth US$1.20. At these rates what is £1 sterling worth in US$?

US$________

Difficulty: .14 Point-biserial .34

This item on currency conversion requires student to use two exchange rates (sterling to Euro and Euro to dollars) in order to calculate a third rate (sterling to dollars). It involves considerable reasoning and would probably be solved using some sort Unitary Method e.g. €1 = $1.25; €0.25 euro = $0.30; €1.25 = $1.50 dollars = £1. It proved to be a very difficult item with just 14% of pupils obtaining the correct answer. The calculations were simple and could be done easily without access to a calculator. The three textbook series give at least two pages to the topic but all the tasks included on the pages are one-step tasks involving conversion from one currency. There were no examples of two step problems, such as item E21 above, in any of the series. The textbooks were obviously keeping strictly to the domain of tasks overtly described by the curriculum objective. This raises the question as to whether the scope of currency conversion should be as narrow in 6th Class as to exclude multi-step problems involving higher order thinking and reasoning.

PSMC Objective: Compare prices to identify value for money

Item G20

During a sale the price of a washing machine was reduced by 30%. The sale price is €455. What was the price before the sale? _____

Difficulty: .12 Point-biserial .49

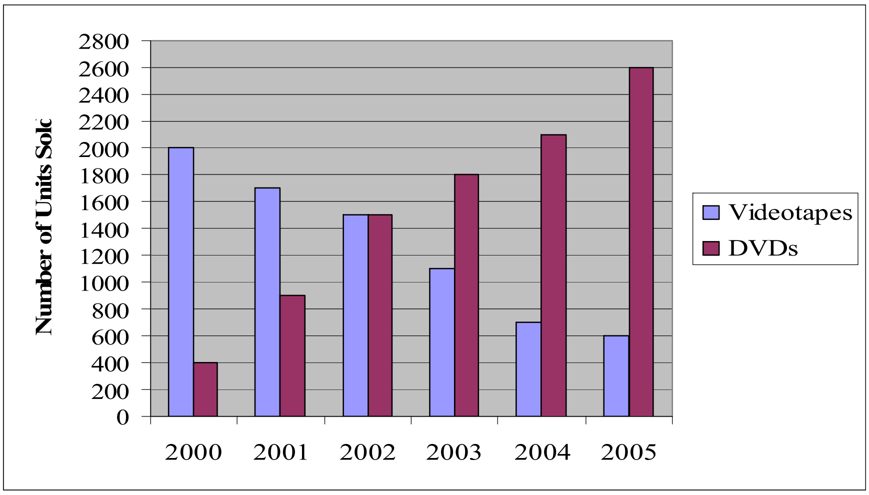
This item involves calculating the original price of an appliance given its sale price and the percentage reduction. Pupils need to have a good understanding of percentage change and calculating methods involved. The key to the solution is recognising that the sale price is 70% and then using that information to calculate 100%. The calculation is then relatively simple particularly if the Unitary Method is used (e.g. 70% = €455; 10% = €65; 100% = €650). Surprisingly, few students seemed to be able to do this question with 12% getting correct answer. There is substantial coverage of percentage increases and decreases in prices of goods in the 6th Class books of the three most popular textbook series, but only one of the three series had questions of the type in Item G20, where pre-reduction prices were asked for.

Data

Two of the items are in the Data strand and one of these is in the strand unit of Chance. Item E24 involves reading multiple bars in a bar chart to identify the 2002 and 2005 sales of CDs (1500 and 2600, respectively). The difference, 1100, is divided by 1500 (100%) and expressed as a percentage, 73.3% of the 2002 sales. This is a complex multi-step problem but should be within the capabilities of the more able 6th Class pupils. However, only 5% got the correct answer (answers between 73% and 74% were accepted) and the item was deleted. The item proved to be more difficult than expected. The PSMC handbook lists this kind data handling task for 5th and 6th classes as: "read and interpret pictograms, single and multiple bar charts, and pie charts".

PSMC Objective: Read and interpret trend graphs and pie charts

Item E24



What was the percentage increase in sales of DVDs from 2002 and 2005? _________

Difficulty: .05 Point-biserial .27

Inspection of the three textbook series showed significant coverage of multiple bar charts but none of them asked questions about the data in the charts which would involve percentage change. This raises the question whether tasks involving the calculation of percentage change in data trends in graphs should be part of the 5th and 6th Class mathematics programme or postponed to post-primary mathematics.

Another factor is that the calculation did not work out 'evenly' and a calculator would have been helpful. Pupils are used to answers that work out 'evenly' and those that got the correct answer, a recurring decimal, may have thought it incorrect and recalculated to get a more acceptable answer. Very few questions on multiple charts in the textbooks required any significant calculation at all.

PSMC Objective: Identify and list all possible outcomes of simple random processes

Item F22

Orla throws two dice together, one black and one white. The two numbers that turn up are multiplied. How many ways can she get a product of 12?

A. 1 B. 2 C. 4 D. 6

Difficulty: .15 Point-biserial .12

Item E25 (Difficulty: .26 Point-biserial .08) tested same objective and was substantially revised for similar reasons

Item F22 involves reasoning about the possible outcomes of throwing two dice and counting the particular outcomes where the two numbers give a product of 12 i.e. 4 ways – 4 and 3; 3 and 4; 6 and 2; 2 and 6. The item, which was deleted from the test, was very difficult with only 15% of pupils obtaining correct response and the low point biserial index of 0.12, suggesting that many of the correct responses were obtained by guessing. The strand to which it belongs, Chance, was not on the mathematics curriculum which preceded the introduction of the 1999 PSMC and teachers may have some difficulty incorporating it into their work. Unusually, no specific help was provided for this new strand in the Teachers Guidelines for Mathematics (DES/NCCA 1999b). Textbooks gave significant coverage to the topic with one series having an almost identical problem to F22.

CONCLUSIONS

The foregoing analysis and discussion of the 13 problematic items revealed that in the case of 8 of them, C9, D12, C7 in Shape and Space; G17, F16, E21 and G20 in Measure, and E24 in Data, poor item statistics could be attributed, in the main, to insufficient or shallow treatment, in the most commonly used textbooks, of the particular content and skills involved in the items. Textbook authors may be taking a narrow interpretation of the specific objectives in the PSMC handbook, particularly where exemplars for these objectives are not adequate or are missing. They focus a lot on tasks providing varied and repetitive practice of basic concepts and skills and not enough on tasks provoking higher order reasoning and more complex problem-solving. This in turn points to over-reliance on textbooks by many teachers who need to bring other resources such as the computer and computer based resources as well as guideline based on recent research into play in preparing and implementing mathematics lessons. This problem has been identified in successive DES reports (Dept of Education, 1980, Dept of Education, 1985) and national assessments (Shiel et al, 2001; Shiel et al, 2006) over the past 30 years and remains a problem today.

The poor item statistics for item C10 in Shape and Space, and F22 in Data could be due to teaching approaches where sufficient consideration is not given to persistent misconceptions or to addressing the development of higher order reasoning such as informal deduction in Shape and Space. The poor performance of G13 in Shape and Space and C17 in measure could be attributed to inadequacies in the diagrams provided with the question as described in the previous section. It is also noteworthy that there were no problematic items belonging to the main strand of the mathematics curriculum, Number, the strand which may be over-emphasised by teachers and textbooks to the neglect of the other strands.

The following are the main issues arising from this analysis of the problematic items of the NAMA pilot study:

- There may still exist a heavy emphasis on lower order thinking and on decontextualised tasks in teaching and in textbooks despite the apparent Constructivist philosophy underpinning the PSMC. Much of what is in textbooks in relation to the topics discussed above consists of brief teaching examples of isolated concepts and skills followed by varied and repetitive practice exercises. These are based on traditional learning theory principles and task analysis rather than on Constructivist theory principles and empirically based developmental learning progressions. There are few tasks in the textbook exercises that provoke significant reasoning, problem-solving or discussion and group work, or that incorporate or connect a range of skills and concepts in practical contexts or situations.

- Following from the above issue there seems to be a tendency by textbook writers to interpret teaching objectives in the PSMC handbook very narrowly. The PSMC handbook itself contributes somewhat to this problem by the limited or patchy exemplars accompanying the teaching objectives. To encourage innovation and improve textbook quality the DES should consider introducing a requirement that textbooks be piloted in schools and then evaluated by DES before being approved for the schools market, a situation that pertained in the past.
- The current teacher guidelines that accompany the PSMC handbook appear to be insufficient for developing lesson plans and lesson sequences for attaining curriculum objectives and lack detail about new topics such as Chance. There is no evidence of the influence of developments in Constructivist mathematics education in the handbook such as guidelines for implementing more experimental approaches to lesson study and learning trajectory design (e.g. Simon, 1995; Clements and Sarama, 2004; Mousley et al., 2004)

Although it is acknowledged that large scale surveys and standardised tests should not drive the curriculum, the analysis of tasks that perform poorly in psychometric terms in trials such as NAMA can help to highlight specific areas of need or neglect in the intended and taught curriculum.

REFERENCES

Clements, D. H., & Sarama, J. (Eds.) (2004). Hypothetical learning trajectories. *Mathematical Thinking and Learning, 6*(2).

Crowley, M.L. (1987) “The Van Hiele Model of the Development of Geometric Thought” In Lindquist, M. M. and Shulte, A. P. (Eds) *Learning and Teaching of Geometry, K-12, 1987 Yearbook.* (pp 1-36). Reston, VA. National Council of Teachers of Mathematics.

Department of Education and Science/National Council for Curriculum and Assessment. (DES/NCCA). (1999a). *Primary school curriculum: Mathematics. Content*. Dublin: Stationery Office.

Department of Education and Science/ National Council for Curriculum and Assessment. (DES/NCCA). (1999b). *Primary school curriculum: Mathematics. Teacher guidelines*. Dublin: Stationery Office.

Department of Education, Curriculum Development Unit, Primary Section. (1980). *Turaisc ar theagasc na matamaitice sna bunscoileanna* (an dara eagrán). [Report on the teaching of mathematics in primary schools, second edition]. Dublin: Author.

Department of Education, Curriculum Development Unit, Primary Section. (1985). *Turaisc ar theagasc na matamaitice i rang VI (1984)*. [Report on the teaching of mathematics in sixth class]. Dublin: Author.

Educational Research Centre (ERC) (2008). *Framework and test specifications for the 2009 National Assessment of Mathematics Achievement – Second and sixth classes* Retrieved June 26, 2009 http://www.erc.ie/documents/nama09_framework.pdf

Mousley, J., Sullivan, P., Zevenbergen, R. (2004) Alternative learning trajectories In I. Putt. R. Faragher. M. McLean. (Eds.), *Mathematics education for the third millennium: towards 2010* Proceedings of Mathematics Education Research Group of Australasia. Conference (pp. 374-381) Merga, Pymble, N.S.W.,

Shiel, G., & Kelly, D. (2001). *The 1999 National Assessment of Mathematics Achievement*. Dublin: Educational Research Centre.

Shiel, G., Surgenor, P., Close, S., & Millar, D. (2006). *The 2004 National Assessment of Mathematics Achievement*. Dublin: Educational Research Centre.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26 (2), 114-145.

MATHEMATICAL KNOWLEDGE FOR TEACHING 3-D SHAPES

Seán Delaney

Coláiste Mhuire, Marino Institute of Education

This paper uses the framework of mathematical knowledge for teaching (MKT) to analyse a lesson on 3-D shapes taught by one teacher to her second class children. Although the teacher had a reasonable score on a test of mathematical knowledge for teaching, she made several mathematical errors when teaching. Close analysis of the lesson reveals that teaching 3-D shapes requires non-trivial knowledge of geometry which can be inferred by studying the practice of teaching using the lens of the MKT framework. MKT consists of at least four subdomains: common content knowledge, specialised content knowledge, knowledge of content and students, and knowledge of content and teaching (Ball, Thames, & Phelps, 2008).

Among the questions which teachers may need to address when teaching and preparing to teach 3-D shapes are: What is a 3-D shape? What is a corner on a 3-D shape? What is a 3-D circle? What shapes in the environment can be classified as 3-D shapes such as cones, cylinders and cuboids? What does it mean for a shape to slide or to roll? Questions such as these are identified using the primary data of the teacher's practice and discussed using the primary school curriculum and relevant mathematics and mathematics education literature.

For the last two decades much attention has been paid to teachers' mathematical knowledge in order to improve teacher education and raise student achievement in mathematics. Several models of teacher knowledge have been proposed (e.g. Ball & Bass, 2003; Davis & Simmt, 2006; Rowland, Huckstep, & Thwaites, 2005), which have improved our understanding of both the content and the nature of teachers' mathematical knowledge (Ball, Lubienski, & Mewborn, 2001, p. 441). Although we have learned a lot in general about the mathematics teachers know, much remains to be learned about what teachers know about particular mathematics topics. In the area of 3-D shapes relatively little is known about what teachers need to know. This paper considers what knowledge is useful for teaching 3-D shapes at primary school level.

Several methods have been used to determine the knowledge that teachers need and use. Policy makers have used methods such as curriculum analysis to draw up lists of specific and general knowledge that teachers must have. Researchers have studied teachers' characteristics (e.g. courses studied) and the knowledge held by teachers. One alternative approach is to study teaching and its mathematical entailments on the basis that it is the mathematical work of teaching that determines what teachers need to know (Ball et al., 2001). This approach has led to the theory of mathematical knowledge for teaching or MKT, which was developed by Ball and her colleagues at the University of Michigan. In this paper I use the framework of MKT to study the work of a teacher who is teaching her second class students about 3-D shapes.

The Irish primary mathematics curriculum is organised under the five strands of number, algebra, shape and space, measures and data. These content areas are underpinned by the skills of applying and problem solving, reasoning, communicating and expressing, integrating and connecting, implementing, and understanding and recalling. Students are expected to learn about the properties of 3-D shapes over the eight years of primary school. By the end of second class it is hoped that students have been enabled to describe, compare and name 3-D shapes, including cube, cuboid, cylinder sphere and cone; discuss the use of 3-D shapes in the environment; solve and complete practical tasks and problems involving 2-D and 3-D shapes; and explore the relationship between 2-D and 3-D shapes. In these classes and in the subsequent four years of primary schooling students are expected to engage with topics that relate roughly to the first three van Hiele levels of visualisation, analysis and abstraction (e.g. van Hiele, 1999).

The construct of MKT in the United States was developed by analysing the work of teaching from a mathematical perspective. This was initiated by collecting records of practice from one year of Ball's (see Ball & Bass, 2003) teaching mathematics to third grade children. Ball and her colleagues hypothesise that teachers' knowledge can be considered in terms of at least four domains of knowledge. Two domains are types of subject matter knowledge: common content knowledge (mathematical knowledge held in common with other educated adults) and specialised content knowledge (mathematical knowledge that is typically not needed for purposes other than teaching such as knowing multiple algorithms). In addition, two domains have been identified which involve combinations of knowledge and are part of pedagogical content knowledge (Shulman, 1986): knowledge of content and students and knowledge of content and teaching (Ball et al., 2008).

Previous studies of teachers' knowledge of geometry analysed teachers' knowledge of geometry in light of the van Hiele levels and found that many teachers had low levels of knowledge of geometry but that an appropriate intervention could boost their knowledge (e.g. Swafford, Jones, & Thornton, 1997). However, the van Hiele levels were developed to study student knowledge and the knowledge teachers need may be different because of the work they have to do (Ball et al., 2001).

Chinnappan and Lawson (2005) have proposed a framework for characterising teachers' mathematical knowledge of geometry and they apply the framework to the focus schema of a square. They used "connectedness maps" – consisting of defining features, related features, relation of the focus schema to other schema, and applications – to illustrate the knowledge of geometry and the knowledge of geometry for teaching held by two teachers. This framework effectively shows the connectedness of the teachers' knowledge of specific geometric concepts. But it provides a level of detail that could be cumbersome as a way of providing an overview of geometric knowledge needed by teachers when applied to other geometric concepts. The model is created by interviewing teachers rather than by observing teaching and it excludes specific reference to mathematical knowledge in relation to knowledge of students or knowledge of teaching.

My interest in this study was to fill in some detail on the model of MKT developed by Ball and her colleagues in Michigan (2008). My specific research goal was to identify the MKT used and needed by one teacher in a second class lesson on 3-D shapes. I addressed this by analysing the teaching using the domains of MKT.

METHOD

This is a case study of one teacher, Veronica[4], teaching one lesson on 3-D shapes with specific reference to cubes, to 12 second class students in a disadvantaged area of a large town in the south of Ireland. Veronica was videotaped teaching four lessons over a two week period all on the topic of 3-D shapes, as part of a larger study of Irish teachers' mathematical knowledge for teaching, including a video study of ten teachers who each taught four lessons (Delaney, 2008). Veronica had been suggested to me as a teacher who might be willing to be videotaped teaching; she was not recommended because of any specific interest, expertise or difficulties in teaching mathematics. The lesson analysed here was the first of the four lessons taught. I asked Veronica to teach a maths lesson as she typically teaches it. She used no textbook in the course of teaching the lesson. I videotaped the lesson focusing mainly on Veronica. Consent was sought to videotape the students.

When Veronica had taught all four lessons, she completed a test of her mathematical knowledge for teaching using multiple choice questions based around teaching scenarios adapted for use in Ireland (Delaney, Ball, Hill, Schilling, & Zopf, 2008). Her overall score on the test of teacher knowledge was 0.4 on a scale where 0 represents the mean score and the scale ranges from -3 to +3 (Delaney, 2008). This score, just slightly above the mean, placed her on the 57th percentile, meaning that her overall performance on items designed to measure MKT is as good as or better than 57% of Irish teachers. This was the seventh highest score among ten teachers videotaped. But when the four lessons taught by Veronica were analysed for the mathematical quality of the instruction (Delaney, 2008; Hill et al., 2008), Veronica scored the lowest of the ten teachers. Whereas other teachers had taught at least two different topics over the four lessons, Veronica taught only lessons related to 3-D shapes. Because teaching students of younger classes is often considered relatively trivial in terms of mathematics content and given that she had scored relatively poorly on the mathematical quality of instruction, I considered that a close analysis of Veronica's teaching would provide important data on mathematical knowledge used and needed when teaching 3-D shapes.

A key premise of the theory of MKT is that the knowledge required is determined by the mathematical work of teaching. In analysing Veronica's lesson I set about identifying the mathematical work in which she engaged. This was done by viewing a video recording of the lesson and reading the transcript of the lesson and identifying teaching tasks that required "mathematical reasoning, insight, understanding and skill"(Ball & Bass, 2003, p. 5). The identified tasks were recorded on a grid organised according to the four domains of MKT that have been elaborated to date: two domains of subject matter knowledge (common content knowledge, specialised content knowledge) and two domains of pedagogical content knowledge (knowledge of content and students and knowledge of content and teaching). The tasks are placed in the domain of MKT that I judged to be most necessary for doing that task. The list of the mathematical tasks of teaching identified is contained in Figure 1. In the Results section I provide warrants for classifying these tasks as I do by providing excerpts

---

[4] Names and identifying details have been changed to ensure anonymity.

from the lesson to demonstrate the kind of knowledge of 3-D shapes that is necessary to teach second class students.

A. Common Content Knowledge is required in order to… (i) Elicit names of 3-D shapes. (ii) Elicit what is meant by the term "3-D shape" (iii) Elicit properties of a cube (iv) Prepare nets of a shape	C. Knowledge of Content and Students is required in order to… (i) Work with students' comments that are incorrect, incomplete or correct. (ii) Present scenarios to reinforce or challenge students' ideas (iii) Select an activity and describe how to do it
B. Specialised Content Knowledge is required in order to… (i) Define a cube in a way that is mathematically accurate and comprehensible by students (ii) Identify shapes in the environment (iii) Elicit examples of cubes in the environment	D. Knowledge of Content and Teaching is required in order to … (i) Choose materials to illustrate a definition or elicit a description (ii) Record work of class (e.g. using a table) (iii) Select an activity to reinforce learning

Figure 1: Mathematical tasks of teaching 3-D shapes identified in one lesson taught by one second class teacher.

RESULTS

Common Content Knowledge of 3-D Shapes

The first category is common content knowledge (CCK): knowledge that teachers hold in common with other educated people and which is used in many settings (Ball et al., 2008). When a teacher asks students to give examples of 3-D shapes – as Veronica did – and students respond with examples such as cubes, prisms and cylinders, the teacher draws on common content knowledge when deciding whether or not the examples are correct (cube), incorrect (circle) or ambiguous ("3-D rectangle"). The teacher's assessment of students' suggestions influences the next move she takes.

When introducing a unit on 3-D shapes, a teacher may want to elicit from students or explain to them some key properties of 3-D shapes. The teacher may consult a dictionary which defines a three-dimensional shape as "having length, breadth, and depth. A three-dimensional figure (solid) can be described in a coordinate system using three variables" (Daintith & Clark, 1999, p. 191). This definition, which includes solid shapes which are polyhedrons and non-polyhedrons, is useful for a teacher to know but is different from the kind of definition that will be useful in second class. When Veronica attempts to elicit from her students what a 3-D shape is, the following exchange takes place.

Veronica: A 2-D shape is a flat shape, but a 3-D shape? Brendan

Student: Is a fat shape.

Veronica: Is a fat shape. Excellent. Great word.

…

Veronica: It's like, it's got a front and a back and a top and a bottom. Alright. It's like a 3-D shape. It's something that you can hold in your hand.

Although it is useful to describe a 3-D shape as something that you can hold in your hand, the non-mathematical term "fat" may give students the mistaken idea that a "slender" shape is not 3-D. Notice that Veronica refers in a circular way to a 3-D shape being "like a 3-D shape" which suggests that she does not have a working definition of 3-D shapes to use with the class. If 3-D shapes are introduced to students with reference to mathematical terms such as length, width, and depth, students will be better able to describe and compare the shapes. Using such terms to introduce 3-D shapes in second class might also help students to recall the term because the "3" can be linked to their understanding of the three dimensions listed.

A teacher uses CCK when eliciting properties of individual 3-D shapes. This includes describing and counting faces, sides and vertices as well as investigating whether shapes can slide or roll. Veronica asked her students how many corners on a cube. Although one student correctly responded that it was eight "obviously" Veronica responds as follows:

> Veronica: Eight, you think only eight. Let's guess, let's think. It's got six faces. How many corners are on each face? One, two, three, four. So it's got four corners on this face, it's gonna have four corners on this face, four corners on this face, four corners on this face, four corners and four corners.

A discussion follows where students disagree on the number of corners based on Veronica's description of corners. Eventually Veronica directs the counting with the students and concludes that there are 24 corners on a cube. In mathematics the term vertex is used to describe "a location where two or more lines or edges meet" (Weisstein, 1999-2009c). The terms vertices and corners can be legitimately used to refer to both 3-D and 2-D shapes but with 3-D shapes it is important to distinguish between corners of the 3-D shapes and corners on the faces of the 3-D shapes. In Veronica's lesson she did not refer to the edges of 3-D shapes at all and she identified only the vertices of the polygon faces of the cube and not to its vertices. This was despite one student's identifying correctly the number of vertices where edges meet on the cube.

For the closing activity in the lesson, Veronica had prepared nets of cubes for the student to work with. Knowing which shapes can be folded to enclose particular 3-D shape requires mathematical knowledge that is not specific to the work of teaching. Architects and engineers use 2-D representations of 3-D shapes in their work. These examples of mathematical knowledge are classed as CCK because such knowledge is typically held by people, other than teachers, who know mathematics (Ball et al., 2008).

Specialised Content Knowledge of 3-D Shapes

The other type of content knowledge identified by Ball is mathematical knowledge that is typically needed only for teaching. This is knowledge that teachers need beyond what they teach their students. This knowledge supports teachers in doing tasks such as "presenting mathematical ideas," "giving or evaluating mathematical explanations" "finding an example to make a specific mathematical point" and "choosing and developing useable definitions" (Ball et al., 2008, p. 400). One definition of a cube is "a solid figure that has six square faces" (Daintith & Clark, 1999, p. 48). When Veronica asked her students what a cube is, possibly hoping for a definition, she received several suggestions. One student said that a cube is "a little thing that has six sides on it"; another said that "it's like a little round thing"; a third student said "it's kinda like a Rubik's cube and … it's just like a square but it has different kinda sides"; and yet another student said that "it's a square but it's 3-D." After some further discussion the teacher summarised that "a cube is like six squares all put together and it's expanded. It's like a, it's a flat square with a top, a bottom, two sides and a back." Missing from this definition is a condition that the shape be solid. "Six squares all put together" could be a definition of a net of a cube rather than of a cube. The definition needed by a teacher is one that is mathematically accurate and understood by students of a given age.

Eliciting and identifying shapes in the environment requires a teacher to draw on specialised content knowledge. One challenge of doing this, however, is that many shapes that incorporate the names of 3-D shapes are poor approximations of the shapes. Ice-cubes are frequently not cube-shaped; traffic cones frequently have a circular top rather than an apex and gas cylinders typically have protruding parts that exclude them from the category of cylinders in a mathematical sense. The teacher and children need to agree criteria to use in order to decide whether or not to accept these shapes as examples of 3-D shapes while the teacher supports students in learning the properties that define these shapes. Although identifying shapes in the environment might appear to require knowledge that teachers hold in common with other people, a teacher has to consider if a given shape in the environment has the necessary properties. Other users of mathematics rarely need to attend carefully to disparities between mathematical properties of shapes and shapes as they exist in the environment, making this specialised content knowledge.

One instance where Veronica used such knowledge is now described. She attempted to elicit examples of cubes in the classroom and the following discussion took place, beginning with a student's example of a cube.

Student 1:	The computer
Student 2:	That's what I was going to say
Veronica:	Now, what did we say about the sides of a com-, what did we say about the sides of a cube? What do they all have to have? What do all the sides of a cube? What are they all?
Student 3:	Eh, the same.
Veronica:	The same what?
Student 3:	The same length.
Veronica:	The same shape. The same length. Now, you're right Donal. Can you look at the board? Alan. Thank you. The face of the computer is a what? The face of the computer is a what? What shape is it? The face of it? The front of it? What shape's that?
Student 4:	A square.

The conversation continues when the teacher turns one side of the computer to the class to show that it is not a square shape and therefore that the computer cannot be a cube. This is an example of how a teacher needs mathematical knowledge to respond to an incorrect suggestion by a student. The computer clearly is not an example of a cube but rather than saying that, Veronica drew attention to a key requirement of the cube, that all faces be squares. Sometimes Veronica refers to the face as a side and sometimes she refers to it as a face. Although both terms are mathematically acceptable, the term *face* is more specific because the term *side* is also used with polygons (Weisstein, 1999-2009a) and it might be better to use one term consistently when students are learning about properties of shapes. The student's response above to say that all the sides of a cube need to be the same length may be attributed to his knowledge of sides of polygons which can vary in length. Although Veronica repeats the phrase "the same length" it seems as if this may be intended to affirm the student's response rather than to give it mathematical legitimacy. But as the terms shape and length are juxtaposed, students might be confused by them. Veronica needed to draw on specialised content knowledge when considering whether a shape had the properties of a cube.

Knowledge of Content and Students

In addition to the two domains of MKT outlined above, which are part of subject matter knowledge, Ball and her colleagues propose two additional domains which are part of what Shulman (1986) calls "pedagogical content knowledge." The first of these additional domains is knowledge of content and students, which is knowledge that combines knowing mathematical ideas and knowing how children think and what they do (Ball et al., 2008). I found several examples in Veronica's teaching where she needed to use such knowledge when teaching her second grade students.

Occasionally students make comments about mathematical objects which make demands on the teacher's knowledge. At one stage in Veronica's lesson, when she is discussing shapes that roll, a student suggests that it would be easier for a circle [than a cube] to roll down a hill. Another student initiates the following exchange:

Student 1: I know the proper name for a 3-D circle?

Veronica: What is it?

Student 1: A sphere.

Veronica: It is a sphere. Excellent, well done. A sphere. But we're not gonna…we're just gonna concentrate on the cubes today. Alright, but well done. Let me see if there's any spheres there.

Student 2: Yeah, there's one there. And there's … and the globe.

Veronica: And the globe is a sphere as well. That's gonna roll down; which is gonna roll down the hill. This is the sphere boys. A sphere is a proper name for a ball, like a football, alright. It's a 3-D circle, ok.

The student claims that a 3-D circle is a sphere and Veronica accepts the claim. The student's contribution is clearly relevant to the lesson because the contribution follows a discussion about whether or not the cube would roll down a hill. In the exchange above, it seemed initially as if Veronica would return the focus promptly to the cube but instead she discussed how the sphere will roll and the cube will not. No explanation is given for why a 3-D circle is a sphere, as opposed to a cylinder. If you imagine a disc rotating on its side, around its diameter, it is possible to visualise the sphere as a 3-D circle. It is also the case that any cross section through a sphere is a circle (Weisstein, 1999-2009b). But a cylinder also has circular cross-sections and must have claims on being classed as a 3-D circle. It is helpful for students to build new knowledge of 3-D shapes on their existing knowledge of shapes but the teacher in turn needs to know about learners and about mathematics in order to know which informal terms – such as 3-D circle – are helpful and which ones could lead to confusion.

At one stage in the lesson a student identified a pyramid as a prism. On another occasion a student described a cuboid as “two squares put together.” These comments arose spontaneously in the lesson and if a teacher is to encourage students to use language more accurately, a teacher needs to understand the kinds of mistakes that students make and know the correct mathematical terms so that students are encouraged to use them.

Sometimes a teacher needs to present scenarios to reinforce or to challenge students’ thinking about a concept. For example, Veronica wants to reinforce the fact that some shapes can roll and some cannot. The following excerpt from the lesson contains the scenario and a student response.

Veronica:	Imagine if you had big square tyres on your car.
Student:	You wouldn’t be able to move. You’d be bumping everywhere. Your head’d be going up in to the ceiling.
Teacher:	Wouldn’t it? Because of all the corners.

Creating scenarios such as this requires knowledge of students and knowledge of mathematics. The teacher must choose situations to which students of the particular age can relate *and* which make sense mathematically. The student’s response above suggests that the chosen scenario was meaningful to the students. But, in creating the scenario Veronica reverts to the language of 2-D shapes, by referring to *square* tyres rather than to tyres that are in the shape of a cube or a rectangular prism.

Finally a teacher often demonstrates and describes to students how to do an activity. The type of description given and the amount of detail that a teacher provides is influenced by the teacher’s knowledge of mathematics and of the students. Veronica took time in her lesson to show students how to cut and fold a net of a cube so that it enclosed a cube shape.

Knowledge of Content and Teaching

The fourth domain of MKT combines knowing mathematics and knowing instruction. Tasks that require such knowledge include choosing suitable examples and representations to illustrate ideas and concepts and choosing questions to ask. At one stage in the lesson Veronica asks a student to select a cube from a box of 3-D shapes. She attempts to elicit from him why a shape – a large cube from a set of Dienes blocks – is an example of a cube. The boy refers to a feature of the particular cube which is not a required condition – and may render it ineligible on the basis of a face of a solid figure needing to be flat (Daintith & Clark, 1999). The boy says that the shape is a cube “because it has all little small squares all put together” referring to the grooves on each face to represent the 100 cubes that are supposed to be visible on each face. Veronica looked for another cube and advised the student not to “worry about those little squares…we’re talking about the big squares.” In selecting examples of cubes for students to use in their explorations, a teacher needs to consider which examples will allow students to learn best the properties of the shape being considered. The teacher needs to consider the objectives for the lesson and then decide whether instruction on the topic can be best enhanced by using folded nets of cubes, pipe-cleaner cubes, solid cubes and empty cubes because each one will bring out different aspects of a cubes.

A teacher uses knowledge of content and teaching when recording outcomes of processes. Veronica was recording the number of corners on each polygonal face of a cube in order to find how many corners on the cube altogether. I mentioned earlier that there are problems with calculating the number of corners on a 3-D shape by counting the number of corners on the 2-D faces, but here I am emphasising *how* the teacher recorded the number of sides. When the teacher had finished the board looked like the illustration in Figure 2.

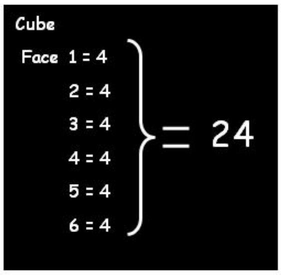


Figure 2: Veronica's record on the board of calculating the number of corners on a cube

Veronica's intention is that her writing on the board records the number of corners on each polygon face of the cube. But there are problems with her use of the equals sign. The equals sign is used to indicate that each face of the cube has four corners, but using it in this way may hamper students' developing conceptions of the sign. Carpenter, Franke and Levi (2003) would say that Face 1 has four corners but that the face itself is not equal to 4. They recommend restricting the use of the equals sign to show the "relation between numbers or expressions representing numbers" (p. 21).

A teacher needs to select activities for students to reinforce their learning. Such activities need to be carefully designed and demonstrated so that they present just enough mathematical challenge for the students. Too much challenge and several students will become frustrated. Too little and they will learn no mathematics. In this lesson Veronica prepared nets of cubes in advance and the students had to cut, fold and stick them together to make cubes. Once a task has been set up, a teacher needs to ensure that students' attention is maintained on the mathematical work and its cognitive demands. A risk of practical activities in mathematics class is that they can turn into cutting and folding activities with little mathematical content. For much of the time when Veronica's students were working on the cubes, their conversation and attention seemed to stray from the curriculum objectives about describing and naming 3-D shapes and about solving practical tasks and problems involving 2-D and 3-D shapes (Government of Ireland, 1999). Teachers' require mathematical knowledge to move such activities beyond the realm of fun (Hill et al., 2008) into ones which have a mathematical pay-off in student learning.

CONCLUSION

Identifying the mathematical knowledge needed by teachers is important for teacher educators, policy makers and providers of teacher professional development. Much has been learned over the last two decades about the mathematics teachers need to know. Although some topics have received substantial attention from researchers – fractions, multiplication and division, and functions, for example – we know less about the MKT needed to teach 3-D shapes.

The MKT framework, which is influenced by Shulman's work on pedagogical content knowledge, highlights knowledge that is used and needed by teachers that might not be observed by analysing curriculum or by applying the van Hiele levels to studying a teacher's knowledge. For example, a study of the curriculum content of 3-D shapes is unlikely to reveal the kind of issues that students raise – such as talking about a "3-D circle" – and which teachers need to respond to. Studying the practice of teaching reveals how knowledge in one area of mathematics – such as the meaning of the equals sign in number or algebra – can be needed when teaching a seemingly unrelated topic. Moreover examining practice reveals how mathematical ideas can be confused, such as the different meanings of corners in relation to 2-D and 3-D shapes.

Sometimes placing a mathematical task of teaching in the cell related to the knowledge needed to do the task was difficult. It could be argued, for example, that identifying shapes in the environment is part of common content knowledge rather than knowledge that is specialised to the work of teaching. The categories are helpful, however, in drawing attention to types of knowledge of 3-D shapes which the teacher used and which might otherwise be overlooked.

This study analyses the teaching of a teacher who exhibits shortcomings in her mathematical knowledge: she accepted an imprecise way to define a 3-D shape; failed to distinguish between vertices of a 3-D shape and vertices of the faces on a 3-D shape; presented students with an incomplete definition of a cube; used the term 3-D circle without clarifying it for students; confused ideas about 2-D and 3-D shapes; incorrectly recorded information about shapes; set up activities with little mathematical or cognitive challenge. Much of the teaching that was studied in developing the theory of MKT was done by Ball, a highly experienced and accomplished teacher. The benefit of studying a teacher such as Veronica is that shortcomings in her use of mathematical knowledge make visible the kind of knowledge that would enhance her performance of the mathematical work of teaching. Furthermore, it serves as a reminder that the kind of mathematics used by teachers is by no means trivial and very often teachers do not have the opportunities to develop it during their initial teacher education or in their subsequent professional development (Delaney, 2005).

This study provides some glimpses into the kind of mathematical knowledge that is needed for teaching 3-D shapes. In a national study of Irish primary teachers, substantial variation was found in the knowledge held by teachers (Delaney, 2008). The glimpses here point to the kind of learning about 3-D shapes that is needed by teachers in pre-service teacher education and by practising teachers: to know names of examples of 3-D shapes; recognise and make nets of 3-D shapes; use criteria to help pupils categorise objects in the environment as belonging or not belonging to sets of mathematical shapes; respond to students' spontaneous comments about 3-D shapes; demonstrate and describe tasks to students and select good examples of shapes to use with students. MKT provides a framework that can inform the mathematical education of teachers. Further study is needed to ascertain how teachers might best acquire the necessary knowledge.

More study is needed too on the mathematical knowledge needed to teach 3-D shapes at primary school level. This paper is based on just one lesson by one teacher at second class level. A similar analysis of other lessons taught by Veronica and by other teachers at various class levels and in diverse school settings, with diverse teaching styles and approaches would enhance the findings presented in this paper about mathematical knowledge for teaching 3-D shapes.

Author note: Although some of the content of this paper may be construed to be critical of Veronica, I wish to record my admiration and appreciation of her as a teacher. She demonstrated many excellent qualities as a teacher. By welcoming me into her classroom, she showed a commitment to advancing the scholarship of teaching. Deficiencies in her mathematical knowledge need to be seen as part of a system-wide problem with how we prepare teachers for the mathematical work of teaching in Irish schools. Veronica's story has, I hope, contributed to our understanding of some of the complexities of the mathematical knowledge of 3-D shapes teachers need.

REFERENCES

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. Paper presented at the the annual meeting of the Canadian Mathematics Education Study Group, Edmonton, AB.

Ball, D. L., Lubienski, S. T., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), Handbook of research on teaching (pp. 433-456). New York: Macmillan.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? Journal of Teacher Education, 59(5), 389-407.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). Thinking mathematically: Integrating arithmetic and algebra in elementary school. Portsmouth, NH: Heinemann.

Chinnappan, M., & Lawson, M. J. (2005). A framework for analysis of teachers' geometric content knowledge and geometric knowledge for teaching. Journal of Mathematics Teacher Education, 8, 197-221.

Daintith, J., & Clark, J., O.E. (1999). The facts on file dictionary of mathematics (3rd ed.). New York: Checkmark Books.

Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. Educational Studies in Mathematics, 61, 293-319.

Delaney, S. (2005). Mathematics professional development for primary teachers: Looking back and looking forward. Paper presented at the Mathematics education in Ireland: A research perspective, St. Patrick's College, Dublin.

Delaney, S. (2008). Adapting and using U.S. measures to study Irish teachers' mathematical knowledge for teaching. Unpublished doctoral dissertation, University of Michigan, Ann Arbor, MI.

Delaney, S., Ball, D. L., Hill, H. C., Schilling, S. G., & Zopf, D. (2008). "Mathematical knowledge for teaching": Adapting U.S. measures for use in Ireland. Journal of Mathematics Teacher Education, 11(3), 171-197.

Government of Ireland. (1999). Primary school curriculum: Mathematics. Dublin, Ireland: The Stationery Office.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J., Phelps, G. C., Sleep, L., et al. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. Cognition and Instruction, 26(4).

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. Journal of Mathematics Teacher Education, 8, 255-281.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. Educational Researcher, 4-14.

Swafford, J. O., Jones, G., A, & Thornton, C. A. (1997). Increased knowledge in geometry and instructional practice. Journal for Research in Mathematics Education, 28(4), 467-483.

van Hiele, P. M. (1999). Developing geometric thinking activities that begin with play. Teaching children mathematics, 6, 310-316.

Weisstein, E. W. (1999-2009a). Side. MathWorld. Retrieved July 5th 2009, from http://mathworld.wolfram.com/Side.html

Weisstein, E. W. (1999-2009b). Sphere. MathWorld. Retrieved July 5th 2009, from http://mathworld.wolfram.com/Sphere.html

Weisstein, E. W. (1999-2009c). Vertex. MathWorld. Retrieved July 5th 2009, from http://mathworld.wolfram.com/Vertex.html

INSIGHT IN PRIMARY MATHEMATICS: TEACHER 'MOVES' THAT FACILITATE THE 'STUMBLING OF IDEAS ACROSS EACH OTHER'

Thérèse Dooley

St. Patrick's College, Dublin and University of Cambridge, U.K.

Mathematics is often associated with certainty and quick recall of facts and thus has suffered more than most subjects from a 'talk and chalk' approach. Whole-class discussion, in particular, is usually characterised by an Initiation-Response-Evaluation (I-R-E) exchange structure that endorses the position of the teacher as sole validator of students' input. However, the increased attention on the role played by social interaction has led to a greater focus on conversational aspect of learning mathematics in which there is a genuine listening to and uptake of students' contributions. In this paper an account will be given of a lesson with students aged 9 – 10 years in which one, David, developed a tentative rule for the solution of the 'handshakes' problem. The teacher moves that facilitated engagement of students with each other's contributions are the particular focus of attention

INTRODUCTION

Although whole-class discussion is common to mathematics lessons, it is often characterized by traditional approaches that endorse the position of mathematics as a kind of received knowledge and the teacher as sole validator of students' contributions. While research shows that such discussion can be fertile ground for higher-order mathematical thinking (Cobb, Wood, Yackel, and McNeal, 1992; O'Connor, 2001), the fast pace with which it is usually associated means that there is little scope for students to make comments and build on each others' mathematical ideas (Hodgen, 2007). One consequence of this is that students become disengaged from the subject, perceiving it to be one in which they have little opportunity for participation (Boaler, 2002).

Although the use of discussion as a means of constructing new mathematical ideas is endorsed in the Irish Primary Mathematics Curriculum (D.E.S./N.C.C.A., 1999), there is evidence that the focus of teacher-student interaction is on lower-order thinking and recall of procedures (Murphy, 2004; Shiel, Surgenor, Close, and Millar, 2006). This is hardly surprising since the orchestration of inquiry-based discussion in mathematics is challenging for teachers. Sherin (2002), for example, alludes to two key tensions experienced by teachers, whereby on the one hand, they are expected to encourage students to share ideas and, on the other, have to ensure that the lesson is mathematically productive. In this paper, consideration is given to the kind of follow-up moves that teachers might make to students' contributions in whole-class discussion to engender the "growth of collective mathematical understanding" (Martin and Towers, 2009: 3).

MATHEMATICS AS CONVERSATION

The traditional classroom interaction structure is the Initiation – Response – Follow-up (I-R-F) model in which the teacher initiates an exchange and the student then makes a contribution and the teacher then makes a follow-up move (Sinclair and Coulthard, 1975). In situations where the follow-up move is 'evaluative', the pattern is described as I-R-E (Mehan, 1979). It is suggested that the I-R-E structure reinforces the asymmetry of power between teacher and pupils (Pimm, 1994) – the teacher retains the locus of control and the students do little more than infer what is in his/her mind. Mathematics because of its association with recall of procedures is particularly susceptible to the I-R-E structure.

The recognition that mathematics needs to be 'co-constructed' by students and teacher has led to interest in how a participatory model of discourse might be developed. In particular, it is felt that a follow-up move other than evaluation might lead to a more conversation-like genre than stems from the I-R-E model. In a study in which four secondary school mathematics teachers were observed and videotaped for two weeks, Brodie (2008) used 'follow-up' as a key category to describe a teacher move. It can refer to a contribution made by a learner either immediately preceding or some time earlier in the discussion. She found several ways that the follow-up move could be used by teachers and developed the following subcategories:

- Insert: The teacher adds something in response to the learner's contribution. She can elaborate on it, correct it, answer a question, suggest something, make a link etc.

- Elicit: While following up on a contribution the teacher tries to get something from the learner. She elicits something else to work on the learner's idea.

- Press: The teacher pushes or probes the learner for more on their idea, to clarify, explain more clearly. The teacher does this by asking the learner to explain more, by asking why the learner thinks s/he is correct, or by asking a specific question that relates to the learner's idea and pushes for something more.

- Maintain: The teacher maintains the contribution in the public realm for further consideration. She can repeat the ideas or ask others for comment or merely indicate that the learner should continue talking.

- Confirm: The teacher confirms that s/he has heard the learner correctly. There should be some evidence that the teacher is not sure what s/he has heard from the learner, otherwise it could be press.

Brodie sees her categories of moves on a continuum of more to less teacher intervention; in 'insert' the teacher makes his/her own contribution while 'confirm' and 'maintain' serve to keep the learner's input in the public domain. For these reasons, 'insert' and 'elicit' are viewed as more traditional than the other subcategories. However the teachers in Brodie's study, although committed to 'inquiry' mathematics were found to use a mixture of follow-up moves in any one lesson.

A particular form of the 'maintain' move that is currently receiving much attention is that of 'revoicing'. Revoicing is described as "the reporting, repeating, expanding, or reformulating a student's contribution so as to articulate presupposed information, emphasize particular aspects of the explanation, disambiguate terminology, align students with positions in an argument or attribute motivational states to students"(Forman and Larreamendy-Joerns, 1998: 106). A key part of the reformulation is the use of 'So you think that' or 'So Tom thinks that'. O'Connor and Michaels (1996) describe this as a 'layering' or 'lamination' of the teacher's phrasing onto the student's contribution. This layering, according to O'Connor and

Michaels (1996), "animates the student as the originator of the intellectual content" (p.79) and "makes possible an expanded and more contrapuntal set of voices and participant roles in constructing an idea than does the IRE" (p.97). Coupled with the attribution ‘you think’, the discourse marker ‘So’ opens up a new slot in the conversational space, giving the student an opportunity to comment on the correctness of the revoiced utterance. The effect of this ‘layering’ is to bring students’ ideas in contact with each other and thus to effect involvement of all children in the conversation (O'Connor and Michaels 1996; Rowland 2000).

As noted above, revoicing is considered to be a more democratic practice than ‘telling’. Although the exploration of revoicing has occurred primarily in the research community, some consideration to its practice in classrooms has been given by a group of eight middle-grade teacher-researchers participating in study group discussions (Herbel-Eisenmann, Drake, and Cirillo, 2009). The teacher-researchers were of the opinion that the multiple forms and functions of revoicing needed to be distinguished. For example, they felt that, while ‘restating’ a contribution (a direct re-utterance of the student’s words) allowed students to maintain ownership and amplified his/her idea, ‘rephrasing’ (addition of new language) shifted control of ideas from the student to the teacher. However, ‘rephrasing’ was felt to be more academically productive than restating. One of the needs identified was that of understanding students’ perspectives as their interpretation of a teacher move could be different from that intended by the teacher. There was strong agreement that, in studying the affects of revoicing, consideration needed to be given to contextual elements.

In this paper I will show how a mathematical conversation dominated by ‘maintain’ (in particular, revoicing) moves by a teacher afforded space for young students to engage with each other’s ideas. It builds on the paper I presented at MEI2 that shows that the coaction of students in plenary discussions is conducive to the production of new mathematical insight (Dooley, 2007).

BACKGROUND

The aim of my doctoral research is to investigate the factors that contribute to the development of mathematical insight by primary school pupils. The methodology is that of ‘teaching experiment’ which was developed by Cobb (2000) in the context of the emergent perspective. In one of three cycles of the research, I taught mathematics to a class of thirty-one 4th class pupils (seven girls and twenty-four boys) aged 9 - 10 years. The school is situated in an area of middle socio-economic status. Many lessons took place over two or three consecutive days, each period lasting forty to fifty minutes. I visited the class on a total of twenty-seven occasions. All phases of lessons were audiotaped. When children were working in pairs, audio tape recorders were distributed around the room. Other data collected included pupils’ written artefacts and digital photographs of blackboard work.

The lesson described here took place on the first of three consecutive visits to the class one week during the Spring term. The context of the activity was the game of chess and the object was to find the minimum number of games that could be played in a competition where each player had to compete with all other participants[5]. In the introductory phase of the lesson consideration was given to the number of games required in the case of one, two, three, four and five entrants. In general pupils used a count strategy by making reference to pupils who were standing at the front of the room as 'models' of the situation. During 'small group' work pupils worked in pair or triads on a worksheet activity in which they were required to solve the problem for one to ten participants. They were also asked to consider the number of games that would apply in the case of twenty participants. Solutions of this worksheet and justification for them were the main subjects of the plenary discussion that took place in the third phase of the lesson. For twenty participants, many pupils erroneously thought that the solution could be found by doubling the number of games required in the case of ten participants. The part of the discussion that is the focus of this paper took place towards the conclusion of the lesson when one pupil, David constructed an explicit formula for twenty participants.

Forman and Ansell (2001) contend that analysis based on isolation and coding of individual turns is too limited to bridge the individual and social. Therefore, I conducted ethnographic microanalysis, which according to Erickson (1992) is especially appropriate when the character of events unfolds moment by moment. The approach adopted was top-down starting with the molar units (lessons) and moving to progressively smaller fragments. I transcribed this lesson in full and, using the software package Nvivo, coded it using a tool that combined 'Underlying Mathematical Principles', 'Epistemic Actions (RBC)' (Hershkowitz, Schwartz, and Dreyfus, 2001), 'Hedges' (Rowland, 2000) and 'Teacher Moves'. In this paper, particular attention is given to the findings that emerged around 'teacher moves' although I also make some reference to 'underlying mathematical principles' and 'hedges'.

DAVID'S MOMENT OF INSIGHT

I was about to complete the lesson with the intention of giving it more time on the following day when Ms. Kelly, the class teacher, observed Brenda[6] using the calculator to add numbers from one to twenty. The following is the transcript of the whole-class discussion that followed from this observation[7]. Teacher 'uptake' moves are coded using Brodie's codes and classifications of revoicing suggested by Herbel-Eisenmann et al (2009). In cases where a student's idea is repeated using his or her name or the pronoun 's/he' rather than the pronoun 'you', it is categorised as 'broadcast' rather than 'repeat'. In some instances suggestions made by students earlier in the lesson are 'rebroadcasted'. If a move is not an 'uptake' it is coded as 'direct'.

---

[5] This is a variant of the 'handshakes' problem. For n people the solution can be found by summing ((n-1) + (n-2) …. + 0) or by application of the formula n(n-1)/2. Tabulating results sequentially from 1 gives rise to the triangular numbers on the l.h.s. column.

[6] Gender-preserving pseudonyms are used throughout the paper.

[7] The following transcript conventions are used: T.D.: the researcher/teacher (myself); Ch: a child whose name I was unable to identify in recordings;…: a hesitation or short pause; […]: a pause longer than three seconds; ( ): inaudible speech; [ ]: lines omitted from transcript because they are extraneous to the substantive content of the lesson.

	Action	***Teacher move***
397 Ms. Kelly: I think Brenda ... she doesn't have the answer but I think she has an idea of how to get to the answer.	Class teacher noticed Brenda working on calculator	
398 T.D.: Right what is Brenda going to do? What is Brenda doing?		Elicit
399 Brenda: I am just going to see on the calculator.		
400 T.D.: And what are you adding on the calculator?		Elicit
401 Brenda: Numbers!		
402 T.D. Numbers (*laughs*). Ok, right.		Repeat insert
403 Ms. Kelly: Tell us the numbers you are adding, Brenda … to get twenty.		
404 Brenda: Em one, two, three, four, five, six, seven all … all the way up to twenty.	Summation (1 – 20)	
405 Ms. Kelly: All the way up to twenty […]		Repeat
406 T.D.: Ok. Fiona?		Confirm
407 Fiona: Well you could em you could do em add one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen up to nineteen.	Summation (1 – 19)	
408 T.D.: Up to nineteen. Anne?		Repeat
409 Anne: You could do twenty multiplied by nineteen.	Multiply 20 by 19	
410 T.D.: You could do twenty multiplied by nineteen. Why do you think twenty multiplied by nineteen? Do you agree with that? Now she's thinking it's twenty multiplied by nineteen? Sorry, we'll come back to you in a moment, Anne		Repeat Press Ask for comment Broadcast Direct
411 Brenda: Well the answer I got was two hundred and ten.	Brenda calculated 210 on the calculator	
412 T.D.: Right[8] and you added up to twenty, did you?		Confirm Elicit
413 Chn: Oh, miss!		
414 T.D.: Hmm. Can I just write down some of the ideas here on the blackboard and we are going to come back to this tomorrow. I am going to take a photograph of all that is on the blackboard. So what will we do to find out. Some people think it's	Suggestions were written on blackboard: 45 x 2	Review

[8] "Right" is not used here as an evaluative comment.

forty-five multiplied by two to give us ninety because the answer for ten is forty-five, for ten it's forty-five so they think for twenty it might be ninety, this is just a question mark - we will put a question mark over here….somebody else said a hundred and ninety and that's because ... what were you adding David, that was David … adding nineteen and seventee - eighteen and he was mentally adding those numbers. I think Brenda added up as far as twenty, she went one plus two plus three, one plus two plus three plus four plus five all the way up to plus twenty. Eh … Anne thinks it is twenty multiplied by nineteen … so … we have got some more people now coming in. Myles?	190 1 + 2 + 3 … + 20 1 + 2 + 3 … + 19 20 x 19	
415 Myles: I think the answer is … what do you call it … a hundred and seventy-nine.	Myles found answer of 179 by adding from 1 – 19.	
416 T.D.: Where are you getting the hundred and seventy-nine from?		Press
417 Myles: Em, cos I added one to nineteen		
418 T.D.: You added one to nineteen and you got a hundred and seventy nine. Ok, that's a possibility … Enda?		Repeat Maintain
419 Enda: Well twenty multiplied by nineteen is eh two hundred and eighty, well it's either three hundred and eighty or two hundred and eighty. I am not sure, three hundred and eighty, I think.	Enda suggested that 20 x 19 was either 380 or 280	
420 T.D.: Hmmm		Confirm
421 Enda: And eh, eh so …		
422 T.D.: You think that might be a bit out?		Elicit
423 Enda: Yeah.		
424 T.D.: David, you were dying to say something there, go on!		Direct
425 David: Em, well if you added twenty again eh that means you would have twenty one.	David suggested that, in finding the number for 20 entrants, 20 should not be included	
426 T.D.: Twenty-one so you don't think Brenda … she should have left out the twenty, should she?		Repeat Elicit
427 David: Yeah.		
428 T.D.: Yeah, ok maybe what I will do is, well I will just take one more. I am always saying I am going to finish up but then I don't finish up! But anyway, yeah?		Direct
429 Myles: I changed my answer to a hundred and ninety.	Myles changed his mind and decided that answer was 190.	
430 T.D.: A hundred and ninety you think. Ok so you have added up, so actually what Myles did, he added … yes, Brenda?		Repeat Broadcast
431 Brenda: Em I got a hundred and ninety cos I left out the twenty.	Brenda also obtained a solution of 190	

432 T.D.: So you got a hundred and …?		Repeat
433 Brenda: … ninety.		
434 T.D.: A hundred and ninety. So by adding nineteen and eighteen and seventeen and so on, you get a hundred and ninety. It seems that the answer is a hundred and ninety. So I wonder what about Anne's … is there a quick way. As Enda said … as Enda said you know it gets very awkward to have to add all the numbers, if you have to add all the numbers it gets really, really awkward…Wouldn't it be very handy if you could find a quick way of doing it?		Repeat Rephrase Rebroadcast Anne (turn 409) Rebroadcast Enda (turn 214) Direct
435 David: Oh!	Exclamation	
436 T.D.: David, you seem to have an idea, what is it?		Direct
437 David: Em, well it might not work all the time but twenty nineteens is three hundred and eighty and half that is one hundred and ninety.	David proposed a formula	
438 T.D.: Nolan, what do you think?		Direct
439 Nolan: I think you () ninety.		
440 T.D.: That's one way of doing it. So David has come up with an idea here, David has an idea… will we come back to it, will we just leave it at that just til tomorrow. I know Brenda, were you going to …?		Insert Rebroadcast
441 Brenda: Eh I was just going to say a good way is to use the calculator.		
442 T.D.: Yes, yeah, it is yeah.		Insert

There is evidence that 'co-acting' (see Dooley, 2007) took place in this lesson. For example, in turn 419, Enda 'reworked' Anne's idea when he suggested that twenty multiplied by nineteen would yield a solution that was too high. In turn 425, David commented on Brenda's idea when he said that twenty should not be included in the summation. Brenda then corrected her own initial idea by summing from one to nineteen (see turn 431). The rule that David expressed in turn 437 ("Twenty nineteens is three hundred and eighty and half that is one hundred and ninety") is an elaboration of Anne's earlier conjecture ("You could do twenty multiplied by nineteen" in turn 409), and of the confirmation given by Brenda and others that the solution was in fact one hundred and ninety for twenty participants.

David's construction in turn 437 was 'partial' –he was able to justify it on empirical grounds but not structurally, i.e., it derived from a particular example rather than an overview of the situation (Rowland, 1999). This partial construct is embedded however in a larger construct of which he seems more certain. His insight is not only about a possible formula but also about the nature of an inductive conclusion. An inductive conclusion is a conjecture because it does not have the status of certain knowledge nor is it an uninformed guess (Rowland, 2000). David's "Well, it might not work all the time" tells something of his disposition. Further, the word ‘might’ is an example of the plausibility shield used by a speaker to express his/her awareness that while the claim being made is a ‘truth estimate’ it may actually be false (Rowland, 2000: 26, 60). Such uncertainty is inherent in coming to know mathematics – "[W]hen mathematics is coming into being in the awareness of an individual, uncertainty is to be anticipated and expected" (Rowland, 2000: 116).

Teacher Moves

When Anne, in turn 409, suggested multiplying twenty by nineteen I endeavoured to maintain her idea in the public arena:

410 T.D.:	You could do twenty multiplied by nineteen. Why do you think twenty multiplied by nineteen? Do you agree with that? Now she’s thinking it’s twenty multiplied by nineteen? [ ]

Here I first repeated Anne’s idea and pressed her for justification. I then asked the rest of the class for an evaluation (“Do you agree with that?”) and broadcasted her idea (“Now she’s thinking …”). It is interesting that I did this for Anne and not for Fiona who suggested a correct solution in turn 407. One factor of course is that Anne’s contribution is approaching the actual formula. Another is that it is a ‘different’ idea and might inject new energy into the lesson that is nearing its conclusion. However, it was not explored in the next turn because my attention was diverted by another pupil.

In turn 417, I responded to Myles’s suggestion that the sum of 1 – 19 was one hundred and seventy-nine by repeating it and also declaring it a ‘possibility’. This served to ‘maintain’ his idea while also giving it a tentative status. In turn 422, I made an ‘elicit’ move in response to Enda’s input (“So you think that might be a bit out?”). One reason for this is that I may be reading between the lines. Enda spoke hesitantly (turn 421) and I may have engaged in some ‘face-saving’ but I may also have wanted to focus on an important mathematical point. In turn 426, I expanded on David’s refutation of Brenda’s strategy (“So you don’t think Brenda… she should have left out the twenty, should she?”). Here my ‘elicit’ move was made for a different purpose. David seemed quite confident in his assertion but may not have wanted to hurt Brenda by mentioning her name. I, on the other hand, had attributed this idea to Brenda in the review (turn 414) and now alerted the students to the fact that David was refuting her strategy. In turn 439, my use of the plausibility shield ‘It seems’ was an avoidance of ‘telling’ students that the solution found by Myles and Brenda (and earlier by David) was one hundred and ninety - this was an effort to maintain a ‘conjecturing atmosphere’ (Mason, 2002: 109). I then rebroadcasted Anne’s earlier conjecture and intimated a connection between it and a ‘quick way’. It is possible that my increased pace was due to the fact that it is nearing the end of the lesson and I feared losing the key to the ‘formula’. My reference to ‘Enda’ in this turn concerns an earlier statement by him that, for higher numbers, “it would take ages” to use the summation procedure (i.e., (n-1) + (n-2) + … + 0). I gave this input little attention at the time because I was concentrating on clarification of the problem ‘conditions’ with the remainder of the class. Here was a more opportune moment to give it the consideration it warranted. Nolan’s input in turn 439 is unclear but he may have been referring to an earlier idea he expressed that the number of games required for twenty was ninety. Even though I did not dismiss that idea (“That’s one way of doing it”), it was David’s conjecture that I emphasised to the class ("So David has an idea here") and I included it with the others on the blackboard as one more possibility.

DISCUSSION

This lesson described here and the teacher follow-up moves are not presented as exemplary. The 'asymmetry of power' between teacher and pupils was not eliminated by my adoption of a less evaluative role – after all, I was in a position to decide on the ideas that were to be reuttered for further consideration. Nonetheless an interesting aspect of the conversation described above was that pupils began to reflect and comment on each other's suggestions and conjectures. Such reflection and commenting does not seem to have been viewed as disrespectful by the originator of the idea. This is evidenced by the fact that Brenda seemed to have little difficulty taking part in the conversation after her peers had discussed her input. It is possible that students do not perceive comments made by other students in plenary discussion as judicial or evaluative. Another factor in the construction of insight is that of *time* to reflect on ideas. When we returned to the lesson on the following day, pupils were able to elaborate on and give a mathematical justification for David's insight. Of particular note also is that not all students who made contributions to the discussion are considered to be high attainers on the basis of results of standardized mathematics tests. It would seem that in order to increase participation by students in mathematics, teachers have to 'let go' of the rigid view of the subject and allow ideas, even those that are 'incorrect', to 'stumble across each other' (Davis and Simmt, 2003).

REFERENCES

Boaler, J. (2002). The development of disciplinary relationships: Knowledge, practice, and identity in mathematics classrooms. For the Learning of Mathematics, 22(1), 42 - 47.

Brodie, K. (2008). Towards a language of description for changing pedagogy. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano and A. Sepulveda (Eds.), Proceedings of the Joint Meeting of PME 32 and PME-NA XXX (Vol. 2, pp. 209 - 216). Mexico: Cinvestav-UMSNH.

Cobb, Wood, T., Yackel, E., and McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. American Educational Research Journal, 29, 573 - 604.

D.E.S./N.C.C.A. (1999). Primary School Curriculum: Mathematics.

Davis, B., and Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. Journal for Research in Mathematics Education, 34, 137 - 167.

Dooley, T. (2007). How the 'stumbling of ideas across each other' facilitates insight in primary mathematics. In S. Close, D. Corcoran and T. Dooley (Eds.), Proceedings of the Second National Conference on Research in Mathematics Education (pp. 43 - 54). Dublin: St. Patrick's College.

Erickson, F. (1992). Ethnographic microanalysis of interaction. In M. D. LeCompte, W. L. Millroy and J. Preissle (Eds.), The handbook of qualitative research in education (pp. 201 - 226). San Diego and London: Academic Press, Inc.

Forman, E. A., and Larreamendy-Joerns, J. (1998). Making explicit the implicit: Classroom explanations and conversational implicatures. Mind, Culture and Activity, 5(2), 105 - 113.

Herbel-Eisenmann, B., Drake, C., and Cirillo, M. (2009). "Muddying the clear waters": Teachers' take-up of the linguistic idea of revoicing. Teaching and Teacher Education, 25(2), 268 - 277.

Hershkowitz, R., Schwartz, B., and Dreyfus, T. (2001). Abstraction in context: Epistemic actions. Journal for Research in Mathematics Education, 32(2), 195 - 222.

Hodgen, J. (2007). Formative assessment: Tools for transfroming school mathematics towards a dialogic practice? . In D. Pitta-Pantazi and G. Philippou (Eds.), Fifth Congress of the European Society for Research in Mathematics Education (pp. 1886 - 1895). Larnaca, Cyprus: University of Cyprus.

Martin, L. C., and Towers, J. (2009). Improvisational coactions and the growth of collective mathematical understanding. Research in Mathematics Education, 11(1), 1 - 19.

Mason, J. (2002). Generalisation and algebra: Exploiting children's powers. In L. Haggerty (Ed.), Aspects of teaching secondary mathematics: Perspectives on practice (pp. 105 - 120). London: RoutledgeFalmer.

Mehan, H. (1979). Learning lessons: Social organization in the classroom. Cambridge, MA: Harvard University Press.

Murphy, B. (2004). Practice in Irish infants classrooms in the context of the Irish Primary School Curriculum: Insights from a study of curriculum implementation. International Journal of Early Years Education, 12(3), 245 - 257.

O'Connor, M. C. (2001). "Can any fraction be turned into a decimal?" A case study of a mathematical group discussion. Educational Studies in Mathematics, 46, 143 - 185.

O'Connor, M. C., and Michaels, S. (1996). Shifting participant frameworks: Orchestrating thinking practices in group discussion. In D. Hicks (Ed.), Discourse, Learning and Schooling (pp. 63 - 103). Cambridge: Cambridge University Press.

Pimm, D. (1994). Spoken mathematical classroom culture: Artifice and artificiality. In S. Lerman (Ed.), Cultural perspectives on the mathematics classroom (pp. 133 - 147). Dordrecht: Kluwer Academic Publishers.

Rowland, T. (1999). 'i' is for induction. Mathematics Teaching, 167, 23 - 27.

Rowland, T. (2000). The pragmatics of mathematics education: Vagueness in mathematical discourse London: Falmer Press.

Sherin, M. G. (2002). A balancing act: Developing a discourse community in a mathematics classroom. Journal of Mathematics Teacher Education, 5, 205 - 233.

Shiel, G., Surgenor, P., Close, S., and Millar, D. (2006). The 2004 national assessment of mathematics achievement. Dublin: Educational Research Centre.

Sinclair, J. M., and Coulthard, M. (1975). Towards an analysis of discourse: The English used by teachers and pupils. London: Oxford University Press.

DO MATHEMATICS TEXTBOOKS OR WORKBOOKS ENHANCE THE TEACHING OF MATHEMATICS IN EARLY CHILDHOOD? : VIEWS OF TEACHERS OF FOUR-AND–FIVE-YEAR OLD CHILDREN IN PRIMARY SCHOOLS IN IRELAND

Elizabeth Dunphy

St. Patrick's College

In recent years considerable consensus has emerged regarding what constitutes high-quality mathematics teaching in early childhood (e.g. Clements, Sarama and DiBiase 2004; Australian Association of Mathematics Teachers and Early Childhood Australia 2006). Pedagogic recommendations suggest, for instance, that mathematical experiences for young children should help them to see and understand the mathematics in everyday activities and conversations; they should elicit children's own mathematical ideas and strategies; they should encourage children to engage in mathematical processes and they should enable the development of dispositions which support mathematical learning and development (Clements, Sarama and DiBiase 2004). In 2007, in a nationally representative questionnaire survey, 95% of teachers of four- and five- year old children attending primary schools in Ireland reported that they used textbooks/workbooks with the children. Approximately four-fifths of these teachers stated that they though that textbooks enhanced their teaching in some respect(s). This paper presents findings in relation why and how teachers use textbooks. Issues related to pedagogy, curriculum and teacher confidence are discussed. Implications for teacher education are also considered.

PEDAGOGICAL GUIDANCE FOR TEACHING MATHEMATICS IN EARLY CHILDHOOD

In considering how best to work with young children to develop their interest and understanding, the advice of Clements and Sarama that "Although mathematics is to be taught, such teaching should be consistent with young children's unique styles of learning" (2000, p. 40), appears to be both sensible and pragmatic. Ginsburg et al. (2005, p. 176) characterise the adult's role in early childhood mathematics education for 3-, 4- and 5-year old children as one of providing what they term *strong adult guidance.* However, they also caution that this involves more than a push-down curriculum i.e. *a curriculum originally designed for older children* (2005, p. 175) and they are clear that what *is* required is that teachers change the way they teach to ensure that the pedagogy used is appropriate for this age-group. Recently, in some countries, there have been substantial efforts to guide early childhood teachers in relation to pedagogy. In Australia there is now a joint statement regarding pedagogic recommendations in respect of early mathematics teaching (Australian Association of Mathematics Teachers and Early Childhood Australia, 2006). This specifies sixteen recommendations which deal with the range of aspects of pedagogy including interactions and communication, planning, resources, assessment, building on children's experiences, key learning and the role of language.

In the United States, seventeen research-based Standards for Early Childhood Mathematics Education were agreed (Clements, Sarama and DiBiase, 2004). In essence these recommendations present a vision for practice in relation to teaching mathematics to children in the age range 2 to 8 years. Eight of the standards pertain directly to pedagogy and these emphasise processes as well as content; mathematical experiences based on children's daily activities and interests; the role of the teacher in helping children to reflect on and extend the mathematics they encounter; planned and informal activity; the role of children's language, culture and outside-school experiences; the understanding of individual children's ideas and strategies in determining teaching approaches; the relationship between conceptual knowledge and skills; and the role of ongoing observation of children as they engage in, and talk about, mathematically-related situations.

PROVISION FOR MATHEMATICAL DEVELOPMENT IN JUNIOR INFANT (JI) CLASSES IN IRELAND

In Ireland there is a long tradition of quite young children attending primary school with about 50% of all four-year-old children and about 95% of all five-year old children enrolled. There is a statutory mathematics curriculum for children attending primary schools (Government of Ireland, 1999a; 1999b). During initial teacher education, all prospective primary teachers undertake a course on the teaching of mathematics. They are also exposed to ideas related to the understanding of learning in early childhood and to key issues related to mathematical development in the early years. Furthermore, to undertake a course in initial teacher education, all candidates must reach a specified proficiency level in terms of mathematics. These are all issues deemed to be critical in determining the quality of teaching and learning of mathematics in early childhood education (Ginsburg and Ertle, 2008).

THE USE OF MATHEMATICS TEXTBOOKS IN EARLY CHILDHOOD

The use of a mathematics textbook with young children is suggestive of a somewhat formal approach to mathematical teaching and learning. While the question of the nature and extent of mathematics textbook use with young children in the first year of school has not been addressed in research, a recent study indicated widespread use of textbooks and workbooks with children aged five-and six-years, in the second year of formal school in Ireland. In that study the mathematics teaching observed by Murphy (2004) was judged to be teacher-directed and focused, and traditional in nature. As described there was very little interactive discussion and a "limited supply of concrete materials" (pp. 253-4), suggesting perhaps that the textbook was replacing or even displacing more desirable child-centred and interactive approaches.

In the United States, Fuson's (2004) analysis of the textbooks on offer to teachers of young children is that there is a common misuse of photos, drawings and cartoons. She also observes that textbooks generally show little understanding of children's progression of methods and they rarely provide sufficient time for children to master particular ideas. Research has shown that children from countries where they start school later overtake English children in their mathematical understanding (Aubrey et al., 2000) and it is suggested that it is the early emphasis on written arithmetic which is detrimental to English children's mathematical competence (Gifford, 2005). Worthington (2007) found that the engagement of children in recording mathematics is an area of confusion for many teachers of children in the age range four- to seven, with few recognising the importance of encouraging children to use their own methods of representing their mathematical thinking.

An important part of provision in the minds of JI teachers appears to be the use of textbooks. In a recent nationwide survey, the vast majority of teachers (96%) reported that they use a maths textbook or workbook with children, aged four- and five-years (Dunphy, 2007). It appears then that for most young children in JI classes, completing mathematical exercises in textbooks and workbooks is a common feature of their mathematically-related activities and experiences. Questions then arise as to why teachers use textbooks and also how they use them. The question of the possible consequences for children's learning and development in the area of mathematics also arises.

KEY RESEARCH QUESTIONS

Arising from the review above the following research questions are stated:

1. To what extent do JI teachers agree/disagree that mathematical textbooks or workbooks enhance their teaching of mathematics?
2. Where teachers report that they use textbooks or workbooks, what are their perceptions of the benefits of using a textbook/workbook with children in JI?

Method

In Spring 2007 a national sample of 346 schools (c. 460 JI teachers) were surveyed about aspects of their mathematics pedagogy. The total number of respondents to the survey was 266. Full information regarding issues such as questionnaire design, sampling and piloting is available elsewhere (Dunphy 2007; 2009).

The participants

The participants were overwhelmingly female (94%) illustrating the extent to which teaching children in the first year of school in Ireland is predominately a female endeavour. Over a quarter of the teachers in the survey sample had less than five years teaching experience at any level in primary schools, and a further quarter had less than 10 years teaching experience.

The majority of teachers (55%) had more than 23 children in their class. About one-fifth (22%) of teachers classified their schools as having "disadvantaged" status (schools designated disadvantaged enjoy lower than average pupil-teacher ratios and many of them enjoy a maximum class size of 15). Only about 8% of teachers reported that they had a classroom assistant. Significantly, in terms of some of the findings reported later in the paper, 47% of teachers were working with either one (20%), two (14%) or three (13%) other classes alongside the junior infant class.

FINDINGS

Teachers were invited to give an extended answer to the question "Do you think a textbook/workbook enhances the teaching of mathematics in JI classes?"

Of the 266 participants, 10 teachers didn't respond to the question at all and 18 responses were ambivalent or didn't address the question as asked. Approximately four-fifths of participants (n=214) stated that they thought that textbooks enhanced their teaching in some respect(s). While some commented that they considered “enhanced” to be too strong a claim, they suggested that textbooks were a “tool” which supported the teaching and learning.

Twenty four teachers (9%) stated that they felt that textbooks did *not* enhance their teaching. Since 96% of participants had indicated that they were using a textbook this implies that a small number of teachers were using textbooks despite their views that they do not enhance teaching. Comments such as the following illustrate their perspectives:

“I feel that textbooks have no benefit at all at junior level and distract from practical concrete activities.” (P28) “I believe that children this young benefit more from games and learning through doing. I think that textbooks can also hold back because [the teacher] can fall into the habit of following the book instead of planning for enjoyment.” (171) “It’s mainly a colouring book.” (P250)

The extended nature of the majority of teachers’ responses provided data related to the ways in which they rationalised their use of mathematical textbooks/workbooks in JI classes. From the analysis of the responses a number of themes emerged. These related to assessment and accountability; perceived benefits of textbooks; and classroom management. Many teachers offered critical comments in relation to the use of textbooks, clearly recognising their limitations. These findings are also presented below.

Assessment and Accountability

Approximately one in six teachers (n=43) stated that in their opinion textbooks were important for assessment purposes. Comments included:

“It backs up the oral lesson and gives the teacher something concrete to assess if the child has grasped the concept.” (P137)
“It provides a good record of work done by children and their level of understanding of a topic (P31)
“I find the textbook helpful at the end of the work to assess what children have learned. It helps me identify any weaknesses I need to work on.” (P193)

Some participants (n=13) specifically referred to their perceptions of the usefulness of textbooks for accountability purposes. These participants appeared to feel the need to provide tangible evidence of work for parents and, to a lesser extent, inspectors. For instance, one teacher stated “I have taught many lessons where we didn’t use the book, but I think parents feel the book needs to be completed, that it is proper work.”(P90) Another stated “It’s expected of you.” (P207)

Benefits for teachers

The benefits for teachers were perceived to be related to consolidating and extending learning; structuring the programme of work; and classroom management.

Consolidating and extending learning

About one-fifth of participants (n=50) spoke explicitly of how they found textbooks useful for moving children from the concrete situation to the abstract; for promoting discussion with children; as a means for children to practise skills such as the writing of numerals. Teachers were generally at pains to emphasise that the textbook was used only *after* the initial exploration of concepts through activity or discussion. As expressed by one teacher "It provides opportunity for the children to consolidate their learning and practise skills." (P71) Another suggested that "a book is essential to consolidate information and material learned. Children also need practice to reinforce concepts they have learned and being able to transfer thought to paper is another skill." (P214).

Structuring the programme

About one-eight of participants (n=32) reported that they used the textbook for structuring the programme of work and for guidance. "The textbook gives a structure to the years work." (P16) Some seemed to follow it closely as evidenced by comments such as the following: "It can act as a guide in structuring your maths programme." (P3); "Number activities are presented in a sequential order …" (P21); "…it does ensure that all areas are covered." (P49); "It gives you a guidance as to what exact concepts should be covered and how far you should go in relation to using appropriate mathematical language for them. You know exactly what they need to know." (P89)

Classroom management

The survey revealed that 47% of teachers were working in consecutive or multiclass situations. Textbooks were seen by many participants (P40) as an important means of coping in these contexts: "I have always used a textbook when teaching JI and find it essential, especially as I work in a multiclass situation." (P117); "I think that in the context of a multiclass situation textbooks are a necessary evil. While I would love to explore more practical elements with junior infants, the reality however is that having multigrade classes puts pressure upon myself in delivering a comprehensive maths programme for all classes." (P221). One teacher commented that the textbook in a multiclass setting was helpful since "it keeps groups busy." (134) Another stated that "It's important in my situation to have some work to set as I move around three different class levels" (P96).

Outside of the multiclass context, teachers commented that the textbook is 'needed in big groups' (P207). One teacher (P135) stated that the textbook was a "great fallback" in dealing with the range of abilities in a large group.

Benefits for children

The benefits for children, as articulated by the participants, were mainly related to affect. Some teachers (n=20) commented on the fact that the use of a textbook appeared to enhance children's enjoyment of the mathematics class. One teacher commented "Children enjoy looking back over work done and often comment on how much better their colouring or numbers are now.'' (P88). Similarly "Children enjoy written and colouring activities …" (P264). Another also focused on non-mathematical content in the comment that "Children like the workbooks and it does provide a task that will develop fine motor skills also … "(P96). Another commented "I do think it keeps them interested. They enjoy the activities..." (P148). There were a few comments (6) related to the children's pride and sense of achievement as a result of their interaction with the textbook: "The children themselves like to have and feel a sense of achievement and pride in completing them." (P34); '"It's very encouraging when they see their work is correct and they can look back at the end of the year and see all the progress they've made…" (P206)

Limitations

The pressure to complete

A small number of teachers (n=13) spoke of the pressure that the textbook imposed on their work with the children. "Textbooks tend to put pressure on teachers to complete them…" (P53) "The teacher feels under pressure to structure to structure plans and lessons around the publisher's schedule." (P266) "I've got two workbooks which I feel I'm killing myself to finish before the end of the year for the parents sake as opposed to the children." (P156)

However not all teachers felt this pressure: "Very often there are useless pages which I leave undone." (P50) "I feel that it is more important that children understand the concepts than have a filled-in workbook at the end of the year."(P57)

The possibility of wasting children's time

A few teachers (n=9) spoke of the detrimental effects of workbooks on the learning experiences of children. In particular they spoke of the waste of children's time " …it can consume too much of the maths time." (P162) "At times in the past I have asked children to do a page in the workbook that maybe useless in regards to learning or perhaps confusing." (P79) "Practically doing tasks is more meaningful but often a question of time." (P82)

Teachers also felt that the textbooks were often unengaging and provided little challenge for children. "There is not enough in the books. Children get bored easily using them, and there is way too much reliance on colouring…"(P181) "Lots of books in print have no challenge at all for the children and are quite boring." (P94) "I do find it provides limited experiences for the child." (P56)

Critical and judicious use of textbooks

A minority of teachers communicated by their comments that they used textbooks in judicious ways. As one teacher stated: "Textbooks do have their place in the classroom but should not be given top priority."(P42) Another teacher described how the textbook was useful "… as guidance and to dip in and out of." (P4) Other discerning comments suggested "…textbooks never cover the whole of any class program." (P32) "I think they keep me on track but I think Maths in Infants should be primarily activity and experience-based …"(P45) "The book doesn't dictate what topic you move onto, the children and the curriculum do." (P95) "The maths book is good as a base. However a lot of extra work is required also …games and hands-on concrete materials are the best though for getting the children to understand."(P240)

DISCUSSION

It appears that mathematics textbooks or workbooks are key tools for teachers of junior infants. Teachers report that they find them useful for a number of purposes and particularly when coping in multiclass situations and with large groups of children. Analysis of teachers' comments provided some insight into key aspects of pedagogy and curriculum. For instance, it appears that some teachers may be working with a restricted view of assessment, seeing it as a process whereby children need to show evidence of mathematical understanding by recording something on paper. Further research would help ascertain the extent to which teachers rely on this strategy, or whether in fact they use other strategies, such as observation and discussion, generally seen as more useful with young children (Dunphy, 2008).

There were also important issues raised about teacher confidence in terms of teaching mathematics. A striking feature of the findings was the extent to which textbooks appeared to be determining the enacted curriculum for children in JI classes. In most instances, participants appeared to consider the textbook and the curriculum as one and the same thing. This suggests that teachers, despite strong advice to the contrary, generally do *not* take also account of children's own interests, concerns, experiences and questions in determining the curriculum. It also suggests that they are not using the statutory curriculum guidance materials in any extensive way. In the only reference to the curriculum, just one teacher commented "I look very closely at the curriculum to make sure that I'm covering every part of every strand." (P242). An important issue then is the way in which most teachers appeared to position themselves in relation to the textbook. Their comments suggested that they very much followed the textbook closely, prioritizing the work in that. In many cases they appeared to put aside pedagogical considerations such as the interests of the children or their dispositional development, even sometimes despite a stated awareness of the limitations of the textbooks. Others appeared to feel constrained by the textbook, but felt compelled to use it anyway. Only a very small minority made comments that suggested a judicious and discerning use of the textbook, these teachers appeared to see it very much as a " tool" rather than a "master".

What emerges then is a picture of a textbook-centred pedagogy rather than a child-centred one. Teachers are concerned that children learn to put their mathematical understandings on paper, and this concern seems to dominate how teachers use the time available for planned mathematically-related learning in JI. If participants were concerned to provide challenging and engaging mathematical activities and to base their pedagogy on genuine mathematical experiences based on children's lives, this did not emerge in the data. However, the suggestion by some, that textbook work is a waste of children's time may be important since in the overall context of the school day there is limited time available to devote to planned activity in mathematics, and that available should be used to the best possible effect. Undoubtedly there are difficulties for teachers in moving from a textbook-dependent culture, especially so in circumstances where a consecutive or multiclass structure applies. A first step then, appears to be one of helping teachers determine those aspects of mathematical development textbooks *can* help with, but particularly to identify those aspects of early mathematics which *cannot* be developed using a textbook approach. Teachers need to appreciate that children may not see either the purpose or value of many of the procedures, conventions and symbols that we introduce to them through the use of textbooks. Nor may they be paying attention to the mathematical aspects of situations featured in such texts and which adults sometimes assume are self-evident. Teachers also need to appreciate that textbooks cannot provide essential first-hand experiences of mathematics; they generally do not relate to concrete meaningful experiences; they cannot engage children in purposeful mathematical problem-solving based on real events in children's lives; they cannot build on, nor recognise, children's informal experiences of mathematics; and they generally assume that children will apply the logic of adults to the situations presented on the page.

All of the above have implications for how children are inducted into mathematical ways of thinking and communicating. In the Irish context, the current situation whereby there is a careful specification of curriculum content for children in the infant classes needs to be accompanied by an equally careful specification of pedagogy. Teachers need to be strongly encouraged to move away from a textbook-centred pedagogy and to explore the potential of play, stories and games in terms of developing mathematical understandings and dispositions. Teachers may need to be reminded that the overall aim is to develop real understanding and lasting interest in mathematics. Guidance for teachers needs to emphasise the importance of focusing on mathematical education rather than on curriculum, or indeed textbook, coverage.

REFERENCES

Australian Association of Mathematics Teachers & Early Childhood Australia. (2006). *Position paper on early childhood mathematics.* Accessed on May 21 2009 at http://www.aamt.edu.au/Documentation/Statements/Position-Paper-on-Early-Childhood-Mathematics-print-friendly

Aubrey, C., Kavkler, M., Tancig, S. & Magajna, L. (2000). Getting it right from the start? The influence of early school entry on later achievements in mathematics, *European Early Childhood Education Research Journal*, 8, 1, 75-85.

Clements, D. & Sarama, J. (2000). Standards for Preschoolers. *Teaching children Mathematics*, September 2000, 38-41.

Clements, D., J. Sarama, & A. DiBiase, Eds. (2004). *Engaging young children in mathematics: Standards for early childhood mathematics education.* New Jersey: Lawrence Erlbaum Associates.

Dunphy, E. (2007). Exploring pedagogy in early years mathematics in the first year of primary school in Ireland: A survey of teachers' views. In S. Close, D. Corcoran, & T Dooley (Eds.), *Proceedings of Second National Conference on Research in Mathematics Education: MEI 2* (pp. 290-307). http://www.spd.dcu.ie/main/academic/education/documents/proceedingsMEI2.pdf

Dunphy, E. (2008) The Framework for Early Learning: A background paper. *Supporting early learning and development through formative assessment.* May 2008. Retrievable at http://www.ncca.ie/

Dunphy, E. (2009). Early childhood mathematics teaching: Challenges, difficulties and priorities of teachers of young children in primary schools in Ireland. *International Journal of Early Years Education*, 17, 1, 3-16.

Fuson, K. (2004) Pre-K to grade 2 goals and standards: Achieving 21st Century mastery for all. In D. Clements, J. Sarama, & A. DiBiase (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 105-148). New Jersey: Lawrence Erlbaum Associates.

Gifford, S. (2005). *Teaching mathematics 3-5: Developing learning in the foundation stage.* Open University Press: UK.

Ginsburg, H. & Ertle, B. (2008) Knowing the mathematics in early childhood mathematics. In O. Saracho & B. Spodek (Eds.), *Contemporary perspectives on mathematics in early childhood education* (pp. 45-6). Charlotte, NC: Information Age Publishing.

Ginsburg, H., R. Kaplan, J. Cannon, M. Cordero, J. Eisenband, M. Galanter & M. Morganlander. (2005). Helping early childhood educators to teach mathematics. In M. Zaslen & I. Marting-Bak (Eds.) *Critical issues-early childhood professional development* (pp. 171-202). Baltimore: Brooks Publishing.

Government of Ireland. (1999a). *Primary School Curriculum: Mathematics*. Dublin: Stationary Office.

____________________. (1999b). *Primary School Curriculum: Mathematics: Teacher Guidelines.* Dublin: Stationary Office.

Murphy, B. (2004). Practice in Irish infant classrooms in the context of the Irish Primary School Curriculum (1999): Insights from a study of curriculum implementation. *International Journal of Early Years Education*, 12, 3, 245-257.

Worthington, M. (2007). Multi-modality, play and children's mark making in maths. In J. Moyles (Ed.), *Early years foundations: Meeting the challenge* (pp. 253-269). UK: Open University Press.

THE USE OF MATHEMATICS TEXTBOOKS TO PROMOTE UNDERSTANDING IN THE LOWER PRIMARY YEARS

Lorraine Harbison

University of Newcastle upon Tyne

The Mathematics Primary School Curriculum was revised ten years ago. Research has shown that mathematics textbooks play an important role in helping teachers implement this curriculum. However, whether it is possible to teach Mathematics for understanding by relying on textbooks as the main method of pedagogy is uncertain, particularly in the early years. Evidence of the use of mathematics textbooks was obtained from questionnaire surveys about practice, completed by 48 classroom teachers across the four class levels from Junior Infants to Second Class. The questionnaire data showed that textbooks are used by teachers in the lower primary years as one of the primary resources in teaching the curriculum. Teachers are satisfied that textbooks adequately cover the content of the curriculum although opportunities for formative assessment are limited and the textbooks generally do not address the question of differentiation. The length of the textbooks was identified by teachers as having a negative impact on teaching. Pressure to complete the textbooks dictated the teaching methods used, the content covered and the sequencing and ordering of activities. The over-all finding of this study indicates that mathematics textbooks are useful in supporting teachers to implement the Mathematics Primary School Curriculum with understanding but with significant limitations.

INTRODUCTION

The revised *Mathematics Primary School Curriculum* (Department of Education & Science (DES), 1999a) is based on a philosophy and psychology of teaching and learning Mathematics with understanding. In order to teach Mathematics for understanding, children must have the opportunity to be active participants in developing their own understandings. Pedagogy that supports the development of mathematical understanding engages children in social interaction, collaborative learning, practical activities and stimulating investigations that require children to use and apply knowledge, previous learning and other experiences. Children who have learnt Mathematics with understanding display a comprehensive knowledge of mathematical facts and concepts as well as a procedural knowledge in which to manipulate this factual knowledge. More importantly, children who understand Mathematics demonstrate an ability to know why certain facts and procedures are used in order to solve mathematical problems in a variety of contexts. The ultimate aim of teaching Mathematics for understanding is that learning is generative, meaning that Mathematics learned in one situation can be transferred to new or different situations (Newton, 2000).

Teaching Mathematics for understanding as advocated by the revised curriculum introduced a shift away from the perception of Mathematics as a fixed body of knowledge to be transmitted from the teacher to the child to that which views the role of the teacher as scaffolding children's learning through guided-discovery activities (DES, 1999a, p.5). It further introduced a movement away from the perception of Mathematics as a subject that involves "doing a range of numerical calculations by hand to a perspective in which children are taught to understand and solve the problems they confront in everyday life, in other school subjects and in the subject of Mathematics itself" (Connolly, 2003, p.54).

Within this context of curriculum reform this research asked a sample of 48 practising teachers to evaluate the extent to which pupils' mathematics textbooks support them in implementing the Mathematics curriculum with understanding in classrooms of the lower primary years in Ireland. This research also provides a window on the varying usage of textbooks and the data generated are useful in suggesting possible rough bounds on emphasis and of demonstrated importance to mathematics textbooks within the first four years of school.

Mathematics textbooks

The *Mathematics Teacher Guidelines* encourages teachers to have a wide variety of textbooks available to children in their class based on the quality of their content in particular strands: Number, Algebra, Shape and space, Measures and Data.

> …consideration should be given to how they reflect the objectives of the curriculum. In general, textbooks should have a balanced treatment of all the strands, varied presentation of problems and an emphasis on the use of manipulatives. They should encourage investigation and provide the child with structured opportunities to engage in exploratory activities (DES, 1999b, p.18).

According to Ó'Dúnlaing (1999), mathematics textbook for children, however carefully prepared, limit children's mathematical learning experiences to pictures and symbols. Textbooks can be attractive to look at, being crammed with pictures, but pictures may not be adequate enough. As indicated by Liebeck (1990, p.16), no book for young children can start where they need to start, namely with experiences and spoken language.

Despite these critiques, research on the use of mathematics textbooks in classrooms following on from the introduction of the revised curriculum demonstrated the importance of mathematics textbooks in supporting the implementation of the curriculum (Surgenor, Shiel, Close & Millar, 2006, p.22). Further evidence on the use of textbooks came from the review of the Mathematics curriculum carried out in 2005 by the National Council for Curriculum and Assessment (NCCA). The data suggested that pupils' textbooks are used widely by teachers to support them in implementing the revised Mathematics curriculum in Ireland.

> Teachers explained that textbooks also serve the function of guiding curriculum implementation. In other words, teachers reported their satisfaction in knowing that by following the textbooks they were addressing the curriculum strand and strand units (NCCA, 2005, p.237).

The research hypothesis

Even though mathematics textbooks play a central role in most Irish classrooms, surprisingly little research has focused on teachers' use of these. Most of the research that has been carried out in other countries looked at the use of textbooks in the upper primary years, the results of which are largely inconclusive as to their value and often conflict on the issue of using textbooks to support the teaching of Mathematics with understanding. Bierhoff (1996), for example, concluded that textbooks are not solely or largely responsible for children's attainment; the way a teacher teaches is equally important.

The present research is an exploratory study which provides an insight into the attempts made by 48 teachers to bridge the gap between the Mathematics curriculum and their classroom practice by using textbooks as their main teaching tool or aid. In order to evaluate the use of textbooks to support mathematical understanding this research asks:

> How does this sample of teachers rate and use textbooks in their classrooms to help them scaffold the development of mathematical understanding in the first four years of primary school in Ireland?

METHODOLOGY

48 teachers took part in this study. Two teacher questionnaires were distributed. Half of the teachers completed the first questionnaire and the other half completed the second.

After an initial pilot, 24 teachers in a cluster sample of five primary schools in a single suburban area of Dublin with mixed socio-economic groups were invited to complete the first questionnaire in a bid to establish what, if any, mathematics textbooks were being used and how they were being used. The questionnaire was modelled on those of the Third International Mathematics and Science Study (TIMSS, 1995, cited in Foxman, 1999) and Surgenor et al. (2006). In addition, teachers were asked to evaluate in detail the design of their chosen textbook in relation to the factors that impede or contribute to readability and legibility.

A second questionnaire was distributed to a further cohort of 24 teachers in a convenience sample (local schools, teaching colleagues, etc.). This questionnaire contained some similar questions to the first, such as asking teachers to indicate on a Likert scale how much they relied on the mathematics textbooks for preparation, planning, teaching and assessment. The second questionnaire asked further questions which arose as a consequence of meeting with publishers of the four most widely used textbooks; CJ Fallon, Folens, EDCO and Carroll/Heinemann. The teachers also had to evaluate the features of the textbooks that they felt contributed to or impeded understanding. A number of these questionnaires were completed by the researcher over the telephone or face-to-face.

Both questionnaires share a number of questions and these have been combined in statistical analysis. Other questions have been analysed independently. Both questionnaires also contained a number of free-response questions to allow teachers the opportunity to qualify and elaborate upon their grades. The free-response questions further helped to identify issues and these were analysed to look for consistent themes and opinions about the mathematics textbooks.

Questionnaires were initially completed by teachers and returned to the investigator. Data were collated to a computerised data base (Excel 2003, Microsoft Corporation, 2003). The analysis was performed using the statistical facilities of Excel 2003 and using a proprietary computerised statistical package (SPSS (version 14.0), Statsoft Corporation, 2005).

There are a number of limitations that need to be taken into consideration when interpreting this data. It was not possible to analyse the data using Chi-Squares as 80% of the cells did not have an expected frequency of 5 or greater and some cells had an expected frequency smaller that 1.0. Therefore, it was necessary to run the Fisher Exact Probability Test by dichotomising the data. Group one combined the negative responses, i.e. “fair/poor” or “never or almost never/rarely”, depending on the question posed. Group two combined the positive responses “good/very good/excellent” and “sometimes/often/very often”, again depending on the question posed. This allowed the investigator to run the Fisher exact probability test for a two row by three or four column contingency table.

The numbers are small, n=48, increasing the risk of type 1 error occurring. That is to say, that due to the small sample size, there is a chance of false-positive results. The small sample size further limits the power of the study to detect small differences between uses of the textbooks. However, it was not felt that such differences were likely to be significant in practice.

RESULTS

All teachers indicated that they used at least one mathematics textbook in their classrooms as their principal resource. These were, in order of the mathematics textbook most widely used, CJ Fallon (46%), Folens (33%), EDCO (17%) and Carroll/Heinemann (4%). Occasionally the teachers referred to the other mathematics textbooks, as recommended as good practice above, for further teaching aids. The EDCO textbooks were only used by teachers who used them as their main textbook, i.e. no teacher used the EDCO textbooks for supplementary teaching ideas. “Mental Maths” by Prim-Ed and “Maths Challenge” by Folens, were cited as extra workbooks used by teachers to give children the opportunity to practise mental strategies.

How the mathematics textbooks are used

The teachers were asked to rate how often they use the pupils’ textbook for deciding what topics to teach and how to present a topic, selecting tasks for class work, home work and assessment; using the Likert scale where 1= never or almost never (Less than once per term), 2 = rarely (Once or twice a term), 3 = sometimes (Once or twice per month), 4 = often (About once per week) and 5 = very often (More than once per week).

Deciding what topics to teach

Similar to the findings of Foxman (1999), there was a close, but not necessarily rigid, reliance on the mathematics textbook by teachers in planning their lessons. 94% of teachers indicated that they used the textbooks at least “Sometimes” to decide what topic to teach. The other 6% of teachers were keener to indicate that textbooks constitute one among several possible kinds of resource. For example, one experienced teacher only used the textbook in the final term, mainly for revision purposes. Nevertheless, on the whole, 83% of teachers followed a mathematics textbook fairly closely, because they felt it provided a sound structure for their teaching.

Deciding how to present a topic

60% of teachers indicated that the mathematics textbook dictated the methods used for teaching “Often” or “Very often”. A further 17% indicated in the qualitative analysis that the extent of this use was not by choice. For example, one teacher disliked using notation boards to teach place value as suggested in the Fallon’s textbook but felt “compelled to do so because of the pressure experienced in getting the textbook finished”.

Selecting problems and exercises for work in class

All teachers indicated that they used the pupils’ mathematics textbook to select problems and exercises for work in class, with just 4% of teachers only using textbooks “Rarely”. 23% of teachers thought that as the children are very young, it is difficult for them to record mathematics in copy books as they have not yet developed fine motor skills. One teacher commented that recording in textbooks teaches children good presentation skills. Another teacher mentioned that it is extremely time consuming to produce teacher designed worksheets and that these cannot compete with the glossy and colourful pages of mathematics textbooks that children are used to.

Selecting problems and exercises for homework.

No teacher of Infants gave homework and only 6% of teachers from First or Second Class gave homework “Very Often”. These teachers used either the Fallon or Folens textbooks, and predominantly homework constituted completing a page of the pupils’ textbook.

Assessment and evaluation

42% of teachers ‘Never” or “Rarely” used the pupils’ textbooks for assessment. Teachers of First and Second Class used the tests and revision sections included in the pupils’ textbooks, the photocopiable tests in the Folens’ teacher manuals and the assessment packs available from EDCO. These tests are summative rather than formative by nature. That is to say that the tests assess factual and procedural knowledge and provide an Assessment of Learning (A*o*L), the results of which are recorded using a simple tick or cross on a teacher designed checklist or those produced by the publishers.

The textbook as a support for teaching the Mathematics curriculum for understanding

The majority of teachers (79%) indicated that the textbook they used helped to support them in teaching Mathematics for understanding. 4% of teachers cited that, teaching for understanding, although an ideal goal, worked well in theory but that the textbooks did not, in general, adequately support them in implementing the curriculum effectively in the classroom. Challenges to teaching for understanding included the large class sizes, lack of resources, cumbersome pupils' textbooks and pressure to be seen to get through the curriculum by completing the textbook.

The teachers were also asked to rate specific factors of the textbooks that contributed to or impeded understanding using the Likert scale where 1 = poor, 2 = fair, 3 = good, 4 = very good and 5 = excellent.

Overall rating of the textbook

The majority of teachers (83%) gave the textbook that they used with the children in their class a positive rating with 4% of those giving an overall rating of "Excellent". However, a recurring theme that was mentioned by the other 17% of teachers was the cumbersome size of the textbooks, clearly cited by a Junior Infant teacher who indicated that "30 books are too heavy to carry around the classroom to distribute".

Legibility

The majority of teachers (87%) gave a positive grade for the overall legibility (i.e. the qualities/factors that affect readability) of the textbooks.

Of these factors, the majority of the teachers (63%) were of the opinion that the pupil textbooks were pleasing to look at. This meant that the cover illustration and visual appearance of the textbook got the attention of the children and made a positive difference to the enthusiasm with which the children approached the textbooks.

The layout of the textbooks was also graded positively by teachers with 33% indicating that it was very easy to find ones way around. The length was cited as causing a difficulty. For example, in Junior Infants children only learn the numbers 1 to 5, yet the textbooks, in general, contain over 100 pages. One teacher commented that it can be very difficult, in particular for the younger children, to locate the correct page of exercises in the textbooks.

All teachers rated the page layout to be at least "Good". Negative comments in relation to the individual textbooks were recorded in the qualitative responses. These included: having a difficulty with the Fallon's textbook as often more than one concept is introduced on a single page; the Folens textbooks could have benefitted from more coloured pictures, diagrams and worked examples to explain to the children how they could approach the exercises; there were too few exercises on each page of the Carroll/Heinemann textbooks and that, "Some pages take seconds to complete."

Meetings with the textbook publishers revealed that they had given a lot of consideration to the script or type style used in the textbooks to keep it as near to what the children were also seeing in their reading books. Despite this, one teacher rated the type style in the Folens textbook as "Poor" although no rationale was given for this response.

Curriculum coverage

All teachers indicated that they were confident that the textbook completely covered the curriculum with 25% of respondents giving the textbook they used an “Excellent” rating.

Frequency of topic-change and progression

The majority of teachers (71%) were happy with the frequency of topic-change and progression for the textbook that they used. The qualitative responses from the other 29% of teachers stated that, in their opinion, the textbooks tended to “jump” from one strand to the next and that the strands and strand units were not treated with increasing depth and complexity before moving on to the next topic. They further argued that the textbooks moved onto more advanced topics too quickly before it could reasonable be assumed that the great majority of the children had mastered the basic concepts.

Stimulating activities and investigations

63% of teachers positively rated the provision for stimulating activities and investigation for the textbook that they used. The qualitative responses indicated that the puzzle pages in the Folens and Fallon textbooks offered children opportunities to apply their mathematical knowledge. The other 37% of respondents thought that the mathematics textbooks did not provide adequate activities to support the transfer of learning from one context to another and that sometimes the contexts for the experiences were a little bit contrived and so were less meaningful to children.

Readability

Although 29% of teachers gave the textbook an “Excellent” rating, overall the readability of the textbooks was not graded as favourably as legibility. This was because teachers (10%) thought that the children were too young to be expected to read the textbooks on their own. This meant that the teachers tended to read the instructions to the class prior to asking the children to complete the exercises.

Strand coverage

34% of teachers thought that the textbooks were weighted in favour of Number. This they qualified as being un-avoidable as “strands such as Measure cannot be taught from a textbook”.

Number and variation of exercises

There was a mixed response to this question. One teacher who used the Fallon textbook thought that there were too many exercises and in some cases too many different types of exercises on a page of text. Other teachers (21%) welcomed the opportunity for children to engage in solving a variety of exercises. A Junior Infant teacher gave the Folens mathematics textbook a “Poor” grade due to the large volume of exercises in the textbook, many of which this teacher deemed unnecessary. The example given was the “eight pages of matching exercises at the start of the Junior Infant textbook”. Others (36%) thought that the textbook allowed the children plenty of opportunities to practise new concepts. No teacher gave the textbooks an “Excellent” grade. The EDCO textbook was viewed by two of the teachers as not having enough drill and practice exercises for the children. They felt the children would benefit from the opportunity to consolidate the facts learnt before moving onto a new concept. Two teachers commented that the children appeared to enjoy the exercises in the textbook and that the exercises were accessible to all.

Extension materials

58% of teachers found the range of extension materials that accompanied the textbook to be at least "Good". Positive comments were made in relation to the additional photocopiable worksheets that were included in most of the teacher manuals and the pre-page ideas in the Folens teacher manuals. Also mentioned were the box of activity cards that accompanied the Folens and EDCO textbooks, the Fallon *Shadow Books* and the EDCO *More Fun and Games Workbook*. However, 13% of teachers gave the range of extension materials a "Poor" rating. They thought that the extension materials were limited to the use of further worksheets and work cards and would have preferred if a variety of extension materials, resources and concrete materials were available.

Hands-on learning

Lack of tangible resources was cited by 42% of teachers as one of the greatest difficulties in teaching Mathematics with understanding. Although Folens provided enough concrete materials free for one group of children, such as a set of compare bears or multi-links, this was seen as inadequate in a classroom based setting. One teacher expressed a difficulty with the Fallon textbook in the use of notation boards to teach place value, hence the "Poor" rating. Two teachers cited having to acquire enough materials to support hands-on learning in the classroom as difficult, costly and time consuming.

Value for money

As there is no universal scheme of free books for primary school pupils, all mathematics textbooks have to be paid for by parents. Small grants are made available to provide books for pupils whose parents have limited financial resources. 8% of teachers indicated that because parents have to purchase the textbooks, there was undue pressure on teachers to get the textbooks completed. These teachers expressed a fear that parents may deduce that should the textbook not be completed, that the curriculum had not been adequately covered. Furthermore, one teacher noted that the Fallon textbooks use a large format so two textbooks are used in Senior Infants but that the advantage of greater durability would have to be offset against the increased expense of two textbooks.

Colour

Teachers indicated that the Carroll/Heinemann textbooks used too much colour and the Folens textbooks, too little. A concern was expressed that the children engaged in tasks that involved little or no mathematical learning but rather appeared to have "no purpose for many of the exercises other than to keep the children occupied colouring in". This could account for the statistically significant difference (p-value of 0.008) between the Folens and Fallon textbooks where the majority of teachers using the Folens textbooks rated the use of colour as "Poor" or "Fair" and those using the Fallon textbooks rated the use of colour in the pupils' textbooks as either "Very Good" or "Excellent".

Social activity and group work

58% of teachers indicated that the provision for social activity and group work was inadequate. The qualitative responses suggested that this was due to two main deficiencies, the lack of adequate resources and classroom management. Teachers (8%) reported the challenges encountered in rotating the tasks for groups to engage in, dealing with groups that complete the task quickly and providing adequate support to the groups that found the task challenging.

Different levels of ability

The majority of teachers (79%) indicated that the mathematics textbook they used did not provide adequate exercises for differentiated learning. The Fallon textbook was reported by one teacher to be too difficult for the lower-attaining children in the class and so certain exercises or even selected pages had to be frequently omitted. Another teacher, on the other hand, felt that the Fallon textbook was "a much more watered down version of their old *Busy at Maths* textbook." The EDCO mathematics textbooks was seen by two teachers as catering exclusively to the lower-attaining children, that it was too basic and contained far too many chapters with not enough variation of level. The difficulty cited with the Folens textbooks (42%) was that there were so many pages to get through there was not enough time to provide additional support materials for the lower-attaining children or extension activities for the higher-attaining children.

DISCUSSION

This study undertook to ascertain how textbooks are used by teachers in the lower primary years to help them implement the *Mathematics Primary School Curriculum* with understanding in their classrooms. The data indicate that textbooks are a principal resource for teachers. In general, teachers rely on one particular textbook and only occasionally look to other textbooks for additional teaching ideas. Although the teachers rely on textbooks for planning, teaching and evaluation, they do not do so uncritically and they do make limited alterations to suit the individual needs of their classes.

Teachers report that they face the dilemma of completing the textbook while also allowing children the time to develop understanding. Focusing on textbook coverage raised a number of concerns as to the shortcomings of using textbooks. Textbooks employ a 'one-size-fits-all' method of pedagogy to teaching the curriculum. Although the teachers are in general happy with their chosen mathematics textbooks, they find that they needed to adapt the textbooks to differentiate for the needs of the children in their class. Textbook coverage does not allow teachers the opportunity to freely develop their own schemes of work and lesson plans, plans that are responsible to the principles of the curriculum and responsive to needs of the children in their class, but rather the textbooks can dictate what and how to teach such as in the example of teaching place value.

The carefully thought out design of the mathematics textbooks is cited by teachers as positively supporting understanding. The exception to the generally positive view of textbook design is the use of colour. The results show that this is the only statistically significant difference between textbooks. Too much colour, as in the Carroll/Heinemann textbooks, is seen as taking up space that could better be used for practice, consolidation and revision exercises. Too little colour, as in the Folens textbooks, is viewed by teachers as engaging children in exercises that do not focus on the learning objective.

Readability is not seen as a huge issue by teachers as children in the lower primary years are too young to be engaging with/reading textbooks on their own. Legibility (the factors that affect readability) is reported by teachers as contributing to making the textbooks more appealing and accessible to children thus helping to motivate them to engage with the textbooks. This is important as, according to Okolo, Bahr and Gardner (1995), children's motivation to learn affects not only how well they learn new skills and information but also how well they use existing skills and knowledge in both familiar and novel situations.

Further challenges to understanding cited by the teachers are that the mathematics textbooks do not adequately provide provisions for hands-on learning, social activity or transfer of learning. Again this is seen as unavoidable as textbooks, by their nature, are designed with solitary independent work in mind rather than leading the way to social activity.

One of the findings of this study is the apparent contradiction between the quantitative and qualitative responses; with teachers recording that they use textbooks less often than the qualitative responses would indicate. As all teachers were given the opportunity in the questionnaires to elaborate upon their responses, and many respondents from the second questionnaire opted to have the researcher record their responses, it was possible to probe in more detail their reasoning for this. This led to the following three observations.

1. The cost of the textbooks, paid for by parents, is seen as placing an onus on teachers to complete the textbooks. As all the textbooks contain a minimum of 94 pages, teachers do not really have time to deviate from the exercises in the textbooks if they hope to get the textbook finished by the end of the academic year.
2. Although the teachers do not wish to have the mathematics textbook dictate when and how mathematical concepts should be introduced, it appeared that the teachers often feel compelled to work through the textbooks page-by-page. Once textbooks are published they cannot be changed. If/when difficulties arise with the use of the textbook in the classroom, teachers do not really have any options left open to them but to use the particular teaching method prescribed, regardless of whether the pedagogy supports understanding or not.
3. Many teachers also indicated that teaching Mathematics through the medium of text is somehow undesirable and that good teachers do not use textbooks but develop their own schemes of work instead. Some teachers also suggest that relying on textbooks is often inescapable as class sizes are large, and it can be quite an overwhelming task to plan for and teach all subjects all day long, in particular as all eleven curriculum subjects have been revised over the past ten years. In order to develop their own lesson plans, the teachers feel that they need to have a comprehensive and flexible understanding of the revised mathematical content to be taught as well as ideas, worksheets and other resources to help children learn it. They therefore argue that relying on mathematical textbooks is a reasonable way to manage; textbooks provide a valuable and fundamental source of information to draw upon and guarantee that by completing the textbook, that the Mathematics curriculum will also be completely covered.

CONCLUSION

Textbooks have their limitations in supporting the teaching of the *Mathematics Primary School Curriculum* for understanding. But acknowledging this limitation does not imply that they are without worth or utility. Textbooks have a role in suggesting a possible pathway for navigating through the strands and strand units of the Mathematics curriculum. Textbooks also serve a function in translating the curriculum objectives into examples of how these principles could be implemented in practice. Furthermore, although textbooks cannot by themselves teach young children to learn Mathematics, they can provide carefully well thought-out explanations to support the understanding of hands-on experience and also incorporate strategies for developing mathematical understanding by providing a range of stimulating exercises.

It is the over-reliance on mathematics textbooks that risks teaching not being well-grounded in understanding. To counteract this, it would be advisable if future pupils' textbooks could have fewer pages with more focused, well thought-out and meaningful exercises. This would help to motivate the children if they saw the tasks as relevant and worth understanding rather than aiming/rushing to get the exercises completed. More comprehensive and supportive teacher manuals to accompany the pupils' textbooks would also help to address this issue.

The constant revision of textbooks puts a further strain on parents' resources. On-line content for children could be a possibility, especially with more computers in classrooms. Further on-line or CD-ROM support materials could also be offered to teachers to include differentiated activities, ideas for extending and generalising concepts, and support in evaluating children's progress and using the results of assessment to guide them in their future teaching. Teachers would therefore have the opportunity to choose appropriate mathematical activities to cater for the various levels of ability in the class. It would also allow publishers the opportunity to revise and update materials based on the feedback from teachers and in the light of new evidence.

REFERENCES

Bierhoff, H. (1996). Laying the Foundations of Numeracy. A Comparison of Primary School Textbooks in Britain, Germany and Switzerland. *Teaching Mathematics and Its Applications.* 15 (4), pp.141-160.

Connolly, E. (2003). *Mathematics Education in Ireland: A Perspective on Teaching for Understanding.* Unpublished thesis, Trinity College, Dublin.

Department of Education & Science (1999a). *Primary School Curriculum: Mathematics.* Dublin: The Stationery Office.

Department of Education & Science (1999b). *Primary School Curriculum: Mathematics – Teacher Guidelines.* Dublin: The Stationery Office.

Foxman, D. (1999). *Mathematics Textbooks across the World. Some Evidence from the Third International Mathematics and Science Study.* Berkshire: National Foundation for Educational Research.

Liebeck, P. (1990). *How Children Learn Mathematics: A Guide for Parents and Teachers.* Great Britain: Penguin.

National Council for Curriculum and Assessment (2005). *Report on Primary Curriculum Review Phase 1.* Retrieved June 16, 2008 http://www.ncca.ie.

Newton, D. (2000). *Teaching for Understanding.* London & New York: Routledge/Falmer.

Ó'Dúnlaing, C. (1999). Mathematics in Primary Schools. *TCD Math Report Series,* 99 (4), May 28.

Okolo, C., Bahr, C. & Gardner, J. (1995). Increasing Achievement Motivation of Elementary School Students with Mild Disabilities. *Intervention in School and Clinic.* 30 (5), May, pp.279-286.

Surgenor, P., Shiel, G., Close, S. & Millar, D. (2006). *Counting on Success. Mathematics Achievement in Irish Primary Schools.* Dublin: The Stationery Office.

WORKING TOWARDS ADDRESSING THE MATHEMATICS SUBJECT MATTER KNOWLEDGE NEEDS OF PROSPECTIVE TEACHERS

Mairéad Hourigan John O' Donoghue

Mary Immaculate College University of Limerick

This study is the result of a perceived need for action within one Irish College of Education. During professional interactions with prospective teachers in both mathematics pedagogy sessions and teaching practice, the author (MH) considered that the number of incidents where prospective teachers demonstrated weaknesses in their mathematics subject matter knowledge was excessive. In response, the first step taken was to seek further insight into the phenomenon both nationally and internationally (Hourigan and O' Donoghue, 2007 a).

Theoretical Framework

International research highlights the importance of adequate mathematics subject matter knowledge among elementary teachers (Shulman, 1986). Consensus exists that rather than achieve extra qualifications in mathematics e.g. study to degree level, elementary teachers of mathematics require a certain type of subject matter knowledge additional to the 'common' subject matter knowledge needed for other numerate professions. It is proposed that 'specialised' knowledge of mathematics is required given the need to 'transform' his/her personal mathematical knowledge as well 'think on ones feet' in order to respond to pupil answers, queries and misconceptions (Department of Education and Science, 2002; Corcoran (2005 b.; Hill et al, 2005). Therefore conceptual understanding of the various mathematical concepts and procedures as well as an understanding of the interconnections between them is essential (Conference Board of the Mathematical Sciences (CBMS), 2001; Ball et al, 2005). There is also agreement in relation to the reality of mathematics subject matter knowledge demonstrated by both prospective and qualified elementary teachers internationally. In many cases, weaknesses are apparent in teachers' conceptual understanding of the relevant concepts and skills, and a tendency to depend on rule-bound knowledge. Shortcomings in procedural knowledge are also reported as well as gaps in knowledge and ignorance of the connections between concepts (Rowland et al, 2005; Ball et al, 2005). Within the Irish context, although this issue has received little attention until the recent past, the findings of the relevant studies reflect many of the international issues (Corcoran, 2005 a, b; Leavy and O' Loughlin, 2006; Delaney 2008 a). This is no surprise, given the highly publicised discontent regarding the mathematical abilities apparent among Irish students entering Third level numerate courses among the relevant parties (Murphy, 2002; Oldham, 2005; NCCA, 2006). The nature of the 'typical' pre-tertiary mathematics experience; that is a teacher-led didactic approach focusing on the rules and procedures likely to be examined; has been found wanting in its ability to develop conceptual understanding among students (Murphy, 2002; NCCA, 2006; Hourigan and O' Donoghue, 2007 b.) Into the future, efforts to 'address the problem where it arises' have begun. In September, 2008, following review and consultation, the phased implementation of the syllabus 'Project Maths' was initiated within a group of pilot schools. This programme seeks to promote conceptual understanding and problem solving within realistic contexts as well as smooth transitions within and between mathematics courses at the respective levels. Undoubtedly the success of this initiative requires ongoing support for teachers and schools in the form of resources and professional development as well as changes to the terminal examination if the intended and implemented curricula are to coincide (Oldham, 2005; NCCA, 2006; EGFSN, 2008). Until the envisaged positive changes associated with the nationwide implementation of such a 'reform' approach to mathematics education at post-primary level become a reality, the predominant pre-tertiary mathematics experience is 'short-changing' entrants to Third level courses, pre-service education included. The author (MH) felt an onus to address the issue as inadequate subject matter knowledge among graduates would have negative implications for the experiences and knowledge of the pupils they teach (CBMS, 2001). Internationally responses developed by initial teacher education to address the specific needs of the population in question consists of one or more of the approaches ranging from the modification of existing courses to the promotion of self study, the development of a peer assisted learning teaching structure or the provision of an 'extra' specialised maths course (CBMS, 2001; Starkings, 2005; Corcoran, 2005 b).

The Study's Context

Despite the importance associated with the development of mathematics subject matter knowledge within initial teacher education courses, the reality of time limitations within the College of Education in question meant that the main-stream mathematics 'pedagogy' classes; which are the sole form of preparation for teaching mathematics; could not begin to explicitly address prospective teachers' mathematics subject matter knowledge (DES, 2002; Corcoran, 2005; Leavy and O'Loughlin, 2006). Therefore it was deemed necessary for some 'extra' provision to be made available to prospective teachers to facilitate the 'audit and remediation' of their mathematics subject matter knowledge. A purposive sample was utilised which consisted of the cohort of second year prospective teachers (Cohen et al, 2000; Mertens, 2005). It was perceived that this cohort would be optimally motivated to partake as their subsequent teaching practice placement was in the senior classes.

The Methodology

The author took a 'pragmatic' approach to the study, believing that approaches should be selected on the basis of 'fitness-for-purpose' (Cohen et al, 2000; Mertens, 2005). As the author (MH) sought to attain further insight into the perceived problem, prior to addressing the issues which became apparent, action research was selected as the most appropriate research methodology (Opie, 2004; Mertens, 2005). The study consists of two cycles of action research (i.e. Cycle 1 or Preliminary study and Cycle 2 or Main study). While the first cycle began in February 2006 and concluded in May 2006 with the selected cohort of prospective teachers, the second cycle, which built upon the learning of the preliminary phase, commenced in February 2007 with the subsequent cohort of prospective teachers. Throughout the necessary ethical obligations were fulfilled (Cohen et al, 2000). Within the preliminary study, the initial idea was 'to address the issue of substandard mathematics subject matter knowledge among prospective teachers'. The reconnaissance stage facilitated the collection of data which shed further light on the nature of mathematics subject matter knowledge apparent among the participating prospective teachers i.e. the characteristics and needs of the population i.e. through the development and administration of a paper-based assessment. The findings reflect both national and international reports, in that a proportion of participating prospective teachers demonstrated mere procedural knowledge without limited awareness of connections. Evidence also existed that a number of students lacked conceptual knowledge of the various concepts in question, relying on memory and procedural knowledge (Corcoran, 2005; Ball et al, 2005) (For details and findings see Hourigan and O' Donoghue, 2007 a.). Subsequently the findings were instrumental in the design of the 'general plan' i.e. the development of a suitable intervention.

Cycle 1: The Intervention

On consideration of the potential intervention approaches, the 'Extra Support' model of intervention was selected as most appropriate for the particular context in light of considerations such as the potential benefits as well as the reality of financial and time constraints (CBMS, 2001; DES, 2002; Murphy, 2002; Ball et al, 2005).

The 'Professional Mathematics' course developed within the College of Education sought to provide support to all interested prospective teachers within the cohort who demonstrated or perceived possessing inadequate mathematics subject matter knowledge. It also strove to develop a deep and connected understanding of the fundamental mathematics concepts. The sessions provided participants with opportunities to experience the 'reform' mathematics for themselves, to challenge and make sense of their previous experiences, perceptions and misconceptions e.g. through sharing ideas or the use of structural materials/other representations (CBMS, 2001; DES, 2002; Oldham, 2005). In terms of the course content, a strong focus was placed on the development of deep understanding of the fundamental principles that underlie the 'very mathematics they are charged with teaching' particularly within the 'weak' areas as gauged by analysis of pre-test findings and student-feedback i.e. number (CBMS, 2001).

Regardless of involvement in the testing phase of the initiative, all prospective teachers within the cohort were invited to attend one of the three weekly alternative Professional mathematics session available to the cohort (Wednesday: 12 noon, 1p.m., 3p.m.). Weekly reminders facilitated prospective teachers to attend sessions they perceived to best meet their needs. In all, 7 Professional mathematics sessions were provided to the cohort by two members of staff (MH and a colleague).

Cycle 1: The Evaluation Stage

Evaluation is defined as 'periodic assessment of the relevance, performance, efficiency and impact (both expected and unexpected) of the project in relation to the stated objectives' (Fort et al, 2001 cited in Mertens, 2005: 47). The author was committed to the evaluation process, open to change and the desire to do things better i.e. 'developmental evaluation'. The findings serve a formative purpose i.e. facilitate the author in sustaining the strong features of the initiative into the future and making appropriate adjustments to the apparent shortcomings in the future (Murchan et al, 2005; Huntley, 2005). In line with Cromptons' (1999) proposal, the author intended that the evaluation would focus on factors such as effectiveness, efficiency as well as satisfaction among the users of the initiative.

Conceptual Framework

Mullan and Travers (2007) report that an evaluation should focus on contexts, mechanisms and outcomes. Reflecting this belief, Shapiro's (1987: 290) framework for the analysis of the evaluation process and findings was adopted. This states that the criteria by which the intervention should be judged include: Treatment effectiveness; Treatment integrity; Social Validity and Treatment Acceptability.

The criterion 'treatment effectiveness' requires insight into the amount of change or improvement evident among the participant group, ideally in comparison with the control group who have not experienced the intervention. The author interpreted change/improvement to refer to both mathematics subject matter knowledge as well as the beliefs/attitudes of all concerned (Mullan and Travers, 2007). The criterion of 'treatment integrity' appraises the extent to which the intervention is implemented as intended across all presentations. Social validity describes the 'effectiveness of the programme' as perceived by the consumers or participants. 'Treatment acceptability' of the intervention determines whether or not the potential participants 'like' the intervention procedure implemented. Shapiro suggests that treatment acceptability is an important criterion because even highly effective interventions fail if judged as unacceptable by the potential 'consumers'. The unintended side effects of the intervention require attention when examining the acceptability.

While this evaluation framework was developed to evaluate the intervention stage only, the author believed that these criteria would also shed light on the effectiveness of the testing phase of the initiative.

Multi-Method Approach

Aware that no one methodological instrument guarantees a holistic grasp of the 'truth' and the wide range of research questions and hypotheses, a multi-method approach was utilised within this evaluation process (Murchan et al, 2005; Crompton, 1999). The sources of data included usage statistics, reflective journal, as well as the administration of a post-test and post-survey. The author believed that narrative information could add meaning to numeric data and vice versa thus increasing the validity and reliability of findings (Murchan et al, 2005).

Throughout the project, the author (MH) systematically recorded all of the relevant events, informal conversations and feedback as well as reflections and interpretations of the situation within a reflective diary (Elliot, 1991). Usage statistics were collected through a weekly log of student participation at the sessions. This data provided insight into the number of student-teachers who partook in the initiative as well as the number of return visits.

Two further instruments were administered after the initiative. As it was necessary to attain feedback from the large cohort of prospective teachers (both initiative participants and non-participants) regarding the accessibility, management and perceived effectiveness of the initiative, the survey was considered most appropriate instrument given its efficiency and the facilitation of anonymity of respondents (Opie, 2004). Although the Likert scale was used for the majority of the survey items within this study, each subsection contained open-ended items which provided respondents with an opportunity to share miscellaneous information or elaborate further on issues they felt strongly about (Draper, 2006). It was deemed necessary to break the survey into 6 subsections addressing the feelings and experiences of the various members of the cohort in relation to the initiative i.e. A: All Students; B: Non-participants in Project; C: Participants in the Diagnostic test; D: Participants in Diagnostic Test but not support sessions; E: Participants in Support Sessions; F: Overall. On becoming accustomed to the possible questions used in similar educational settings e.g. Huntley (2005); Murchan et al (2005); a draft survey was developed. Methods utilised to enhance the reliability and validity of the survey included consultation with a 'jury of experts' regarding the 'content validity' of the survey and the subsequent piloting to a small group of initiative participants from a different cohort (n=17) (Opie, 2004). During the administration of the survey to the cohort (Monday/Tuesday 24th/25th of April (Week 11)), standard conditions were established. The sample size was 282 students, 147 of whom had participated in some stage of the initiative. The numbers who completed Sections A-F of the survey were 282, 135, 120, 39, 106 and 282 respectively.

It was also decided to administer a mathematics post-test at the end of the initiative as an evaluation tool, focusing on participants who completed both the pre- and post- tests. The use of the same instrument for the post-test was deemed inappropriate due to the threat of 'testing' to internal validity (Mertens, 2005). Therefore a new instrument was developed for the post-test. During pre-test creation, the author developed a large number of items from which to construct equivalent forms. The 'equivalent form reliability' was tested by administering both pre and post-tests to the same prospective teachers who were non-participants in the initiative (n= 8) at the same time in a bid to check the correlation of scores i.e. the Guttmann Split Half method. The coefficient of equivalence was .959 suggesting that the score received by an individual was about the same on both forms (McMillan and Schumacher, 2001). The post-test in the preliminary study was administered in Week 12 (Thursday 4th May: 10 a.m.-1p.m), following notification and information the previous week. Seventeen student-teachers took the post-test, 8 of who had completed the pre-test. In terms of analysis, pre-test/post-test comparison was done through the use of a paired sample t-test measure (Mc Millan and Schumacher, 2001).

Findings and Conclusions

The multi-method approach proved effective in adding depth and breadth to the findings given the complementary nature of the various approaches. The author became more fully informed regarding the participants' beliefs and experiences, thus facilitating reflection regarding the successes of the existing service and the potential for modification. Sharipo's (1987: 290) four criteria for effectiveness were central to the evaluation of the initiative.

The first component of the evaluation framework addresses the 'Treatment effectiveness' i.e. the evidence of change or improvement within participants of the intervention (Sharipo, 1987). The comparison of pre- and post-test performance found a statistically significant improvement (p= 0.01) from the small group of intervention participants involved (n= 8). As the preliminary phase of the action research did not have a control group, there is no way of knowing how these same student-teachers would have fared if they had not undertaken the support. The reality of a very small sample is also an issue. Unfortunately, the effectiveness of this aspect of the evaluation was hampered by the problem of 'experimental mortality'

(Mertens, 2005). The author believed that this was due to the late timing of the post-test within a prolonged initiative. Also the timing of the post-test may not have been convenient for all prospective teachers. The author, aware that such results should be interpreted cautiously sought to present such findings in light of other findings e.g. post-survey (Crompton, 1999).

In relation to the survey outcomes, positive feedback comes from the fact that 86.3% of the intervention participants (n= 82) reported feeling *'…more prepared to teach mathematics at senior primary level'* (a/sa) (item 46). In terms of confidence, the finding that 78.2% (n= 79) reported gaining *'…confidence about teaching mathematics at senior level since participating in the sessions'* (item 42) is also an encouraging indicator. Similar feedback was provided directly to the author (MH): *"From the students' perspective, it is very beneficial… they are very content with the sessions and are already feeling more confident…"* (Reflective Journal (R.J.), Week (W) 9: Wed). However while these findings were to be welcomed, there is no objective data suggesting that such changes in attitude/confidence had occurred. The author intended to introduce an appropriate pre-post survey within the main study.

The second element of the evaluation framework refers to the 'integrity of the intervention' i.e. the extent to which the planned or intended initiative is implemented. A number of issues came to the fore under this category, which required attention. While the author was content that in general the initiative was implemented as intended (e.g. communication of information, the administration of pre-test) a number of unsatisfactory aspects became apparent. Firstly, with regard to the accessibility of the initiative to all student-teachers, the survey found that 18.4% (n=51) of respondents (participants/non-participants) disagreed (d/sd) that **"***The times…allocated for testing/follow up sessions facilitated me to attend if I chose to"* (item 8). In fact almost half of the prospective teachers who did not partake in any aspect of the initiative (47.3%-n= 62) reported **"***I did not partake due to other timetable commitments***"** (a/sa) (item 11). However the qualitative statements suggest that in many cases rather than being physically unable to attend one of the three sessions in place, prospective teachers' workload and prioritisation of commitments was the main source of this difficulty: "*I had too much work to do to be going to extra classes"*. It also became apparent that many students had difficulty with the day in question e.g. *'Don't have all sessions on the same day as some days we only have one hour off during the day anyway'.* During the intervention stage, the author (MH) became aware of this issue: *"…Because a number of students have 6 hours on Wednesday, their only free session is 1p.m. Therefore in order for them to attend this session, they have no lunch break. …undoubtedly, the 'final' timetable does not represent the demands made on this cohort"* (R.J., W9: Wed). The author envisaged the formal allocation of timeslots on the cohorts' timetable labelled 'Professional Mathematics- Elective' would go some way to avoiding the timetabling of tutorials at the respective times. Also a request was made for the sessions to be spread over more than one day to facilitate students to avoid their 'busy' day.

The timing of the initiative within the semester was also perceived by the author as unsatisfactory. While 86.5% (n= 103) of those who completed the test reported being *'…content with the time which lapsed between test completion and feedback'* (a/sa) (item 19), those who reported discontent or uncertainty reflected the author's beliefs on the issue. It was perceived that the time lapse between test administration and feedback of three weeks was unsatisfactory and that it limited many prospective teachers opportunities to attend support sessions as course work and exams began to exert pressure from week 8 onwards, peaking at week 10: *"I became aware today that ...students have projects/coursework due over the next few weeks"* (R.J. W10, Wed). The usage statistics also provide further support. While 92 prospective teachers attended the initial session (week 5), attendance decreased substantially as the weeks went on with the attendance for the three weeks from week 10 onwards were 45, 36 and 25 respectively. "*Overall I feel that a lack of continuity (Week 6- no session) and late start in the semester as well as timetabling issues had a negative impact on attendance*" (R.J., W13: Wed). In terms of timing, the various findings suggest that it is necessary that subsequent initiatives must be focused early in the semester thus avoiding a clash with times where students have very heavy workloads i.e. end of semester. This can only be achieved through prompt feedback and timely commencement of intervention. Unlike the preliminary stage, in light of this ambition, the author (MH) felt it necessary to correct all the tests herself with just a single week of a turnover.

When questioned about the type of feedback received, almost one third (31%) of responding pre-test participants declined to concur that they were '*...satisfied with the nature of feedback...*' (20 (d/sd), 16 (n)) (item 20). The subsequent items provided additional information regarding the issue. Only 54.6% (65 students (a/sa)) of respondents believed that '*The detail received in the test feedback gave me clear insight of my mathematical strengths/weaknesses'* (item 21) (22.7% d/sd (n= 27); 22.7% n (n=27)). The author was acutely aware of this shortcoming: *"Only one or two people on receiving their preliminary feedback have requested individual meetings. Unfortunately this means that the majority have a very vague sense of their strengths and weaknesses*" (R.J. W5, Mon). A number of students communicated sentiments such as: *'I would have liked to know the questions I got wrong in the test'*. While one student reported *'The fact that I didn't get adequate feedback was my own fault'*, the author was determined to address this issue in the main study. In terms of feedback, the author envisaged providing all pre-test participants with optimum opportunity to view their work and gain insight into their errors and strengths/weaknesses. The absence of a cut-off point was also an issue, reflected in the fact that one student teacher reported that *'deciding whether/not to attend the follow-up sessions because of the results of the test'* as his/her **'greatest challenge'** (item 48). In fact 85.9% (n=103) of those who had completed the pre-test concurred (a/sa) that '*Students should be given a cut-off point e.g. 20 correct, below which attendance at follow-up sessions is 'strongly advised''* (item 22). The introduction of this 'cut-off point' would serve to encourage low achievers to avail of the support available, in light of the fact that 31% (n=9) of those who achieved 0-20 (out of 41) in the pre-test attended no support sessions. However decisions were required regarding the position of the cut-off point.

Another issue which influenced the integrity of the intervention was the fact that it was possible that the approach taken by the respective presenters within the intervention stage differed, providing participants with varying experiences depending on the sessions they attend. The author intended that in the main study, she would assume sole responsibility for intervention preparation and presentation to ensure that the focus and emphasis proposed were consistently implemented across all sessions.

'Social Validity' of an intervention refers to the participants' beliefs regarding the success of the initiative in achieving its goals (Sharipo, 1987). There were high levels of satisfaction among respondents with the overall organisation and value of the initiative. While 93.5% (n= 121) agreed that *'The project was organised and arranged well' (item 44), 95% (n=132) acknowledged that 'The project offered a valuable service to students'* (item 45)). However dissatisfaction regarding the constraints e.g. time, workload, other commitments was also communicated. While a number of intervention participants contested the suggestion that they possessed *'...adequate mathematics subject matter knowledge to teach mathematics effectively to senior classes'* (item 5) (11%- 14 students), the percentages reporting feeling better prepared and more confident as outlined above speak for themselves. Also 86.4% (n= 89) of the intervention participants reported that their *'... mathematical needs were met by the follow-up sessions*' (a/sa) (item 38). The qualitative comments ('greatest success' of the initiative/'overall comments') further support quantitative findings e.g. '*greater understanding of topics'*; *'consolidating knowledge'.*

The final criterion 'acceptability of the intervention' focuses on the degree to which the cohort 'likes' the means of intervention i.e. the procedures. The popularity of the initiative is reflected in the fact that a total of 210 prospective teachers within the cohort participated in some aspect of initiative. The usage statistics reported that 121 prospective teachers within the cohort partook in the intervention stage. The perceived benefits of the service are illustrated by the fact that many of the students chose to return to attend more than one session. While the mean attendance was 2.98 sessions, 64.5% of the intervention group attended two or more sessions. In fact while 33.1% of participants (n= 40) attended 4-5 sessions, one tenth (9.1% (n= 11)) of this group either attended six or seven sessions. Although the initiative timeslots proved problematic for a sub-group of the cohort, there was strong overall support for the timing of the initiative within the course i.e. 75.9% (n= 205) reporting it was 'appropriately **placed'** (item 9). While dissatisfaction with the communication of pre-test performance has been previously discussed, the feedback in relation to the **'content'** of the pre-test (item 18) and **'***time allocated'* to its administration (item 17) reflected high levels of satisfaction among participants (88.2% (n= 105) a/sa and 91.5% (n=109) a/sa respectively). In terms of the characteristics of intervention the feedback was extremely affirmative in relation to amount (item 28: 89.1% (n= 90) **'about right'**) and nature of content (item 35: 93.3% (n= 97) a/sa). However over a quarter (26.4%) did not believe *'the time allocated to the course (weeks/class duration) was adequate*' (item 41). The author was also dissatisfied with this characteristic and intended that the intervention programme would be extended in the main study. In terms of the approach taken within sessions, 82.1% (n= 83) reported finding **'…the sessions interesting'** (a/sa) (item 36). Almost two thirds of respondents believed that they **'...***had an opportunity to be active learners'* (item 36) (65.4% (n= 67) a/sa) and that '*During the sessions there was a discursive atmosphere'* (item 37) (61.6% (n=61) a/sa). In light of the fact that 28.7% (n= 29) of the intervention only attended one or two sessions and were not in a strong position to comment on the approach taken in the support programme, the author was extremely satisfied with this response. Overall the evidence from the various sources suggests that with some exceptions, the nature of the initiative was well received. The fact that the author would present all of the support sessions meant that there was a guarantee this in the main study.

While the evaluation process suggested that the intervention did cause positive change, it also allowed the author to learn a number of valuable lessons regarding constraint and shortcomings of the initiative within the initial cycle and indicated potential improvements. The author was '*...anxious that the project will build on the lessons learned from the first cycle'* (R.J. Summer).

REFERENCES

Ball, D.L., H.C. Hill and H. Bass, 2005. 'Knowing Mathematics for Teaching- Who Knows Mathematics Well Enough to Teach Third Grade, and How can we decide?' *American Educator*, Fall: 15-46.

Cohen, L., L. Manion and K. Morrison, 2000. *Research Methods in Education*, 5th Edition. London and New York: Routledge Falmer Academic Publishers.

Conference Board of the Mathematical Sciences (CBMS), 2001 a. *The Mathematical Education of Teachers*. Providence RI and Washington DC: American Mathematical Society and Mathematical Association of America, [Accessed on-line], http://www.cbmsweb.org/MET_Document/index.htm , Last accessed on 17/09/2008.

Corcoran, D., 2005 a. 'An Exploration of the Mathematical Literacy of Irish Students Preparing to be Primary School Teachers Knowledge of Irish Primary Pre-Service Teachers.' In: *Proceedings of the First National Conference on Research in Mathematics Education (MEI 1)16th, 17th September, 2005: Theme: 'Opening Doors'*, S. Close, D. Corcoran and T. Dooley, eds. Dublin: St. Patrick's College of Education, 220-233.

Corcoran, D., 2005 b. 'Mathematics Subject Knowledge of Irish Pre-Service Primary Teachers', Paper presented at European of Educational Research, UCD, Dublin, 7- 10 September, 2005, [Accessed on-line], http://www.leeds.ac.uk/educol/documents/144080.doc, Last accessed on 12/02/2008.

Crompton, P., 1999 'Evaluation: A Practical Guide to Methods', for LDTI: the Learning Technology Dissemination Initiative, [Accessed on-line], http://www.icbl.hw.ac.uk/ltdi/implementing-it/eval.htm, Last accessed on 12/05/2007.

Delaney, S., 2008 a. 'Unacknowledged Expertise-Irish teacher's mathematical knowledge for teaching. *InTouch*, October: 41-45.

Delaney, S., 2008 b. 'Knowledge for Practice: The mathematical demands of primary teaching'. *InTouch*, November: 40-42.

Department of Education and Science (DES), 2002. *Preparing Teachers for the 21st Century: Report of the Working Group on Primary Preservice Teacher Education*. Dublin: Stationary Office.

Draper, S., 2006. 'Observing, Measuring, or evaluating courseware: A conceptual introduction', [Accessed on-line], http://www.icbl.hw.ac.uk/ltdi/implementing-it/measure.htm, Last accessed 27/02/2006.

Elliot, J., 1991. *Action Research for Educational Change*. Milton Keynes: Open University Press.

Expert Group on Future Skills Needs (E.G.S.F.N.), 2008. *Statement on Raising National Mathematical Achievement*, Dublin: EGFSN, [Accessed on-line], http://www.forfas.ie/media/Raising%20National%20Mathematical%20Achievement%20-%20Final%20WEB.pdf, Last accessed on 02/06/2009.

Hill, H.C., B. Rowan and D.L. Ball, 2005. 'Effects of Teachers' Mathematical Knowledge on Student Achievement'. *American Educational Research Journal*, 42 (2): 371-406.

Hourigan, M. and O' Donoghue, J., 2007 a. 'Mathematical Under-Preparedness: The Influence of the Pre-tertiary mathematics experiences on students' ability to make a successful transition to tertiary level mathematics courses in Ireland'. *International Journal of Mathematical Education in Science and Technology*, 38(4): 461-477.

Hourigan, M. and O' Donoghue, J., 2007 b. 'The Challenges Facing Pre-service Education: Addressing the Issue of Mathematics Subject Matter Knowledge among Prospective Primary Teachers.' In: *Proceedings of Second National Conference on Research in Mathematics Education (MEI2) Theme: 'Walking the Talk'-Using Mathematics Education Research*, Dublin, 14-15 September, S. Close, D. Corcoran and T. Dooley, eds. Dublin: St Patrick's College, 323-338.

Huntley, R., 2005. 'An Evaluation of Primary Trainees' view of the Subject Knowledge Audit Process.' In: *Proceedings of the British Society for Research into Learning Mathematics* 23(2) University of Oxford, 7 June, J. Williams, ed, 63-68, [Accessed on-line], http://www.bsrlm.org.uk/IPs/ip23-2/BSRLM-IP-23-2-Full.pdf, Last accessed on 08/04/2009.

Leavy, A. and N. O' Loughlin, 2006. 'Preservice Teachers Understanding of the Mean: Moving Beyond the Arithmetic Average'. *Journal of Mathematics Teacher Education*, 9: 53-90.

McMillan, J.H. and S. Schumacher, 2001. *Research in Education: A Conceptual Introduction*, 5th Edition. New York: Longman.

Mertens, D.M., 2005. *Research and Evaluation in Education and Psychology: Integrating Diversity with Quantitative, Qualitative, and Mixed Methods,* California: Sage Publications.

Mullan, Y. and J. Travers 2005. 'Educational Disadvantage and Mathematics: The Impact of the Numberworld intervention programme on the number knowledge of children in a Junior Infant class.' In: *Proceedings of the First National Conference on Research in Mathematics Education (MEI 1)16th, 17th September, 2005: Theme: 'Opening Doors'*, S. Close, T. Dooley and D. Corcoran, eds. Dublin: St. Patrick's College of Education, 64-79.

Murchan, D., A. Loxley, K. Johnston, M. Quinn and H. Fitzgerald, 2005. *Evaluation of the Primary Curriculum Support Programme (P.C.S.P.): Final Report*. Dublin: Education Department, University of Dublin, Trinity College.

Murphy, M., 2002. *An Investigation into Mathematical Under-preparedness among Third Level Entrants: The Possible Contribution of the Second Level Mathematics Experience*, Unpublished M.Sc. Thesis, University of Limerick, Ireland.

National Council for Curriculum and Assessment (NCCA), 2006.*Review of Mathematics in Post-Primary Education: Report on the Consultation*. Dublin: Stationary Office.

Oldham, E., 2005. 'Solving Problems with Student-Teachers: Reactions and Reflections 1995-2005.' In: *Proceedings of the First National Conference on Research in Mathematics Education (MEI 1)16th, 17th September, 2005: Theme: 'Opening Doors',* S. Close, T. Dooley and D. Corcoran, eds. Dublin: St. Patrick's College of Education, 250-264.

Opie, C. (ed.), 2004 . *Doing Educational Research: A Guide to First Time Researchers*. London: Sage Publishers.

Shapiro, E.S., 1987. 'Intervention Research Methodology in School Psychology'. *School Psychology Review*, 16 (3): 290-305.

Shulman, L., 1986. 'Those who understand: knowledge growth in teaching'. *Educational Researcher,* 15(2): 4-14.

Starkings, S., 2005. 'Numeracy and Teacher Training'. *MSOR Connections*, 5(2): 1-3.

TEACHING FRACTIONS IN PRIMARY SCHOOL: HOW IS A TEACHER'S KNOWLEDGE COMMUNICATED TO PUPILS?

Bodil Kleve

Oslo University College,

Through classroom observations and focus-group meetings with four mathematics teachers in 5th year primary school in Norway, I have been studying how teachers draw on their knowledge in mathematics and mathematical didactics in their teaching. In this paper I present a lesson with a teacher, Berit, in which the topic is fractions. As theoretical framework and analytical tool I have used the Knowledge Quartet (Rowland, Huckstep, & Thwaites, 2005), I discuss the course of the lesson and suggest why difficulties with illustrating fractions bigger than one whole took place. This emphasizes the need to focus on different aspects of mathematical topics in teacher education.

BACKGROUND AND INTRODUCTION

In my doctoral work I studied how mathematics teachers in lower secondary school interpreted a curriculum reform in Norway. I studied the relation between teachers' beliefs about teaching and learning mathematics and their teaching practices visible for me as an observer in their classrooms. I pointed out factors constraining the teachers' teaching practice and suggested that it is not the curriculum but the teacher's knowledge and competence in mathematics which is important for what mathematics teaching children meet in school (Kleve, 2007, 2009).

An important question to address within teacher education is what knowledge is required for the teaching of mathematics. This question has been widely discussed within mathematics educational research, both with regard to what comprises the knowledge, and how this mathematical knowledge is made accessible to others. In this paper I discuss how different aspects of a mathematics teacher's knowledge became visible in a 5th grade lesson about fractions and what implications that may have for what kinds of mathematical knowledge on which it is important to focus in teacher education.

Through classroom observations and focus-group meetings with four mathematics teachers in 5th grade elementary school in Norway I have been studying how teachers draw on their knowledge in mathematics and mathematical didactics in their teaching. I have followed four mathematics teachers over a period of 5 weeks in their work with fractions. The lessons were video recorded. Based on the analysis of data from video transcripts, I address what aspects of the concept of fractions are emphasised; what concept and knowledge of fractions the teachers demonstrate they have from their own educational background and how different aspects of their knowledge are visible in their teaching practice. Before presenting the analysis of a lesson with the teacher Berit, I will report some research about mathematical knowledge for teaching which I use in my study, and also briefly report research about fractions which suggests some factors explaining why pupils' concepts of fractions only become partly developed.

MATHEMATICAL KNOWLEDGE FOR TEACHING

In his article "Those who understand: Knowledge growth in teaching" Shulman (1986) discussed how student teachers in the US in the 18th century were examined in factual and procedural knowledge in subjects which they were educated to teach. Mathematics was no exception. According to Shulman this was in great contrast to how students have been evaluated and tested since the 1980s where most weight has been put on student teachers' abilities to teach and not on the subject to be taught. Shulman also criticised research studies of how pupils' learning can increase which focuses on organisation, evaluation, individual and cultural differences, educational policies, administration and management. He claimed that one central aspect of classroom life thus is ignored: the subject matter (p. 6). A central issue for Shulman was that little attention was paid to how subject matter should be transformed from being teacher's knowledge to something being taught. He reports this as "the missing paradigm [which] refers to a blind spot with respect to content that now characterizes most research on teaching" (pp 7-8). Research within teacher education focusing on content to be taught was searched for. Where do teachers' explanations and questions come from? How does the teacher use his/her own knowledge in the subject?

Against this background Shulman searched for a theoretical framework concerning areas and categories for content knowledge. How are teachers' thinking with regard to content knowledge? How do teachers think about content knowledge related to pedagogical knowledge? Shulman suggested distinguishing among three categories of content knowledge: Subject Matter Content Knowledge, Pedagogical Content Knowledge and Curricular Knowledge. *Subject Matter Content Knowledge* (SMK) refers to the knowledge the teacher has in mind. It does not only require knowing facts and concepts of a subject, but also understanding both substantive and syntactic structures. Teachers do not only have to tell what a correct solution is, but also why. *Pedagogical Content Knowledge* (PCK) goes beyond knowledge of the subject and refers to content knowledge for teaching. "It is the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogical powerful" (Shulman, 1987, p. 15). It represents a mixture of content and pedagogy linked to an understanding of how special topics and problems are organised and includes an understanding of why special topics are easy or difficult to learn. It comprises illustrations, examples, explanations, demonstrations and analogies, in other words ways in which the subject matter is made comprehensible to others. It also incorporates knowledge about misconceptions and what aspects of a subject are more difficult to learn and why. Therefore teachers need knowledge about strategies in order to recognise learners' understanding and how misconceptions can be ground for further learning. *Curricular Knowledge* is both lateral and vertical. Lateral curriculum knowledge is how the teacher is able to relate the content and issues discussed in his/her subject to that being discussed in other subjects. Vertical curriculum knowledge is about what has been taught in earlier lessons (and years) within a subject as well as what is relevant to be taught in the next lessons.

Building on Shulman's categories for content knowledge, Ball, Thames and Phelps (2008) carried out a study. Through the analysis of problems which arise in mathematics classrooms, they investigated mathematical knowledge for teaching. They distinguished between *Common* *content knowledge* which is mathematical knowledge possessed not necessarily for the purpose of teaching, and *Specialised content knowledge for teaching* which is about being able to see patterns in pupils' wrong answers and finding out if a non orthodox method of solving a problem is working generally. The latter is also about how mathematical topics in the curriculum are linked together and the teacher's way of 'unpacking' mathematics which is neither necessary nor desirable for others to do. Ball et al suggested dividing Shulman's (1986) PCK into two components: *Knowledge of content and students* (KCS) and *Knowledge of content and teaching* (KCT). KCS is knowledge about what wrong answers, among several possible, it is most likely pupils give, and also the ability to listen and interpret pupils' incomplete thinking which requires interaction between mathematical understanding and knowledge about pupils' mathematical thinking. The other combines knowledge about teaching and knowledge in mathematics. It involves advantages and disadvantages of different approaches and examples within a mathematical topic.

With Shulman's categories of knowledge and the development of these made by Ball et al as a background, I will use "The Knowledge Quartet" (KQ) developed by (Rowland, et al., 2005) as a theoretical framework for the purpose of analysing data in my study. Like Ball et al, Rowland et al based their work on Shulman's categories of knowledge. However, unlike Ball et al who refined Shulman's categories of knowledge in order to develop instruments to measure teachers' SMK and PCK, Rowland and Turner (2008) argue that in the Knowledge Quartet as a theoretical framework

> the distinction between different kinds of mathematical knowledge is of lesser significance than the classification of the situations in which mathematical knowledge surfaces in teaching. In this sense the two theories may each have useful perspectives to offer to the other (p.2)

In order to make the classification of situations Rowland et al (2005) carried out video studies of student teachers towards the end of their education. The goal of the research was to develop an empirical based theoretical framework to provide "a means of reflecting on teaching and teacher knowledge with a view to developing both" (p. 257). Through a grounded approach to the data from the video studies, the Knowledge Quartet was identified. The quartet has four broad dimensions; foundation, transformation, connection and contingency. *Foundation* is the mathematical knowledge the teacher has gained through his/her own education, it is knowledge possessed and which can inform pedagogical choices and strategies. *Transformation* focuses on the teacher's capacity to transform his or her foundational knowledge into forms which can help someone else to learn it. The third category, *Connections*, binds together distinct parts of the mathematics and concerns the coherence in the teacher's planning of lessons and teaching over time. *Contingency* is the category which concerns situations in mathematics classrooms that are impossible for the teacher to plan for; the teacher's ability to deviate from what s/he had planned and the teacher's readiness to respond to pupils' ideas are important classroom events within this category.

Based on this I found the KQ useful as an analytical tool in categorising classroom situations where Berit's (the teacher on whom I report here) mathematical knowledge became evident and based on the analysis of Berit's lessons to reflect on what is needed for teachers' knowledge in mathematics and teachers' teaching practices to be developed.

FRACTIONS

As with decimals and percentages fractions occur with different meanings. These meanings can also be seen in everyday life. A fraction can be a part of a whole, a place on the number line, an answer to a division calculation or a way of comparing two sets or measures (Anghileri, 2000; Breiteig & Venheim, 1999; Keijzer, 2003). According to Anghileri (2000) much of the focus when working with fractions in school is identification of fractions as part of a whole. She claims that success in working with fractions depends on the ability to see the fraction both as representing a number and a ratio which reflects the procedure for finding the number. ¾ represents a number between ½ and 1 on the number line and three parts of four when the whole is divided in four. Furthermore, ¾ can be a result of a calculation. Being able to see the fraction as a result of a calculation procedure is of decisive importance to understand that ¾, 6/8 and 15/20 all represent the same number and also to be able to identify ¾ with 0.75 and with 75%. She writes:

> Research suggests that an approach to fractions which identifies each as numbers to be located on a number line, without emphasizing the way of partitioning a whole, will help to establish the equivalence with decimals and percentages (p.115)

She thus warns against emphasizing fractions as parts of a whole in schools. This is in accordance with Askew (2000) who claims that if one focuses on fractions as part of a whole so that becomes a social convention, possibilities for a obtaining a well developed fraction concept are limited. To emphasize this Askew suggests that if teachers and pupils are asked what a rectangle divided in five where three are shaded illustrates, the majority will answer 3/5 or some may say 2/5. But, he says, it is also possible to read the diagram in several other ways: 1 2/3, 2 ½, 1 ½ or 2/3. He writes:

> The reason it is almost universally read as 3/5 is not to do with the diagram per se, nor to do with pupils' ability to perceive the fraction within the diagram. Three-fifths is taken as the common reading because this is a well established common practice: everyone else from textbook writers to teachers to parents 'reads' the diagram as three-fifths. A social practice is at the heart of reading the diagram (p 139, my emphasize).

BERIT'S LESSON

With the components of the Knowledge Quartet and suggestions from educational research about difficulties with regard to development of pupils' concept of fractions in mind, I will present a 5th grade mathematics lesson about fractions with the teacher Berit. The goals for the lesson were written on the board and the teacher started the lesson reading them:

- To know how to compare fractions with different denominators
- To know how to find out which is the biggest and which is the smallest of fractions with numerator like 1.

Berit emphasised that it was the first one of the two goals they were going to concentrate on. She then showed a video (available on the internet from the publisher of the textbook) as a 'brush up' of what they had worked with so far about fractions. On the video ½=2/4=4/8 was shown and illustrated with circles.

The numbers of the tasks from the textbook on which the pupils should work individually were then written on the board. The first one was: Write the correct sign: <, > or = :

2/4[ ]1/2,   3/5[ ]1/2,   4/7+4/7[ ]1,   6/9-3/9[ ]1/3

Then there were some number sequences where the task was to write the next three fractions. The one discussed in this paper was:

> What three fractions are the next ones in the row: 2/10, 4/10, 6/10?

When the pupils had worked on their own for about 15 minutes, Berit announced:

> We shall now do something together; we shall do some thinking together – on the board. Because, suddenly some of you discovered that now we are on a task where the numerator is bigger than the denominator. That is on what we are now. What happens when we have to go higher than a whole number? Let us think together about that. [ ] Let us think together on the task 6.51 b which says: 2/10, 4/10, 6/10.

After having written the number sequence on the board, Berit asked the pupils about the next and why. She accepted their answers about why when pupils said that they just added two on the numerator and kept the denominator because "the denominator is the same". To illustrate Berit suggested showing on a figure. She drew a rectangle on the board and asked for contributions from the pupils about what it should look like. Nina suggested:

> First, you can divide in two and then divide each in two and then each in two and then each in two.

Berit divided the rectangle into two with a horizontal line saying:

> Yes, how is, now I have divided it in two, how is my whole? How can I know that? I'd wished I could draw a cartoon strip showing you how it goes from two tenths to four tenths (pointing to the fractions she had written on the board). How can I know what the first figure will look like?

Based on a suggestion from Sigrid, Berit divided the rectangle in tens using vertical lines and again she emphasized that it had to be ten because ten was the whole. Then she shaded the squares two by two while pointing to the sequence of numbers: 2/10, 4/10, 6/10, 8/10, 10/10=1, emphasising that 10/10 is the same as one whole

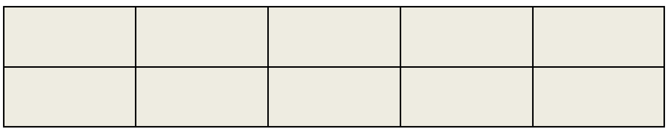


Fig 1, rectangle drawn on the board

The transcript below shows how the dialogue developed as they were searching for the next fraction in the number sequence after 10/10=1 and it also reveals the problems which occurred while trying to illustrate the fractions on the figure.

1	Berit:	Help! What shall I do now? Where shall I shade, what am I supposed to do? Can anybody help me a little? Martin?
2	Martin:	It is like when we learned this. We had it in class with 'hooded pullovers' and several more came in. Then the denominator became bigger and bigger as we became more and more [1]
3	Berit:	mmm (confirming). Remember that?
4	Martin:	Then the denominator became more and more because we became more. So it is that way. If the denominator becomes bigger, or, if the numerator becomes bigger, then the denominator becomes bigger as well. If numerator is above denominator...
5	Berit:	mmm..mmm (confirming). Then it was the whole that changed. Has our whole actually changed now? Do we talk about, have we started talking about something else than tenths, you think? We have talked about ten tenths now (pointing to 10/10 on the board). Magnus?
6	Magnus:	Twelve twelfths!
7	Berit:	Yes, twelve twelfths (confirming). If I put twelve twelfths (writes 12/12 after 10/10=1. (On the board it now stands: 1/10, 4/10, 6/10, 8/10, 10/10=1, 12/12)
8	Berit:	What am I then supposed to do here? (points to the figure in which all ten squares are shaded). Eh Sigrid?
9	Sigrid:	Add two
10	Berit:	Yes, can I do that? Can I change it like that? (Sketches two merged squares on the figure)

11	Pupil:	Yes
12	Berit:	Okay?
13	Pupils:	Cannot just do that!
14	Berit:	No? Now we disagree a little…… What do you think, Marius?
15	Marius:	If it were a cake, you couldn't just take yet another piece of cake!
16	Berit:	No, one cannot do that. (She erases the two merged squares). Now we are talking about part of a whole. What we talked about when wearing hooded pullovers, it was the whole set, the whole entirety, it might come in extra people [ ]. It was a very good example, Marius. We cannot just put on pieces of cake. We cannot change the cake like that, Thora?
17	Thora:	Can divide it into more pieces.
18	Berit:	Yees.. , but will the *piece* of the fraction then be the same? If I started to divide it into more pieces, why wouldn't that be correct? Thora?
19	Thora:	Because it will be the numerator which becomes bigger.

20	Berit:	Yes, then all of it becomes bigger, agree? If, if two tenths (pointing to 2/10), if I started dividing yet another time (pointing to the rectangle and sketching vertical lines in the air) getting twenty pieces, then two tenths wouldn't be the same ...(pause)... any more, perhaps? No? (points to a pupil).
21	Pupil:	You'll have to decrease the pieces or make them smaller.
22	Berit:	Yeees..
23	Pupil:	That will give you some space, or you can make yet another one like that (pointing to the rectangle on the board)
24	Berit:	Or can I start talking about yet another whole (draws a new rectangle and divides it into ten pieces) consisting of ten? If we imagine getting more, but not wanting to change this (points to the number sequence), if we think about this as an example, how would our next number have been? Now we have one whole. Next number if I wanted ten in my denominator, Sigurd?
25	Sigurd:	No.
26	Berit:	Nina?
27	Nina:	I think perhaps two out of twenty or twelve out of twenty or ten or something like that.
28	Berit:	Nooo... Sigrid?
29	Sigrid:	Or if you want to continue with the denominator, it becomes twelve tenths.
30	Berit:	Yes, then it becomes twelve tenths (shades two squares in the new rectangle). You all agree?

ANALYSIS AND DISCUSSION

First, I will analyse this episode from the perspective of the four components in the Knowledge Quartet to classify the situations in which Berit's knowledge was seen. Second, and based on the use of the KQ in analysing the episode, I will discuss aspects of the fraction concept emphasised in this lesson and how that may infer the development of pupils' fraction concept. This, in turn, creates a background for a discussion of what mathematical knowledge for teaching is needed in teacher education.

Foundation: How did Berit's foundational knowledge become visible in this episode? In turn 24 (after it had been initiated by a pupil in turn 23), Berit suggested talking about yet another whole consisting of ten. She also emphasized that the denominator did not change, something that had been discussed earlier. This was also repeated by Siri who (finally) came up with a correct answer (turn 29). In turn 16, Berit emphasised that what they now were talking about, was part of a whole and that was different from the example or exercise they had had in class about the hooded pullovers. She thus demonstrated knowledge about the difference between the example about hooded pullovers and the task they were working on in this lesson. After this lesson, Berit expressed frustration and when discussing the lesson in a focus group discussion together with the other three 5th grade teachers after the lesson, she said:

> Very shortly, I was not very well prepared that there was a task (in the textbook) where the answer was more than one whole and I hadn't thought about how to present that for the pupils. I started drawing and started only with one unit and I would have needed more.

This may suggest that she knew and was aware that she would have needed another unit but that it was her planning of the lesson which had not been sufficiently thought through. Another suggestion can be that she did not have sufficient foundational knowledge how to illustrate the sequence of numbers since she did not foresee the necessity of yet another rectangle. However, through the dialogue between her and the pupils she learned it. With regard to what fraction was the next one after 10/10 in the sequence, she first confirmed 12/12 (turn 7) but later she reconfirmed that (turn 24). In that case as well, the foundation of her knowledge seemed to be too weak.

Transformation: How did Berit make her knowledge about fractions accessible to her pupils? Berit chose an interactive dialogic approach. She invited pupils to participate (turn 1), listened to and took the pupils' contributions into account. Thus she encouraged pupils' contributions and attention. She also chose to illustrate the fractions with a figure (rectangle), which was not required in the task presented in the textbook. Furthermore, she suggested a cartoon strip as an illustration of the relation between each number in the sequence and the shaded squares in the rectangle. I suggest that although Berit after all had the foundational knowledge to realise that 12/12 was not the next fraction in the number sequence, she did not seem able either to communicate that to the pupils or why 12/12 was not correct.

Connection: How was the connection within mathematics, within the lesson and between this lesson and earlier lessons? Through the video with which Berit started the lesson, *links to previous work with fractions were made*. Thus the pupils were put in 'fraction mood' and previous knowledge about fractions got a 'brush up'. Although the textbook did not ask for illustrations of the number sequences, Berit chose to illustrate with a figure and she shaded squares while pointing to the belonging number. That way she emphasised a *connection within mathematics*; the link between fraction as a number and as part of a whole. However, neither illustrating the fractions on a number line nor as ratios or answers to calculations, can be seen as 'a missing link' with regard to the connection aspect of Berit's knowledge. Another weakness which became visible was the *connection within the lesson*. She started writing the goals for the lesson on the board, however, only the first task from the text book which only took 5-10 minutes dealt with 'comparing fractions with different denominators'. During the rest of the lesson (45 minutes) they worked on a number sequence of fractions exceeding one whole and how to illustrate fractions bigger than one whole on a figure. None of the other tasks from the textbook written on the board dealt with either of the two goals for the lesson which she had written on the board.

Contingency: How did Berit respond to pupils' contributions which had been impossible to plan for? In turn 2, Martin linked to the exercise about hooded pullovers with which they had started their work on fractions. This gave the teacher the opportunity, which she used, to elaborate how the example with the hooded pullovers deviated from the task on which they now worked (turn 5 and turn 16). In turn 9 Sigrid suggested adding two more squares. Berit responded to this by sketching two more squares, but erased them after protests from other pupils (turns 13 and 15). However, she never explained why they could not just add two more squares to the rectangle. In the focus group conversation after this lesson she said she was not well enough prepared for this lesson to respond to the pupils' contributions to the extent she had wished.

The way the teacher in this example communicated with the pupils, especially when - according to what she said in the focus group – she felt she had lost control, (she said: "For about ten minutes I felt *help, I am losing control*") reveals Berit's ability to teach. She orchestrated the class and commented on pupils' contributions in a way which demonstrated her Pedagogical Content Knowledge (PCK). She spent the time between and during the pupils' suggestions thinking how to get out of the situation. She said: "It worked in hundred and twenty up here (pointing to her head) while responding 'yes - mmm' to the pupils". One suggestion may be that it was the comment from a pupil in turn 23 which finally made her realise that she would have to draw yet another rectangle. Little subject matter knowledge was visible during the ten minutes she referred to as having lost control. Referring to Ball et al's division of PCK into KCS (Knowledge of Content and Students) and KCT (Knowledge of Content and Teaching), it seems that Berit had much knowledge of students, KS, and much knowledge of teaching, KT. However, the C, the content, was not very visible. She demonstrated being an experienced teacher where organisation and management of pupils are concerned, and she used that to get out of the situation in which she had lost control of the mathematical content being dealt with. Her ability in classroom management rescued her from losing face as a mathematics teacher and helped her to solve a mathematical challenge. This aspect of a teacher's competence is not a component of the Knowledge Quartet. However, having used the Knowledge Quartet with its four components as an analytical tool made me identify this as a strong aspect of Berit's teaching competence. Relating back to Shulman and his 'missing paradigm', it seems that Berit's educational background had provided her with knowledge of classroom management but not sufficient knowledge in the subject matter, which in this case was mathematics. This suggests that still in teacher education in Norway too little weight is put on subject matter knowledge in mathematics. The next question then to discuss is: What "Specialised content knowledge for teaching" is needed within mathematics?

In this lesson the mathematical topic was fractions. Throughout the whole episode to which I have referred in this paper, the teacher emphasised fraction as part of a whole. This was also the case in the other lessons I observed both with Berit and her colleagues. Over and over again they emphasised that a fraction was part of a whole. They did not illustrate a fraction on a number line, nor did they refer to a fraction as an answer to a calculation. This is in great contrast to, Anghileri (2000) who claims that to succeed in work with fractions, it is necessary to be able to see fractions both as points located on the number line and as results of a calculation. Askew (2000) goes further in his caution, in suggesting that focusing on fraction as part of a whole can be a hindrance in the development of the fraction concept. Based on this, I suggest that Berit's choice to focus on the fraction as part of a whole, may have caused some of the difficulties she fell into when searching for and trying to illustrate 12/10. If she had used a number line, the problem of missing a unit (rectangle) had not come up. This suggests that in teacher education more focus has to be put on "Specialised mathematical knowledge for teaching". Important components in this specialised mathematical knowledge are the different aspects and representations of mathematical topics. Also knowledge about suggestions based on educational research about what can increase (or create hindrances for) pupils' learning of mathematics ought to be part of teachers' pedagogical knowledge in mathematics.

NOTES

1. He referred to an activity in class the week before where they expressed how big a part of the class had hooded pullovers. While doing this two more pupils came into the room who also wore hooded pullovers. Then both the whole and the number of pupils wearing hooded pullovers changed. It started with 10/17 of the pupils wearing hooded pullovers, and then it became 11/18 and 12/19.

REFERENCES

Anghileri, J. (2000). *Teaching number sense*. London: Continuum.

Askew, M. (2000). What does it mean to learn? What is effective teaching? In J. Anghileri (Ed.), *Principles and Practices in Arithmetic teaching*. Buckingham Open University Press.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching? What makes it special? *Journal of Teacher Education 59*(5), 389-407.

Breiteig, T., & Venheim, R. (1999). *Matematikk for lærere 1* (3. utg., 2. [rev.] oppl. ed.). Oslo: Tano Aschehoug.

Keijzer, R. (2003). *Teaching formal mathematics in primary education, fraction learning as mathematising process*. Utrecht: Wilco, Amersfort.

Kleve, B. (2007). *Mathematics teachers' interpretation of the curriculum reform, L97, in Norway.* Doctoral Thesis, Høgskolen i Agder, Kristiansand.

Kleve, B. (2009). Mathematics teachers' beliefs about teaching and learning mathematics and constraints influencing their teaching practice. In C. Winsløw (Ed.), *Nordic Research in Mathematics Education, Proceedings from NORMA08 in Copenhagen, April 21-April 25, 2008* (pp. 167-174). Rotterdam: SensePublisher.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary Teachers' Mathematics Subject Knowledge: The Knowledge Quartet and the Case of Naomi. *Journal of Mathematics Teacher Education, 8*(3), 255-281.

Shulman, L. S. (1986). Those Who Understand: Knowledge Growth in Teaching. *Educational Researcher, 15*(2), 4-14.

Shulman, L. S. (1987). Knowledge and Teaching: Foundations of the New Reform. *Harvard Educational Review, 57*(1).

Turner, F., & Rowland, T. (2008). The Knowledge Quartet: A Means of Developing and Deepening Mathematical Knowledge in Teaching? http://www.maths-ed.org.uk/mkit/MKiT5_Turner&Rowland.pdf

EXPLORING THE USE OF NUMICON IN A MAINSTREAM PRIMARY CLASSROOM IN THE REPUBLIC OF IRELAND

Marie Lane

University College Dublin

The focus of this study was to examine the impact of **Numicon,** *as an early intervention programme, on the mathematical attainment of children in a designated disadvantaged urban school in the Republic of Ireland.* **Numicon** *is a UK (England) research-based programme which was first developed in the late 1990s in order to meet the needs of children who were not reaching age-related expectations for numeracy skills.* **Numicon** *is a multi-sensory approach designed to develop systematic mental arithmetic capability in children, to develop mathematical language and to apply their arithmetic to real-life problems.*

The **Numicon** *programme was used in a mixed gender, Senior Infant classroom of twenty four children, between five and six years old. Prior to this study, the outcome of a range of school based assessments provided the evidence to indicate that many of these children in this sample presented with delays in mathematics after one year of formal schooling. This baseline data indicated that many had not received the type of learning experiences and opportunities necessary for them to construct the mathematical understandings needed to successfully engage with the school mathematics curriculum. The challenge, therefore, in this study was to create an appropriate learning environment by providing mathematics instruction using the* **Numicon** *programme.*

THEORY UNDERLYING THE NUMICON PROGRAMME

Numicon grew out of a classroom based research project funded by the Teacher Training Agency in England, carried out between 1996 and 1998 by Ruth Atkinson, Romey Tacon and Dr. Tony Wing. Wing and Tacon (2007) contend that number ideas are abstract and complex and therefore children need to have these presented to them in a wide variety of ways. In adherence with this theory, Wing and Tacon set out to establish whether using visual structured imagery would support children's arithmetic understanding. Through the research a programme of teaching activities was devised using *Numicon* imagery in a multi-sensory way, following a series of structured lessons and using specific resources that would be easily followed by teachers. This programme has formed the basis of the *Numicon* programme. The *Numicon* approach builds upon the work of Catherine Stern in the 1940s and Calab Gattegno in the 1950s who advocated the use of visual images in developing children's understanding of number and how visual images could be used to develop arithmetic capability.

Numicon as a Multi-Sensory Approach

Numicon materials are structured visual representations to make the number system both visual and tactile and to make clear the stable order of the number system (that the 'next' number is 'one more') and how different numbers are related (6=3+3, or 6=1+5). One of the key features of the programme is that it provides children with visual representations of whole numbers in different shapes and colours which help to develop a mental imagery for numbers, thus supporting mental arithmetic (Wing and Tacon, 2007).

As children's understanding develops they gradually cease to rely on the concrete *Numicon* imagery using their own mental imagery of number and relationships between numbers and arithmetic operations. This is achieved through the use of structured multi-sensory play activities outlined in the Teacher's Manual. According to Willis and Johnson (2001) understanding Gardner's theory of Multiple Intelligences and how people learn results in a deeper and richer understanding of mathematical concepts. In recognition of Gardner's theory (Gardner, 1993), *Numicon's* visual, auditory and kinaesthetic approach appeals to different learning styles. Children learn through both seeing and physically touching *Numicon* patterns relating how they connect with each other. Children need the abstract brought to the concrete level for understanding. The use of manipulatives permit children to become active participants of their own learning and in their formation of idea concepts. When children can manipulate and experience conceptual information through activities, only then, will they learn and retain information more readily (Willis and Johnson, 2001). Although this type of learning style is used throughout life, it becomes less dominant as the visual and auditory modalities develop. Clements and McMillen (1996) contend that manipulatives have shown to be beneficial in mathematics. Children who use manipulatives in their mathematics classes outperform those who do not. Accordingly, the increase in performance is evident in all class levels, ability levels and topics.

This multi-sensory approach can assist children struggling with number, irrespective of how old they are. The National Council of Teachers of Mathematics (NCTM, 2000) highlight that many of the children experience difficulties when the curriculum moves away from using concrete and visual activities. Resources that support a child's numeracy work becomes less readily available as the child progresses up through the classes and as the curriculum become more challenging, pupils are expected to focus on activities that are more abstract in nature. This occurs at the middle class stage in English primary schools and the NCTM (2000) warn that if early concepts around number are not securely embedded this is where barriers to learning in mathematics and many error and misconceptions become more prominent.

Hughes (1986) and Gifford (1997) contend that representation, both through language and visual images, has been emphasised for mathematical learning, including children representing and recording in their own ways. This strongly supports the theory underpinning the *Numicon* programme. Carpenter et al (1987) indicate that in their early invented solutions for addition and subtraction problems, children need direct and complete representation before gradually developing an understanding of the abstract. Children use counting onwards and backwards strategies and finally they build up number facts to derive strategies. Representation of external objects and the manipulation of objects or their representations, or symbols, is central to mathematics. Kaput (1987) concluded that the idea of representation is 'co-terminous' with mathematics itself. Representations may reflect and support personal understandings about mathematics, while formal mathematical representation serves as a means of communication to others.

Many studies indicate consistently that children use imagery in the construction of mathematical meaning. One such study comes from Thomas et al (2002) who argue that the further developed the structure is of a child's internal representational system for the counting numbers, including kinaesthetic, auditory and visual/spatial representation of the counting sequence, the more coherent and well-organised the child's externally produced representations will be and the wider his or her range of numerical understanding.

METHODOLOGY

The purpose of the research was to accurately measure and record the different ability groups that existed within the classroom and to analyse the overall impact a multisensory structured intervention programme would have on the children's numeracy. The three month study was broadly experiential in its design where pre and post tests were carried out using the Basic Number Diagnostic Test (Gillham, 2001). This test was considered the most suitable as it was age appropriate and it tested the area of number only, which was in keeping with number concept development of the Numicon model. Furthermore, at the time of research there was no known Irish designed test for number in the Early School years. Given that the teacher acted as researcher in this study and that the ultimate objective was to bring about "an improvement in practice" (Mc Niff, Lomax and Whitehead, 1996) action research was deemed to be the most suitable research method. Triangulation, a concept applied to qualitative inquiry by Denzin (1978), was used in this research in order to gain a more complete insight in to the effect of this relatively new and previously unexplored programme in an Irish mainstream classroom. Such a multi-method approach included whole class teaching, independent group work and focus teaching, observation and questioning, individual student portfolios, written records, pictorial recordings together with audio recordings. Mathematics achievement, measured by comparing the gain from the pre-test to post test scores, was the dependent variable and the method of instruction was the independent variable. The differences between the two groups were examined to see if the type of instructional approach had an effect on achievement.

FINDINGS

Pre-Test Findings

Two thirds of the children were found to be at risk of encountering persistent difficulties with number. There was no difference in gender with respect to those more at risk of persistent difficulties with number. Cognisance must also be taken of the fact that while the remaining eight children were not ascribed to the 'at risk' category they were nevertheless underachieving for their age.

Common Errors

Table 1. Common Errors and Misconceptions, pre-test January, 2008

Task	**Common Errors and Misconceptions**
Reciting Numbers	• '20 10' instead of 30. • Reverting to teen numbers
Counting Bricks	• Not yet able to state the sequence of number names to 20. • Failure to synchronise the recitation of the numbers with finger-checking. • Difficulty in counting. Can only begin counting at one, an absence of systematic approaches. • Making counting mistakes when using teen numbers.
Selecting Bricks	• Rote counts the number sequence to at least 15, but is not yet able to reliably count a collection of that size. This is evidenced in comments such as *"What number is it again?"* and *"There's too many."* • Counting up unreliably - still counting the smaller number to get one too many in the answer.
Copying over and underneath Numerals	• Reversal of single digit numbers. • Reverse order of double digit numbers.
Writing Numerals in Sequence	• Reversal of single and double digit numbers. • Difficulty conceptualising teen numbers as reflected in the following comments: *"What does 15 look like?" "Don't know that one", "How do you write…?"*
Addition with Numerals	• Difficulty combining two single digit numbers together where the answer exceeded ten.
Subtraction with Numerals	• Difficulty in partitioning. • Failure to understand the minus sign.

Post-Test Findings

Analysis of the pre and post tests strongly support the view that in the very early years of school, boys and girls may use different strategies for solving mathematical problems, but there is no difference in the overall level of performance. As substantiated by the findings the effect of gender becomes negligible as mathematical difficulties are equally common in boys and girls. The chronological age of boys was 6.3 yrs and for girls 6.5yrs.

Fig 1

Comparison of Male Number Ages Pre-test and Post-test

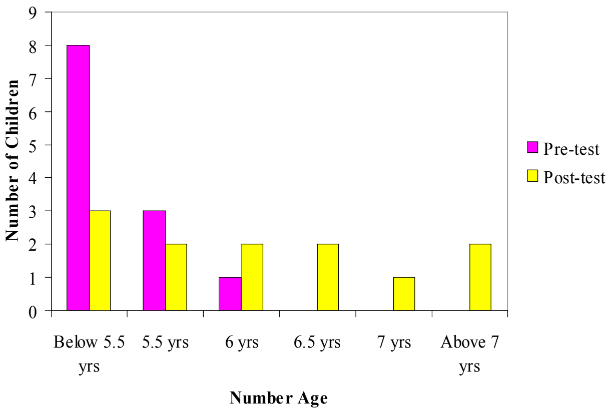
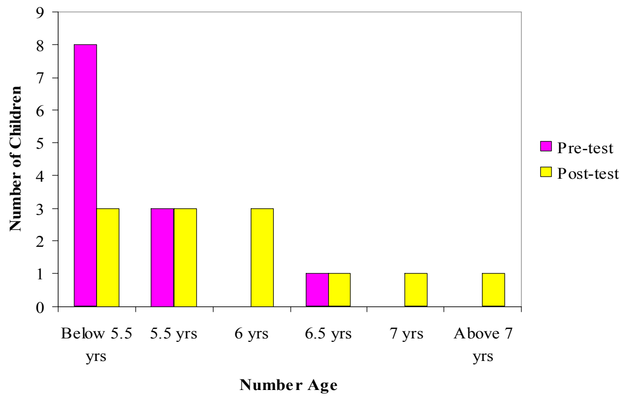


Fig 2

Comparison of Female Number Ages Pre-test and Post-test



This part of the study includes brief descriptions of the arithmetical performance of three children. These three children are the representative samples of the high, average and low arithmetical ability within the class as measured on the Basic Number Diagnostic Test. Findings synthesise a range of evidence for each child gathered both during planned assessment tasks and routine classroom activity.

Table 2 Profile of Anna Pre and Post-test

Pre-test – January, 2008	**Post-test – April, 2008**
• **Score:** 34 • **Age:** 6 yrs 0 mths • **Number Age:** 5 yrs 5 mths • **Strengths** ➢ Co-ordinated the recitation of the number names with the action of pointing at the objects. ➢ Used the counting on strategy. ➢ Counted on from the one number to find the total of two collections. • **Key Issues/Difficulties** ➢ ’20 10’ instead of 30. ➢ Incorrect number formation of some single and double digit numbers. ➢ Reversal of double digit numbers. ➢ Reversed order of double digit numbers. ➢ Reverted to fingers and counters. ➢ Subtraction tasks: Failed to understand the concept of ‘take away’. ➢ Hesitant and uncertain throughout tasks.	• **Score:** 50 • **Age:** 6 yrs 2 mths • **Number Age:** Above 7 yrs • **Identified Development** ➢ Could read, write, interpret and order single digit and double digit numbers. ➢ Applied known number facts. ➢ Applied knowledge of the ‘doubles’. ➢ Counted on from the larger number to find the total of two addends. ➢ Counted fluently to 30 and beyond. ➢ Could combine and partition sets without difficulty. ➢ Confident throughout all tasks. ➢ Applied newly acquired mathematical language. ➢ Deeper understanding of number concepts and had the ability to execute procedures that embodied those concepts. *“2 plus 8 makes 10 and 6 plus 4 makes 10”, “8 is one less than 9”, 10 is 8 more than 2”, “6 is 3 less than 9”, “20 take away 10 equals 10”.*

Table 3 Profile of David Pre and Post-test

Pre-test – January, 2008	**Post-test – April, 2008**
• **Score:** 24 • **Age:** 5 yrs 9 mths • **Number Age:** Below 5 yrs 5 mths • **Strengths** ➢ Could count only to 12. ➢ Could name numerals to 10. • **Key Issues/Difficulties** ➢ Could not name any teen numeral. ➢ Incorrect formation of single digit numbers. ➢ Difficulty writing teen numbers and numbers ending in -ty. This is reflected in the following comments: *"I don't know 13/20…"* *"I don't know that one".* For the number '20 he asked *"Is that a 0 and 2?"* For the number '50' he asked *"Is that 1 and 5?"* ➢ Failure to count on from one number to find the total of two collections. ➢ Reverted to fingers and counters but was still unsuccessful. ➢ Subtraction tasks: No concept of 'take away'.	• **Score:** 50 • **Age:** 5 yrs 11 mths • **Number Age:** Above 7 yrs • **Identified Development** ➢ Counted confidently beyond 30. ➢ Could name all numbers 1-20 including the numbers 26, 31, 36, 44 and 48. ➢ Correct formation of single and double digit numbers. ➢ Applied the counting on strategy. The following reflects his advanced understanding: When asked to show '18' counters he reached for the 15 counters he had counted from the previous task and added on 3 more. He grinned and said *"I cheated"*. When asked what did he mean by this he replied *"I counted on 3 more.* ➢ Could write all numbers in sequence to 24. ➢ Applied known number facts. ➢ Applied knowledge of the 'doubles'. ➢ Subtraction: Each task successfully completed with the aid of counters.

Table 4 Profile of Jason Pre and Post-test

Pre-test – January, 2008	**Post-test – April, 2008**
• **Score:** 18 • **Age:** 6 yrs 2 mths • **Number Age:** Below 5 yrs 5 mths • **Strengths** ➢ Could count only to 12. ➢ Could name numbers 1-14. ➢ Could sequence numbers 1-10. • **Key Issues/Difficulties** ➢ Could not name numbers beyond 14. ➢ Incorrect formation of single and double digit numbers. ➢ Reversals of single digit numbers. ➢ Reversed order of double digit numbers. ➢ Rote counted to 11 then said *"50".* ➢ Could not reliably count a collection of counters to the values 7/11/17 or 22. ➢ Could not count on from one number to find the total of two numbers when the answer exceeded 10. ➢ Reverted to fingers and counters but was still unsuccessful ion the completion of tasks. ➢ Difficulty in conceptualising numbers as captured by the following comments: *"I don't know how to draw 11", "I don't know that one".*	• **Score:** 41 • **Age:** 6 yrs 4 mths • **Number Age:** 6 yrs • **Identified Development** ➢ Confident counting to 60. ➢ Could name numbers 1-20. ➢ Could read, write, interpret and order numbers 1-20. ➢ Could confidently and competently count a collection of objects to the values 5/ 9/ 15/18 and 24. ➢ Could write numbers in sequence from 1-24. ➢ Applied knowledge of number facts to solve simple addition. ➢ Counted on from the larger number to find the total of two numbers. ➢ Could write all teen numbers including 28/34/39/41 and 50 to dictation. ➢ Subtraction of objects successfully completed without reverting to fingers or counters. ➢ Subtraction with Numbers: He could recognise the inverse sign and was able to explain that when we take away the answer *"gets smaller".*

Findings in this study strongly recognise the integral role played by manipulatives in the teaching of mathematics where an increase in performance was observed in all ability levels. The materials and methods supported the development of early number concepts and in particular the ability to calculate, such as in addition and subtraction. For many children using *Numicon* enabled them to develop these skills for the first time.

DISCUSSION

Visual Imagery

The research findings showed striking evidence of children, from all ability groups, responding positively to the visual structured images and the related programme of activities. The teaching approach deliberately played to the children's strength as visual learners and to their strong sense of pattern, through seeing and doing. The imagery supported the children's memory for number facts and their understanding of number relationships.

Mathematical Language

It was found that the imagery provoked ideas for the children on position, pattern, colour, shape, as well as the value of numbers. In this supportive mathematical context the children were helped to understand much mathematical language. Children needed specific teaching of number language, for example 'bigger than', 'greater than', 'more than', 'smaller than' and 'less than'. It was ostensible that the depth of the children's concept image was increasing as a result of using the materials and that the materials were also facilitating the development of language providing information about position and relative sizes.

Recording Arithmetic

Children were not required to record their arithmetic on paper until they had shown understanding in a practical context of arithmetic symbols and knew addition and subtraction facts to 10 in practical activities. However, when the children started to record arithmetic they were able to do this quickly and accurately. This recording of arithmetic was not simply confined to the remit of the mathematics lesson. Some children displayed a keen interest towards the recording of their own arithmetic that often times they would prefer to write than eat during their lunch break. Numbers ending in –ty and –teen were often confused by the children and this problem was addressed using the *Numicon* materials. When the children used the shapes to represent the numbers, the teacher could identify when they were not hearing or not understanding the difference between numbers such as 14 and 40. In addition, the use of the shapes by the children to represent their answers to mathematical problems indicated understanding which was not always clearly articulated in their spoken responses.

The *Numicon* programme is in keeping with the Primary Curriculum: Mathematics (Government of Ireland, 1999) in the Republic of Ireland which advocates that children be afforded the freedom to record number work content not merely in the traditional written form but also in a variety of different ways including concretely, orally, pictorially, to mention but a few. This study found that for one boy with autism, he benefited from the more practical tasks like *Numicon.* Children must have the opportunity to gain knowledge and attitudes needed to become lifelong learners of mathematics. To achieve this goal, more time needs to be spent on *understanding mathematical concepts* and less time on how to do paper-and-pencil computations.

Attitudes to Mathematics

During the implementation of the *Numicon* programme the following behaviour patterns indicating a growth in numerical confidence were observed and recorded:

- Children displaying a tendency to self correct.
- The propensity displayed by the children to record their own arithmetic through the medium of the magnetic board, the whiteboard and pencil and paper work.
- Children recording their own arithmetic in self made and designed 'booklet of numbers'.

Mathematics and the Environment

In this study children's quantitative thinking and understanding of number relationships are reflected in the following comments *"13 take away 3 equals 10"* and *"Two 15s take away one 15 equals 15".* Towards the end of the programme a girl with speech and language difficulties and whose pre-test score had identified her to be at-risk of encountering persistent difficulty with number, was able to volunteer the following information unaided during whole-class teaching: *"Four 5s make 20"* and *"two 5s plus two 4s makes 18".* An awareness of number transferred into children's learning of other key strand units such as time and money through the use of *Numicon.*

CONCLUSION

The focus of this study was on the response of children to a twelve week intervention programme to address numeracy difficulties. As indicated in the post-test findings mathematical difficulties can be addressed through appropriate intervention. The multi-sensory learning with maximum use of structured equipment and everyday materials helped to address the cross gender learning styles within the classroom and to model mathematical concepts. The *Numicon* programme promoted active learning and forged connections with familiar and relevant contexts. Attention to mathematical language was fostered with key vocabulary modelled and highlighted throughout the programme. *Numicon* stimulated, motivated and sustained children's attention. A further noteworthy outcome was the experience of success which allowed children build confidence in themselves as learners. However, cognisance must be taken of the limited time frame of three months in which this study was conducted. The aim of this paper is to offer an honest and accurate reporting of the underachievement of children in the area of number in a designated disadvantaged urban school and how through careful intervention and application of the Numicon model these children began to achieve success. The clear teaching objectives of the programme inform ed the thinking, practice and teaching of the **teacher.** Numicon changed the way in which the **teacher** taught and how **children** learned and understood number. Prior to this study there was little known research on the use of *Numicon* in the Republic of Ireland, apart from its use as an approach to teaching number to children with Down Syndrome. If the aim of mathematics education is to create numerate citizens then there needs to be an awareness and acknowledgement of the fact that what works with children with special educational needs *can* equally be effective in *all* mathematics classrooms.

REFERENCES

Clements, D.H. & McMillen, S. (1996). Rethinking concrete manipulatives. *Teaching Children Mathematics,* 2 (5), 270-279.

Carpenter, T.P., Fennema, E., Peterson, P.L. & Carey, D. (1987). Teachers' pedagogical concept knowledge in students problem solving in elementary arithmetic. *Journal of Research in Mathematics Education,* 19 (5), 385-401.

Denzin, N.K. (1978). *The research act: A theoretical introduction to sociological methods* (2nd ed.). New York: McGraw Hill.

Gardner, Howard. (1993) "Multiple Intelligences: The Theory In Practice." New York: Basic Books.

Gifford, S. (1997). When should they start to do sums? In I. Thompson (Ed.), *Teaching and learning early number* (pp. 75-88). Milton Keynes: Open University Press.

Gillham, B. (2001). *Basic number diagnostic test: Individual assessment, diagnosis and follow-up in basic number skills* (3rd ed.), Teacher's Manual, Department of Psychology, University of Strathclyde.

Hughes, M. (1986). *Children and number: Difficulties in learning mathematics.* Oxford: Blackwell.

Kaput, J.J. (1987). Representation systems in mathematics. In C. Janvier (Ed.),*Problems of Representation in the Technical and Learning of Mathematics*. Hillsdale, NJ: Erlbaum.

McNiff, J., Lomax, P. & Whitehead, J. (1996). *You and Your Action Research Project.* London: Routledge.

National Council for Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics.* Reston VA: National Council for Teachers of Mathematics.

The Primary Curriculum: Mathematics, (1999) Government of Ireland, Dublin: Stationary Office.

Thomas, N.D., Mulligan, J.T. & Goldin, G.A. (2002). Children's representation and structural development of the counting sequence 1-100. *Journal of Mathematical Behaviour,* 21, 117-133.

Willis, J. & Johnson, A. (2001). Multiply with MI: Using multiple intelligences to master multiplication. *Teaching Children Mathematics,* 7, 260-270.

Wing, T. & Tacon, R. (2007). Teaching number skills and concepts with Numicon materials. *Down Syndrome Research and Practice,* Vol. 12 (1).

A PICTURE IS WORTH A THOUSAND WORDS: INSIGHTS INTO GRAPHICACY SKILLS OF PRIMARY PROSPECTIVE PRESERVICE TEACHERS

Aisling M. Leavy

Mary Immaculate College

Graphs are an integral component of primary and secondary level mathematical experiences as a part of the probability and statistics strand. A review of curriculum implementation in primary mathematics revealed that, relative to other areas of mathematics, teachers found the data strand least useful in the planning for and teaching of mathematics. Furthermore, it was ascertained that data received the least classroom attention leading to the recommendation that research is needed to help 'develop guidance to support teachers in implementing this aspect of the Mathematics Curriculum' (NCCA, 2005b, p.7). This paper presents an analysis of the content knowledge skills of 456 preservice teachers in the area of graphical representations in an effort to identify their professional development needs. Participant responses on a survey of statistical knowledge are analyzed and discussed in an effort to identify strengths and weaknesses in graphicacy skills. Tasks were derived from the OECD Programme for International Student Assessment (PISA, 2003) and the National Assessment of Educational Progress (NAEP), the largest nationally representative assessment of school mathematics in the United States. Analysis of the data indicates wide-ranging difficulties with graph construction, graph selection, and understandings of data type.

INTRODUCTION

Understanding of statistics is becoming an increasingly important skill in today's society. The importance of enabling people to become statistically literate in an information-laden society is highlighted in national and international educational initiatives and curriculum documents (e.g., UNESCO, 1990; European Commission, 1996; National Council of Teachers of Mathematics, 2000). Graphical representations display quantities of measured data geometrically and are used to describe, summarize and explore data through the combined use of numbers, words, and pictures. It was nearly 200 years ago that William Playfair first employed the use of graphs in examining data. Since then, graphs have become pervasive for the processing of information and in the interpretation and analysis of data (Tukey, 1977; Pittenger, 1999).

The important function of graphs as tools in the making of meaning has been identified by a number of researchers (Cobb, 1999; Lajoie, 1993; Scaife & Rogers, 1996; Zhang, 1997). Tufte (1983) describes excellence in statistical graphics as consisting of "complex ideas communicated with clarity, precision, and efficiency". The use of graphs allows observation of trends that occur in the data, trends that may be missed with the use of descriptive statistics. As Tukey (1977) noted, "the greatest value of a picture is when it forces us to notice what we never expected to see." Tufte (1983) describes graphics as *revealing* data and states that graphics can be superlative to statistical computations in revealing information about data.

THEORETICAL PERSPECTIVE

Modes of representation Across multiple disciplines, there have been calls for greater attention to be paid to the use and combination of symbols in mathematical representational systems (Kaput, 1987). Specifically, the translation processes involved in moving between modes of representations is one fundamental use of symbolism that has been overlooked (Janvier, 1987). The importance of translation processes has been asserted by several researchers, notably by Lesh (1979) and Burton (1979) who assert their importance in problem solving, and Bell (1979) who declares them as playing a crucial role in mathematical modeling. In the case of statistics, the establishment of connections among modes of representation of data is critically important for developing understanding of data (Bright & Friel, 1998; Janvier, 1987). Similarly, the field of science depends very much on the transformation of nature into mathematical representations, within science contexts these transformations are referred to as transcriptions (Roth & McGinn, 1998). Shah & Hoeffner (2002) in their review of graph comprehension research identified the benefits of engaging learners in the activity of translating between representations. Experiences in translating between representations, they contend, may enhance the ability to see connections between quantitative information and the associated (graphical) visual features thus developing graph comprehension skills.

Figure 1, adapted from Janvier (1987) illustrates the translation processes involved in moving between three modes of representation: verbal descriptions, tables, and graphs; and identifies six different processes utilized when translating between representations. On examination of any pair of representational variables, there are two translations that facilitate movement between the representations, the appropriate translation to use being determined by the “target point of view” (Janvier, 1987, p. 29). For example, when considering the representational variables: table and graph, if we wish to translate from a table to a graph (table → graph) then the translation process of interest is *plotting*. If however, we wish to translate in the other direction from graph to table (graph → table) then the translation process of interest is *reading off*.

To / **From**	Situations, Verbal Descriptions	Tables	Graphs
Situations, Verbal Descriptions		Measuring	Sketching
Tables	Reading		Plotting
Graphs	Interpretation	Reading off	

Figure 1: Translation processes for moving between modes of representation

Within the context of primary statistics education, the focus on engaging students in the process of statistical investigations necessitates a complete cycle of translation processes from situation descriptions that identify variables of interest and the measurement of those variables (situation descriptions → measuring), to the construction of representations (table → graph), followed by analysis and verbal descriptions of underlying patterns (graph → verbal descriptions), which can in turn be related back to the situation descriptions.

Research examining the translation processes *table → graph* and *graph → table*

Most cognitive research on translations and graphical representations has focused on graph comprehension - where comprehension refers to reading and interpreting graphs. Four critical factors that affect graph comprehension have been identified by Friel, Curcio & Bright (2001) as purposes for using graphs, task characteristics, discipline characteristics, and characteristics of the reader. Despite what studies have contributed in terms of identifying factors which should be considered when making graphs, graph comprehension has been shown to be a complex and demanding activity fraught with systematic errors (Carpenter & Shah, 1998; Leinhardt et al., 1990; Shah & Carpenter, 1995). Work pertaining to graph comprehension at the school level has made valuable contributions in terms of identifying aspects of statistical literacy in the interpretation of graphs. Research carried out on school students' interpretations of graphs (Curcio, 1987) culminated in a commonly used typology of types of graph interpretation: *reading the data, reading between the data, and reading beyond the data.* Recommendations for the sequence of instruction for the introduction of graphs at schools have been made by Friel et al (2001). For a detailed and comprehensive review of the research on graph comprehension refer to the review of graph comprehension research carried out by Shah & Hoeffner (2002) and Friel et al (2001).

In contrast, much less work has been carried out on graph construction. Friel et al. (2001) use the term *graph sense* to refer to the skills associated with reading, interpreting, and constructing graphs in addition to choosing the graph best suited to particular situations. This construct of graph sense provides a framework within which certain behaviors associated with graph sense are identified. Within the construct of graph sense, they identify three abilities which directly relate to graph construction: (1) To recognize the components of graphs, the interrelationships among these components, and the effect of these components on the presentation of information in graphs; (2) To understand the relationships between a table, a graph, and the data being analyzed; and (3) To recognize when one graph is more useful than another on the basis of the judgment tasks involved and the kind(s) of data being represented (p. 146).

TEACHER KNOWLEDGE AND STATISTICS EDUCATION

Over the past several decades, educational research has focused on identifying the elements that contribute to good teaching, in particular the relationship between subject matter knowledge and pedagogy. Ball & McDiarmid (1990) suggest that three outcomes derive from subject matter learning: *substantive knowledge of the subject* (understanding the information, ideas and concepts associated with a specific field), *knowledge about the subject* (being knowledgeable about the fundamental activities of the field) and, d*ispositions toward the subject* (developing preferences for particular areas of the field). Although a necessary prerequisite for teaching, subject matter knowledge is not sufficient. Pedagogic knowledge, defined as 'the most useful from of analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject in order to make it comprehensible to others' (Shulman, 1986, p.9), is also necessary. Good teaching, it is generally agreed, requires both subject matter knowledge and pedagogic knowledge, and depends on 'the capacity of *a teacher to transform the content knowledge he/she possesses* into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students' (Shulman 1986, p. 9). While this study acknowledges the role that both subject matter knowledge and pedagogical content knowledge play in teaching, the focus of the study is on the subject matter content knowledge (Shulman, 1986) of preservice teachers. Referring back to the framework posited by Janvier (1987) this study will focus exclusively on the translations: $table \rightarrow graph$ and $graph \rightarrow table$.

The influence of subject matter knowledge on everyday pedagogical decisions, such as posing questions and setting tasks, has been established (Grossman, Wilson & Schulman, 1989; Hashweh, 1987; Thompson, 1984). The importance of teachers possessing a deep understanding of specific mathematical concepts and ideas (Ma, 1999) cannot be overemphasized. Poor teacher mathematical content knowledge often results in an emphasis on the richness and meaningfulness of mathematical concepts being replaced by a focus on engagement in mathematical activities without the corresponding focus on meaning (Heaton, 1992; Putnam, 1992). The importance of attending to preservice teachers' statistical knowledge, in particular, is becoming increasingly evident. As Heaton and Mickelson (2002) note:

> If statistical education is to be addressed seriously in elementary education … specific focus needs to be placed on the learning of teachers … We cannot seriously attend to children's understanding of statistics without simultaneously attending to teachers' understandings.

A number of studies have examined how preservice and practicing teachers read and interpret graphical representations. Teacher attitudes about graphs have been found to be positive and teachers are confident when it comes to teaching graphs. This confidence was evident in a study of preservice teachers carried out by Watson (2001). Administration of an instrument designed to profile teachers' competence and confidence to teach probability and statistics found that pre–service teachers reported greater confidence in teaching graphical representations than other statistical concepts (median, average, data collection). This confidence relating to graphical representations was also a finding of a study of secondary preservice teachers (Gonzalez & Pinto, 2008). The authors found that pre-service teachers underestimated the complexities of the mathematical ideas fundamental to statistics education, and these same teachers who appeared confident in their understandings of graphical representations were identified as having limited knowledge and treatment of graphical representations (Gonzalez & Pinto, 2008). In contrast, the presentation of media graphs to preservice teachers demonstrated critical thinking skills in the interpretation of data presented on the graphs (Monteiro & Ainley, 2006). It should be noted that the context of reading *media* graphs, the authors contend, may be different to school graphs and both contexts may draw on different kinds of knowledge when interpreting the data. Taking this into consideration, it is difficult to ascertain to what extent the critical thinking skills relating to media graphs as demonstrated by pre-service teachers in this study extend to school-graphing contexts.

Other studies have examined how pre-service teachers work with graphical representations within pedagogical contexts. A study of preservice teachers carried out by Heaton and Mickelson (2002) provides some insight into the processes that occur between data collection and data construction. The authors observed that graph construction often became the endpoint of statistical investigations, with the predominant focus being on technical aspects of graph construction. The preservice teachers tended not to engage children in reasoning about the data through examination of graphical constructions, nor did they use the constructed graphs to coordinate a refocus on the initial purpose of the statistical investigation i.e. the research question. A similar study of elementary preservice teachers carried out by Leavy (2006) engaged participants in two semester long statistical investigations. The study revealed the tendency to focus on descriptive statistics when examining data. Participants did not recognize graphical representations as important tools in supporting the description and comparison of distributions of data. Furthermore, those participants who created graphs, at the outset of the study, did not depict the individual data values. Rather they used the graphs to report descriptive statistics. Reliance on descriptive statistics as opposed to graphs was persistent throughout the study. Leavy argues that descriptive statistics are appropriate as comparative measures, absolute reliance on descriptive statistics is limiting as they provide merely one perspective on the data, that of centers, and do not take into account other features of the data e.g. shape, variability.

Fewer studies have examined the skills of pre-service teachers when *constructing* graphs i.e. the *table* → *graph* translation, which is the focus of this study. Little is know about pre-service teachers' own practices in constructing representations from data. Examination of the translation processes of pre-service science teachers during science investigations sheds particular light on the table → graph translation (Bowen & Roth, 2005). Two different levels of difficulty were identified. The first difficulty related to pre-service teachers not knowing if a graph should be used. The second difficulty was fundamental to the construction of representations and related to not knowing how to structure the data and choose inscriptions that were appropriate for the context chose. All these studies highlight difficulties understanding the role of graphical representations within the process of statistical investigation.

METHOD

Participants

Participants were entry-level primary education students and tested in their first week of Year 1. They were given 50 minutes to complete the survey. Participation was voluntary and anonymous. Tests were administered within groups of 50. In all, 456 of 480 participants completed the survey.

Task design and administration

This was a large-scale survey administration of 12 statistical items. Five tasks relate to graphical literacy, this study reports on three of those tasks. Table 1 outlines the categorization of all five tasks, survey placement, and the number of participants administered the task. The two PISA tasks will not be discussed in this paper.

Identifier	GAISE Code	Source	Survey version	Sample size
2a	AD1	Designed for study	A	118
2b	AD1	Designed for study	B	110
2c	AD1	Designed for study	C	119
2d	AD1	Designed for study	D	113
3	AD3	Designed for study	C, D	231
4	AD4	PISA	A, B, C, D	456
5	AD4	PISA	A, B, C, D	456
6	AD4	NAEP	A, B	225

Table 1: Categorization of survey tasks

The framework used to categorize tasks is the GAISE (2005) framework outlining the *processes of statistical investigation*: formulating a research question (FQ), collecting data to answer the question (CD), analysing the data (AD). Tasks were then located within the Irish primary curriculum (DES, 1999) and the National Council of teachers of Mathematics (NCTM) *principles and standards for school mathematics*. Given the number of tasks, four versions of the survey were administered (A, B, C, D).

RESULTS

Task 1: Constructing graphs [The paperclip task]

The purpose of this task was to identify graph construction skills. Participants were presented with raw data and asked to translate the data to a graph (table → graph) using the process referred to as *plotting* (Janvier, 1987); different test versions requested the construction of different graphs (bar charts, pie charts, histograms, and most appropriate graph). Pie and bar charts were included due to their positioning within the Irish Primary School Curriculum. The histogram was chosen due to its inclusion in the secondary school curriculum and participants' familiarity with the representation. The fourth variant allowed the participant the choice of any graph to construct based on their determination of the way to best represent the data. This latter open-ended question involving an element of choice is not a common practice in Irish curricula but is a skill identified by Friel et al. in their definition of *graph sense*.

Task 2The paperclip task

The following values represent the distances (in cm) that a group of 30 students blew a paperclip. Each value represents each participants' best attempt. Graph the data using the following graph: pie chart/histogram/bar chart/most appropriate graph.

2	10	17	22	22	22	23	25	27	27
28	34	40	50	50	50	52	54	55	60
60	64	66	68	73	76	84	85	150	180

As can be seen from table 2, 34% of participants correctly constructed a *pie chart.* The traditional taught strategy was the least prevalent – only 3 of 118 participants used this strategy. The most prevalent approach was division of the circle into the total number of data values and shading the sub parts in accordance with the frequencies displayed in the data. This is not a traditionally taught strategy and has been found with children who have not been taught traditional methods of constructing pie charts (Leavy, 2007). 11% of participants accurately constructed a *histogram* (version b), a representation prevalent in secondary school. Interestingly, 26% erroneously completed a bar chart rather than a histogram. 52% accurately constructed a *bar chart* (version c). The majority of bar charts were case value charts (47%) and the remaining were frequency bar charts. 12% of (incorrect) responses involved the construction of a histogram rather than a bar chart. When permitted to construct *any graph of their choice* (version d) 40% of participants constructed a graph that accurately presented the data. Most popular were frequency line plots (12%), bar charts (12%), and histograms (12%). Pie charts (1%) and line plots (1%) were least popular. The most prevalent response in this category, 36%, was the construction of a trend graph. However, a trend graph indicates that the data values are linked by time, in other words that the values have a chronological order. The presented data set has no relationship or connection with time; hence the construction of this representation is inappropriate for this data set.

	Correct	Incorrect
Version a: Pie chart	34%	66%
Version b: Histogram	11%	89%
Version c: Bar chart	52%	48%
Version d: Most appropriate	40%	60%

Table 2: Performance on the graph construction task

Taken in its entirety, less than one third (30%) of participants provided a complete and accurate response to the task. Furthermore, the number of incomplete or non-responses accounted for 36% of all responses. The bar chart was the representation constructed with most success. However, confusion existed relating to bar charts and histograms with almost 10% of all participants confusing the representations. The low popularity of pie charts on versions d, when taken in conjunction with the low success rates with pie charts on version a, indicates that pie charts pose considerable challenges for preservice teachers – which in turn raises questions about their suitability for inclusion on the primary curriculum.

Task 2: The relationship between data type and graphical representations [The ice cream task]

The purpose of this task was to ascertain if an understanding exists of which graphs are appropriate for display of categorical data. Relating data type and associated graphical representation is important in that certain data types can be represented only on particular graphs. Otherwise, what emerges is a skewed, and at times, inaccurate representation of the data. The options presented were derived from commonly used graphs, in NCTM and GAISE documents, for use with preservice teachers. The GAISE document defines teacher knowledge necessary to complete the task as 'Teacher possesses conceptual/relational understanding of how certain data types can be represented only on specific graphs'. The task addresses the NCTM objective 'In grades 3–5 all students should recognize the differences in representing categorical and numerical data.' There is not an equivalent objective in the Irish Primary Curriculum.

Task 2: The ice cream task

The following are responses of 12 children when asked to identify their favourite ice cream flavour:

Mint	Strawberry	Banana	Mint	chocolate	chocolate
chocolate	Banana	Mint	Vanilla	Banana	Banana

Please circle each of the following graphs that may be appropriate to use to graph the data:

pie chart	histogram	bar chart	bar-line	box and whisker plot
pictogram	line plot	dot plot	ogive	stem and leaf plot

Given the lack of attention to data types and their associated graphs in the Irish Primary curriculum, it is not surprising that 9% of participants correctly identified the graphs suitable for display of categorical data. 65% over estimated the number of graphs suitable for displaying categorical data. It should be noted that given the limited menu of graph options presented in Irish secondary school curricula, there is the possibility that participants were not familiar with all options presented in the task.

Task 3: NAEP election data

An election involving four candidates for mayor has been held. Of the following, which is the best way to present the percentage of votes each candidate received?

A) Circle graph

B) Line graph

C) Box plot

D) Scatterplot

E) Histogram

Please justify your answer:

____

____

____

____

The purpose of this task was to ascertain whether participants could choose a suitable graph to illustrate patterns in data. The task is a released NAEP task from the 2005 administration to 12th grade students. Within the NAEP classification, this item is classified as *low complexity* as it 'relies heavily on the recall and recognition of previously learned concepts and principles. Items typically specify what the student is to do, which is often to carry out some procedure that can be performed mechanically. It is not left to the student to come up with an original method or solution'.

Although this was one of the better-answered questions, with 48% of participants getting the correct solution, more than half answered the item incorrectly. The results are particularly worrying when the results are compared with those of American 12th graders who answered the same task. 60% of US 12th grade students chose the correct graph to represent the data – a significantly higher number than pre-service teachers in this sample. One reason which may account for low performance on this task is that the Irish statistics curricula generally prescribe the graph to be constructed – classroom instruction then focuses on the construction and interpretation of the graph. Students are rarely, if ever, involved in discussion regarding the selection of an appropriate graph to display data. This task focused specifically on determining if the respondent could select the graph most suitable to illustrate aspects of a situation. The reasoning skills involved in making these decisions are poorly developed in Irish curricula.

DISCUSSION

The purpose of this study was to examine the graphicacy skills of primary pre-service teachers. The study was carried out on entry to the College of Education and prior to any instruction in mathematics education pedagogy. In this sense, the content knowledge understandings on exit from secondary school were the focus of study. The five tasks administered to participants can be categorized in terms of difficulty level by reference to the graph complexity organizer posited by Friel et al (2001) in addition to the appearance of the statistical concepts in school curricula (NCTM, 1999; DES, 1999).

The first two tasks, the paperclip task and the ice cream task, are of comparable difficulty as both draw on content knowledge understandings expected from primary level children. Performance on graph construction in task 1 is extremely concerning, with only 30% correct responses across all versions of the task, considering the extent of participants exposure to the representations over their school lives. Task 2 showed the lowest success rate with only 9% correct responses in determining which representations best suit categorical data. These results highlight the need for Irish curricula in the area of data handling and statistics to better reflect best practices evident in international curricula. The results highlight the need for greater emphasis to be placed on the nature of data, in particular numerical and categorical distinctions. Furthermore we need to question the suitability of pie charts in primary level curricula in light of the fact that (a) two-thirds of pre-service primary teachers themselves demonstrate profound difficulties with these representations, and (b) less than 1% used this representation when provided with a menu of graph options. The critical need to highlight the distinctions between graphs and their uses, particularly the bar chart versus histogram distinction, is also clearly evidenced in this study.

The NAEP Election Data task, task 3, was defined as a 'low complexity' task as it relies on recall of previously known facts. Almost half of the responses correctly identified a circle (pie) chart as the representation most appropriate for the presentation of percentage of electoral votes. It is very disconcerting that pre-service primary teachers fared considerably less well than 12th grade American students. Moreover, the sample of Irish pre-service teachers represents a relatively high achieving academic group as compared to a random sample of US students. The fact that 52% incorrectly responded to this task highlights poor ability to discern the functionality and use of particular graphs, calling to question the statistical literacy of the pre-service teachers in this study.

In conclusion, the poor performance on graphicacy tasks calls into question the statistical literacy of pre-service primary teachers. Participants in this study performed poorly on basic and important skills represented in school mathematics. The results of this study have implications for the design of instructional activities and curricula at the primary and secondary level. The results presented here also highlight the need for focused research involving integrated cycles of instructional design and analysis of student understanding in an effort to better support the development of students' statistical literacy, reasoning and thinking at all levels.

REFERENCES

Ball, D. L. & McDiarmid, G. W. (1990). The subject-matter preparation of teachers. In W. R. Houston (Ed.), *Handbook of Research on Teacher Education.* New York: Macmillan, 437-449.

Bell, M. (1979). What Most People Need From Mathematics: Teaching Mathematics so as to be Useful. In C. Janvier (Ed), *Translation Processes In Mathematics Education. Problems of Representation in the Teaching and Learning of Mathematics*. Hillsdale, N.J.: L. Erlbaum Associates.

Bowen, G.M. & Roth, W-M. (2005). Data and graph Interpretation Practices among Preservice Science Teachers. *Journal of Research in Science Teaching,* 42(10), 1063-1088.

Bright, G. & Friel, S. N. (1998). Graphical Representations: Helping Students Interpret Data. In S. Lajoie (Ed.), *Reflections on Statistics Learning, Teaching, and Assessment in Grades K-12*. Lawrence Erlbaum Associates.

Burton, L. (1979). The Classification of Problem-Solving Skills and Procedures - Presentation of an Inventory. *Proceedings of the Third International Conference of PME*. Warwick.

Carpenter, P. A., & Shah, P. (1998). A Model of the Perceptual and Conceptual Processes in Graph Comprehension. *Journal of Experimental Psychology:* Applied, 4, 75-100.

Cobb, P. (1999). Individual and Collective Mathematical Development: The Case of Statistical Data Analysis. *Mathematical Thinking and Learning,* 1, 4-43.

Curcio, F. (1987). Comprehension Of Mathematical Relationships Expressed In Graphs. *Journal for Research in Mathematics Education*, 18(5), 382-393.

Department of Education and Science (1999). *Primary School Curriculum- An Introduction*. Dublin: The Stationery Office.

European Commission. (1996). *White paper on education and training: Teaching and learning-towards the learning society.* Luxembourg: Office for official publications of the European Commission.

Friel, S.N., Curcio, F.R. & Bright, G.W. (2001). Making Sense of Graphs: Critical Factors Influencing Comprehension and Instructional Implications. Journal *for Research in Mathematics Education,* 32(2), 124-158.

Gonzalez, T. & Pinto, J. (2008). *Conceptions of four pre-service teachers on graphical representation.* In C. Batanero, G. Burrill, C. Reading & A. Rossman (Eds.), Joint *ICMI*/IASE Study: Teaching Statistics in School Mathematics. Challenges for Teaching and Teacher Education. Proceedings of the ICMI Study 18 and 2008 IASE Round Table Conference.

Grossman, P.L., Wilson, S.M. & Schulman, L.S. (1989). Teachers of Substance: Subject Matter Knowledge for Teaching, In M.C. Reynolds (Ed.), *Knowledge Base for the Beginning Teacher.* Pergamon: Oxford.

Hashweh, M. (1987). Effects of Subject-Matter Knowledge in the Teaching of Biology And Physics. *Teaching and Teacher Education*, 3(2), 109-120.

Heaton, R.M. (1992). Who Is Minding The Mathematics Content? A Case Study of a Fifth Grade Teacher. *Elementary School Journal,* 93, 153-162.

Heaton, R.M. & Mickelson, W.T. (2002). The Learning and Teaching of Statistical Investigation in Teaching and Teacher Education. *Journal of Mathematics Teacher Education,* 5, 35-59.

Janvier, C. (1987). *Problems of Representation in the Teaching and Learning of Mathematics.* Hillsdale, N.J.: L. Erlbaum Associates.

Kaput, J. J. (1987). Towards a theory of symbol use in mathematics. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 159-195). Hillsdale, NJ: Lawrence Erlbaum Associates.

Lajoie, S.P. (1993). Computer Environments As Cognitive Tools For Enhancing Learning. In Lajoie & Derry (Eds.), *Computers as Cognitive Tools.* Lawrence Erlbaum Associates: London.

Leavy, A.M. (2006). Using data comparison to support a focus on distribution: Examining preservice teacher's understandings of distribution when engaged in statistical inquiry. *Statistics Education Research Journal,* 5(2), 89-114.

Leavy, A.M. (2007). Coordinating student learning and teacher activity – the case of Savannah: Motivating an understanding of representativeness through examination of distributions of data. In S. Close, T. Dooley and D. Corcoran (Eds.), *Proceedings of the*

Second National Conference on Research on Mathematics Education in Ireland –MEI II. St. Patrick's College Dublin.

Lehrer, R. & Romberg, T. (1996). Exploring Children's Data Modeling. *Cognition And Instruction*, 14(1), 69-108.

Leinhardt, G., Zaslavsky, O., and Stein, M. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research,* 60(1), 1-64.

Lesh, R. (1979). Some Trends In Research And The Acquisition And Use Of Space And Geometry Concepts. In C. Janvier (Ed.), *Translation Processes In Mathematics Education. Problems of Representation in the Teaching and Learning of Mathematics*. Hillsdale, N.J.: L. Erlbaum Associates.

Ma, L. (1999). *Knowing and Teaching Elementary Mathematics: Teachers' Understanding Of Fundamental Mathematics in China and the United States.* Mahwah, NJ: Erlbaum.

Monteiro, C. & Ainley, J. (2006). *Student teachers interpreting media graphs*. ICOTS-7. National Council of Teachers of Mathematics (NCTM) (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

OECD (2003). *PISA 2003 Assessment Framework: Mathematics, Reading, Science and Problem Solving Knowledge and Skills*. Available at http://www.oecd.org/dataoecd/46/14/33694881.pdf**.**

Oldham, E. (2006). The PISA mathematics results in context. *Irish Journal of Education*, *37*, 27–52.

Pittenger, D.J. (1999). Teaching Students about Graphs, In M.E. Ware & C.L. Brewer (Eds.), *Handbook for Teaching Statistics and Research Methods*. Lawrence Erlbaum Associates.

Putnam, R.T. (1992). Teaching the 'How' Of Mathematics for Everyday Life: A Case Study of a Fifth Grade Teacher. *Elementary School Journal,* 93, 163-177.

Roth, W. M., & McGinn, M. K. (1998). Inscriptions: Toward a theory of representing as social practice. *Review of Educational Research*, 68(1). 35-59.

Scaife, M. & Rogers, Y. (1996). External Cognition: How Do Graphical Representations Work? *International Journal of Human-Computer Studies, 45*, 185-213.

Schulman, L.S. (1986). Those Who Understand: Knowledge Growth in Teaching. *Educational Researcher,* 15(2), 4-14.

Shah, P., Carpenter, P. A. (1995). Conceptual limitations in comprehending line graphs. *Journal of Experimental Psychology: General, 124*, 43–61.

Shah, P., & Hoeffner, J. (2002). Review of graph comprehension research: Implications for instruction. *Educational Psychology Review, 14,* 47-69.

Thompson, A. (1984). The Relationship of Teachers' Conceptions of Mathematics and Mathematics Teaching To Instructional Practice. Educational Studies in Mathematics, 15, 105-127.

Tufte, E.R. (1983). *The Visual Display of Quantitative Information*. Cheshire: Graphics Press.

Tukey, J.W. (1977). *Exploratory Data Analysis*. Addison-Wesley Publishing Company.

UNESCO (1990). *Final Report on the World Conference on Education for All* (Jomtien, Thailand). Paris: Author.

Watson, J.M. (2001). Profiling Teachers' Competence and Confidence to Teach Particular Mathematics Topics: The Case of Chance and Data. *Journal of Mathematics Teacher Education*, 4, 305-333.

Zhang, J. (1997). The Nature of External Representations in Problem Solving. *Cognitive Science*, 21(2), 179-217.

FOUR YEARS LATER

Yvonne Mullan

National Educational Psychological Service

This paper examines the maths achievement of 9 children who took part an early intervention programme four years ago during the academic year 2004-2005 when they were aged between 4 and 5 years. The children, who completed 3rd Class in June 2009, were tested using standardised tests of literacy and mathematics. Test scores from 2004 and 2005 are presented along with 2009 test scores. The results are reassuring with regard to arithmetic ability. The results also indicate that children who failed to make significant progress during the 2004-2005 intervention continue to have difficulty in certain areas and that mathematical difficulties can often be interwoven with concentration difficulties and lack of support for learning at home. Research suggests that numeracy failure starts early and becomes entrenched if it is not tackled early. The only solution is good quality individualised early intervention.

INTRODUCTION

Understanding basic mathematics is an important life skill. According to a recent study by the National Audit Office in Britain (NAO, 2008), pupils who master mathematics in their early years are in a good position to go on to further studies and those who do not, are likely to be disadvantaged in the labour market (NAO, 2008). This study, like many that preceded it, identifies a strong link between succeeding early and continuing to succeed.

The question of how best to help children to succeed early in mathematics is the subject of much debate. In recent years, mathematics curricula have been developed that draw from students' understandings, build on them and progressively move towards abstract and formal mathematical processes - a movement referred to by the Dutch as "progressive mathematisiation" (Zevenbergen, Dole and Wright, 2004, p.4) In parallel with this development has been the development of constructivism theory and the general awareness that children actively construct meanings from their experiences (Zevenbergen et al, 2004). Constructivist approaches are central to Ireland's Primary School Mathematics Curriculum (National Council for Curriculum and Assessment, 1999). However, despite these welcome developments many children get left behind in mathematics early on in their school lives. There is a particular concern here in Ireland about the children who get left behind in low-income communities (Sheil and Kelly, 2001; Weir, 2003; Surgenor, Shiel, Close and Millar, 2006).

One of the problems faced by teachers, when attempting to build on the understandings of children aged between 4 and 5 years, is that within one class group, children's understandings and mathematical experiences vary enormously. Evidence suggests that there can be a three year differential in achievement levels between children in early mathematics knowledge as they begin school (Griffin, Case and Siegler, 1994; Mullan and Travers, 2007). The best way of addressing this differential is early intervention.

The subject of this paper is the mathematics ability of 9 children who took part in an early intervention programme, Number Worlds (Griffin and Case, 1997), during the academic year 2004/05 when the children were aged between 4 and 5 years. In June 2009 these children and were aged between 8 and 10 years and had just completed 3rd class. At the time of the Number Worlds intervention in 2004, there were relatively few mathematics interventions available to designated disadvantaged school in Ireland. Since then, and thanks to the Social Inclusion programme Delivering Equality of Opportunity in Schools (DEIS) (Department of Education and Science, 2005) the Mathematics Recovery intervention (Wright, Martland and Stafford, 2000) and the Ready Set Go intervention (Pitt, 2001) have been offered and are widely used in DEIS schools. The Number Worlds intervention was based on Central Conceptual Structure theory (Griffin et al., 1994) and was used successfully to close the number knowledge gap between children in schools in low-income, high-risk communities and their more affluent peers in Massachusetts (Griffin and Case, 1997) and in Dublin (Mullan and Travers, 2007). The intervention involved a mixture of whole-class teaching and scaffolded small-group work. There was a heavy emphasis on counting and language skills in order to help children to gain a representation of number akin to a mental counting line. The intervention aimed to teach children to generate verbal labels for each number, recognise written numerals 1- 10, understand 1-1 correspondence, understand that each verbal label has a set size, which has a certain canonical perceptual form, and understand that movement from one set to the next involves addition or subtraction of one unit. According to Griffin et al. (1994) the absence of this knowledge constitutes the main barrier to arithmetic ability later on in school. This paper examines the current arithmetic ability of the children who took part in the intervention four years ago.

METHOD

School

The 9 subjects attended school in a growing suburb of Dublin. In the year of the intervention (2004/2005) the Junior School and the adjoining Senior School had disadvantaged status. Subsequently the Department of Education and Science reorganised support for disadvantaged schools into the DEIS initiative (Department of Education and Science, 2005) and the school did not qualify for disadvantaged status. The school opened in 1985 in a Greenfield site with 7 pupils. There are currently almost 1000 pupils on campus between the Junior and Senior Schools.

Subjects

The subjects were 9 children, 4 boys and 5 girls, in third class in the Senior School. The breakdown of nationalities seen in Table 1 is typical of the current population of both Junior and Senior schools. Over 50% of the children attending the school are international. The children ranged in age from 8 years 8 months to 10 years 6 months.

Table 1: Age, gender and nationality of children

Pupil	Age	Sex	Nationality
1	9'5	male	Nigerian
2	9'7	male	Nigerian
3	9'2	male	Ghana
4	9'9	male	Lithuanian
5	10'6	female	Pilipino
6	9'8	female	Irish
7	9'2	female	Irish
8	8'9	female	Irish
9	8'8	female	Irish

Tests

The Numerical Operations subtest of the Wechsler Individual Achievement Test Second UK Edition (WIAT-II 2^{uk}) (The Psychological Corporation, 2005) was administered by the author in order to observe children's written arithmetical skills. Unlike most standardised mathematics' tests used in schools which require certain language and literacy abilities, Numerical Operations is a test of children's ability to work with numbers and symbols only. Thus, it was hoped that children's literacy or language skills would not affect their scores and that a clear picture of children's arithmetic ability would evolve. The MICRA-T standardised test of literacy Level 2 (Wall and Burke, 1990) and the SIGMA-T standardised tests of mathematics Level 3 (Wall and Burke, 1992) were administered by the class teacher and were collected by the author.

Interview

The class teacher was interviewed to get her opinions about the children's concentration skills, their progress in mathematics, and the teacher's perceptions of the existence of learning support at home.

Limitations of Study

Ideally it would have been better to review the progress of all 21 children from the original Number Worlds class. This was not possible because of time constraints and because only 9 children from the original class still attend the school in which the Number Worlds Intervention took place.

FINDINGS AND DISCUSSION

Test results

On the Numerical Operations test of arithmetic ability all 9 children achieved standard scores (S.S.) of 85 or higher. 7 of the 9 children achieved standard scores in the Average range (S.S. =90-109) and 2 children achieved scores within the Low Average range (S.S.=80-89).

2 of the 9 children scored higher than the Average range (S.S. >109) On the SIGMA-T. 4 children scored in the Average range (S.S. 90-109) and 3 children scored in the Low range (SS 70-79).
On the MICRA-T all scores apart from one (Child 3) were average or higher (S.S.>90). A score of 79 was achieved by Child 3.
Pre- and post- test scores on the Number Knowledge Test (NKT) (Griffin et al., 1994) from 2004-2005 can be seen in the shaded columns to the left of Table 2.
2009 standard scores for each of the children on the SIGMA-T, MICRA-T and Numerical Operations subtest can be seen in the white columns to the right of Table 2.

Table 2. 2004 and 2005 NKT scores and 2009 SIGMA-T, MICRA-T and Numerical Operations scores.

		Pre-test Nov 2004		Post-Test April 2005		June 2009 Standard Scores			
Child	Age	NKT score	Age Equivalent	NKT score	Age Equivalent	Age	SIGMA-T	MICRA-T	Numerical Operations*
1	4’10	6	3-4	13	5-6	9’4	73	91	98
2	4’8	2	2-3	8	4-5	9’2	77	97	89
3	5’1	1	2-3	7	4-5	9’7	78	79	85
4	5’3	10	5-6	-	-	9’9	114	99	101
5	5’11	9	5-6	17	6-7	10’5	>130	116	98
6	5’2	7	4-5	13	5-6	9’8	93	104	107
7	4’8	2	2-3	14	5-6	9’2	92	100	100
8	4’5	5	3-4	12	5-6	8’9	91	114	104
9	4’2	7	4-5	14	5-6	8’8	92	110	114

**Standard Scores for SIGMA-T are class- based. All other standard scores are age-based.*

Differences between SIGMA-T and Numerical Operations scores

7 of the 9 children achieved higher scores on the test of Numerical Operations than on the SIGMA-T. While standard scores may not be compared reliably because one set (Numerical Operations) was age-based and the other set (SIGMA-T) was class based, the differences between scores are noteworthy. It is possible that the lower SIGMA-T score were due to the added chore in the SIGMA-Test of reading word problems and directions. This may have lowered scores for any child who had literacy or language difficulties. For example, Child 3’s

lower SIGMA-T score (S.S. 78) may be explained by his literacy difficulties (MICRA-T S.S. 79). However it is also possible that Child 3's achievements are linked to concentration difficulties or to the level of support for learning at home (Table3).

Gender, Attendance and Teacher's Views

Each child's gender and the number of days missed from school during the school year 08/09 can be seen in Table 3. Class teacher's perceptions of parental support and children's concentration skills can also be seen in Table 3.

The class teacher reported the following: Three of the four boys have concentration difficulties (children 1, 2 and 3) and two of the boys do not appear to receive support for learning at home (children 1 and 3). Children 1, 2 and 3 need a lot of help with mathematics in class. Children 1 and 2 are daydreamers. Child 3 has serious literacy difficulties. Child 4 is a very good student who gets anxious when he is not completely sure of things. Child 5 is very quiet in class and is an excellent student. Child 6 is a very good student. Child 7 was unsure of mathematical concepts at the beginning of the year but she is really improving. Child 8 is very good and is an independent worker. Child 9 is competent enough and seems to grasp concepts once they have been explained thoroughly.

Table 3. Attendance, home support and concentration

Child	Sex	Absent 08/09	Home Support	Concentration
1	Male	5	No	Poor
2	Male	5	Yes*	Poor
3	Male	6	No	Poor
4	Male	11	Yes	Good
5	Female	16	Yes	Good
6	Female	4	Yes	Good
7	Female	10	Yes	Good
8	Female	6	Yes	Good
9	Female	20	Yes	Good

* *Support was not good during the year but has improved in recent weeks.*

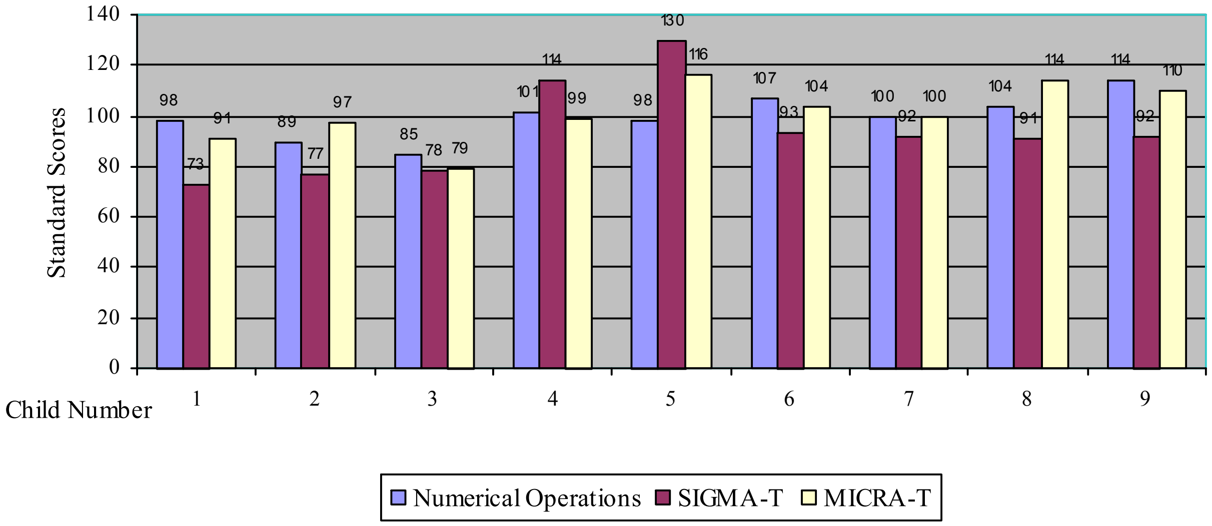


Figure 1. Numerical Operations, SIGMA-T and MICRA-T Standard Scores

Standard scores for all three tests can be seen again in Figure1 above. Two children scored in the Low Average range on the Numerical Operations subtest and in the Low range on the SIGMA-T. These two boys, (Children 2 and 3) who are now 9 years old also achieved low pre and post test scores on the NKT four years previously (Table 2). One of the boys is reported to have serious literacy difficulties (Child 3). Child 1, who achieved a standard score of 73 on the SIGMA-T, achieved a standard score of 91 on the Numerical Operations subtest. This discrepancy between scores may indicate a language or a literacy difficulty. All three children (Children 1, 2 and 3) are international, all are reported to have concentration difficulties and all three receive extra support in school for mathematics.

CONCLUSION

It is reassuring to see that all 9 children achieved standard scores of 85 or higher on a test of arithmetic ability. It is also reassuring to see that some of the children's arithmetic ability is higher than average. It is less reassuring to see that 3 of the 9 children achieved low scores on a test of mathematical achievement which included arithmetic ability *and* mathematical reasoning, and which required certain levels of literacy and language skills.

It is difficult to disentangle the reasons behind any one child's success or failure in mathematics as there are a myriad of intermingling individual and environmental influences on children's learning. In the cases of the 3 children who seem to be falling behind in this study, literacy skills, language skills, poor concentration and perceived lack of support for learning at home seem to play a part. In the case of two of these children, their difficulties were observed four years ago.

Early intervention is essential in order to prevent negative attitudes developing about mathematics so that mathematical difficulties will not negatively affect other curricular areas.

Recent research on mathematics interventions (Williams, 2008; Dowker 2009) indicates that the most effective mathematics' interventions occur daily in 1-1 sessions or possibly within a group of three or four children, with a specially trained teacher, for a whole school term. Williams (2008) suggests that best practice includes careful selection of children, detailed assessment of their strengths and weaknesses, parental consultation, dedicated resource rooms, materials, multi-sensory tools and IT. Many of these practices are part of Mathematics Recovery (Wright et al, 2000) intervention which is available in DEIS schools. However, Mathematics Recovery begins in First Class, which is often two full years after the first differences in mathematical ability have been noticed and sadly the programme is not available in all schools.

While the principle of early intervention is widely accepted, the practice of early intervention is not yet widespread in Irish schools (Travers, 2007). There has been progress over the last four years but we still have a long way to go in order to ensure that every child receives the right support early enough to make a difference. Early intervention needs to be embedded in all of our schools, not just in our DEIS schools. Early mathematics interventions need to focus not just on number concepts but on language, literacy skills and mathematical reasoning. We need resources, professional development and reviews of interventions. It will be expensive (Every Child a Chance Trust, 2009) but in the long term, the returns will be worth the investment.

REFERENCES

Dowker, A.D. (2009) What Works for Children with Mathematical Difficulties? Stationery Office: London.

Eunice Pitt, E. (2001) Ready, Set, Go, Maths: A guide for teachers to help children who find mathematics difficult experience a secure start in early number, Inter Board Numeracy Group: Belfast.

Every Child a Chance Trust (2008), The Long Term Costs of Numeracy Difficulties, www.everychildachance.org

Griffin, S. and Case R. (1997), Rethinking the Primary School Curriculum: An Approach Based on Cognitive Science in Issues in Education, Vol. 3 (1), pp. 1-49.

Griffin, S., Case, R. and Siegler, R. (1994) Rightstart: Providing the Central Conceptual Prerequisites for First Formal Learning Of Arithmetic to Students at Risk for School Failure in McGilly, K (ed) Classroom Lessons: Integrating Cognitive Theory and Classroom Practice, pp 24-49, MIT Press: Cambridge, MA.

Griffin, S., Case, R. and Siegler, R. (1994) Number Knowledge Test (NKT) in Rightstart: Providing the Central Conceptual Prerequisites for First Formal Learning Of Arithmetic to Students at Risk for School Failure in McGilly, K (ed) Classroom Lessons: Integrating Cognitive Theory and Classroom Practice, pp 24-49, MIT Press: Cambridge, MA.

Mullan Y. And Travers, J. (2007) A Team Approach to Number Development, In S. Close, T. Dooley and D. Corcoran (eds) Proceedings of the Second National Conference on Research in Mathematics Education, St. Patrick's College, Dublin, September, 2007.

National Audit Office (2008), Mathematics Performance in Primary Schools: Getting the Best Results, Stationery Office: London.

National Council for Curriculum and Assessment (1999) The Primary School Curriculum, Mathematics, Stationery Office: Dublin.

Sheil, G. and Kelly, D. (2001) The 1999 National Assessment of Mathematics Achievement: A Study Carried out by the Educational Research Centre in co-operation with the Inspectorate of the Department of Education and Science, Education Research Centre: Dublin.

Surgenor, P., Shiel, G., Close, S. and Millar, D. (2006). The 2004 National Assessment of Mathematics Achievement, Educational Research Centre: Dublin.

Wall, E. and Burke, K. (1990) MICRA-T Reading Attainment Tests Level 2, C J Fallon: Dublin.

Wall, E. and Burke, K. (1992) SIGMA-T Mathematics Attainment Test, Level 3, C J Fallon: Dublin.

Weir, S. (2003) The Evaluation of Breaking the Cycle: A follow-up of the Achievement of 6th Class Pupils in Urban Schools 2003: Report to the Department of Education and Science, Education Research Centre: Dublin.

The Psychological Corporation (2005), Wechsler Individual Achievment Test, Second UK Edition (WIAT-II [uk)] Harcourt Assessment: London.

Travers, J. (2007) Learning Support for Mathematics in Irish Primary Schools: A Study of Policy, Practice and Teachers' Views, A dissertation submitted as part of the requirements for the Degree of Doctor of Education (EdD) in the School of Education, Queen's University Belfast Sept. 2007.

Williams, P. (2008) Independent Review of Mathematics' Teaching in Early Years Settings and Primary Schools, Nottingham: DCFS Publication.

Wright, R., Martland, J. and Stafford, A. (2000) Early Numeracy: Assessment for Teaching and Intervention, Chapman: London

Zevenbergen, R., Dole, R. and Wright, R.J. (2004) Teaching Mathematics in Primary Schools, Allen and Unwin: Australia.

ENDEAVOURING TO TEACH MATHEMATICAL PROBLEM SOLVING FROM A CONSTRUCTIVIST PERSPECTIVE

John O'Shea

Mary Immaculate College

This study focuses on a primary teacher and her teaching of mathematics particularly as she endeavoured to explore mathematical problem solving from a constructivist perspective with 6th class students over a school term following her engagement with constructivist theory in professional development. Through the medium of case study and utilising semi-structured interview, group interview and observation, the researcher engaged with this teacher throughout the period of research researching the teaching of mathematics from a constructivist perspective in the classroom. As the teacher and her students renegotiated the norms associated with the mathematics class and became fully immersed in mathematical problem solving, various issues and dilemmas emerged that affected the exploration of mathematical problem solving from a constructivist perspective. The experience was a learning experience for the teacher as she attempted to implement constructivist theory, evident in the curriculum yet perhaps less so in the average primary classroom and brought her underlying assumptions and beliefs about the teaching of mathematical problem solving to the fore. By revealing this teacher's story one can appreciate the difficulty in undertaking what is a radical shift to utilising constructivism as a basis for instruction.

INTRODUCTION

Any attempt to teach mathematical content to students without seeking to make it relevant to them has little hope of succeeding in the long term. Constructivist theory offers an alternative to traditional methods of teaching. Constructivism shares the metaphor of carpentry, architecture or construction work. Von Glasersfeld (1989, p.182) explains that '*knowledge is not passively received but actively built up by the cognising subject'*. A constructivist approach to the teaching of mathematics involves teaching for understanding, it holds that the current knowledge and experiences of pupils as the foundation blocks for future constructions that constructivists try to enable children to build.

A constructivist approach to teaching can be effective but are we sure of its implications for the classroom? Teachers need to have a good understanding of constructivism. The basic principle of constructivism is that children construct their own knowledge. This is a broad sweeping principle and the implications for the traditional classroom as it exists are far reaching. Do we have enough evidence upon which to build solid curricula and prepare primary teachers for employing constructivist principles in the classroom so that they may move away from what has characterised the very essence of teaching for decades? Research would suggest we have not moved far in a constructivist direction to date (Surgenor, Shiel, Close and Millar, 2006; Windschitl, 2002). Airsian and Walsh (1997) reveal that constructivism is a theoretical framework, which broadly explains human activity of knowing but unfortunately, it offers teachers very little detail in the art of teaching.

Proponents of educational reform view the process of getting to know mathematics as a social endeavour that happens during the interactions within the classroom (Ball, 1993; Bauersfeld, 1995; Cobb, Yackel and Wood, 1993; Lampert, 1990). This is an emerging perspective on constructivism. Such interactions are characterized by having students think, talk, agree and disagree about mathematics that is relevant to them. Both the NCTM (2000) Principles and Standards for School Mathematics and the Primary Mathematics Curriculum (Government of Ireland, 1999a, 1999b) articulate that rather than explaining and demonstrating, teachers move towards a non-traditional way of teaching in the classroom by allowing students to meaningfully explore their own mathematical ideas, express them and to explore the thinking offered by others. Fullan (1993) and Joseph, Bravmann, Windschitl, Mikel and Green (2000) explain that effective forms of constructivist teaching depend on nothing less than the reculturing of the classroom but the features that make constructivist classrooms effective complicate the lives of teachers, students, administrators and parents. Teaching contexts, teacher characteristics, teacher thinking and their interactions are influential factors in attempts to implement classroom reform (Gess-Newsome and Lederman, 2003). Many reform initiatives have arrived at the classroom doors of teachers but Cuban (1988) noted that reforms that seek to change fundamental structures, cultures and pedagogies are difficult to sustain and progress. Enacting classroom practices that support discourse in the mathematics classroom poses challenges for teachers as they bear little resemblance to their current practices (Nathan and Knuth, 2003).

We have been engaged in reform in Irish primary classrooms for the past ten years. We know a lot about teaching and learning in the primary school and our curriculum has undergone reform and evaluation (NCCA, 2008) but, to what extent is this knowledge and reform impacting upon teaching practices within the primary mathematics classroom? Worryingly, particularly in relation to mathematics education, the National Council of Curriculum and Assessment's (2008) review of the implementation of the Primary Curriculum has revealed that teachers still feel challenged by methods of teaching, particularly group teaching, espoused by the Primary Curriculum (1999). Various reports and research conducted since 1999 have revealed that Irish primary students can perform basic mathematical skills quite well, they know their mathematical facts, but these students compare poorly to similar cohorts in other countries in relation to higher level mathematical process such as reasoning, analysing and solving problems and analysing solutions (Mullis, Martin, Beaton, Gonzalez, Kelly and Smith, 1997; Surgenor, Shiel, Close and Millar, 2006). This can be expected of students that have come through an education system that places significant emphasis on direct instruction and less emphasis on alternative forms of instruction.

This research attempts to investigate a typical teacher's utilisation of constructivist practices in the senior primary classroom following her engagement with constructivism and mathematical problem solving from a constructivist perspective in a professional development initiative delivered during the autumn of 2007.

REVIEW OF LITERATURE

While the Primary School Mathematics Curriculum (Government of Ireland, 1999) stems from its predecessor, Curaclam Na Bunscoile (Government of Ireland, 1971), it advocates greater use of resources and practical activities at all class levels to develop mathematical concepts. In its description of a child centred curriculum, the curriculum encourages the use of constructivist approaches.

> Constructivist approaches are central to this mathematics curriculum. To learn mathematics children must construct their own internal structures. As in reading and writing, children invent their own procedures (Government of Ireland, 1999a, p.3).

The curriculum (Government of Ireland, 1999a; 1999b) acknowledges that children must experience formal mathematics instruction, 'we accept that children must go through the invented spelling stage before they begin to develop a concept of the structure of spelling.

The same is true of mathematics' (Government of Ireland, 1999a, p.3), yet purports that ultimately, the child should be encouraged to experiment with personal strategies, refine them through discussion and engage in a wide variety of tasks (Government of Ireland, 1999a). The curriculum advocates that the children must be encouraged to operate in small groups or pairs to facilitate constructivist learning. Through involvement in these situations, children are expected to engage in the discussion of mathematical problems and their solutions, while supporting and helping other students.

The Mathematics Curriculum (Government of Ireland, 1999a) encourages children to adopt models of problem solving behaviours. "Children need to work out when to use a particular plan, what they want to achieve and the actual procedure needed to complete the task" (Government of Ireland, 1999a, p.4). This coincides with literature that emphasises the significance of applying models of problem solving behaviour during activity (Lester and Garofalo, 1985; Polya, 1945; Shavelson, Mc Donnell and Oakes, 1989). The curriculum acknowledges the importance of focussing on the process as opposed to the product, as a medium of developing individual learning strategies. It also emphasises the use of open-ended problems, where considerable emphasis is placed on discussion and the acquisition of skills and not just the achievement of the correct answer. This is in alignment with current curricular thinking in the United States as outlined in Principles and Standards for School Mathematics (Carpenter and Gorg, 2000).

METHODOLOGY

In this study, a teacher utilised Polya's (1945) four stage problem solving procedure, understand the problem, devise a plan, carry out the plan and reflect with primary mathematics students for a period of a term as engaging children in such a process is closely aligned with a constructivist approach to teaching. Through case study and utilising semi-structured interview, group interview and observation the endeavours of the teacher were revealed. Data is specifically drawn from semi-structured interviews with both students and teachers and mathematical problem solving lessons conducted from a constructivist perspective by classroom teachers. In the following description of one teacher's case, data is taken from groups of students' mathematical explorations.

SUSAN'S CASE

Susan is an energetic teacher, eager for her students to learn. Her school is a large suburban primary school where children come from middle class backgrounds and resources are plentiful. The school has very little difficulties in terms of parental support, resources or teaching space. Susan has experience and a high level of education behind her and is committed to helping students achieve their best. From working with Susan, I conclude that her teaching reflects the principles of the Primary Curriculum (1999) but she teaches for understanding and the structure of her lessons discourage in depth explorations of a students understanding. In Susan's opinion, this is due to the various learning styles present in her classroom. Susan is eager for children to experience all areas, strands and strand units of the mathematics curriculum but her particular situation, and the children that she teaches this year, have required Susan by her own admission, to take a traditional approach to the teaching of mathematics. It becomes evident that Susan's assumptions about learning are grounded in learning being that of memorisation and practice rather than activity.

Susan's didactic teaching style

Susan identifies herself strongly as a good learner of the mathematics. She also has a passion and enthusiasm for the subject and it was her teachers that engendered this within her. Susan has adopted a traditional approach to her teaching of mathematics and freely admits that this approach stems from her own experiences as both a primary student and secondary student. She recalls clearly her days as a student of mathematics and has revealed that they have had a significant impact on how she teaches mathematics. Susan enjoyed her time at primary school and was challenged by her teachers by their provision for her of difficult textbooks and mathematical problems. At primary level, Susan's teachers challenged brighter pupils by supplying them with mathematical problems from textbooks such as *Figure it Out* and *Busy at Maths.* Susan challenges her own high achieving mathematics pupils by supplementing their daily assignments with mathematical problems taken from other textbooks. Susan has great respect for her own teachers of mathematics and places significant merit on such methodologies and strategies utilised by those teachers. It is evident that Susan employs similar teaching methodologies that she herself experienced as a pupil. She places significant emphasis on direct instruction. According to students, Susan spends a significant amount of time utilising direct instruction during mathematics classes 'we correct our homework first and then she would explain something and ask us to do questions on it. She does things on the board loads of times and then we go and do it ourselves'.

Susan's focus on computation

Students in Susan's classes must have a significant understanding of and experience in the operations addition, subtraction, multiplication and division before progressing, for example, to exploring mathematical problems in association with their peers. In fact, students spend the majority of time in Susan's mathematics classroom working alone. Susan's students rarely spend time working together during typical daily mathematics lessons.

Susan is critical of the achievements of today's primary mathematics students when recalling the accomplishments of students of *Curaclam na Bunscoile* (1971). She attributes achievements of students of *Curaclam na Bunscoile* to methods of teaching mathematics that can be described as traditional. Children need to 'have copies where they repeat and repeat their sums'. In her analysis of constructivist teaching and learning Susan even went on to say 'it is very valuable but it has to be used in conjunction with the rote learning, the chalk and talk and the teacher directed learning'.

Susan's mathematics classes exuded traditional conceptions of mathematics teaching and learning. She used new materials and ideas yet conducted exercises in a thoroughly traditional fashion. This is evident in the manner Susan conducted the mathematical problem solving lessons with her students. Susan regularly utilises direct instruction in her exploration of mathematical problem solving by launching into an explanation of a problem before children have the opportunity to decide on an appropriate solution or strategy or explain such strategy themselves. In the following instance, Susan asks a student to explain a solution to a problem, the student is hesitant in describing her solution to the problem therefore Susan proceeds to explain the solution to the problem for the student to the rest of the class (Problem: In how many ways can the carriages of a three car train be arranged?).

Teacher:	Student X – will you explain to us please what you did.
Student X:	I can't really remember by looking at this.
Teacher:	What Student X is trying to say to us is that her group named the carriages 1, 2 and 3. You could get 1, 2, 3 you could get 1, 3, 2. Then you might put number 2 first and get 2,1,3 2,3,1 3,1,2, and 3,2,1. They are all the different ways they can be arranged so let's count them – 6. I think most groups got that, Good job, well done.

Susan teaches mathematics for understanding and the didactic format of her teaching inhibits pupils' own explanation or exploration of pupil ideas. This is due significantly to her experience of such methodologies at school. She declared that third level courses on methodology were less than informative.

Susan has a traditional view of what counts as mathematical prowess and her conviction about her approach was plain. Students in Susan's classroom study the foundations of operations, algorithms and procedures in significant detail. Students spend significant amounts of time doing mathematical operations that are supplied by the teacher and by textbooks. Mathematical knowledge is broken into clearly defined units particularly for those students who may be having difficulty with the subject. This became apparent as Susan explained constructivist philosophy could only be employed in her particular classroom on a 'topic by topic' basis. In fact, Susan explained, 'forget problems that require a number of concepts or operations'.

Susan's emphasis on the rote memorisation of number facts

Traditional conceptions of mathematics teaching and learning are clearly evident in her strong belief in the need for the rote memorisation of number facts or, as they are frequently referred to by Susan, tables. Susan has a strong conviction that students need to be fluent in the operations. When Susan was a student at primary school, her teachers placed a significant emphasis on the rote memorisation of number facts or tables. The rote memorisation of such facts is at the heart of Susan's teaching of mathematics and Susan is unwavering in her belief that every student must have a comprehensive understanding of tables. Susan requires all of her students to memorise their tables and spends a portion of her allocated time for mathematics on the examination of these tables on a regular basis. During one visit to Susan's classroom, Susan had spent 15 minutes examining tables before proceeding to explore a mathematical problem. She explained in interview, 'we need to go back a little bit to the old style where it was drill'. Susan's conviction is such that her students do not progress very far beyond simple computation and simple problem solving if they have not a comprehensive knowledge of their tables. She explains that this is particularly the case in this academic year as she regards her students as mathematically weak. Susan rationalised that students must have a comprehensive understanding of tables if they are to succeed in mathematics. She highlighted the fact that students will need to have a capacity to do mental mathematics in nearly all avenues of life including, as she outlined, simple shopping expeditions. It is here she warns of overusing the calculator at primary level. Susan is convinced children can become over reliant on the calculator and consequently do not develop the ability to become proficient at mental mathematics.

Susan's difficulty with a large pupil teacher ratio

At the outset of engaging in semi-structured interview, Susan asked if she could be controversial. One issue that Susan feels very strongly about is the lack of resources and procedures that enable teachers to teach from a constructivist perspective in the Irish Primary school context. Susan speaks from her own perspective and in particular feels very strongly that the pupil teacher ratio in her school, and she highlights that similar ratios exist in schools of her colleagues, prevent in particular the employment of constructivist methodologies in her teaching. The current ratio is 28:1 but Susan explains that in reality, this is always greater when numbers of teachers in Learning and Language Support are taken into account. Observing Susan's classroom, it is clear that exploring mathematical problem solving in group situations is difficult in that the amount of space in her classroom is limited. The population of the area where Susan teaches has exploded in recent years and this is reflected in the number of pupils enrolled at the school. Susan feels that this restricts her in approaching mathematical problem solving from a constructivist perspective.

Susan finds it difficult to explore basic concepts with students of varying abilities in her mathematics classes because of the numbers of students involved. During a typical lesson, Susan endeavours to teach a particular concept to all students through direct instruction and then segregate the class into groups assigning work to one group as she continues to teach another who may have difficulties in understanding. She explains it is 'next to impossible' to teach that many children. Susan cannot see how one would have children problem solve from a constructivist perspective on regular basis given the difficulties that arrive during classes involving direct instruction with large groups of students.

Susan's use of group work

Susan's groups were used for instructional purposes in a distinctive way. During lessons, Susan directed the students to answer questions related to a problem rather than facilitated students in their quests to identify questions and problems related to the construction of an appropriate problem solving method. Susan was using a traditional approach to the teaching of mathematics while combining a reform based approach to facilitate her teaching. This is clearly evident in the following (Problem: How many addition signs must be put between the numbers 987654321 to make a total of 99?).

Teacher:	But would they add up to 99 if you used 8 addition signs? 9 + 8 is 17 plus 7 is 24 plus 6 is 30 plus 5 is 35 plus 4 is 39 + 3 is 42 plus 2 is 44 and plus 1 is 45. So no, try and put some of the numbers together.
Student B:	What about 1 + 2 + 4?
Student C:	But it will still give you 45?
Student A:	We have to be careful of how we use the big numbers
Teacher:	How about joining your 7 and 6 together maybe?
Student A:	You put the 2 and the 1 together that's 21 and 9 is 30
Student B:	There are lots of ways the 3 and the 5 together is 35
Student A:	We have to be careful and keep the 9 and the 8 separate. Careful how we use the big numbers
Student B:	Are we allowed move around the numbers or do they have to be like that?
Teacher:	Start at the beginning and work it out.

Susan gives students opportunities to interact with each other but is very clearly directing students towards a strategy of solution clearly identified by the teacher in advance.

Susan's approach to teaching pupils with different learning abilities

Susan teaches sixth class students who were identified as mathematically weak by the school by using a standardised test. Susan describes her particular situation as 'different from others'. Susan explained that as her students did not have a firm understanding of operations consequently exploring mathematical problem solving from a constructivist perspective was going to be particularly challenging for them from the outset. From the initial stages of the project until its conclusion, Susan believed that it is unrealistic to approach mathematics from a constructivist perspective with low achieving students of mathematics and this restricted her students from experiencing the subject as discussed during professional development. This is evident in her explorations of particular problems with students. Susan offers students the solutions and strategies for obtaining solutions to the problems while giving students little opportunity to reach a conclusion or design an appropriate problem solving strategy themselves. Susan's lessons were regularly brought to an abrupt conclusion by her exploration of the problem at the blackboard in front of the whole class. This is illustrated in the following (Problem: A farmer has pigs and chickens. She counted 140 eyes and 200 legs. How many pigs and how many chickens were there?)

Student A: This is complicated. We need the teacher

Teacher: Can I give you a hint, some people have worked out that if there are 140 eyes in total, there are 70 animals altogether as each animal has 2 eyes.

Students A, B and C: Oh, 70 animals.

Teacher: So now we have to figure out all the different ways of making 70 and see which would make sense. Take a guess, 30 chickens, so 2 legs each is 60 legs and then there would be 40 pigs and 4 by 40 is 160 – so it is 230, could that be right.

Class: No

Teacher: It's all trial and error. That's what we have to do make guesses and check them out.

Teacher: Don't rub out any of your answers. Remember the eyes are sorted and that it is the legs that we need to work on. Have we an answer?

Student A: 30 pigs and 40 chickens, 30 pigs and 40 chickens have 140 eyes so that is right, 30 pigs will have 120 legs and 40 chickens will have 80 legs.

Teacher: That is 200 legs altogether, that's right, well done.

Susan's belief in the mathematical abilities of the children in her care is highlighted in the following quotations. They also explain Susan's significant involvement in the students' attempts to construct a strategy to solve the problem.

> 'I think to be honest because they were particularly weak, having an idea or putting an idea about something forward would have caused difficulty in any subject area not to mind maths. They need the teacher as a crutch. They couldn't even put an argument together in English, one sentence and that was it'.

Susan is open to students solving problem solving from a constructivist perspective but students must have particularly strong background knowledge of mathematical concepts and operations as illustrated by the following.

> 'Some of them have even difficulties adding hundreds tens and units and some of them had some idea about for example the addition of fractions so a very mixed bag indeed. Constructivism is great and I will do it next year where I know my class will enjoy it more and get more benefit out of it but this year is particularly hard'.

Susan continued to elaborate explaining 'I can really see it working well with more able students.

Susan described the interpersonal skills of the students as weak. According to Susan, these students did not have the required skills to work appropriately in group situations. Throughout their primary school years, due to a poor understanding of basic mathematical concepts, this group of children, according to Susan, had very little experience of mathematical problem solving. Susan explains 'it was a lack of problem solving, they hadn't experienced enough of it but where do you go if you can't add, subtract or multiply?' Significantly, Susan believed that her students had particular difficulties with memory explaining that students, when they returned from a break or holiday period, would act like they had never seen the material before. This would suggest Susan would have to revert to exploring basic number facts and operations with students repeatedly. These students were at a disadvantage, because of teachers consistently labelling them as mathematically weak, they experienced little, if any, constructivist approaches to mathematics in the later years of their primary schooling.

Susan's constructivist approach to learning

Susan's classroom was organised for cooperative learning but her instructional strategies cut across the grain of this organisation. The class was conducted in a highly structured and classically teacher centred fashion as illustrated in the mathematical problem solving episodes. Susan has considerable experience as a teacher and also has experience as a teacher educator and has therefore a sound understanding of the implications of approaching mathematical problem solving from a constructivist perspective. Her reservations for exploring mathematics from such a perspective have not evolved from a lack of understanding of constructivist theory but rather from a distinct belief in its appropriateness relative to the occasion. This is clearly evident as Susan acknowledges the value and purpose of a constructivist approach to learning but 'it has to be used in conjunction with the rote learning, the chalk and talk and the teacher directed learning'. Also, the mathematical abilities of students are taken into account before Susan employs particular methodologies in her lessons.

She describes a constructivist approach to teaching as 'about problem solving, finding out where the students are at and then building upon it'. She continues 'it's about giving a little bit more ownership to the students. It is going away from directed learning'. This is a firm description of a constructivist approach to learning. She admits however that 'more guidance is required' and that 'the material isn't there to facilitate the teacher'. Susan believes the vast majority of teachers would not approach the teaching of mathematical problem solving from a constructivist perspective due to a lack of pedagogical knowledge, 'It may say it in the curriculum, but I don't think many teachers would be familiar with how to go about doing it in the classroom'. This reinforces a finding by the Primary Curriculum Review Group: Phase 2 (2008) which found that teachers are challenged in developing a child's higher level thinking skills and that whole class teaching strategies are the most frequently utilised teaching strategies in primary classrooms.

CONCLUSION

Susan's story reveals the complex ideologies that are brought to classrooms that impact on how primary mathematics is explored. Her story highlights the complex nature of the Irish primary classroom and the day-to-day challenges faced by primary teachers as they implement the curriculum. Fullan (1993) and Joseph et al. (2000) explain that effective forms of constructivist teaching depend on nothing less than the reculturing of the classroom but the features that make constructivist classrooms effective complicate the lives of teachers, students, administrators and parents. This has certainly been revealed in Susan's case. Endeavouring to explore primary mathematics from a non-traditional perspective presents the teacher with many challenges which may be difficult to overcome but not if sustained professional support across the teaching continuum were in place to reconceptualise the teaching of mathematical problem solving.

REFERENCES

Airsian, P.W. & M.E. Walsh, (1997). Constructivist Cautions. *Phi Delta Kappan*, 78(6), 444-449.

Ball, D. L., (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93*(4), 373-397.

Bauersfeld, H., (1995). The structuring of the structures: Development and function of mathematizing as a social practice. In: L. Steffe and J. Gale (eds.), *Constructivism in education*, Hillsdale, NJ: Lawrence Erlbaum Associates, 137-158.

Cuban, L., (1988). A Fundamental Puzzle of School Reform. *Phi Delta Kappan*, 69, 341-344.

Fullan, M. (1993). *Change Forces: Probing the Depths of Educational Reform*. London: Falmer Press.

Garofalo, J. & F. Lester, (1985). Metacognition, Cognitive Monitoring and Mathematical Performance. *Journal for Research in Mathematics Education* 16: 163-176.

Gess-Newsome, J., S.A. Southerland, A. Johnston & S. Woodbury, (2003). Educational Reform, Personal Practical Theories, and Dissatisfaction: The Anatomy of Change in College Science Teaching. *American Educational Research Journal, 40*(3), 731-767.

Government of Ireland, (1971). *Curaclam na Bunscoile*. Dublin, Stationary Office.

Government of Ireland, (1999a). *Primary School Mathematics Curriculum: Teacher Guidelines.* Dublin: Stationary Office.

Government of Ireland, (1999b). *Primary School Mathematics Curriculum*. Dublin: Stationary Office.

Joseph, P. B., S.L. Bravmann, M.A. Windschitl, E.R. Mikel & N.S. Green, (2000). *Cultures of curriculum.* Mahwah, N.J: Lawrence Erlbaum.

Lampert, M., (1990). Connecting inventions with conventions: The teachers' role in classroom communication about mathematics. In: L. Steffe & T. Wood (eds.) *Transforming Early Childhood Mathematics Education.* Hillsdale, NJ: Erlbaum, 253-265.

Mullis, I.V.S., M.O. Martin, A.E. Beaton, E.J. Gonzalez, D.L. Kelly & T.A. Smith, (1997). *Mathematics Achievement in the Primary School Years: IEA's Third International Mathematics and Science Study (TIMSS).* Chestnut Hill, MA: Boston College.

Nathan, M.J. & E.J. Knuth, (2003). A study of whole classroom mathematical discourse and teacher change. *Cognition and Instruction* **21**(2): 175–207.

NCCA, (2008). Primary Curriculum Review: Phase 2. www.ncca.ie , accessed 10-05-2009.

Polya, G., (1945). *How to Solve It*. NJ: Princeton University.

Shavelson, R., L. McDonnell & J. Oakes, (1989). *Indicators for Monitoring Mathematics and Science Education.* Santa Monica, CA: Rand Corporation

Surgenor, P., G. Shiel, S. Close & D. Millar, (2006). *Counting on Success: mathematics Achievement in Irish Primary Schools*. Dublin: The Stationery Office.

Von Glasersfeld, E., (1989). Constructivism in Education. In: T. Husen and N. Postlethwaite (eds.) *International Encyclopedia of Education.* Oxford: Pergamon, 162-163.

Windschitl, M., (2002). Framing Constructivism as the Negotiation of Dilemmas: An Analysis of the Conceptual, Pedagogical, Cultural, and Political Challenges Facing Teachers. *Review of Educational Research*, 72(2), 131-175.

Yackel, E., T. Wood & P. Cobb, (1993). Discourse, mathematical thinking, and classroom practice. In: E. Forman, N. Minick, & C. Addison Stone (eds.) *Contexts for learning: sociocultural dynamics in children's development.* Oxford: Oxford University Press, 91-119.

LEARNING SUPPORT FOR MATHEMATICS: LESSONS FROM 100 LESSONS

Joseph Travers

St. Patrick's College, Drumcondra

This paper reports on an analysis of the teaching practices of 50 learning support teachers for mathematics based on tutor observation reports of 100 lessons. The practices are compared to those recommended in the literature. Overall, the quality of instruction observed was very good with some exemplary practice observed. At the heart of this was the ability to individualise instruction and utilise an eclectic range of empirically validated teaching and learning strategies to promote learning. However, there was also evidence of shortcomings such as inadequate planning, failure to differentiate, non-use and inappropriate use of concrete materials, non-use and poor use of mathematical software and not realising the potential of small groups working collaboratively on shared tasks and learning from each other.

INTRODUCTION

A diverse range of strategies have been highlighted in the literature for supporting pupils who experience difficulties in learning mathematics. In terms of raising standards in mathematics for pupils with general learning disabilities Porter (2003) stresses the following: tailoring the learning context to the pupils' needs and interests, connecting the abstract with the practical, linking skills with understanding, reducing the emotional impact, scrutinising and adapting the language of instruction, paying attention to old as well as new learning, providing contexts for consolidation and generalisation and using visual cues to reduce the load on working memory.

A key theme of research in this area is that the range of strategies which proved effective crosses pedagogical philosophies incorporating constant time delay, peer tutoring, time trials, direct instruction, strategy instruction and using concrete materials (Butler *et al.,* 2001). Learners benefited from "interventions stressing frequent feedback, explicit instruction, and ample drill-and-practice" (p.29). At the same time "strategy instruction promoted student independence in addition to increasing mathematics performance" (p.29).

Baker *et al.* (2002, p.67) having set up their "gold standard" (Whitehurst, 2003) criteria maintain that: "Although this is not a large body of research, four findings are consistent enough to be considered components of best practice." First, providing students and teachers with specific information on how each student is performing seems to improve mathematics achievement, "raising scores, on average, by 0 .68 SD units" (p. 67). Second, using peers as tutors or guides enhances achievement. This is confined to computational abilities and "holds promise as a means to enhance problem-solving abilities" (p.68). Third, based on two studies, providing "specific, objective, and honest" feedback to parents of low achievers and detailing "successes (or relative successes) as opposed to failures or difficulties" have the potential to enhance achievement (p. 68). All of these seem very low cost measures on the surface, but the level of knowledge and skill involved in diagnosing strengths and needs to be used for feedback and further teaching should not be underestimated (Pitt, 2001). The fourth finding is that "in terms of curricula, a small body of research suggests that principles of direct or explicit instruction can be useful in teaching mathematical concepts and procedures" (Baker et al., 2002, p. 68).

The literature among other areas also highlights the benefits of targeted early intervention based on diagnostic assessment (Dowker, 2004), cognitive/metacognitive problem solving strategies (Xin and Jitendra, 1999), while Conway (2002) argues the benefits of a socio-cultural perspective, which would entail a shift from a psychology of individual differences to building learning communities with more attention paid to the social context and participation structures. The present study sought to ascertain the actual pedagogical practices of learning support/resource teachers of mathematics in Irish primary schools.

METHODOLOGY

Two of the courses I teach on in the Special Education Department in St. Patrick's College involve observation of experienced teachers teaching in a learning support or special educational context. Detailed field notes are taken on these observations to facilitate both formative and assessed feedback to the teachers. While the field notes are not compiled for research purposes, they represent a unique data source in relation to actual pedagogical practice and permission was sought for their use from 65 teachers. Just over 50 teachers replied granting access, giving a total of 100 lessons observed by six tutors, which could be analysed.

The pro forma for the lesson observation included detailed information on size of groups, classes taught, resources used, topics taught, descriptions of lessons and evaluative comments on pedagogy observed. The main focus of the observation was on the teaching and learning situation. Thus, a detailed description of the lessons observed was gained, incorporating both teacher and pupil behaviour. All pupils were experiencing difficulties in mathematics ranging from mild to those assessed with a general learning disability. While practice was judged against recommendations from the literature the reports are inevitably affected by the subjective views of the tutors.

TEACHING AND LEARNING

In relation to the lesson evaluations teachers had a mean number of 3.6 pupils in their groups with a mode of 4 (38%). The lessons covered all primary classes except junior infants (first year in school) with 81% of the lessons in classes between 1st and 4th (third year to fifth year in school). While all aspects of the mathematics curriculum were covered, the topics most frequently covered were aspects of number (observed in 45 lessons), word problem solving mainly for addition and subtraction (observed in 20 lessons) and fractions and place value (observed in 18 and 17 lessons respectively).

Lesson evaluation notes were analysed firstly in terms of positive descriptions made by the tutors informed by criteria identified in the literature as supportive of pupils experiencing difficulties in mathematics. Ninety-one of the lessons contained evaluations that were interpreted as positive comments. These were then further analysed for key themes and issues in practice. The following emerged as the areas most positively evaluated and interpreted as best practice: the quality of planning and an eclectic range of appropriate teaching strategies.

Planning

The quality of the planning particularly when tailored to pupil needs and linked to weekly and termly plans were highlighted (DES, 1999; 2000). Also evident in the planning was subject competence:

> *...high level of subject competence was displayed in the planning and teaching of the lesson. (Tutor evaluation of lesson 28)*

Other features mentioned included lesson cohesion, organisation and systematic planning.

There was a strong emphasis on teachers developing pupils' conceptual understanding of the mathematical material:

> Strong emphasis on conceptual understanding and making linkages, and monitoring understanding; very good use of mental strategies and concrete materials to promote understanding, very good pace of instruction; well planned in five different parts; engaging manner with the pupils. (Tutor one evaluation of lesson 69)

This emphasis on conceptual understanding goes to the heart of all recent reform efforts in mathematics education (NCTM, 1989, 2000).

Appropriate teaching strategies

Teaching strategies that were highlighted as being appropriate for the mathematics' lesson objectives were of an eclectic nature. They included the development of metacognitive skills (Bley and Thornton, 2001), direct instruction (Baker *et al.,* 2002), peer tutoring (Dowker, 2004) and teacher modelling of thinking strategies by "thinking aloud" (Conway and Sloane, 2006):

Inculcation of metacognitive skills, through guided reflection, on part of the pupil e.g. What method do you have to remember your tables? Do you just learn them like a poem? Do you see them in your head? Do you count on up from the one that you are sure of? (Tutor one evaluation of lesson 9)

…superb teacher questioning/comments. Self-talk was effectively used to show understanding of both the sequence and concept. (Tutor one evaluation of lesson 46)

Direct teaching was used when necessary but you always posed questions that first enable the pupils to reason and problem solve and when successful expressed delight and praise for their successful efforts. (Tutor four evaluation of lesson 18)

The explicit modelling of strategies by thinking aloud, though validated in the literature is not observed all that often:

Considerable research suggests that teachers rarely use think alouds and other strategies that model and make explicit complex and expert problem-solving. This is especially troubling as such strategies have been demonstrated to be effective with lower-achieving students in both primary and post-primary settings (Conway and Sloane, 2006, pp.101-102).

Other strategies observed which have been validated in the literature include helping pupils make connections (Askew *et al.,* 1997) and appropriate use of concrete materials:

…content very well sequenced, new knowledge was related to prior knowledge. (Tutor six evaluation of lesson 36)

…related decimal fractions to fractions, excellently grounded in continuous "thinking aloud " strategy that helped pupils understand each step, use of song to reinforce link between fractions and decimal fractions. (Tutor four evaluation of lesson 42)

Pupils guided from the concrete, to pictorial and symbolic representation, appropriately paced and pitched allowing for challenge and progression, performance was monitored, all actively engaged." (Tutor three evaluation of lesson 17)

Maintaining attention through active engagement of the pupils in purposeful and appropriate activities also strongly featured. Giving the pupils opportunities for application, practice, reinforcement and over learning (Baker *et al.,* 2002; Lerner, 2006):

> *…and lots of practical applications so the children had lots of opportunities to grasp and practice concepts. Reinforced with lovely rap. (Tutor six evaluation of lesson 85)*

Monitoring and assessing understanding before moving on (Westwood, 2007) was a key component of many lessons:

> *Works through the problem orally, making sure the children are ready before proceeding, thorough rigorous teaching, well planned, children really becoming competent in this difficult area of maths. (Tutor five evaluation of lesson 92)*

Attention to the language of mathematics and the direct teaching of mathematical terms was a feature of many lessons (Porter, 2003):

> *…attention to language of the four operations; excellent focus on language and on using problem solving strategies; very good use of discussion to elicit thinking. (Tutor four evaluation of lesson 53)*

Linking learning to pupil experience and environment which is one of the principles of the revised primary curriculum (DES, 1999) was evident in many lessons:

> *Very good use of personal benchmarks to illustrate cm and metre, good emphasis on hands-on work and actual process of measuring and monitoring understanding. (Tutor one evaluation of lesson 83)*

> *Cue cards, my personal challenge card, multiplication rap (made up by pupils), attention to language of the four operations activity; excellent focus on language and on using problem solving strategies, very good use of discussion to elicit thinking. (Tutor one evaluation of lesson 80)*

The individualising of instruction to challenge the pupils at an appropriate level was also evident (Dowker, 2004):

> *…individual needs catered for, differentiation through questioning and through acquired tasks. (Tutor four evaluation of lesson 18)*

Excellent use of overhead projector as a material resource, monitored the group to ensure task engagement, most impressive aspect was the gradual inbuilt increases in level of challenge for each individual, motivating activities with a great element of fun were a feature. (Tutor two evaluation of lesson 16)

There was also extensive use of a range of concrete materials and visual cues to facilitate understanding (Porter, 2003). The range of materials used can be seen in Table 1. Materials relating to place value, number games and various cards featured most often and, on the whole, were used appropriately. The low use of the calculator can be partly explained by the age range of the classes but the low use of software cannot.

Table 1 Use of mathematical resources in 100 lessons

Resources used in observed lessons	Number of lessons	Types of the resource observed
Place value materials	21	Diene's blocks, lollipop sticks, place value boards, unifix cubes
Maths Games	20	Number bingo, dice games, snake and ladders, dominoes, number snap, lucky dip, numero, fraction game
Cards	17	Number cards, playing cards, fraction cards, word cards, arrow cards, cue cards, number chain cards, flash cards with number facts
Number line	15	Plastic number lines, number "washing line", number strips on tables, number ladder
Whiteboard/blackboard	13	
Counting materials	13	Blocks, cubes, teddies, spindle box, objects
Materials for time	11	Clock, calendar
Fraction materials	10	Fraction wall, paper folding, cutting apples, game and cards
Hundred Square	9	Magnetic, blank, colour- coded, large walk on
Magnetic board	7	Magnetic numbers, 100 square, magnetic fruit
Shapes	7	Two and three dimensional and fraction shapes

Money	7	Coins, notes, shop and catalogue, price list
Materials for length and weight	5	Metre stick, balance, scales, objects
Pegs	5	Pegboard, 100 square and number line
Charts	4	Flip chart, problem strategy chart, place value chart
Computer software	3	Numbershark and Destination maths
Songs	3	Counting songs
Other	2	Drill, mitre saw, dado rail, holiday brochure
Calculator	1	
Overhead projector	1	

Pupil learning strategies observed and praised included self-checking and monitoring, using recall strategies for learning basic facts and problem solving strategies (Bley and Thornton, 2001; Xin and Jitendra, 1999). There was some evidence of the use of real life contexts for problem solving. These included using holiday brochures, shopping catalogues, rail and bus timetables and TV schedules as contexts for realistic and genuine problems. The use of rich non-mathematical contexts for problem solving is a feature of the realistic mathematics education movement, which has its roots in the work of Hans Freudenthal (Conway and Sloane, 2006).

Opportunities given for the pupils to ask questions were also observed. A small number of observer comments highlighted small groups working collaboratively as a group as opposed to individually in a group. Such collaborative work has been advocated as a very appropriate way to develop pupils' mathematical language and reasoning skills (Haylock, 1991).

Aspects of teaching style, which received positive comments, included intensity in the teaching, high-energy fast paced instruction, and respectful and positive behaviour management techniques (Heward, 2003). The quality of teacher-pupil interaction and relationship were also valued.

SHORTCOMINGS IN OBSERVED LESSONS

Areas of lessons where shortcomings were identified were analysed across the lesson evaluations. Forty-six of the lesson evaluations contained comments that could be interpreted as representing a shortcoming. Comments made were grouped according to similar patterns and then into categories. These shortcomings emerged as the opposite or weaker approximations of many of the practices observed in the more exemplary lessons. The categories which emerged from the analysis were: inadequate planning, failure to differentiate, shortcomings in teaching strategies, non-use and inappropriate use of concrete materials and other issues grouped under other shortcomings.

Inadequate planning

Shortcomings at the planning stage of lessons were identified. Some of the plans had an over emphasis on content to be covered as against how the content would be used to develop pupil skills and strategies. Some appeared like isolated units or were not integrated with longer-term plans. Others included a lack of attention to teacher actions to meet lesson targets and a failure to differentiate in the plans for different pupil needs:

> *Notes and planning were minimum. Does not have a grasp of writing targets and objectives. (Tutor two evaluation of lesson 77)*

Failure to differentiate

At the heart of special education is personalising instruction to meet individual needs (Heward, 2003). Inadequate attention to this was a feature of critical comments. This can be traced back to the planning phase and can manifest itself with regards to content, materials and teaching and learning approaches (Lewis and Norwich, 2005). Failure to differentiate can result in material being too difficult or some pupils not being sufficiently challenged. Illustrative comments included:

> *This work needed to be differentiated for both more able and less able children in term of content and teaching strategies; not enough challenge for some and others were lost. Needed to think about individual children. (Tutor five evaluation of lesson 71)*
>
> *Children could have been working on more challenging concepts and skills. (Tutor six evaluation of lesson 33)*

Shortcomings in teaching strategies

Difficulties with sequencing, structure, consolidation and monitoring of understanding were evident in the evaluations:

More structure needed, necessary for the child to put the blocks in a line before counting them; more time needed explaining the commutative property of addition; child needed strategies for counting on and counting back apart from the number line. (Tutor three evaluation of lesson 3)

Too many strategies for tables introduced at the one time, needed to consolidate understanding more fully, moved too quickly to discrete units with fractions. (Tutor one evaluation of lesson 6)

Insufficient time devoted to consolidation, need to monitor learning and adjust to individual needs. (Tutor three evaluation of lesson 38)

Inadequate attention to reviewing and linking with prior knowledge also featured. This has been identified in the literature as a weakness in practice (Dixon *et al.,* 1992):

Required much greater structure and preparation. It would have benefited from links with prior knowledge and review of tens and units material covered to date. (Tutor four evaluation of lesson 58)

Non-use and inappropriate use of concrete materials

Some lessons had shortcomings in use of concrete materials which featured in the following ways: not used when they should have been; not linked to the representational and symbolic; removed too quickly before pupil had mastered the skill or understood the concept; premature progression to other materials; appropriate materials offered to pupils but used inappropriately; and inappropriate materials used to try and support conceptual understanding. While the importance of using concrete material and linking their use to representational and symbolic understanding has long been a feature of the literature in mathematics education (e.g. Bruner, 1966), there is considerably less on inappropriate use. Evidence of this from the lesson observations included using a number line to develop the skill of counting on while allowing the pupils to use counting all, in which case the concrete is not supporting the mental operation (Fuson and Fuson, 1992). The movement from proportional models of place value to non-proportional models like the abacus and money was done too quickly for some pupils, as was the progression from continuous models of the fraction concept, using for example paper strips, to discontinuous models using for example discrete objects. These have been highlighted by Reys *et al.* (1998) in relation to place value and Behr *et al.* (1998) in relation to fractions.

Excerpts from the evaluations included:

…link concrete and symbolic in mathematics. Ensure mastery before moving on, pace too fast for some, need further consolidation with base ten materials. (Tutor one evaluation of lesson 37)

More support from concrete materials would be helpful e.g. set out cubes when singing counting songs and add or remove cubes as the song progresses. (Tutor five *evaluation of lesson 29)*

Other shortcomings

Not covering enough content, not making connections, inadequate attention to mathematical language, not engaging all pupils in the lesson, lack of attention to problem solving and too slow a pace were also picked up as shortcomings:

> *Important problem solving section barely touched on. (Tutor four evaluation of lesson 65)*

> *Relevant links were not made between the content presented and the students' life experience. (Tutor four evaluation of lesson 58)*

The non-use and poor use of mathematical software and the calculator also received critical comment.

CONCLUSION

Overall, the quality of instruction observed was rated 'very satisfactory'. At the heart of this was the ability to individualise instruction and utilise an eclectic range of empirically validated teaching and learning strategies to promote learning (Heward, 2003). Lewis and Norwich (2005, p. 218) also emphasise this point: "An underlying theme, meshing with the notion of the *intensification of common pedagogic strategies*, is the skilfulness required to apply a common strategy differentially."

Giving pupils time and attention on an individual basis was no guarantee that instruction was individualised. This could result simply in more of the same. The evidence from the observations supports the notion of "high intensity" within a continua of pedagogic strategies (Lewis and Norwich, 2005, p. 6). This could be seen in the references to more explicit, direct teaching; using smaller steps; giving more examples; teacher monitoring of understanding; carefully scaffolding instruction; teacher modelling of thinking strategies and increased use of concrete hands-on equipment.

One of the concerns in the literature on pedagogy from socio-cultural theorists is that there is an over emphasis on individualised teaching and not enough on fostering communities of learners (Conway and Sloane, 2006; Brown, 1994). While there was some evidence of peer tutoring and paired work, the opportunities to exploit small groups working collaboratively on shared tasks and learning from each other were not fully realised.

While the principles of fostering communities of learners are primarily focused on classroom settings, the potential to enact them may be more feasible in small group withdrawal sessions initially. Such settings should be conducive to an emphasis on classroom discourse, active participation and self regulated learning (Brown, 1994). Learning support teachers might then be in stronger position to support class teachers in developing such practices in mainstream classes.

REFERENCES

Askew, M., Brown, M., Rhodes, V., Wiliam, D. and Johnson, D. (1997). *Effective Teachers of Numeracy*. London, King's College London.

Baker, S., Gersten, R. and Dae-Sik, L. (2002). A synthesis of empirical research on teaching mathematics to low-achieving students, *The Elementary School Journal*, 103(1), pp 52-73.

Behr, M.J., Wachsmuth, I. and Post, T. (1998). Rational number learning aids: Transfer from continuous models to discrete models, *Focus on Learning Problems in Mathematics*, 20(1), pp 63-82.

Bley, N.S. and Thornton, C.A. (2001, 4th edn). *Teaching Mathematics to Students with Learning Disabilities.* Austin, PRO-ED, Inc.

Brown, A.L. (1994). The advancement of learning, *Educational Researcher*, 23(8), pp 4-12.

Bruner, J. (1966). *Towards a Theory of Instruction*. Cambridge, Mass., Belknap Press.

Butler, F.M., Miller, S.P., Kit-hung, L. and Pierce, T. (2001). Teaching mathematics to students with mild-to-moderate mental retardation: A review of the literature, *Mental Retardation*, 39(1), pp20-31.

Conway, P.F. (2002). Learning in communities of practice: Rethinking teaching and learning in disadvantaged contexts, *Irish Educational Studies*, 21(3), pp61-91.

Conway, P.F. and Sloane, F. (2006). *International Trends in Post-Primary Mathematics Education: Perspectives on Learning, Teaching and Assessment.* Research report commissioned by the National Council for Curriculum and Assessment (NCCA). Dublin, NCCA.

Department of Education and Science (1999). *Primary Curriculum Mathematics Teacher Guidelines.* Dublin, The Stationery Office.

Department of Education and Science (DES) (2000). *Learning-Support Guidelines*, Dublin. The Stationery Office.

Dixon, R., Carnine, D.W. and Kameenui, E.J. (1992). *Curriculum Guidelines for Diverse Learners*. Eugene, OR, University of Oregon, National Center to Improve the Tools of Educators.

Dowker, A.D. (2004). *Children with Difficulties in Mathematics: What Works?* London, DfES.

Fuson, K.C. and Fuson, A.M. (1992). Instruction supporting children's counting on for addition and counting up for subtraction, *Journal for Research in Mathematics Education*, 23 (1), pp 72-78.

Haylock, D. (1991). *Teaching Mathematics to Low Attainers, 8-12*. London, Paul Chapman.

Heward, W.L. (2003). Ten faulty notions about teaching and learning that hinder the effectiveness of special education, *Journal of Special Education,* 36(4), pp 186-205.

Lerner, J. (2006). *Learning disabilities and Related Disorders, Characteristics and Teaching Strategies.* Boston, Houghton Mifflin.

Lewis, A. and Norwich, B. (Eds) (2005). *Special Teaching for Special Children? Pedagogies for Inclusion.,* Maidenhead, Open University Press.

National Council of Teachers of Mathematics (NCTM) (1989). *Curriculum and Evaluation: NCTM Standards for School Mathematics*. Reston, V. A., NCTM.

National Council of Teachers of Mathematics (NCTM) (2000). *Principles and Standards for School Mathematics.* Reston, V.A., NCTM.

Pitt, E. (2001). *Ready, Set, Go Maths: Report on action research to investigate the most effective teaching approaches for children who find mathematics difficult*. Northern Ireland, Inter Board Numeracy Group.

Porter, J. (2003). Raising standards in mathematics, in C. Tilstone and R. Rose (Eds) *Strategies to Promote Inclusive Practice*. London, RoutledgeFalmer.

Reys, R., Suydam, M., Lindquist, M. and Smith, N. (1998). *Helping Children Learn Mathematics.* Boston, Allyn and Bacon.

Westwood, P. (2007). *Commonsense Methods for Children with Special Educational Needs*, (5th edn). London, Routledge.

Whitehurst, G. (2003). The Instititute of Education Sciences: New Wine, New Bottles, paper presented at the conference of the *American Educational Research Association.*http://www.ed.gov/offices/IES/speeches/aera/03.html (Last accessed 3rd July, 2007).

Xin, Y.P. and Jitendra, A.K. (1999). The effects of instruction in solving mathematical word problems for students with learning problems: A meta-analysis, *The Journal of Special Education,* 32(4), pp 207-225.

"I THOUGHT THIS WAS A TRICK QUESTION!" — REALISTIC MATHEMATICAL MODELLING: A CLASS STUDY

Ronan Ward,

Education Department, St. Patrick's College, Drumcondra, Dublin.

This classroom study is based on the research article "Teaching Realistic Mathematical Modeling in the Elementary School: A Teaching Experiment with Fifth Graders" by Verschaffel and De Corte, 1997. The authors argued that elementary school children tended to neglect real-world knowledge and realistic considerations during mathematical modelling of word problems in school arithmetic. This study provides support for the hypothesis that it is possible to develop a disposition toward more realistic mathematical modelling in elementary school pupils. The effect of two experimental teaching learning units and their possible transfer to items representing the same mathematical modelling difficulties as those encountered during the lesson, but embedded in a considerably different context, are reported on.

BACKGROUND

The confusing and arbitrary nature of word problems has often been the subject of satire. Gustav Flaubert, back in the mid 19th century, satirised word problems in mathematics with the following:

> "A ship sails the ocean. It left Boston with a cargo of wool. It grosses 200 tons. It is bound for Le Havre. The mainmast is broken, the cabin boy is on deck, there are 12 passengers aboard, the wind is blowing East-North-East, the clock points to a quarter past three in the afternoon. It is the month of May. How old is the captain?"

In the late seventies French and German researchers tested 97 elementary school children's disposition toward unrealistic mathematical modelling using absurd problems such as "There are 26 sheep and 10 goats on a ship. How old is the captain?" (Selter, 1994). Seventy-six children combined the numbers, coming up with the result, for instance, that the captain had to be 36 years of age. Thus, almost 80 per cent of the pupils solved an unsolvable problem by connecting pieces of data that had no relevance to each other, showing absolutely no common sense.

THEORETICAL FRAMEWORK

Realistic Mathematical Education (RME) is a Dutch instructional approach to mathematics that emphasises the context in which mathematics is taught. According to Freudenthal (1968), mathematics is the activity of solving problems and looking for problems, and, more in general, the activity of organising all the information you have about a problem situation which he calls "mathematising". Treffers (1987), later, distinguished "horizontal mathematisation" and "vertical mathematisation". Freudenthal (1991) referred to the former as going from the world of life to the world of symbols, and to the latter as moving within the world of symbols. Another characteristic that is closely related to mathematisation is what is referred to as the "level principle" of RME: "Students pass through different levels of understanding on which mathematisation can take place: from devising informal context-connected solutions to reaching some level of schematisation, and finally having insight into the general principles of a problem and being able to see the whole picture" (Van den Heuvel-Panhuizen, p.105). Models play a powerful role in achieving rises in levels. According to Streefland (1993) a model is constituted very closely connected to the problem situation at hand. Later on the context-specific model is generalised over situations and then becomes a model that can be used to organise related and new problem situations and to reason mathematically. The process of model development is guided by the teacher.

Problem-solving: Suspension of real-world knowledge

Vershaffel and De Corte assert that children's suspension of real-world knowledge and realistic considerations in solving word-problems develops as a result of schooling. It has been argued that school arithmetic problems are perceived as artificial tasks unrelated to the real world (Davis, 1989; De Corte and Vershaffel, 1985; Freudenthal, 1991; Greer, 1993; Kilpatrick, 1987; Nesher, 1980, Nunes, Schliemann, and Carraher, 1993). The American Alan Schoenfeld reported on a piece of research conducted by the Swiss psychologist Kurt Reusser, who observed pupils from Grades 1 through 5 working on 'age-of-the-captain' problems. Reusser obtained similar results, on which Schoenfeld (1991, 316f.) comments as follows:

> "The students he interviewed not only failed to note the meaninglessness of the problems as stated but went ahead blithely to combine the numbers given in the problems and produce answers.... There is reason to believe that such suspension of sense-making develops in school, as a result of schooling".

Radatz (1983) concludes that pupils' behaviour is decisively influenced by the amount of mathematics teaching they have already received. The hypothesis seemed to be confirmed that "arithmetic ... is seen as a kind of play with artificial rules and without any particular link to reality" (p.215). Selter (1994) however argues that they had learned during their educational socialisation that every mathematical problem has to have a result, which can be definitely determined. Accordingly, no problem is really insoluble; there are only problems one cannot work out due to lack of ability. Interestingly, he also found that if the interview began with a reference to some problems being soluble and others being insoluble, fewer pupils tried to work out the latter. He concluded that these children knew that they actually could not just combine the numbers in order to get the result, because it would not be a realistic one. On the other hand, they were sure that all problems could be worked out and consequently had the feeling that the solution must have been hidden somewhere.

STUDY

Sixteen pupils participated in the present study, six girls and ten boys, all pupils in a sixth class suburban school. They were formed into mixed ability groups of four, based on scores attained in the Drumcondra Primary Mathematics Test (D.P.M.T.) at the end of their 5^{th} class. Each group consisted of one pupil who had scored highly, two pupils who had average scores and one pupil who had low scores in the test. Over two days the pupils were presented with worksheets of word problems (Appendix A, Appendix B). These became the teaching learning units (TLU). The topic of the first TLU focused on problems where it was not immediately clear whether addition or subtraction was involved and whether the answer involved one more or one less. The topic of the second TLU was based on the "Soldier's Day" set of problems in the article, and concerned interpreting the outcome of a division problem involving a remainder. The pupils were invited to solve the problems individually first, to make comments regarding the problems and then to provide an agreed answer from each group to each question (Phase 1). Phase 2 involved a whole class discussion, during which pupils shared their comments and strategies. Following each TLU, pupils were given worksheets with similar problems to solve during class time and for homework.

The Verschaffel and De Corte test was administered to the same 16 children one week later. This became the post-teaching learning unit test. The test consisted of 10 word problems and 5 traditional word problems, which were constructed by the class teacher. Two versions of the test were produced in order to prevent copying. Thus, pupils sitting at the same table received a different version. In Version 1 (cf. Appendix 3), the ten items were ordered in such a way that the pairs of items with the same underlying modelling difficulty were separated by at least five other problems. The five traditional items acted as buffers and were numbered 1, 4,8,10 and 13. In Version 2, the fifteen items were presented in reverse order.

RESULTS

Teaching Learning Unit (TLU) 1

Table 1: % individual scores attained in word problems

Questions:	% correct:	% incorrect:
1	25%	75%
2	37.5%	62.5%
3	25%	75%

Table 2: % group scores attained in word problems

Questions:	% correct:	% incorrect:
1	50%	50%
2	75%	25%
3	25%	75%

The mean percent correct for the three questions was 29%, with 71% incorrect. All pupils who had scored highly in the D.P.M.T. answered correctly. The majority of pupils subtracted in order to solve the three problems. Following discussion amongst the pupils, the mean percent correct for the three questions was 50%, with 50% incorrect.

Teaching Learning Unit (TLU) 2

Table 3: % individual scores attained in "A soldier's day"

Questions:	% correct:	% incorrect:
1	25%	75%
2	37.5%	62.5%
3	62.5%	37.5%
4	50%	50%

Table 4: % group scores attained in "A soldier's day"

Questions:	% correct:	% incorrect:
1	50%	50%
2	50%	50%
3	100%	0%
4	87.5%	12.5%

The mean percent correct for the four questions was 44%, with 56% incorrect. Following discussion amongst the pupils, the mean percent correct for the four questions was 72%, with 28% incorrect.

Post-Teaching Learning Unit Test

The post-teaching learning unit test was administered one week after the teaching learning units. A class discussion preceded the test, during which issues that had emerged during the TLU's were explored and debated. It was explained to the pupils that the post-teaching learning unit test comprised of questions that might require more than a routine solution ("It might not be a matter of adding, subtracting, multiplying or dividing. You might have to think about it a little more!"). Pupils worked as individuals, and they were invited to write down difficulties encountered, and/or comments in the boxes provided.

Table 5: % scores attained in the Verschaffel and De Corte test

Number	Question	% correct:	% incorrect:
1	1.	100%	0%
2	1a.	75%	25%
3	2a.	0%	100%
4	4.	87.5%	12.5%
5	3a.	69%	31%
6	4a.	6%	94%
7	5a.	0%	100%
8	8.	100%	0%
9	1b.	69%	31%
10	10.	100%	0%
11	2b.	0%	100%
12	3b.	69%	31%
13	13.	87.5%	12.5%
14	4b.	25%	75%
15	5b.	0%	100%

DISCUSSION

The Teaching Learning Units

Question 3 in the TLU1 caused the greatest problem for pupils, irrespective of whether they were working individually or in groups. Pupils justified their use of subtraction as a means of solving the problem by saying that it was either too difficult or too awkward trying to count from 524 to 616 to find the difference!. While pupils performed better in groups than individually, it was evident to the class teacher, from comments made, that the weaker pupils looked to the better pupils for guidance on how to approach the problems. These findings, coupled with the observations that the teacher made while the pupils worked in groups, seem to indicate that:

- The pupils were not accustomed to discussing word problems in groups. Discussion in groups seemed to be confined mostly between the pupils who were better at mathematics.
- Although most pupils got 37.5 as an answer to all the problems in the "Soldier's Day" problems, the better pupils pointed out the unrealistic nature of the answer and so the weaker students followed the leadership of the better students. When the better pupils had decided on appropriate answers, the rest of the group wrote down the same answer.

These findings were confirmed during the class discussion that followed. The better pupils were first to respond, pointing out realistic considerations such as “You can’t have 0.5 of a jeep” and “They cannot move a crate with 4 men”. However, other pupils who did not involve themselves initially in the discussion showed a better understanding when questioned by the teacher afterwards. One pupil’s response of “They are all almost the same” led the pupils to reflect on how the same operation was used in all four word problems, but that something extra had to be applied before deciding on an answer. This “something extra” was what most pupils neglected: real-world knowledge and realistic considerations.

The Post-Teaching Learning Unit Test

Unsurprisingly, the pupils scored best on the traditional word problems, items 1, 4,8,10 and 13, which demanded the application of mathematical operations only. Of all the problem items presented, the pupils scored best on Questions 1a, 1b, 3a and 3b (Numbers 2, 9, 5 and 12). These were the problems most closely aligned to the problems posed in the TLU’s. This concurs with Verschaffel and De Corte’s hypothesis that the positive effect of the experimental program would transfer to the Verschaffel and De Corte items representing the same mathematical modelling difficulties as those encountered during the training, but embedded in a considerably different context. One pupil commented on the fact that item 2b was very like item 2a, indicating a recognition of same type problems. Some pupils, while aware that the problems were not “straightforward”, could not make the connection with real-world problems, and so find plausible solutions. Comments such as “You need to think”, “I’m not so sure” and “I thought that this was a trick question” illustrate this point.

All pupils failed to score on questions 2a, 2b, 5a, and 5b. Failure to deal with these particular word problems highlights a lack of realistic mathematical modelling on the part of the students. They treated these particular items as traditional word problems, only dealing with the mathematical operations involved, thus demonstrating a strong tendency to exclude real-world knowledge when confronted with the problematic versions of the problems.

It is evident from an analysis of these findings that results on this test were firstly affected by the fact that pupils were not involved in group work on this task. Unlike the first test, they could not tease out the word problems together and so reverted to dealing with the word problems in a traditional way. The only comments made by 5 pupils to all the problems were “I added”, “I subtracted”, “I multiplied” and “I divided”. Secondly, all but one made comments on how they approached the post-teaching learning unit test problems, whereas only three commented during the teaching learning units test. This would seem to indicate that having had the experience of being invited to comment in a previous test, pupils now were more aware of and more willing to express difficulties encountered.

There may be other factors which affected pupils' poor responses. Pupils' failure to give a realistic reaction may have been caused by lacking knowledge about the context involved in the problem e.g. "Even though 'Panoramic' has nothing to do with the sum, it's easier if you know all the words" (Pupil 1, Q.1b). Another problem facing students was that it was difficult to give an appropriate numerical answer to items such as items 2a and 4a. One pupil (Pupil 15) answered 9 to the rope question, commenting that the ropes would lose a bit on each side when they were tied. Another pupil's comments (Pupil 3) best summarises the difficulties pupils have of dealing with realistic mathematical modelling- "The sum was very straight forward because there were no remainders" and "He can saw 10 planks out of all the planks, but I'm not so sure, it could also be 8, if you don't add the halves" (Question 3a and 14b).

CONCLUSION

Although the test was not designed to elicit responses from the class teacher, it is worth noting that the teacher felt that the pupils had benefited from working in groups prior to having a class discussion and had benefited from experiencing working with realistic word problems. Pupils and teacher felt that the realistic mathematical modelling of word problems provided more interesting and challenging activities in comparison to the class textbook. Some pupils reported the involvement of parents in problem solving assignments that pupils were given for homework. Although this study was not of the scale of the Vershaffel and De Corte study, it did, however, show how the use of realistic non-routine problems can motivate pupils and provide more positive attitudes to Mathematics. It would also seem to indicate that, through practice, pupils can improve at solving related and new problem situations and to reason mathematically.

REFERENCES

Davis, R. B. (1989). The culture of mathematics and the culture of schools. *Journal of Mathematical Behaviour, 8*, 143 – 160.

De Corte, E, and Vershaffel, L. (1985). Beginning first graders' initial representation of arithmetic word problems. *Journal of Mathematical Behaviour, 4,* 3 – 21.

Freudenthal, H. (1968). Why to teach mathematics so as to be useful? *Educational studies in mathematics. 1,* 3 – 8.

Freudenthal, H. (1991). Revisiting Mathematics Education. *China lectures*. Dordrecht: Kluwer Academic Publishers.

Greer, B. (1993). The modelling perspective on wor(l)d problems. *Journal of Mathematical Behaviour, 12*, 239 – 250.

Kilpatrick, J. (1987). Problem formulating: Where do good problems come from? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematical education* (pp. 123-147). Hillsdale, NJ: Erlbaum.

Nesher, P. (1980). The stereotypical nature of school word problems. *For the learning of Mathematics, 1*, 41 – 48.

Nunes, T., Schliemann, A. D., and Carraher, D. W. (1993). *Street mathematics and school mathematics.* Cambridge, England: Cambridge University Press.

Schoenfeld, Alan. (1991). On mathematics as sense-making: An informal attack on the unfortunate divorce of formal and informal mathematics. In: James F. Voss et al. (eds.) *Informal Reasoning and Education.* Hillsdale, NJ: Lawrence Erlbaum, 311-343.

Selter, C. (1994). How old is the captain? University of Dortmund: *Strategies*, 5(1), 34-37.

Streefland, L. (1993). The design of a mathematics course. A theoretical reflection. *Educational Studies in Mathematics, 25,* 109-135.

Treffers, A. (1987). Three Dimensions. A Model of Goal and Theory Description in Mathematics Instruction-the Wiskobas Project. Dordrecht: Reidel Publishing Company.

Van den Heuvel-Panhuizen, M. (1999). Context problems and assessment: ideas from the Netherlands. In I. Thompson (Ed.), *Issues in teaching numeracy in primary schools*. Buckingham, UK: Open University Press.

Verschaffel, L and De Corte, E (1997). Teaching Realistic Mathematics Modeling: A Teaching Experiment with Fifth Graders. *Journal Research of Mathematics Education. Vol 28(Issue 5)*: 577-601

Appendix A: Teaching Learning Unit 1

John went on holidays on June 17th and returned home late in the evening on June 25th. How long were his holidays?
Answer:
Comment:

There is one year between each of 5 children in a family. The eldest child is 10. What age is the youngest child?
Answer:
Comment:

The numbers on the first and last tickets sold at a swimming pool were 524 and 616. How many tickets were sold on that day?
Answer:
Comment:

Appendix B: Teaching Learning Unit 2: "A Soldier's Day"

300 soldiers must be transported by jeep to their training site. Each jeep can hold 8 soldiers. How many jeeps are needed?
Answer:
Comment:
At the training site, the soldiers are brought to a hangar. This hangar is filled with a large number of heavy crates that need to be transported to another hangar. These crates are so heavy that it requires 8 men to lift them. How many crates can be transported at a time by these 300 soldiers?
Answer:
Comment:
Back in the barracks, all soldiers are very hungry. The cook has prepared 300 litres of stew. Therefore, he needed 8 big, completely filled kettles, all of the same size. How many litres of stew does one kettle contain?
Answer:
Comment:
In the evening the soldiers have to participate in a military parade. They have to form rows of 8. How many soldiers are left after having made a maximum number of rows?
Answer:
Comment:

Appendix C: Version 1 of the Verschaffel and De Corte test

1. John has 16 Pokemon cards and Jill has 22 cards. How many have they altogether?
2. 1180 supporters must be bussed to a soccer stadium. Each bus can hold 48 supporters. How many buses are needed?
3. At the end of the school year, 50 elementary school children try to obtain their P.E. certificate. To get their P.E. certificate, they have to succeed in two tests: running 400m in less than 2 minutes and jumping 1.5m high. All the children participated in both tests. 9 children failed the running test and 12 children failed the jumping test. How many children did not get their certificates?
4. 480 children are divided evenly into 15 classes. How many children will be in each class?
5. Some time ago the school organised a farewell party for its principal. He was the school's principal from 1 January 1959 until 31 December 1993. How many years was he the principal of that school?
6. A man wants to have a rope long enough to stretch between two poles 12m apart, but he has only pieces of rope 1.5m long. How many of these pieces would he need to tie together to stretch between the poles?
7. John's best time to swim the 50m breaststroke is 54 seconds. How long will it take him to swim the 200m breaststroke?
8. A car travels 55 miles in one hour. How far will it travel in 3 hours, travelling at the same speed?
9. 228 tourists want to enjoy a panoramic view from the top of a high building. The building has only one elevator. The maximum capacity of the elevator is 24 persons. How many times must the elevator ascend to get all the tourists on the top of the building?
10. The schoolyard is 30m long and 24m wide. What is the perimeter of the yard?
11. Kevin and Anne are classmates. Kevin has 9 friends he wants to invite for his birthday party, and Anne 12. Because Kevin and Anne have the same birthday, they decide to give a party together. They invite all their friends. All their friends come to the party. How many friends are there at the party?
12. This year the annual school concert was held for the 15th time. In what year was this concert held for the first time?
13. Michael is 7 years old. What age will he be in 5 years time?
14. Steve has bought 4 planks of 2.5m each. How many 1m planks can he saw out of these planks?
15. This flask is being filled from a tap at a constant rate. **(Picture of a cone-shaped-shaped flask)** If the water is 4cm deep after 10 seconds, how deep will it be after 30 seconds?

"APPLICABLE MATHEMATICS" IN SENIOR CYCLE MATHEMATICS EDUCATION: SELECTED RESULTS OF AN IRISH RESEARCH PROJECT

Brian Carroll and John O'Donoghue

NCE-MSTL, University of Limerick

In recent years there has been considerable concern about the low level of mathematical skills of students emerging from second-level education and, in particular, of those proceeding to third-level education (Gill, 2006; NCCA, 2005; O'Donoghue, 1999).The authors believe that there is a clear need to address the issue and to facilitate the transition from Second-Level to University mathematics education in Ireland. The authors propose a teaching intervention entitled: "Support for Applicable Mathematics", which is primarily aimed at the teaching and learning of senior- cycle mathematics in Ireland. The intervention involves the use of ICT within a distinct modelling approach aimed at highlighting the potential of mathematics in relation to real-life contexts. Previously researchers such as Bajpai (1975) and the Harvard Calculus Consortium (1991) devised approaches to improve the teaching of third-level undergraduate mathematics. Following a thorough analysis the authors have adapted these approaches so as to design and implement the teaching intervention. The results of the Exploratory Phase of the research project will be presented, with particular emphasis on what mathematical topics Irish Senior-Cycle students would consider applicable to real-life contexts.

INTRODUCTION

The intervention presented in this paper is specifically designed to make the students aware of the relevance of mathematics to their everyday lives and hence improve their interest and attitudes towards mathematics. The results of the Exploratory Phase of the author's doctoral study will be presented in this paper, with particular emphasis on what mathematical topics the students would consider applicable to real-life contexts. These results are based on the administration of a student questionnaire to the students participating in the teaching intervention.

The author identified a gap in the research literature regarding the teaching and learning of senior cycle mathematics in Ireland, with respect to facilitating the transition from second-level to university mathematics courses. The significance of the 'Mathematics Problem' combined with growing concerns that higher education graduates of engineering, science, business and computing are lacking the required levels of mathematical proficiency for economic development are issues of concern both in Ireland and worldwide (Lawson, 1997). In recent years there has been considerable concern about the falling standards of mathematical skills of students emerging from second-level education and, in particular, of those proceeding to third-level education (Gill, 2006; NCCA, 2005; O'Donoghue, 1999). There is a growing need for more students to make a successful transition from second-level mathematics education to third-level mathematics education so that improvements in quality, retention, and completion rates, can be achieved in higher education courses that contain a significant mathematics component.

The research problem outlined by the authors is to investigate the use of applications in the transition from second-level to university mathematics. The study aims to design, develop, implement and evaluate a teaching intervention aimed at the teaching and learning of upper second-level mathematics which involves the use of ICT. The mathematics focus is aimed at highlighting the potential of mathematics in relation to real-life contexts within a distinct modelling approach. The author proposes a teaching intervention entitled: "Support for Applicable Mathematics", which is primarily aimed at the teaching and learning of Senior- Cycle mathematics in Ireland. This research will draw on insights into the factors determining the level of understanding and attitudes of students towards learning school mathematics at upper-secondary level garnered from the literature review and the author's personal experience as a secondary school mathematics teacher.

RESEARCH BACKGROUND

Many mathematics educators have devised alternative approaches to the teaching and learning of mathematics (Bajpai, 1975; Freudenthal, 1968; Hughes-Hallett, 1991; Meyer & Ludwig, 1999; Ormell, 1972). Two of these approaches were examined and analysed by the author: The Integrated Approach developed by Bajpai and his colleagues at Loughborough University (1975) and the Harvard Approach (Hughes-Hallett, 1991) devised by the Harvard Calculus Consortium in the early 1990's. These approaches were examined and analysed as the author felt that an adoption of these approaches would ensure that the students would be provided with opportunities to appreciate the fact that often a combination of techniques are used when solving problems arising in real-life contexts, both in industry and everyday experiences.

Exploratory Phase

The Exploratory phase of the teaching intervention was developed to explore the issues involved which were highlighted through a survey of the literature and fieldwork. The fieldwork involved was based on the data collected from two separate Senior-Cycle classes. The student profile of each class will be discussed in the next section. Field data was collected through questionnaires to assess the students' attitudes towards mathematics, their attitudes towards their studies and their usual way of studying, while also exploring their views on applicable mathematics; participating teachers' reflective journals and semi-structured interviews. This phase was developed so as to ensure the initial adaptation of the research framework APOS theory. The author proposed an adaptation of APOS theory, developed, as a result of the work of Dubinsky and his colleagues in the Research in Undergraduate Mathematics Education Community (RUMEC), to suit the needs of the learners. It is intended that the questionnaire will inform the author of the existing perceptions of students regarding applicable mathematics. Based on the analysis of these findings the authors will reassess the appropriateness of the theoretical analysis and instructional design and make modifications where necessary for the Implementation Phase of the teaching intervention.

STUDENT QUESTIONNAIRE

The questionnaire was selected as one of the primary data collection instruments of the exploratory phase as it was a required method of observation outlined by RUMEC with respect to the theoretical framework of APOS theory (Asiala et al, 1996), which is the theoretical foundation of the author's larger study.

It provided the author with demographic information including the student's age, Junior Certificate grade and level of mathematics study. Following the initial demographic information, the questionnaire was divided into three distinct sections. Section A was concerned with determining the students' attitudes towards mathematics. Section B was concerned with determining the students' approaches to learning and studying. Section C was aimed at discovering which mathematical topics the students considered applicable to real-life contexts. Sections A and B were statements based on a five point Likert scale.

In Section A there were 40 statements based on Tapia's (2002) Attitude Towards Mathematics Inventory. Section B consisted of 20 statements based on Biggs et al.'s (2001) Revised Two-Factor Study Process Questionnaire. Strong feelings could be indicated on either side of the scale and there was an option for respondents who were unsure of statements:

The first part of Section C consisted of two-open ended questions to determine the students initial reactions to what they deem to be applicable mathematics. In question 3 the students were asked to tick the mathematical topics (from a list of 18) according to what they considered applicable to real-life contexts. The students could tick more than one answer. Question 4 took the same format as Question 3, where the students were asked to tick the mathematical topics (from the same list of 18 topics as the previous question) according to what they considered **least** applicable to real-life contexts. This section is based on a study undertaken in Vienna by Humenberger et al. (2000) in which students, student teachers and in-service teachers were questioned on their opinions in relation to applications in mathematics education.

The results of Section C will be presented in this paper by the authors. In order to analyse Section C, Excel was used.

Student Profile

The questionnaire was adminstered to a purposive sample, and was designed to examine the attitudes of students in Senior-Cycle mathematics who are preparing for third level education. Therefore, the cohorts selected to take part in this study were two classes of Higher-Level Leaving Certificate mathematics students. It was also required that the teachers involved must also be teaching mathematics at Senior-Cycle level.

A fifth year class of twenty-one students in a community college participated in the Exploratory Phase. The age range of students was 16-17 years, with the average age of 17 years. All of the students are currently studying Higher Level mathematics at Leaving Certificate level in Ireland. Also, a transition year class of fourteen students in an all-girls convent participated in the Exploratory Phase. The age range of students was 15-16 years, with the average age of 16 years. All of the students intend to study higher level mathematics at leaving certificate level in Ireland.

PERCEPTIONS OF APPLICABLE MATHEMATICS

The students were first asked to write down their initial feelings regarding what mathematical topics they would consider applicable to real life contexts (see Appendix). Their spontaneous reaction to the question was recorded. Arithmetic was referred to 17 times, Statistics 15 times and Trigonometry 10 times. Basic Operations (addition, subtraction, division and multiplication) were referred to in some form by 9 different students. There were a total of 53 references, with 7 different mathematical topics referred to. One student of the 21 did not answer this question while at the other end of the spectrum one student stated: “All of the maths topics could be used in real-life situations”.

Given that the students were then asked to select from a list of 18 mathematical topics (see Appendix) according to what they considered applicable to real-life contexts, the data showed that when the mathematical topics are listed Statistics is the mathematical topic in the Leaving Certificate syllabus considered to be most applicable to real-life contexts. Table 1 shows the top 5 references of applicable mathematical topics according to the participating students.

There were a total of 167 responses because the students were allowed to tick more than one topic. Arithmetic was placed 2nd, with 26 references. The high number of references to Statistics is perhaps due to the fact (or at least partly) that this was a recent topic covered in their mathematics lessons. Topics such as Differentiation, Integration, Functions, and Linear and Quadratic equations, had a combined total of 13 references (half the number of references to Arithmetic alone). Considering the wealth of applications these topics encompass it seems remarkable that topics that such a low number of references.

Rank	Mathematical Topic	Number of References
1	Statistics	29
2	Arithmetic	26
3	Geometry	24
4	Probability	24
5	Trigonometry	17

Table 1: Most applicable mathematical topics - by students' opinion

The students were also asked to tick from the same list of 18 mathematical topics according to what they considered **least** applicable to real-life contexts. It is unsurprising to see Logarithms score one of the highest with 15 references. Linear Equations had most references with 21 separate references, followed closely by Quadratic Equations with 20 references. Table 2 shows the top 5 references of applicable mathematical topics according to the participating students.

Rank	Mathematical Topic	Number of References
1	Linear Equations	21
2	Quadratic Equations	20
3	Algebra	16
4	Logarithms	15
5	Inequalities	14

Table 2: Least applicable mathematical topics - by students' opinion

There were also a total of 167 responses because the students were allowed to tick more than one topic. It is notable that Algebra has 16 references from students when asked about what they consider least applicable to real-life contexts. However in the previous question (topics the students considered applicable mathematics), Algebra had 14 references. Arithmetic and Statistics had one reference each, while Probability had 6 references.

Furthermore, one student of the 21 did not answer this question. However, the same student ticked each mathematical topic when asked what mathematical topics they considered to be most applicable. This is perhaps due to the student considering all mathematics to be applicable.

Rank	Subject	Number of References
1	The Sciences	29
2	Business Studies	25
3	Design and Communications Graphics	7
4	Geography	6
5	Construction Studies	4
6	Computers	4
7	Engineering	3
8	Art	2
9	Music	1

Table 3: Most applicable subjects- by students' opinion

Another aim of the student questionnaire was to clarify what other subject areas (which the students study in school) they would consider provides the opportunity to use mathematics (see Appendix). In total there were 83 different responses, with Table 3 above showing the breakdown of these responses. One student failed to provide at east one response to the question. In total 9 different subject areas were mentioned, with the Sciences being the most referred to with 29 references.

Design and Communication Graphics (which is the new revised technical Graphics syllabus) had a total of 7 references. Geography, Construction Studies, Computers and Engineering had a total of 6, 4, 4 and 3 references respectively. Art had 2 references, while Music had 1 reference.

Rank	Breakdown of Science References	Number of References
1	Chemistry	9
2	Science	9
3	Physics	5
4	Biology	3

Table 4: Breakdown of Science references- by students' opinion

Notably, a breakdown of Science responses, which is presented in Table 4, shows Chemistry had 9 distinct references, Science *per se* also with 9 references. Physics had 5 references and Biology had 3 references. Also, Business Studies had a breakdown of responses, which is presented in Table 5, where Accounting had 11 responses and Business Studies had 10 responses. Economics had a total of 4 responses.

Rank	Breakdown of Business Studies References	Number of References
1	Accounting	11
2	Business Studies	10
3	Economics	4

Table 5: Breakdown of Business Studies references- by students' opinion

DISCUSSION

It appears from the outset that Irish Senior-Cycle students consider the most obvious mathematical topics applicable to real-life contexts i.e. Arithmetic, Geometry, Statistics and Trigonometry.

This is perhaps due to the instructional approaches practised by teachers in the Irish education system. Teachers are affected by the continued emphasis placed by pupils on acquiring as many points as possible, where they are required to help pupils achieve these goals. They are forced to adopt a teach-to-the-examination approach in order to maximise points, as opposed to teaching for understanding (Gill, 2006).

Furthermore, time allocated to any one subject at senior cycle is low in international terms (NCCA, 2005), thus, time allocated to teaching-for-understanding is minimal in Irish mathematics classrooms, where emphasis on rote-learning and cramming is paramount (O'Donoghue, 1999).

The intervention has shown that many students are unable to appreciate the role of mathematics in everyday life, where they fail to understand or realise the influence that mathematics can exert on one's future education and/or work-life. Current practices in the teaching and learning of mathematics in senior cycle schools in Ireland generally fail to make the necessary connections between mathematics and its place in real-life, as documents from the NCCA and the Chief Examiners Report have shown (NCCA, 2005; State Examinations Commission, 2005).

CONCLUSION

It is clear that there are real concerns affecting the teaching and learning of mathematics in senior-cycle curricula in Ireland. To ensure these concerns are alleviated, acceptable pedagogical experiences in the mathematics classroom must be available. Furthermore, there is a growing need for more students to make a successful transition from second-level mathematics education to third-level mathematics education so as improvements in quality and in retention and completion rates can be achieved in higher education mathematics courses.

There is a need to examine and maximise the possible contribution of applications in both the curriculum and assessment process within the senior-cycle level, thereby addressing an important aspect of the ongoing 'Mathematics Problem'. Without such intervention, we will continue to provide a schooling experience that is conducive to the under-preparedness of our students entering third-level mathematics courses.

REFERENCES

Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D. & Thomas, K. (1996) A Framework for research and Curriculum Development in Undergraduate Mathematics Education. IN *CBMS Issues in Mathematics Education.* 1-32.

Bajpai, A. C., Mustoe, L. R. & Walker, D. (1975) Mathematical education of engineers: Part 1. - A critical appraisal. *International Journal of Mathematical Education, Science and Technology,* 6, (3)**,** 361-380.

Biggs, J., Kember, D. & Leung, D. Y. P. (2001) The revised two-factor Study Process Questionnaire: R-SPQ-2F. *British Journal of Educational Psychology,* 71, 133-149.

State Examinations Commission, (2005) Chief Examiner's Report Leaving Certificate Examination – Mathematics, Athlone.

Cohen, L., Lawrence, M., & Morrisson, K., (2000) Research Methods in Education, 5th Edition, London, Routledge

Freudenthal, H. (1968) Why to teach mathematics as to be useful? *Educational Studies in Mathematics,* 1, (1)**,** 3-8.

Gill, O. (2006) What Counts as Service Mathematics: An Investigation into the 'Mathematics Problem' in Ireland. *University of Limerick.*

Hughes-Hallett, D. (1991) Visualization and Calculus Reform. IN Zimmermann, W. & Cunningham, S. (Eds.) *Visualization in Teaching and Learning Mathematics.* USA, Mathematical Association of America

Humenberger, H. (2000) "Applicable Mathematics" in Mathematics Education - Selected Results of a Viennese Research Project. *International Journal for Mathematics Teaching and Learning,* 2, (2).

Tapia, M. & Marsh, G. (2002) Confirmatory Factor Analysis of the Attitudes toward Mathematics Inventory. *Annual Meeting of the Mid-South Educational Research Association.* Chattanooga.

Lawson, A. (1997) What can we expect from A-Level Mathematics Students? *Teaching Mathematics and its Applications,* 16, (4)**,** 151-156.

Meyer, M. R. & Ludwig, M. A. (1999) Teaching mathematics with MiC: An Opportunity for Change. *Mathematics Teaching in the Middle School,* 4, (4)**,** 264-269.

NCCA (2005) Review of Mathematics in Post-Primary Education: *a discussion paper*. IN Oldham, E. (Ed.) Dublin.

O'Donoghue, J. (1999) An Intervention to Assist 'At Risk' Students in Service Mathematics Courses at the University of Limerick. *U.L. Teaching Fellowship Report.*

Ormell, C. P. (1972) Mathematics, Science of Possibility. *International Journal of Mathematical Education, Science and Technology,* 3, 329-341.

APPENDIX

Section C

Section C is aimed at discovering which mathematical topics you (the student) consider applicable to real-life contexts.

1. What mathematical topics would you consider applicable to real-life contexts? i.e. mathematics that can be used in everyday situations

____
____
____
____
____
____

2. What other subject areas (that you study in school) would you consider that provides the opportunity to use mathematics?

____
____
____
____

3. Please tick the following mathematical topics according to what you consider applicable to real-life contexts. You may tick more than one answer.

Arithmetic	
Algebra	
Linear equations	
Quadratic equations	
Inequalities	
Logarithms	
Complex numbers	
Matrices	
Geometry	
Vectors	
Trigonometry	
Sequences and Series	
Functions	
Curve Sketching	
Differentiation	
Integration	
Statistics	
Probability	

WHO AM I AND HOW DID I GET HERE?: EXPLORING THE MATHEMATICAL IDENTITY OF STUDENT TEACHERS

Patricia Eaton — Stranmillis University College

Maurice O Reilly — St Patrick's College

ABSTRACT

This paper reports on the results of a SCoTENS funded project examining the mathematical identity of student teachers who have chosen to specialise in mathematics in their B.Ed programme. Students were drawn from two institutions, one in Northern Ireland, one in the Republic. Mathematical identity is considered as the multi-faceted relationship that an individual has with mathematics, including knowledge and experiences, perceptions of oneself and others. Understanding student teachers' mathematical identity is of critical importance because, as is widely indicated in the literature, it is likely to have a major impact on their behaviour as a teacher.

Our research method uses narrative as a tool to access this identity. The narrative material was gathered in two stages. The first stage involved a questionnaire with open-ended prompts giving respondents the opportunity for their mathematical story to flow freely. This was followed by use of focus groups, one in each institution, to elicit further narrative development. Our analysis of these narratives gives some insight into the formation of mathematical identity amongst such a select well-motivated group. We expect this to inform good practice more broadly in mathematics education.

INTRODUCTION

Much has been written for over a decade on the links between the beliefs about, attitudes towards and self-efficacy in mathematics, on the one hand, and classroom practice, on the other (Ernest, 1988; Fang, 1996). This discourse has flourished as understandings of the shift from a didactic to a constructivist approach to teaching mathematics have deepened.

The experiences that a student has during his/her own formative years in the classroom as a pupil have been shown to have a major impact on their behaviour as a teacher (Ernest 1988, Ball 1988, Hill 2000). Student teachers have a tendency to revert to models of teaching that they themselves experienced rather than try the often new and unfamiliar models that they study during their teacher education programmes (Borko et al. 1992). It is sensible, therefore, to try to access, unpack and understand this 'baggage' that student teachers carry.

In the study about to be described, student teachers' mathematical identity was examined in some detail. By 'mathematical identity' is understood the relationship an individual has with mathematics, including knowledge and experiences as well as perceptions of oneself and others (Wenger, 1998). Gee (2000) lays out a broad framework drawn from an anthropological perspective for using identity as an analytic lens for research in education. Focusing on how identity and agency allow access to different mathematical 'worlds', Boaler and Greeno (2000) identify two contrasting 'figured worlds' for learning mathematics, one that is "structured, individualized, and ritualized", while the other is "relational, communicative, and connected". In the former 'world', students viewed the task of learning mathematics as mastering the performance of procedures, while, in the latter, the emphasis was more on thinking and conceptual understanding.

A tool used increasingly in recent years to help access identity has been that of narrative (Clandinin & Connelly, 2000; Kaasila, 2007). It is hoped that by providing an opportunity for students to tell the story about their own relationship with mathematics, identity will emerge and themes will come to the fore that give insight into their thinking and attitudes.

The study was carried out in February 2009 involving participants from two colleges of education, one in Dublin and one in Belfast. All participants were pre-service primary school teachers in the third year of their B.Ed. programme, having chosen to specialise in mathematics. Data was gathered using a questionnaire (with, mainly, open-ended questions) followed by focus groups, involving the same participants on each campus, five in Dublin (but only four of whom participated in the focus group) and four in Belfast. Moreover their mathematical sophistication was significantly higher than is typical amongst pre-service primary school teachers in Ireland (cf. Corcoran, 2005). In particular, 2.8% of the B.Ed. cohort in Dublin and 15% of the Belfast cohort chose to take mathematics to degree level. These two factors afforded the opportunity to explore two mathematically motivated sub-populations in some detail, although, in this paper, no attempt will be made to distinguish between the characteristics of the two groups.

This paper begins by giving a detailed account of the methodology used in the study. Particular attention is paid to the challenge of gaining access to students' mathematical identities through narrative by stimulating recollections but without being directive in doing so. The analysis that follows begins by focussing on the most immediate themes arising from the questionnaires, and then considers the significant issues emerging from the complete set of field texts (questionnaire responses and focus group transcripts).

METHODOLOGY

In designing the questionnaire the aim was to balance the need for some direction with the need to allow respondents to make open-ended responses that were indicative of their personal mathematical identity. A draft questionnaire for participants was prepared some ten weeks in advance of gathering data. This timing enabled it to be piloted amongst research colleagues from five institutions at a meeting in NUI Maynooth. In addition to some fine changes to the text of the questionnaire to avoid ambiguities, this group made the recommendation that the data might be collected without direct involvement of either of the researchers with his/her own students. The researchers acknowledge with appreciation the attention to detail of these colleagues. In due course, the questionnaire was administered on each campus by the researcher from the other institution.

The questionnaire began with two questions on participants' backgrounds followed by some questions on their attitudes to mathematics (using Likert scales). This paper does not address the quantitative data gathered in that section. Participants were then prompted into revealing their mathematical identity by being asked: "Think about your total experience of mathematics. Tell us about the dominant features that come to mind." The aim of these initial sentences was to allow the most prominent recollections to emerge without giving students explicit directions as to which recollections *should* be most prominent. It was felt that had this starter contained explicit reference to events or individuals or stages of study this would suggest that respondents should focus on these. Instead, by having an open-ended initiator, it was anticipated that the most dominant feature would emerge first and that the nature of this initial response would in itself be indicative of powerful influences on mathematical identity. After completion of this initial section, lasting approximately ten minutes but allowing all respondents to complete as fully as they desired, a second page was distributed, this time with more direction to encourage students to reflect on a wider range of features:

"Now think carefully about <u>all stages</u> of your mathematical journey from primary school (or earlier) to university mathematics. Consider:

- Why you chose to study mathematics at third level
- Influential people
- Critical incidents or events
- Your feelings or attitudes to mathematics
- How mathematics compares to other subjects
- Mathematical content/topics

With these and other thoughts in mind, describe some further features of your relationship with mathematics over time."

It was expected that these prompts would encourage respondents to consider areas that may have been influential but which did not spring immediately to mind, rather than act as list of questions each of which was to be answered in turn.

The texts from the questionnaire were analysed for recurring themes and seven clusters of issues were identified to give the subsequent focus group discussions some direction:

- Reflections on the questionnaire

- The changing nature of maths as experienced from early childhood to now
- The balance between challenge and interest
- Critical events
- Attitudes of other people
- Ways of studying maths
- Persistence/perseverance with maths

The discussion in the two focus groups (one on each campus facilitated by the researcher from the other institution) was directed largely by these issues, while maintaining an informal conversational atmosphere.

ANALYSIS

The complete set of field texts consisting of questionnaire responses and transcripts of focus groups were coded to highlight patterns, threads and themes (Clandinin and Connelly, 2000). These themes, described in detail below, lead to a new metastory, created from the narratives provided by the respondents (Riessman, 1993) in which the different strands have been re-weaved to produce a coherent overview.

Considering students' responses to the initial open-ended prompt reveals that the feature most frequently described was that concerning the nature of the subject. All of the nine students described aspects of the subject with comments such as:

> No subject can stimulate the mind as much as Maths can.
>
> It was made to seem like a challenging subject but I always found it interesting and stimulating.

Two of the students identified the importance of the influence of teachers, for example:

> The view of the teacher concerning maths has a real effect on success of children.

Another common initial response was the journey from primary school to university level mathematics with a focus on the changing nature of the mathematics studied:

> Once reaching University level it was more about proving everything that you have been taught and focused less on problem solving.

A complete review of all the field texts led, as described above, to the identification of the following key themes: harnessing student teachers' mathematical identity as a tool for self-reflection, the role played by key figures in the formation of mathematical identity, ways of working in mathematics, how learning mathematics compares with learning in other subjects, the nature of mathematics, 'right' and 'wrong' in mathematics and mathematics as a rewarding subject. We now consider each of these in turn.

Harnessing student teachers' mathematical identity as a tool for self-reflection

In two recent studies, the authors discussed specific aspects of the data in some depth. One (Eaton & OReilly, 2009a) focused on how student teachers' mathematical identity can be harnessed as a tool for self-reflection:

> I think in third level when we're learning why there's certain proofs and why [a] certain thing is the way it is, make us much better teachers because we can actually show the kids why there's a certain formula instead of just, "Here, learn it off." So I think it's our understanding of maths has really improved. And I really think it'll benefit the kids we teach.

Here, we see how a participant re-evaluated the teaching she herself experienced. She expresses a clear commitment to promoting understanding in teaching mathematics.

The role played by key figures in the formation of mathematical identity

The second study (Eaton & OReilly, 2009b) addressed the role played by key figures such as family, peers and teachers in the mathematical life of the students. Through self-reflection on how influential particular teachers were in their own formation, it was envisaged that the students would engage in a meta-analysis of the impact on their pupils of their own teaching. When pressed to articulate why certain teachers had been so influential students expressed views more about the contagious love and enthusiasm for the subject rather than any particular teaching style or technique:

> Observing their enthusiasm and enjoyment for maths like my own, has encouraged and influenced me greatly in wanting to also share my own interests in it with other people.

An analysis of the role played by other significant figures can provide insight into societal mathematical engagement and reveal the influences on those choosing to continue to study the subject to a higher level. An understanding of this can help the students to identify key formative critical factors for their own pupils, in particular the importance of peer pressure both inside and outside the classroom:

> But first impressions from people you don't know is usually, "Oh no."

> People …are more impressed when you say that you do maths, but they're also more critical in a way of your social abilities, I think. Like, "You big nerd. You're doing maths."

Ways of working in mathematics

Students commented on their approach to engaging in mathematics and in particular the balance between the individual pursuit of the subject and the opportunities and rewards to be gained from collaborative working:

> What I find is very helpful is even during our classes as well, we all do a question in a group. We all help each other with the question, you know.
>
> You know, we would be, well, I think quite helpful to each other, or try to be, anyway.
>
> We would definitely get together or somebody would be like, "Oh, she knows how to do it." So everybody would be all like, "Right, tell us how to do it like."… It's teamwork.

The team approach taken at third level was contrasted with the individual style experienced during second-level education:

> I just think that because, as A said, it's so small, we really do work together. In secondary school, it was very much individual based because you were kind of afraid to ask the person beside you because they were able to do it and you couldn't. And I don't know, it was looked down upon. Like we were never encouraged to help each other, ever.

How learning mathematics compares with learning in other subjects

The differences between mathematics and other subjects were highlighted (Lyons, Lynch, Close, Sheerin & Boland, 2003, pp 225-251). Maths was observed to be a more demanding subject than others because of the volume of work involved and yet satisfying by virtue of the clarity in having achievable criteria:

> Although I feel there is more work involved in maths than other subjects I feel its a fairly marked subject, unbiased and what you put in you get out!

Compared to other subjects in which reading background texts is of paramount importance mathematics was appreciated for its practical nature:

> To me certain other subjects are very book orientated whereas maths is practical and engaging. I don't find enjoyment in reading so maths appeals greatly to me as it is practical.

The nature of mathematics

The nature of mathematics as encountered in school, both North and South, reflects a procedural approach to the subject, while in university a richer perspective emerges. A student in Belfast expressed wonder when studying Mersenne primes, epitomising the unexpected immensity of mathematics:

> It's the first time you've really any experience of, "Oh, that hasn't been found. We don't know how to do that" sort of thing. Or I can't just plug this in and I'll get an answer. I mean it sort of make[s] you think more about it. I think, you start to think about maths is a separate language and a separate world, almost to everything else.

Another student, this time in Dublin, remarked on a change in the nature of enquiry from procedure to meaning in mathematics:

> Secondary school is just about learning off your formulas and just plugging in your answers, and that was it. But now in college, it's kind of coming back to why you're using the formulas. So there is kind of a deeper meaning into it.

'Right' and 'wrong' in mathematics

Participants recalled almost nostalgically that whether maths was 'right' or 'wrong' was recurring theme in their school experience:

> That was always a great thing when you got the right answer, when you got the same answer as everybody else.

In giving individual lessons ('grinds'), however, another participant tried to resist this simplistic dichotomy (cf. Bolhuis & Voeten, 2004):

> Just encouraging them to try it out, basically. Like instead of focusing on whether it's right and wrong all the time, get them to try it at least. And just keep trying it.

Yet this same student still harbours the security of the 'right' procedure when faced with understanding the 'discomfort' of proofs:

> I personally like the procedural ones because I know that it's right, and I know that's a terrible thing to say, but the proofs, it just gets me so frustrated that you could spend ages at it and get nowhere with it.

Mathematics as a rewarding subject

Students not only engage with mathematics, but also persist with it because they enjoy it. This fundamental notion that it can be a rewarding subject was widely emphasised. In reflecting on why she persevered with mathematics, a participant explained:

> I think the enjoyment of it. I always enjoy maths, no matter what. Even if it is too difficult for me, I enjoy trying to work out, you know, trying to get over the difficulty of it. I think that's what's made me stick with it. Just enjoyment of it.

Others drew attention to the experience when all becomes clear:

> Yeah. I love the moments where you're like, "Oh, I got it."
>
> Yeah, that's the moment that you're really enjoying maths, when it clicks like that. … The enjoyment, yeah, has the wow factor at the end.

The fact that such moments of understanding occur also in primary school emphasises how important it is that teachers are prepared to challenge children and give them space to make significant independent learning (cf. Borko, Eisenhart, Brown, Underhill, Jones & Agard, 1992):

> In primary school, I think that my real love of maths came to the fore when we were dealing with fractions. The teacher challenged the class with a question about how you divide fraction one by another. I really felt a sense of achievement when I was able to figure it out on my own. I think that this is a key aspect of what maths is about.

It appears to take time to develop a sensibility for the satisfaction that mathematical challenge brings:

> Over time my experience with maths has become better as I now appreciate the challenges it brings and have satisfaction overcoming them.

CONCLUSION

The students surveyed have all chosen to study mathematics at a higher level as part of a bachelor of education programme and it is perhaps unsurprising to find that they hold a positive view of their experiences of mathematics. The journey undertaken to reach this point however has been a complex and personal one and yet common themes seem to emerge which can inform practice as we seek to understand what it is that has encouraged these students to stay with mathematics. Findings on the impact of attitudes of peers and family will have resonance with those seeking to attract more adherents to mathematics, as the role played by the attitudes of society to what is traditionally thought of as a difficult and challenging subject is uncovered.

Students' insights into the nature of the subject and how it is delivered can be used to inform teaching with a view to increasing the number of students choosing to participate in further study. For example, Esmonde (2009) has argued for "providing students access to the means to construct … positive mathematical identities". She advocates a learning environment in which students cooperate in a community of practice and where those who are positioned marginally in this community are supported to move to a more central position. Thus the role of the teacher includes fostering change for the better in students' mathematical identity. To effect such change, requires knowledge about students' mathematical identity in the first instance. From the experience of this survey, we maintain that narrative is an efficient method of finding this knowledge and, indeed, of drawing attention to wider issues in teaching mathematics. In the context of continuing professional development, we recommend workshops with teachers adapting the approach of this study.

Moreover at a meta-cognitive level we contend that this act of exploration is an extremely valuable exercise particularly for student teachers. To remember how they were taught, to discuss their memories and in doing so to tease out and distil important issues in the complexity of how they learned mathematics, will bring greater awareness to the professional practice which lies ahead of them. Further longitudinal study would be required to discern whether or not the students who participated incorporate the insights gained into their practice and, if so, how. In the context of a greater emphasis on teachers as reflective practitioners (GTCNI, 2007) this process can be seen as effective preparation for a career in which self-reflection is central to good practice. This exercise is also of value to lecturers in teacher education contexts, in getting to know the formative context of students and leading them to a deeper understanding of their relationship with mathematics and how this impacts on its learning and teaching (cf. Smith, 2006).

ACKNOWLEDGEMENT

This paper arises from a study entitled "A cross-border comparison of Student Teachers' Identities relating to Mathematics" (MIST) and supported by the Standing Committee on Teacher Education, North and South (SCoTENS). The authors are grateful for this support.

REFERENCES

Ball, D.L. (1988). Unlearning to Teach Mathematics, *For the Learning of Mathematics*, 10 (2), 10-16.

Bolhuis, S. & Voeten, J.M. (2004). Teachers' Conceptions of Student Learning and Own Learning. *Teachers and Teaching: theory and practice,* 10 (1), 77-98.

Boaler, J. & Greeno, J. (2000). Identity, agency and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning*, Stamford, CT: Ablex, 171-200.

Borko, H., Eisenhart, M., Brown, C., Underhill, R., Jones, D. & Agard, P. (1992). Learning to Teach Hard Mathematics: do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education,* 23(3), 194-222.

Clandinin, D.J & Connelly, M. (2000). *Narrative Enquiry: Experience and Story in Qualitative Research*. San Francisco: Wiley.

Corcoran, D. (2005). An exploration of the mathematical literacy of Irish students preparing to be primary school teachers. In S. Close, T. Dooley & D. Corcoran (Eds), *Proceedings of First National Conference on Research in Mathematics Education* (MEI 1), Dublin, 231-247.

Eaton, P. and OReilly, M. (2009a). Exploring mathematical identity as a tool for self-reflection amongst pre-service primary school teachers: "I think you have to be able to explain something in about 100 different ways". Presented for *Proceedings of 10th Mathematics Education into the 21st Century Project* conference, Dresden.

Eaton, P. and OReilly, M. (2009b). What other people think and why it matters? An investigation of key influences on mathematical identity. Presented for *Proceedings of 34th Annual Conference of the Association for Teacher Education in Europe*, Palma de Mallorca.

Ernest, P. (1988). The Attitudes and Practices of Student Teachers of Primary School Mathematics. In A. Borbas (Ed.), *Proceedings of 12th International Conference on the Psychology of Mathematics Education, Hungary,* Vol. 1, 288-295.

Esmonde, I. (2009). Ideas and Identities: Supporting Equity in Cooperative Mathematics Learning. *Review of Educational Research*, 79(2), 1008-1043.

Fang, Z. (1996). A review of research on teacher beliefs and practices. *Educational Research,* 38, 47-65.

Gee, J.P. (2000). Identity as an analytic lens for research in education. *Review of Research in Education*, 25, 99-125.

General Teaching Council for Northern Ireland (GTCNI) (2007). *Teaching: The Reflective Profession*. Retrieved 29 June 2008. http://www.gtcni.org.uk/uploads/docs/GTCNI_Comp_Bmrk%20%20Aug%2007.pdf.

Hill, L. (2000). Theory, Practice and Reflection: a Pre-Service Mathematics Education Programme, *Teachers and Teaching: Theory and Practice,* 6 (1), 23-39.

Kaasila, R. (2007). Using Narrative Inquiry for Investigating the Becoming of a Mathematics Teacher. *ZDM-International Journal of mathematics Education*, 39 (3), 205-213.

Lyons, M., Lynch, K., Close, S., Sheerin, E. and Boland, P. (2003). *Inside Classrooms: the teaching and learning of mathematics in social context*. Dublin: Institute of Public Administration.

Riessman, C.K. (1993). *Narrative Analysis* (Vol.30). Thousand Oaks, London: Sage Publications.

Smith, T. (2006). Self-Study Through Narrative Inquiry: Fostering Identity in Mathematics Teacher Education. In P. Grootenboer, R. Zevenbergen & M. Chinnappan (Eds.), *Identities, cultures, and learning spaces*, MERGA, Adelaide, 471-478.

Wenger, E. (1998). *Communities of Practice: Learning, meaning and identity.* Cambridge: Cambridge University Press.

THE IMPORTANCE OF PRE-SERVICE TEACHERS' CONCEPTIONS OF MATHEMATICS AND APPROACHES TO LEARNING FOR THE FUTURE OF MATHEMATICS EDUCATION IN IRELAND

Miriam Liston John O'Donoghue

National Centre for Excellence in Mathematics and Science Teaching and Learning

A study was carried out by the author (ML) at the University of Limerick to gain an understanding of pre-service teachers' conceptions of mathematics, and test the hypothesis of a relationship between conceptions of mathematics and approaches to learning. Both quantitative and qualitative research was conducted for the author's PhD thesis. The studies focus on three groups of first year students on Service mathematics programmes (degree courses where mathematics plays a part in the students' studies but is not the main focus) at UL, at the beginning of the university academic year 2006/2007 and includes a number of pre-service teachers, including pre-service mathematics teachers. There is little or no research in Ireland, focused specifically on pre-service secondary teachers' conceptions of mathematics and approaches to learning, and such research contributes to our knowledge of pre-service teachers particularly in an Irish context.

INTRODUCTION AND BACKGROUND TO THE STUDY

A factor that often widens the gap in the transition to university mathematics is one's conceptions of mathematics (Marton, 1988 and Crawford, Gordon, Nicholas, and Prosser, 1994). This 'gap' refers to the mathematical preparedness, or lack of, and the standard of students' mathematics coming from second-level. Evidence suggests also that the approaches students adopt to learning at university will almost certainly have an effect on their academic success (Greasley, 1998). These two such issues, conceptions of mathematics and approaches to learning, form strong elements of this research study, which aims to understand the relationships between these two constructs and add clarity to their interactions. For example, the literature reports on the existence of a relationship between conceptions of mathematics and approaches to learning and we wished to test if the same was true for this sample.

The authors hypothesise that students, both in this research and in general, encounter many mathematical difficulties in the transition to university mathematics. Reports, such as the NCCA's (2005) *Review of Mathematics in Post-Primary Education*, provide evidence of these difficulties. Some important findings were also identified at UL in relation to students following this transition. Through the use of diagnostic testing, it was found that the Leaving Certificate Ordinary Level mathematics syllabus is inadequate preparation for Service mathematics courses at UL (Gill, 2006). Results have shown over

30% of new students scored 20 or less (out of 40) and although Higher Level students performed better on the diagnostic test, this did not shed much light on their mathematical preparedness because the test is set at Ordinary Level Leaving Certificate standard and is focused on basic skills and knowledge. This study includes pre-service teachers, as the authors believe that such students have an important part to play in making a move away from the problems just mentioned, and in working towards building future students' understanding of mathematical concepts, as well as enabling them to see how mathematics can be related to everyday life.

Conceptions of mathematics open a window to students' understanding of mathematics. That is, it provides us with an insight into their deep understanding of the subject. Crawford et al. (1994, p.343) conducted a study on university students' conceptions of mathematics and how it is learned concluding that "students' conceptions of mathematics are formed by their approaches to learning it and also form their approaches". The role of the teacher in changing students' conceptions is emphasised by Thompson (1992, p.132) who defines a teacher's conception of the nature of mathematics as "that teacher's conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics". She believes that if changes are to occur in the mathematics classroom, that is, changing teachers' approaches to teaching, then teachers' conceptions must be addressed.

Anthony (2000) says students' conceptions of learning have an onward effect on the way they approach their studies and in turn affects the quality of their learning. The type of approach to learning that students adopt is a strong deciding factor in whether students' transition to university is successful or not. Deep-level versus surface-level, reproduction versus comprehension and relational versus instrumental understanding are vitally important terms according to many researchers (e.g. Marton and Saljo, 1976). Marton and Saljo's work was one of the first, and most important, to focus on students approaches to learning and they identified two processes, deep-level and surface-level. It was Marton and Saljo's work that influenced that of Biggs et al.'s (2001), which also divides approaches to learning into deep level versus surface level and was used in the authors' research. This study aims to classify students as leaning more towards a surface or deep approach to learning but the authors' are also aware of approaches to learning that may fall between the surface and deep approach to leaning. For example, Case and Marshall (2004) identified two intermediate approaches between the classic surface and deep approaches. These are referred to as 'procedural surface' and 'procedural deep' approaches to learning.

This paper will describe a quantitative and qualitative study carried out in the academic year 2006/2007. One aim of the study was to investigate the impact of conceptions of mathematics and approaches to learning on pre-service teachers, including pre-service mathematics teachers, in order to establish the extent of the problem that mathematics education may face in Ireland. Details of the study, including the research instrument, data collection, research sample and analysis of data are provided. Firstly, an overview of the theoretical framework used in this study is provided.

THEORETICAL FRAMEWORK

For the overall research study a number of factors were investigated. The focus in this paper is on conceptions of mathematics and approaches to learning which are an integral part of the study. In relation to these two constructs, the theoretical frameworks followed include the work of Crawford et al. (1994, 1998), which was inspired by the work of Marton (1988) and Biggs et al. (2001) (which followed on from the work of Marton and Saljo, 1976). There are many definitions among researchers as to what exactly are conceptions and approaches to learning. It is important therefore, to provide a definition or explanation of how each factor was interpreted for use in this study.

Conceptions of mathematics can be viewed as either fragmented (when the learner focuses on parts rather than the whole), or cohesive (when the learner concentrates on the whole picture rather than just the constituent parts) (Crawford, 1998).

Approaches to learning can be divided into a surface learning approach, where the main focus is reproduction of knowledge, or a deep learning approach, which aims for comprehension (Biggs et al., 2001).

METHODOLOGY

Quantitative research was carried out through a questionnaire and following the statistical analysis of this quantitative data, qualitative research was conducted in the form of semi-structured interviews. Both studies are outlined now.

Research Instrument

Quantitative study

A questionnaire consisting of 78 statements based on attitudinal scales was designed and implemented. Attitudes to mathematics, beliefs about mathematics, mathematical self-concept, conceptions of mathematics and approaches to learning were measured. Conceptions of mathematics and approaches to learning are discussed in this paper and were investigated using Crawford et al.'s (1998) 'Conceptions of Mathematics' and Biggs et al.'s (2001) 'Revised two-Factor Study Process Questionnaire'. The questionnaire was divided into 3 sections. The first section contained demographic information about the student e.g. Leaving Certificate grade and level, etc. In part of Section A, there are 19 statements examining conceptions of mathematics. This scale is divided into ten 'fragmented statements' (FCM) and nine 'cohesive statements' (CCM). Section B consisted of 20 statements and looked at students' approaches to learning. Students responded to both Section A and Section B in accordance with a Likert scale indicated above the items. For Section A, strong feelings could be indicated on either side of the scale and there was an option for respondents who could indicate that they were unsure of statements (i.e. 1= strongly disagree, 2= disagree, 3=unsure, 4=agree, 5=strongly agree). The Likert Scale for Section B of the instrument consisted of 1 = never or only rarely true of me, 2 = sometimes true of me, 3 = true of me about half the time, 4 = frequently true of me, 5 = always or almost always true of me.

Qualitative study

Fifteen semi-structured interviews were conducted with five randomly selected students from each of the three Service mathematics courses. Questions for the interviews originate from the scales used in the quantitative study, as well as from other areas in the mathematics education literature. For example, students were asked about their general transition to university given that reports have shown that making the transition from secondary school to university is a difficult time for students, both academically and socially (e.g. Kantanis, 2000).

Data Collection

Quantitative study

As mentioned, the questionnaire was distributed to the following three Service mathematics modules - Technological Mathematics 1, Science Mathematics 1 and Engineering Mathematics 1. Each of these modules is made up of students from a number of different degree programmes. 607 questionnaires were completed in full and returned to the author. The focus in this paper is on the 197 pre-service teachers who participated in the quantitative study (see table 1 below).

Teacher Education Course	Number of students who participated in quantitative study	Number of students who participated in qualitative study
Science Degree with concurrent Teacher Education	59	2
Technology Degree in the teaching of Materials and Construction	79	1
Technology Degree in the teaching of Materials and Engineering	50	0
Physical Education and Mathematics with concurrent Teacher Education	9	2

Table 1:Breakdown and number of teacher education participants in the quantitative and qualitative studies

Qualitative study

Fifteen semi-structured interviews were carried out in April 2007 – May 2007. Five of these were with pre-service teachers (see table 1). A plan for each section of the interview was constructed. Interviews were recorded with interviewee’s permission using an IC Recorder. After the interview the recording was transferred to Voice Editor 3 (voice editing PC software). The interviews were then transcribed and saved as a word document. This facilitated their importation into NVivo – software designed for the analysis of qualitative data.

Research Sample

Quantitative study

The research sample consisted of students making the transition to university mathematics and who are studying Service mathematics courses at the University of Limerick. The authors chose to focus on students from Science, Engineering and Technology (SET) disciplines due to the increasing dependency within Ireland on these areas of the economy e.g. ISA, (2008). The large sample size and various groups within the sample allowed for much diversity in ability. The focus of this paper is on the pre-service teachers in the sample. Two out of the three groups selected include pre-service teachers who may or may not teach mathematics at second level in the future. That is, all Physical Education students who elected mathematics as their academic subject will most likely teach the subject upon obtaining their degree. The three other teacher education courses; Science Degree with concurrent Teacher Education; Technology Degree in the teaching of Materials and Construction and Technology Degree in the teaching of Materials and Engineering, are not qualified to teach mathematics in school but according to sources, such as Royal Irish Academy (2008), many often end up in a situation where they will teach mathematics at some level in secondary schools in Ireland.

Qualitative study

Within the three groups; Science, Engineering and Technology, two groups (Science and Technological Mathematics 1) were again divided into teacher education and non-teacher education groups. This was done so as to ensure that pre-service teachers were included in the sample. Interviewees were selected using random number tables. In total there are five interviewees from each module including five pre-service teachers.

Data Analysis

Quantitative study

The data resulting from the questionnaire was analysed using the statistical package of SPSS (Version 15 for Windows). The reliability of each of the scales was analysed using Cronbach's Alpha scores. While the scales used in the questionnaire are those tested in the literature, a confirmatory factor analysis was carried out on all items of each scale. Two scales, the Conceptions of Mathematics scale and the Approaches to Learning scale, and results stemming from analysis of these scales, are discussed in this paper. Descriptive statistics revealed the mean and standard deviation for all items. Further, more in-depth analysis of the data looked at paired-samples t-tests, correlations (Pearson's), and regressions.

Qualitative study

An interview plan guided the 15 interviews that took place (5 of these interviews with pre-service teachers are specifically addressed in this paper) and lasted approximately 30 minutes each. The data were coded based on a list of starter nodes drawn up for each section of the interview as outlined above. After careful analysis, the nodes were re-categorised into new nodes, which emerged from the data. Such re-structuring of categories provided any confirming or disconfirming evidence of the nodes identified or any relationships between them.

FINDINGS AND DISCUSSION

The main findings in relation to teacher education students' conceptions and approaches to learning, obtained from the data collected and analysed in both the quantitative and qualitative studies, are now presented and discussed. Teacher education students are compared to their non-teacher education peers. The implications of these findings are very important for mathematics education in Ireland.

Main findings for teacher education students and specialist second level pre-service mathematics teachers (quantitative study)

Independent t-tests revealed a statistically significant difference ($p < .05$) between the teacher education students' and non-teacher education students' fragmented and cohesive means. The mean score on the fragmented conceptions of mathematics scale was higher for teacher education than non-teacher education groups. This suggests that pre-service teachers in this sample have a more disjointed view of mathematics than their peers. Worryingly the teacher education students also had slightly lower mean scores on the cohesive scale that the non-teacher education students. The authors are aware however, that because of the large sample size, again small differences can be significant. Therefore, confidence intervals were also examined to see where the true difference between the means for teachers and non-teachers lie. For the cohesive mean, the confidence interval is [-0.20, -0.01] which are all negative values and indicate that again non-teachers tend to score higher on average. For the fragmented mean, the confidence interval is [0.04, 0.18] which are all positive values indicating that the teachers are scoring higher on average on this scale than non-teachers. The fact that there is a statistically significant difference between the two means on both these scales is similar to the Independent t-test.

The relationship between conceptions of mathematics and approaches to learning was also analysed for both these groups. The relationship between cohesive conceptions and deep approaches to learning was positive and statistically significant for both teacher education students ($r = .34$, $p< 0.01$) and non-teacher education students ($r =. 31$, $p< 0.01$). There was a positive statistically significant but weak correlation ($r = .14$, $p< 0.01$) between fragmented conceptions and surface approaches to learning for the non-teacher education group. No statistically significant relationship was found between fragmented conceptions of mathematics and surface approaches to learning for the teacher education group.

Responses from PE and Mathematics students were analysed specifically, and compared to the responses from students in the other three teacher education courses, as these are the students in the sample who elected to study mathematics to degree level and who will almost certainly go on to teach mathematics in secondary school. Results showed that students on all teacher education courses had a relatively high fragmented conception of mathematics mean (3.3). PE and Mathematics students had the highest cohesive conceptions of mathematics mean of 3.7(SD = .22) but again it was not statistically different from that of the students in all three other teacher education courses. In relation to conceptions of mathematics, PE students' fragmented and cohesive scores were similar to their peers. It was worrying however, that despite their relatively high mean on the cohesive conception of mathematics scale, PE students had the lowest deep approach mean [25.8(SD = 4.15)] compared to their pre-service teacher peers. Researchers, e.g. Ball (1990, p.141), have explored pre-service elementary and secondary school teachers' mathematical knowledge and understanding and in her study on pre-service teachers' understanding of division, she concluded that their understanding is based on rules and reproduction and founded more on "memorisation than on conceptual understanding". On a more positive note however, PE students had the lowest surface approach to learning mean [22(SD = 4.15] of the teacher education students. Case and Marshall's (2004) continuum of approaches to learning (described earlier) may explain why these pre-service mathematics teachers have low deep approach scores and low surface approach scores. The qualitative study allows additional conclusions to be drawn in relation to these issues. Also, high surface approach mean scores for both of the technology degree programmes with concurrent teacher education of 24.9(SD = 6.41) and 25(SD = 6.09) out of a possible 40 is a situation that may cause difficulties for mathematics education should these students teach mathematics at second level in the future.

Main findings for teacher education students and specialist second level pre-service mathematics teachers (qualitative study)

The aim of the qualitative study was to both gain a deeper understanding into why the respondents responded the way they did in the quantitative study, and to follow up results obtained from the statistical analysis of this study. Table 2 below also identifies the mean scores, obtained by the pre-service teachers interviewed, on each of the scales in the quantitative study.

NAME	Deep Approach	Surface Approach	Cohesive Conception of Mathematics	Fragmented Conception of Mathematics
Emily	29	21	35.1	32
Greg	26	16	34.2	35
Frances	31	23	33.3	34
Peter	23	26	32.4	30
Vinny	34	22	35.1	24

Table 2: Interviewees' (pre-service teachers) mean scores on the approaches to learning and conceptions of mathematics scales in the quantitative study

In terms of conceptions of mathematics and in order to gain a deeper understanding of how students perceive mathematics, students were asked what they believe to be the most important aspect when learning mathematics. A range of responses led the authors to categorise the findings for each of the pre-service teachers as follows: *to understand* (Peter – a Technology Degree in Materials and Construction student and Greg – a Physical Education and Mathematics student), *focus on procedure* (Emily – a Physical Education and Mathematics student and Vinny – a Biological Science student), *to reproduce material* (Frances – a Biological Science student).

Two of the pre-service teachers viewed procedure and learning off procedure as imperative to do mathematics successfully (one PE/Mathematics and one Science Education student).

Emily: Probably following a certain procedure that once you learn it properly then you'll be able to apply it so learning the procedure.

Mason and Spence (1999) stress the failings of rehearsal and practice of techniques. They believe it is more effective for a teacher to divert the focus of their teaching away from rehearsal of a skill.

One pre-service mathematics teacher, Greg, emphasised the need to understand the mathematics but despite this, he still refers to rote-learning methodologies rather than an understanding of the concepts involved.

As already discussed in this paper, students' conceptions of learning mathematics affects the way they approach their studies and in turn affects the quality of learning (Crawford et al., 1998 and Anthony, 2000). In the interviews, the researcher queried the respondents on their approaches to learning mathematics at university. All interviewees felt that the way in which they approach their mathematics learning is positive at university. First impressions in the interviews suggested to the authors that the two pre-service mathematics teachers adopted approaches that were linked to their claim that understanding mathematics and relaying its applications to their future students is very important. Both pre-service mathematics teachers mentioned their desire to understand mathematics in order to learn effectively. However, further analysis of the findings in this section of the interview, imply and not for the first time, that some students beliefs about, and conceptions of, mathematics are not always consistent with the approach they adopt to studying. Greg is a case in point. Methods employed by Greg when studying include practice, procedure and learning formula. He relies heavily on formulae.

Greg: Well if there's formulas involved, learn the formulas and know what it is you need to put into formulas or get out of it. Then if there are not formulas just practice the questions basically.

He previously stressed how vital a deep understanding is for effective learning yet he relies on procedures, formula and repetition to study mathematics at university. Such an insight into Greg's approaches to learning is somewhat surprising given his low surface approach to learning mean of 16 in the quantitative study, although his deep approach mean score of 26 was also among the lowest of the students interviewed (see table 2). Such low surface and deep scores may again suggest use of procedural approaches to learning. Possibly Greg leans towards a 'procedural deep' approach where his intention is towards understanding but his strategy does not include concepts.

The type of approach to learning adopted by students in this study focuses predominantly on exam papers, practice and repetition of questions. These are mostly procedural or surface strategies than quite often fail to develop the students' conceptual understanding. The most common approach or method to studying mathematics, by the pre-service teachers interviewed, was to do exam papers and practice questions. For example,

Vinny: Eh just do the past papers and kind of study them and just kind of get the four questions or whatever.

Peter: Em…well what you've to learn to do you've to learn by doing so I suppose sit down and take out your maths book and keep on doing questions until you're hand is sore and you can't do no more!

Often these approaches take the form of rote learning, which is particularly detrimental to understanding mathematics. Porter and Masingila (2000) claim that rote learning can interfere with students' procedural ability while also preventing an understanding of mathematical concepts. The role of the teacher and the importance of pre-service undergraduate training are of utmost importance in tackling issues such as this reliance on rote-learning and procedural knowledge.

CONCLUSION

It was the authors' objective to use the above findings, based on trainee teachers in this sample, to develop a profile of these pre-service teachers' conceptions of mathematics, in particular.

Three out of the five pre-service teachers in this sample would be comfortable teaching mathematics (two of which are pre-service mathematics teachers) while the other two pre-service science teachers would not feel confident or comfortable teaching mathematics at secondary school. As previously mentioned, quite often teachers in Ireland will end up teaching a subject that they may not have a degree in, a situation that must be addressed.

In summary, the quantitative and follow-up qualitative study revealed the following profile: a lack of consistency between pre-service teachers' conceptions of mathematics and their learning approaches or strategies; the existence of often poor, fragmented conceptions coming from secondary school and the general consensus among teacher education students, that procedures are extremely important in mathematics. There is a fear, that a reliance on a traditional style of teaching in Ireland will be replicated by future mathematics teachers should these issues not be tackled in the immediate future.

The move away from rote-learning and a traditional style of teaching may be addressed in Ireland under a new initiative for 2008 called 'Project Mathematics' which will see much greater emphasis being placed on student understanding of mathematical concepts and relationships and an increased use of contexts and applications that will enable students to relate mathematics to everyday experiences (NCCA, 2008).

If we are to extend students' mathematical knowledge and understanding beyond algorithms, to include understanding of its' relationship to other disciplines and real-life problems, change must begin with teachers. The author, in her PhD study, makes recommendations and proposals for new teaching policies in mathematics education. One such suggestion is that teacher education programmes should include a constructivist approach to teaching and learning so that students play an active part in developing their knowledge and in turn apply such learning strategies to their own classroom practices.

There is need for further research on the implications of conceptions of mathematics and approaches to learning mathematics for prospective teachers of mathematics. These issues subsequently influence learners of mathematics in Ireland, and future research is necessary to emphasise the importance of this issue and to create possible solutions to combat the root of the problem.

ACKNOWLEDGEMENTS

This research has been carried out in conjunction with the National Centre for Excellence in Mathematics and Science Teaching and Learning (NCE-MSTL).

REFERENCES

Anthony, G. (2000) 'Factors influencing first-year students' success in mathematics', *International Journal of Mathematical Education in Science and Technology*, **31**(1), 3-14.

Ball, D.L. (1990) 'Prospective elementary and secondary teachers' understanding of division', *Journal for Research in Mathematics Education,* **21**(2), 132-144.

Biggs. J., Kember, D. and Leung, D. (2001) ‘The revised two-factor study process questionnaire: R-SPQ-2F’, *British Journal of Educational Psychology*, **71**, 133-149.

Case, J. and Marshall, D. (2004) ‘Between deep and surface: procedural approaches to learning in engineering education contexts’, *Studies in Higher Education,* **29**(5), 605-615.

Crawford, K., Gordon S., Nicholas, J. and Prosser, M. (1998) ‘University mathematics students' conceptions of mathematics’, *Studies in Higher Education*, **23**(1), 87-94.

Crawford, K., Gordon S., Nicholas, J. and Prosser, M. (1994) ‘Conceptions of mathematics and how it is learned: the perspectives of students entering university’, *Learning and Instruction*, **14**(4), 331-345.

Gill, O. (2006) *What Counts as Service Mathematics? An Investigation into the ‘Mathematics Problem’ in Ireland*, unpublished thesis (PhD), University of Limerick.

Greasley, K. (1998) ‘Does gender affect students' approaches to learning?’ in Brown, S., Armstong, S. and Thompson, G., eds., *Motivating Students*, London: Kogan, 105-112.

Irish Software Association (2008) *The Indigenous Software Sector: Mainspring of Ireland’s Knowledge Economy, Budget Submission 2008* [online], available: http://www.software.ie/Sectors/ISA/ISADoclib3.nsf/wvICCS/A90BD147A25198AE8025737C003C65FF/$File/Budget+submission+2008+final.doc, [accessed 28 November 2007].

Kantanis, T. (2000) ‘The role of social transition in students' adjustment to the first-year of university’, *Journal of Institutional Research*, **9**(1), 100-110.

Mason, J. and Spence, M. (1999) ‘Beyond mere knowledge of mathematics: the importance of knowing-to act in the moment’, *Educational Studies in Mathematics*, **38**, 135-161.

Marton, F. (1988) ‘Describing and improving learning’ in Schmeck, R.R., ed., *Learning Strategies and Learning Styles*. New York: Plenum, 53-82.

Marton, F. and Saljo, R. (1976) ‘On qualitative differences in learning I. Outcome and process’, *British Journal of Educational Psychology,* **46**, 4-11.

National Council for Curriculum and Assessment (NCCA) (2008) Leaving Certificate Mathematics Revised Syllabus Strands [online], available: http://www.projectmaths.ie/system/files/LC%20Maths%20Syll%200708.pdf, [accessed 31 August 2009].

National Council for Curriculum and Assessment (NCCA) (2005) *Review of Mathematics in Post-Primary Education* [online], available: http://www.ncca.ie/uploadedfiles/MathsReview/MathsDiscusPaperEng.pdf, [accessed 15 May 2006].

Porter, M.S. and Masingila, J.O. (2000) 'Examining the effects of writing on conceptual and procedural knowledge in calculus', *Educational Studies in Mathematics*, **42**, 165-177.

Ramsden, P. (1992) *Learning To Teach In Higher Education*, London, New York: Routledge.

Royal Irish Academy Committee of Mathematical Sciences and Chemical & Physical Sciences (2008), *Response to the Proposal to offer Bonus Points for Maths*, Dublin: RIA.

Thompson, A. G. (1992) 'Teachers' beliefs and conceptions: A synthesis of the research', in Grouws, D.A., ed., *Handbook of Research on Mathematics Teaching and Learning*, New York: Macmillan, 127-146.

WHY IT'S DIFFERENT WITH MATHEMATICS: PROSPECTIVE TEACHERS' REFLECTIONS ON WHAT MAKES TEACHING POST-PRIMARY MATHEMATICS UNIQUE

Maria Meehan — University College Dublin

Catherine Paolucci — NUI, Galway

This paper will discuss findings from a survey of 210 Postgraduate Diploma in Education (PGDE) students from five universities in Ireland. It will focus specifically on the reflections of these students regarding the differences between teaching mathematics and teaching other subject areas. The responses offer insight into these prospective teachers' conceptions of the subject of mathematics as well as their sense of what should take place in a mathematics lesson and a mathematics classroom. These results have important implications for the preparation of post-primary mathematics teachers in Ireland.

INTRODUCTION

In September 2007 we surveyed 210 prospective second-level mathematics teachers who were embarking on a one-year, full-time Postgraduate Diploma in Education (PGDE) at five universities in Ireland. The study was intended to examine these students' beliefs about mathematics teaching and learning. The discussion in this paper focuses on the responses to one particular question on the survey which asked the student-teachers whether they believe that teaching mathematics is different from teaching other subjects. We feel that the responses to this question are particularly worth examining because the prospective teachers were able to articulate their ideas through comparing and/or contrasting teaching mathematics with teaching other subjects. In this context, they revealed elements of their conceptions of the nature of mathematics as well as some of the pedagogical challenges that they believe are faced by those who teach the subject at second level.

LITERATURE REVIEW

In a 3-year longitudinal study, Boaler (1999) followed two groups of students from two second-level schools in the UK as they progressed through their mathematics classes. In one of the schools, Amber Hill, the students were taught mathematics using a traditional, didactic approach – methods and techniques were explained by the teacher to the class, and then students practised these methods by doing lots of exercises from a textbook. In the other school, Phoenix Park, students were taught using a more exploratory, project-based approach, where group-work and discussion were encouraged. Not surprisingly, given the environment in which they were taught mathematics, many of the students from Amber Hill believed that the subject consisted of rules and formulas to be memorised, and was "incompatible with thought" (p. 263). On the other hand, the students from Phoenix Park believed that "mathematics was a thinking, flexible subject" (p. 264).

Similar findings emerged from a study conducted by Boaler and Greeno (2000) in six high-schools in the US. Eight students from each of the schools were interviewed, and all 48 were taking the Advanced Placement (AP) Calculus course. Based on students' descriptions, four of the schools took a traditional approach to teaching mathematics, while the other two schools encouraged student discussion and collaboration in the mathematics classroom. The students from the four "traditional" schools expressed the view that little or no thought is required in mathematics and that it is a subject that is all about procedures, formulas and rules.

> The students suggested that the procedural presentation of mathematics they encountered forced them to become passive receivers of knowledge – with a narrowly defined role that was one of memorisation (Boaler & Greeno, 2000, p.181).

When these students were asked if they enjoyed mathematics, 18 out of the 32 said they did, and gave reasons that indicated they were happy to be "received knowers". They explained that they liked mathematics because "there were only right and wrong answers, and because they did not have to consider different opinions or ideas, or use creativity or expression" (Boaler & Greeno, 2000, p.185). The remaining 14 students gave the same reasons for *not* enjoying mathematics. They saw themselves as creative individuals with opinions, and felt that as a result, mathematics was not the subject for them.

The two studies described above were carried out in the UK and USA respectively. Does a traditional, didactic approach to teaching mathematics also exist in second-level classrooms in Ireland? The study of mathematics teaching and learning at the Junior Certificate Level, conducted by Lyons, Lynch, Close, Sheerin, & Boland (2003) suggests that it does. They found that a traditional, "drill-and-practice" approach to teaching mathematics was prevalent in all ten schools that took part in the study. The review of mathematics in second-level schools in Ireland conducted by the National Council for Curriculum and Assessment (2005) and the Chief Examiner's Report (2005) on the Leaving Certificate Mathematics Examination further support the findings of Lyons et al. (2003).

The Teaching Council of Ireland requires that all prospective second-level mathematics teachers must study mathematics as a major subject in their undergraduate degrees "extending over at least three years and of the order of 30% at a minimum of that period" (p. 24). Thus even if an aspiring second-level mathematics teacher emerges from his or her own secondary studies believing that mathematics is "structured, certain, and nonnegotiable" (Boaler & Greeno, 2000, p. 196), he or she will have to study mathematics to quite a significant level at university before qualifying to teach the subject in Ireland. This leads to the question of how these university studies will influence the view of mathematics held by this student.

Ball and McDiarmid (1990) make the point that not only is the second-level phase of study (five-six years in Ireland) longer than the third-level phase (three-four years), much of the content that second-level teachers will be expected to teach, they have learned themselves at second level and not at university. Therefore, even if the prospective teacher appreciates the conceptual and problem-solving nature of mathematics, is this enough to prevent him or her from reverting back to the "drill-and-practice" view of the subject once they are back in the second-level classroom? It is also of importance to consider whether it is this "drill-and-practice" view of the subject that attracts mathematics graduates to become second-level teachers. While it is not possible to address all of these questions in this paper, the findings do shed light on some of these issues and these are addressed further in the discussion of results.

RESEARCH METHODOLOGY

The questionnaire employed in this study was designed to gain information about prospective second-level mathematics teachers' educational backgrounds, personal experiences studying mathematics, and perspectives on mathematics teaching and learning. Although there were several items on the questionnaire, the discussion in this paper focuses specifically on the qualitative analysis of participants' responses to the following question:

> Do you believe that there are significant differences between teaching mathematics and teaching other subjects? Please explain.

Students in the PGDE are required to study teaching methodologies for at least two subject areas. The questionnaires were distributed to students in Ireland's five full-time PGDE programmes (in 2007) who had chosen mathematics as one of their subjects. This was done during a visit to the mathematics teaching methodology class in each university at the start of its programme. As a result, the population for this study includes both students who had identified mathematics as their primary teaching subject and those who had chosen mathematics as their second or third subject.

Throughout the discussion of this study, those with mathematics as their primary teaching methodology will be referred to as *majors* (noted with an *M*) and those who chose it as a second or third option will be referred to as *minors* (noted with an *m*). This distinction is made significant by the fact that students are advised to choose their primary teaching methodology in a subject in which they have enough third-level study to meet the Irish Teaching Council's requirements for teaching that subject. Therefore, although it is not the case that all *minors* have weak backgrounds in mathematics, it can be assumed that any participant who is referred to as a *major* does have a significant third-level background in mathematics.

While some majors had as much as a masters degree in mathematics, some of the minors had not formally studied mathematics since their Leaving Certificate Examination. While the distributions of majors and minors in the mathematics methodology classes varied, less than half of the subjects in each institution were majors. Overall, out of 210 participants, 77 were majors and 133 were minors. Some of the participants' other methodology choices included various science subjects, business studies, English, history, religion, music, and geography. It is also important to note that all but 4 participants (98.1%) studied second-level mathematics in Ireland and sat the Irish Leaving Certificate Mathematics Examination.

This study's inclusion of students from the five full-time PGDE programmes provides a uniquely comprehensive look at the majority of the nation's cohort that would have become eligible to teach second-level mathematics at the end of the 2007-2008 academic year. It is reasonable to expect that a number of the subjects are now practising teachers in schools throughout Ireland. Thus, implications from this study must be considered in the context of both initial teacher education and professional development for practising teachers.

The analysis of the responses to this particular question was predominantly qualitative and conducted in accordance with the procedures for developing grounded theory as outlined by Strauss and Corbin (1998). The researchers worked independently to code the responses and identify overarching categories which emerged from the responses. They then met to compare, discuss and debate their findings. Some codes were refined, others merged and the researchers independently returned to the data with the new codes. This process was repeated until consensus was reached. We now present the major findings from this analysis by outlining six main categories that emerged and a series of themes from within these categories.

STATEMENT OF FINDINGS

All but one of the 210 participants in this study responded to the question: "Do you believe that there are significant differences between teaching mathematics and teaching other subjects? Please explain." Of these, 171 subjects (81.4%) responded that there are differences, 16 (7.6%) wrote that there are no differences, and 15 (7.1%) felt that there are both differences and similarities. Additionally, 4 respondents weren't sure and 3 of the responses were not clear enough to categorise. Overall, only 3 subjects simply responded "Yes" or "No" without explanation, providing the researchers with 206 elaborated responses. Given that the majority of participants answered "yes," the analysis of these responses will be the focus of this paper. We note at this point that when quoting a response directly, we will label it with an *M* or *m* to denote whether the respondent is a major or minor respectively, and also with a number so that the reader can identify whether responses are from the same, or different, respondents.

In many cases, the "yes" responses did not specifically address *teaching* mathematics, rather they attended more to learning mathematics and the perceived nature of the subject. This is especially true of the responses in Categories 1 – 4, which are described below.

Category 1: Maths is a "doing"/"practical" subject. (40 respondents – 12M & 28m)

Respondents in Category 1 indicated that mathematics is different from many other subjects in that it involves doing a lot of questions and examples and practising mathematical techniques.

> Yes. Maths requires much more emphasis on doing questions, whereas other subjects do not. (m1)

> Yes, as maths is a much more practical subject than others. I believe if you practice mathematics problems over & over again you will learn how to solve the problem. (m2)

The use of the word "practical" here seems to refer more to practising newly learned techniques than real-world applications. Discrepancies in the use of this word arise throughout the data, but most comments referring to mathematics as practical fall into Category 1. Elsewhere in the data, eight respondents say that mathematics is "practical" in that it can be used to solve real-world problems. Also, three respondents describe mathematics as *not* being a practical subject, when compared to subjects such as Chemistry or Physics.

In addition to the inconsistencies in the use of "practical," questions also arose surrounding the use of the word "understand". While "understanding" a concept would normally suggest a certain depth of meaningful knowledge, the term seemed to be used more loosely in a number of responses. Category 2 emerged out of these responses, and although further cross-analysis of these responses is necessary to flesh out the intended meaning of the word "understand" in this context, it is sufficient to say that it refers to learning that involves at least some level of processing beyond memorisation and regurgitation of facts.

Category 2: One must understand maths.

> Yes I do. A lot of other subjects can be learnt without understanding and regurgitated in an exam. This is not possible for maths. (M1)

In explaining why mathematics is different, a combined total of 20 responses (13M & 7m) from Categories 1 and 2 articulated the following implication for learning mathematics: Because mathematics is a doing/practical subject and/or a subject which must be understood … it is not possible to "learn maths off by heart".

> You can't learn off maths like you can for other subjects you have to understand it. (M2)

> Yes. It requires lots of examples & lots of practice by the students. You can't learn maths off by heart, you need to fully understand what is going on. (M3)

> Yes. Maths is all about practicing [sic] & understanding. A lot of other subjects is centred on learning information by heart. (m3)

> Well yes, it's not so much a learn off by heart subject, it requires practice, and more practice of methods rather than the droll of learning it off. (M4)

In their comparisons to several other subjects, Accountancy and Physics are mentioned by one student as being similar to mathematics in the sense that "you can't just learn them off," while subjects described as being unlike mathematics because they *can* just be learned off include History, Geography (twice), Biology (twice), and Business (twice).

In addition to respondents focusing on the fact that mathematics is not a subject that can be "learned off by heart," they pointed out a series of other implications of mathematics being a predominantly "doing" subject. These include the perception that there is little or no theory in mathematics; it is not necessary/sufficient to read about mathematics; and, one can't "lecture" mathematics.

> Yes. There is very little theory involved in maths. It's all about the student practicing [sic] in order to understand. (m4)

> Yes. Few people can understand maths after a reading a textbook. It requires a lot of work (i.e. solving problems) other subjects don't require this. (m5)

> Yes. Maths is more practical. You have to work to solve problems, it's always better to work with the students, make it like your [sic] learning too. With economics, I feel like I lecture them. There is no engagement like I have with maths. (M5)

In summary, the responses in Categories 1 and 2 suggest that many of the prospective teachers view mathematics as a subject with little or no theory, in which reading is not a necessary or sufficient means of learning, and lecturing is not an effective means of teaching. They believe that this is the case because mathematics is a subject that cannot be learned off by heart – it must be practised and understood.

Two additional overlapping categories emerged to reflect further ways in which prospective teachers perceive mathematics to be different from other subjects. They are as follows:

> **Category 3:** Maths is either right or wrong. (20 respondents – 7M & 13m)

> **Category 4:** Maths has a method/process to it. (31 respondents – 13M & 18m)

The respondents in Category 3 believe that mathematics is either right or wrong and the emphasis in teaching and learning mathematics is on finding the one right answer.

> Yes, there is a right, specific answer, and that's what they aim to achieve. (m6)

The responses in Category 4 paint a picture of mathematics being a process-orientated subject that can be "broken-down" and taught in a "step-by-step" fashion. The emphasis for teaching and learning mathematics is on illustrating and practising "methods" respectively.

> Yes, mathematics I feel is easier to teach as there are numerous techniques/methods for easy learning/teaching. (M6)

> Yes – maths has a set structure that can be taught on a step-by-step basis. (m7)

Several students from within Categories 3 and 4 further elaborated on the implications of mathematics being a clear cut subject that is right or wrong and/or has a method or process to be followed. The most prominent implication that was articulated by 19 respondents (9M & 10m) respondents was the following: Because mathematics is either right or wrong and/or has a method or process to be followed … there is little opinion/debate involved. Of the 19 responses that highlighted this characteristic of mathematics, 12 were part of Categories 3 and/or 4 and an additional 7 were stand-alone statements. Some of these respondents made the point that because there is always a right or wrong answer, there is no opinion required when *grading* mathematics.

> There is a method to maths that once you know and constantly practise you will rarely go wrong. It is a subject that is not opinionated. There is always a right answer & should you get it then you will get full marks – nothing can be taken away from you and students have the capabilities to get top marks in it. The same cannot be said for the majority of subjects. (M7)

Others indicated that because there are methods to be followed, students' opinions don't play a role in *teaching and learning* mathematics.

> Yes. Maths is very black & white, there is a method that's how you do it & there are no grey areas. With other subjects, students own opinions matter & there is a lot of independent thought involved. (m8)

Based on these implications, mathematics is considered different from English (mentioned by nine respondents), History (twice), Religion, Geography, Physics and Languages.

Two similar but less prominent implications arising from Categories 3 and/or 4 included little creativity/imagination and little feelings/emotion involved in teaching and learning mathematics. These respondents expressed their views that mathematics is not a creative subject and/or there is no room for a student to be creative in the subject. This distinguishes it from English, Geography and Art, but similar to Physics, Applied Mathematics, Science, Economics and Business.

> Maths is different from the more creative subjects. It follows theories and rules. (M8)

> Yes there aren't so many ways for students to be creative in maths. (M9)
>
> Yes maths is straightforward, other subjects like English, art etc need imagination etc. (m9)

The lack of feelings and emotion in mathematics was another way in which subjects felt it differed from English and Irish was mentioned here as well. One respondent explained that "There is almost a dehumanised aspect" of mathematics, while another wrote the following:

> Yes, English for example requires getting emotional. Maths is very precise and usually doesn't require an opinion from the pupil. (M10)

Although some responses fit into both Categories 3 and 4 the two categories consisted of a combined total of 46 distinct responses (approximately 22% of those surveyed).

Another significant set of responses indicated that mathematics is distinctly difficult to teach.

Category 5: Mathematics is a more difficult subject to teach.

The 66 responses in Category 5 focus on the various reasons why mathematics is more difficult to teach than other second-level subjects. This category consists of 31.4% of the responses from which a number of themes emerged. Each theme is meant to follow the statement *Maths is more difficult to teach because…*

Theme 5A: ...Some people are more mathematically minded, and others just don't get maths.
(14 respondents – 4 M & 10 m)

The most notable thing about this theme is that, while some of the responses focus on the difficulty of helping students who believe they just aren't good at mathematics, others clearly reflect that the prospective teachers themselves actually believe that some students just can't do mathematics. The following two responses help to illustrate this:

> Yes, maths is a subject that students from a young age believe that there either good at or not. So when they start second level they are either willing or unwilling to try & understand the concepts ... (m10)
>
> Yes –sometimes the students just seem to get it or they don't. I know it is my responsibility to ensure they do, but it can be hard. It can be that the student may have a maths mind or not! (m11)

In responses such as the second, which reflect the prospective teacher's concern with students' mathematical mindedness, some (such as the one above) attribute it to natural ability while others seem to believe that it is a consequence of students' experiences. For example,

> Not really. Some people "get" very easily and others have a complete mental block. This happens more in maths than in other subjects but I believe this is primarily a function of students not being shown the "easy" bits they can do. For those who don't have a mathematical mindset maths is very hard - but their main issue is that they go against the subject early and stop trying (& the teacher stops trying to reach them!) (M11)

Overall, the responses fitting this theme seem to identify students' low mathematical self-efficacy as a reason why mathematics is difficult to teach. The respondents suggest that both students' and teachers' beliefs that students are or aren't mathematically minded can impact students' confidence in their mathematical abilities, students' willingness to give up on mathematics early in their education, and the persistent influence that any bad experience with mathematics can have on students' attitudes toward their future learning.

Theme 5B: …It is difficult to make interesting and attractive to students (32 respondents – 4M & 28m)

This theme is developed as the underlying consequence of a variety of issues presented in this series of responses. The issues raised include limited and unappealing resources, limited opportunities for exploration and experimentation, and difficulty finding real-world applications and making it relevant to students' lives. The following responses help to illustrate each of these issues:

> Yes, so far in my PGDE, there has been huge emphasis on using different resources, but I'm getting it hard to see what I could use in maths, whereas in most other subjects, there is a world of material that could be used. (m12)

> Yes. It's probably more difficult as it's all in your head. All I remember bout learning maths in secondary was white chalk on blackboard – no colour, experiments, models, nothing to see or touch or feel. That's why I believe some students fear it. (m13)

> For example, teaching French the priority is to have students learn vocab. Grammar may be lost/forgotten/rust, but w/ vocab they can communicate with French speakers in real situations. Goal is to communicate. Maths would not necessarily have a "social" goal. (m14)

As part of this theme, a few responses also raised the issue that it's difficult to get students interested because it's a compulsory subject.

Theme 5C: …Mathematics is a cumulative subject (6 respondents – 2 M & 4 m)

While this theme consisted of only a small number of responses, it has particular relevance in connection with the first theme that discussed students giving up on mathematics early and having 'bad' early experiences with mathematics. As a result, the concern arises in this theme that they won't know the basics that they need to continue or build on their learning.

> Yes. I think in Mathematics it is difficult for a weak student to progress since Maths is really built like a pyramid – if you can't master Algebra, for example, you'll be lost in several other areas. This puts pressure on teachers. With other subjects e.g. History, you can "start fresh" on one section without relying on another one. (M12)

The final category consists of comments from the small set of respondents that answered no.

Category 6: No difference between teaching mathematics and teaching other subjects.

This category emerged from 16 responses (7.6%) claiming that there is no difference between teaching mathematics and teaching other subjects. One subject simply responded "No" but the other 15 explained their reasoning and two themes emerged from their responses.

Theme 6A: Teaching is teaching, regardless of the subject. (6 respondents – 4M and 2m)

> No, You should be able to apply the same methodologies to all subjects. (m15)

Theme 6B: Teachers should teach a student rather than a subject. (9 respondents – 4M & 5m)

> No, you are not teaching a subject, you are teaching a student. No matter what the subject, you will have to change your way of teaching to suit them. (M13)

Overall, the responses in this category convey a belief that there are no special techniques or strategies necessary for teaching mathematics - good teachers can teach anything.

DISCUSSION OF FINDINGS

In their effort to explain the differences between teaching mathematics at second level and teaching other subjects, the prospective teachers revealed a great deal about their perceptions of the nature of second-level mathematics and their beliefs about teaching it.

A predominant view that emerged from the responses in this study is that mathematics is a "doing"/"practical" subject, a subject that is right or wrong, and a subject that has a "method" or process to it. A number of respondents indicated that mathematics does not require creativity, opinions or debate. This view of mathematics is very similar to the views held by the students from the studies of Boaler (1999) and Boaler & Greeno (2000), who were taught mathematics using a traditional approach. In one way this is not surprising given that all but four of the 210 respondents studied second-level mathematics in Ireland, and the reviewed literature offers compelling evidence that suggests a "drill-and-practice" approach to teaching mathematics is prevalent in Irish second-level classrooms (Lyons et al., 2003; NCCA, 2005; State Examinations Commission, 2005).

What is surprising is that this view of mathematics was expressed not just by minors, but also by several majors, because in almost every case, these are students who had met the Teaching Council's requirement that at least 30% of their undergraduate studies (extending over a minimum of three years) consisted of mathematics. Is it the case that their undergraduate study of mathematics has failed to alter their view of the subject from second-level? Or, is it that they possibly view second-level mathematics as a different subject to third-level mathematics? Or is it, as Ball & Diarmid (1990) suggest, that when they think about teaching second-level mathematics, they think about it in the context in which they were taught? We believe these to be very interesting and relevant questions and hope that a more extensive analysis of our survey data will help to shed some light on these issues.

Additional concern arises from the identification of mathematics as a subject void of creativity, opinion and debate, since Boaler and Greeno (2000) noted that in some cases this perception prevented students from enjoying their mathematical studies. In this regard, the lack of responses characterising mathematics as a creative, discussion-based subject suggests that students with this perception are either not continuing their mathematical studies beyond second-level, or possibly choosing to pursue a different career. This raises questions about the capacity for many prospective teachers to employ anything other than a procedure-oriented approach to teaching mathematics that allows little room for original thought.

Consequently, it is hardly surprising that a significant number of respondents felt that mathematics is a difficult subject to make interesting and relevant for students. Nor is it surprising, among respondents who do not see a need for creativity in mathematics, that this challenge was partially attributed to a lack of appealing resources for teaching mathematics. A number of respondents suggested that while the prospective teachers are familiar with a wide variety of engaging teaching resources, they do not consider them appropriate for teaching and learning mathematics. It is unlikely that this would be the main concern of someone who believed that innovative, exciting strategies should be employed in teaching the subject.

Overall, the combined themes emerging in Categories 5 and 6 highlight the importance of mathematics teaching methodology courses in the preparation of future mathematics teachers. Furthermore, the fact that the cohort in this study completed the PGDE in May 2008 means that many are currently practising teachers. It is therefore also relevant to note the potential benefit of similar focus on innovative mathematics-specific teaching resources and strategies in professional development opportunities for practising teachers.

REFERENCES

Ball, D. & McDiarmid, G. W. (1990). The subject matter preparation of teachers. In W. R. Houston (Ed.), *Handbook for research on teacher education* (pp. 437-449). NY: Macmillan.

Boaler, J. (1999). Participation, knowledge and beliefs: a community perspective on mathematics learning. *Educational Studies in Mathematics*, 48, 259-281.

Boaler, J. & Greeno, J. (2000). Identity, agency and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 171-200). Westport, CT: Ablex Publishing.

Lyons, M., Lynch, K., Close, S., Sheerin, E. & Boland, P. (2003). *Inside classrooms – the teaching and learning of mathematics in social context.* Dublin, Ireland: Institute of Public Administration.

National Council for Curriculum and Assessment (NCCA) (2005). *Review of mathematics in post-primary education – a discussion paper.* Dublin 2, Ireland.

State Examinations Commission (2005). *Chief Examiner's Report: Higher, Ordinary and Foundation Levels – Mathematics Leaving Certificate Examination 2005.* Retrieved June 30, 2009 www.examinations.ie.

Strauss, A. & Corbin, J. (1998). *Basics of qualitative research: techniques and procedures for developing grounded theory.* Thousand Oaks, CA: SAGE Publications, Inc.

Teaching Council. *General and special requirements for teachers of recognised subjects in mainstream post-primary education.* Retrieved July 2, 2009. www.teachingcouncil.ie.

REFORM VIA TEXTBOOKS: LESSONS FROM A CROSS-NATIONAL COLLABORATION

Pamela Moffett

Stranmillis University College, Belfast

Since 1971, the Freudenthal Institute in the Netherlands has developed a theoretical approach towards the teaching and learning of mathematics known as 'Realistic Mathematics Education' (RME). The Dutch reform of mathematics education was called 'realistic' not simply because of its connection with real-life, but because of the emphasis placed on offering students problem situations which they can imagine.

In the 1990s, the Freudenthal Institute collaborated with the University of Wisconsin to produce a mathematics curriculum for American middle schools based on RME principles. Manchester Metropolitan University trialled the materials in a number of English classrooms in 2003 and has been collaborating with the Freudenthal Institute since 2007 to develop curriculum materials based on RME principles for use in post-primary schools in England. Given that the recently revised Northern Ireland Curriculum (2007) promotes the use of meaningful contexts for learning mathematics, the paper investigates the impact of implementing RME principles within a United Kingdom setting. What are the main obstacles? Are the benefits worth pursuing? What can Northern Ireland learn from the Dutch-English curriculum collaboration?

INTRODUCTION

Realistic Mathematics Education (RME) and its underlying educational theory is the Dutch answer to the need, felt worldwide, to reform the teaching of mathematics (Van den Heuvel-Panhuizen, 2003). The development of RME began in the early 1970s and was strongly influenced by Freudenthal's views on mathematics. He believed that mathematics must be taught 'so as to be useful' (Freudenthal, 1968, p. 3). As in most approaches to mathematics education, RME aims to enable students to apply their mathematical knowledge and understanding in real life situations. In RME, however, this link with reality is not only recognisable at the end of the learning process, in the area of application; reality is also conceived as a source for learning mathematics (Van den Heuvel-Panhuizen, 1996, 2000). Freudenthal argued that if children learn mathematics in an isolated fashion, divorced from real experiences, then they will quickly forget it and be unable to apply it (Van den Heuvel-Panhuizen, 1996, 2000).

In practice, the Dutch reform mathematics education depended largely on the introduction of textbooks which reflect the principles of RME. More than three-quarters of primary schools in the Netherlands now use a mathematics textbook that was inspired to some degree by the reform movement (Van den Heuvel-Panhuizen, 1996). It is not just the Dutch primary school mathematics curriculum that has been influenced by the new approach to mathematics education. In the 1990s, the Freudenthal Institute (FI) in the Netherlands collaborated with the University of Wisconsin-Madison (UW) in the United States to produce a mathematics curriculum for American middle schools based on RME principles (Romberg, 2001). Manchester Metropolitan University (MMU) trialled the materials in a number of English classrooms in 2003, and has been collaborating with the FI, since 2007, to develop curriculum materials based on RME principles for use in post-primary schools in England (Eade, Dickinson, Hough & Gough, 2006).

Northern Ireland (NI) introduced its new curriculum for primary schools in 2007. Although the content in Mathematics and Numeracy – one of the six Areas of Learning in the revised curriculum – has not changed considerably from the previous version (CCEA, 1996), there is a stronger emphasis on the application of mathematical knowledge, skills and understanding in a variety of contexts. In particular, the curriculum recommends that ideal contexts are ‘relevant real life situations that require a mathematical dimension’ (CCEA, 2007, p. 6). Not only does the NI curriculum support the application of mathematics in real life situations, greater importance is also attached to the introduction of mathematical ideas and activities through contexts that are meaningful to children. The programme for the Foundation Stage (Years 1 and 2) states that ‘mathematical activities should be presented through contexts that have a real meaning for children’ (CCEA, 2007, p. 23), and at Key Stages 1 and 2 (Years 3-7) the programme states that ‘mathematical ideas should be introduced to children in meaningful contexts’ (CCEA, 2007, p. 57).

It is important that adequate support and guidance is provided for teachers as they begin to implement the revised primary curriculum for Mathematics and Numeracy. As Gravemeijer (1994, p. 22) points out, ‘It makes no sense to try to rouse teachers’ enthusiasm for new ideas if there are no suitable instructional materials available.’ Given that the use of real life contexts is one of the key features of the RME approach to mathematics education, this paper investigates the impact of implementing RME principles to mathematics education within a UK setting. The paper discusses insights gained from the pilot of RME textbooks in an English classroom. The main benefits and barriers are discussed.

THE DUTCH REFORM OF MATHEMATICS EDUCATION

RME is rooted in Freudenthal’s views on what mathematics is, how children learn mathematics and how mathematics should be taught (Van den Heuvel-Panhuizen, 1996, 2000, 2003). Freudenthal believed that mathematics must be connected to reality, stay close to children’s experience and be relevant to society if it is to be of human value (Van den Heuvel-Panhuizen, 1996, 2000, 2003). The Dutch reform of mathematics education was called ‘realistic’ not just because of its connection with the real world, but because of the emphasis placed upon offering students problem situations which they can imagine

(Van den Heuvel-Panhuizen, 2000, 2003). In the RME approach, a problem presented to students can be based on a context from the real world but this is not always necessary; the fantasy world of fairy tales and even the formal world of mathematics can offer other suitable contexts, provided they are real in the student's mind (Van den Heuvel-Panhuizen, 2000, 2003).

Freudenthal regarded mathematics not as a body of subject knowledge but as a human activity (Van den Heuvel-Panhuizen, 1996, 2000, 2003). In his opinion, the process of 'mathematizing,' that is, the activity of organizing matter from mathematics or from reality, should be the main focus in mathematics education (Van den Heuvel-Panhuizen, 1996, 2000, 2003). Within RME, instead of being receivers of mathematics as a closed structure, students are treated as active participants in the educational process, in which they develop a range of mathematical tools and concepts themselves. Freudenthal posited that mathematics lessons should give students the 'guided' opportunity to 're-invent' mathematics by doing it (Van den Heuvel-Panhuizen, 1996, 2000, 2003).

At the heart of Freudenthal's concept of learning mathematics, and closely related to the process of mathematizing, lies the level theory of learning (Van den Heuvel-Panhuizen, 1996, 2000, 2003). As students learn mathematics, they pass through various levels of understanding, 'from the ability to invent informal context-related solutions, to the creation of various levels of short cuts and schematizations, to the acquisition of insight into the underlying principles and the discernment of even broader relationships' (Van den Heuvel-Panhuizen, 2000, p. 5). The activity of mathematizing that occurs on one level can be subjected to analysis at the next level. As a result of this reflection, students' informal organizing activities gradually become more formal. Within RME, models are seen as important vehicles in eliciting and supporting the level raising process. In order to bridge the gap between informal context-related mathematics and more formal mathematics, models must be rooted in realistic, imaginable contexts and, at the same time, be flexible enough to be used also in higher levels of mathematization (Van den Heuvel-Panhuizen, 1996, 2000, 2003).

Another important characteristic of the RME approach is that 'mathematics, as a school subject, is not split into distinctive learning strands' (Van den Heuvel-Panhuizen, 2000, pp. 7-8). The solution of rich contextual problems often involves the application of a broad range of mathematical concepts. For example, calculating the cost of a car journey may incorporate aspects of measure as well as number. Van den Heuvel-Panhuizen (2000, p. 8) contends that this 'inter-twinement principle … renders coherency to the curriculum.'

Teachers and educational programs have a key role in how students learn mathematics in the RME approach (Van den Heuvel-Panhuizen, 2000, 2003). They 'steer the learning process, but not in a fixed way by demonstrating what the students have to learn' (Van den Heuvel-Panhuizen, 2000, p. 9). Teachers need to create a learning environment that enables students to develop mathematical tools and insights by themselves. Mathematics is considered to be a 'social activity' within RME (Van den Heuvel-Panhuizen, 2000, p. 8). By providing time for students to interact and discuss their ideas and strategies with one another, students have the opportunity to reflect upon their work, and this reflection enables them to progress to higher levels of understanding. There is therefore a strong preference for keeping the whole class together within RME lessons (Van den Heuvel-Panhuizen, 2000). This does not mean that all students do exactly the same work at the same time. Instead, students are considered as individuals, each on their own learning journey (Van den Heuvel-Panhuizen, 1996, 2000). By providing problem situations that can be solved on different levels of understanding, the education can be adapted to suit the range of abilities within the class. Clearly, the choice of problem scenario is of great significance. The context problems that are presented to students within RME programs allow the emergence of models and their further evolution to occur in a natural manner (Van den Heuvel-Panhuizen, 2000, 2003).

REFORM VIA TEXTBOOKS

The reform of mathematics education in the Netherlands has been largely stimulated by the development of textbooks which reflect the RME approach (Gravemeijer, 1994). Now, over three-quarters of Dutch primary schools use textbooks inspired by RME. The reform proceeded in an informal way without government influence; developers, researchers, teacher educators, school advisors and teachers collaborated to develop teaching activities and learning strands which were later included in the new textbooks (Gravemeijer, 1994; Van den Heuvel-Panhuizen, 2000). 'Different textbooks lead to different instruction, and different instruction leads to different learning results' (Gravemeijer, 1994, p. 137). Van den Heuvel-Panhuizen (2000, p. 10) believes that textbooks are the 'most important tools' guiding teaching content and teaching methods. However, Gravemeijer (1994) points out that reform in mathematics education does not depend solely on the introduction of new textbooks. Regardless of the detail within a textbook's guide, the actual instruction in the classroom remains the ultimate responsibility of the teacher. At the heart of RME is the idea that students should be given the opportunity to reinvent mathematics under the guidance of the teacher. This approach 'does not fit with the idea of a teacher proof curriculum' (Gravemeijer, 1994, p. 13). RME lessons are highly interactive, with the teacher building upon the ideas that the students bring to the fore. For this to happen, the teacher needs to interpret the textbook's guide flexibly and adapt the education to suit newly acquired insights and experiences.

THE DUTCH-ENGLISH COLLABORATION

In the 1990s, research and development teams from the FI and the UW collaborated over a period of 6 years to develop a comprehensive mathematics curriculum for American middle schools: Grades 5-8 (Romberg, 2001). The resulting curriculum was based on RME principles and became known as 'Mathematics in Context' (MiC; National Center for Research in Mathematical Sciences Education & Freudenthal Institute, 1997-1998). In 2003, The Centre for Mathematics Education at MMU purchased a set of MiC materials and trialled them with Year 7 classes in local schools as part of a research project funded by the Gatsby Foundation (Eade, Dickinson, Hough & Gough, 2006). A summary of the project's findings highlights positive outcomes for both teachers and students: students demonstrated a significant improvement in aspects of mathematical understanding, especially in relation to problem solving; and there was evidence that teachers were more focused on engaging students in discussion in order to develop mathematical thinking rather than simply demonstrating the mathematics to be learned (Anghileri, 2009). Following the success of this project, a team of mathematics educators at MMU, in collaboration with the FI, began, in 2007, to write 'Making Sense of Mathematics' (MSM). The MSM materials are based on the principles of RME and are intended for use with students working towards the new Foundation level GCSE. They have been developed in consultation with teachers and are currently being trialled in a number of schools in Manchester.

METHODOLOGY

The aim of the research is to investigate what can be learned from the implementation of RME principles within an English setting. A range of qualitative methods were used: an unstructured non-participant observation of an RME lesson in an English classroom followed by a semi-structured telephone interview with the class teacher.

The observation of the RME lesson in an English post-primary school was conducted in July 2008. The school was chosen because of its close links with MMU. It is an all-ability catholic college for 11-16 year olds in the Trafford local education authority. The college has received a Specialist Status Award for mathematics and computing and it was judged as outstanding in every area in its Ofsted inspection in 2008. The class teacher has been implementing RME principles in her teaching for about five years and has been involved in the MSM project from its inception. She was observed working with a low-ability Year 10 class on activities from a trial version of the MSM unit: 'Fitting In: Area and Volume.' The lesson lasted about one hour. There were nine students in the class and they had been working with the RME materials for one year. The purpose of the observation was to gather information about how the textbook, based on RME principles, was being used to develop students' mathematical knowledge, skills and understanding. Brief notes were recorded in situ and later analysed.

The class teacher was then interviewed in order to gain a deeper understanding of the impact of implementing the new materials. An interview schedule was prepared, based on the findings from the classroom observation of the RME lesson and from readings on the RME approach to mathematics education. The schedule consisted of a range of open-ended questions. The teacher gave her permission to have the interview recorded and she was presented with the questions a fortnight in advance. The interview took place in May

2009 and was digitally recorded. A written transcription was then prepared and emerging themes were identified. The following section sets out the key findings.

FINDINGS

The implementation of the RME approach in the post-primary class was initiated mainly through the introduction of MSM textbooks. Key differences between the MSM textbooks and the textbooks that the teacher was using prior to the project are identified below. The new textbooks required a radical change in teaching style and this is also explained. Finally, notable benefits and challenges of the new approach are highlighted.

Differences in textbooks

Perhaps the most significant difference between the MSM textbooks and the traditional English textbooks relates to the use of context. Most of the sections within the MSM textbooks are based on a realistic context and more time is devoted to developing the context used, with some questions based solely on contexts. This is hardly surprising, given that the 'reality' principle is one of the determining characteristics of the RME approach to mathematics education (Van den Heuvel-Panhuizen, 2000). Although real-life contexts do feature in English textbooks, they tend to be used to introduce a new topic or idea and very little reference is made to the actual context itself. As the teacher notes, "A lot of traditional textbooks pay lip-service to context really." Another difference relates to the way the textbooks are organised. For the most part, English textbooks are set out in separate sections with each section focusing on a particular mathematical topic. By contrast, the various sections within the MSM textbooks are based on a theme or context and tend to incorporate a broader range of topics, thus reflecting the RME principle of 'inter-twinement' (Van den Heuvel-Panhuizen, 2000). In the words of the teacher, students are "learning lots of different topics at the same time rather than sort of focusing on one topic and then shifting and focusing on another topic and so on. It's much more inter-related than that." Overall, the teacher believes that the RME textbooks are excellent. However, she points out that "it's really the style of teaching that goes with them that's so good … rather than the actual textbooks themselves." This is consistent with Gravemeijer's (1994) view that teachers cannot rely solely on new textbooks if they wish to improve their teaching.

A new style of teaching

The teacher recognises that the MSM materials offer a new approach to teaching mathematics and not simply new lesson content. Her classroom practice has "changed drastically" since involvement in the project. She describes the traditional English approach as "fast-paced" with a focus on delivery and covering as much material as possible at the expense of student understanding: "The aim is to *cover* as much material as possible, almost with less regard about real-life contexts or any kind of context that the pupils will understand." Now, she is much more conscious of the need to find out about students' prior understanding or experience at the beginning of each lesson: "I think I've realised that no matter how well I try to explain something, that explanation is pretty useless really unless we're talking on the same level to start with, unless there's some sort of common ground established beforehand." For example, in the lesson observed, a video clip of a builder at work was used to engage students in thinking about his everyday tasks and how they incorporated aspects of mathematics. The main focus was tiling and this was the realistic context for the work on area which was to follow.

One of the most important revelations for the teacher is that "pupils learn maths from looking at the world around them … [they] don't learn maths and then apply it to the world." She has come to appreciate the value of "carefully chosen contexts." In traditional lessons on area, students are taught the formula for finding area; once they grasp the formula, they are then given various problems to solve which require the application of the formula to a real-life situation, such as finding the area of a garden or a car park. By contrast, in the lesson observed, students were presented with real-life tiling patterns as a starting point from which they would gradually develop formal methods for finding area. Initially, students simply counted the squares covered in diagrams of tiling patterns. As the lesson progressed, they developed their own informal strategies. This is an illustration of the RME 'activity' principle at work (Van den Heuvel-Panhuizen, 2000). Although the same problems were given to all students, they could be solved on different levels of understanding. Some students relied on drawing the squares inside their shapes to calculate the total area; others used multiplication as a short cut to determine the area of each individual part before finding the overall total. This demonstrates the preference in RME for keeping the whole class together and at the same time adapting the education to suit the different levels of ability (Van den Heuvel-Panhuizen, 2000).

While whole-class discussion had always played an important part in her mathematics lessons, the teacher finds that this has become much more central since involvement in the project. Furthermore, there has been a significant change in the nature of this discussion. Previously, the classroom talk was oriented around students explaining a taught method; now it's more about students explaining their own ideas and understandings. The 'interaction' principle (Van den Heuvel-Panhuizen, 2000), another characteristic of RME lessons, was clearly evidenced throughout the lesson observed. Students were encouraged to share and discuss their own strategies for calculating the area of various composite shapes. This interaction evoked reflection which in turn enabled students to progress to higher levels of understanding as they developed more formal methods for calculating area, an illustration of the 'level' principle (Van den Heuvel-Panhuizen, 2000) of RME.

Since the solution of many RME contextual problems involve the application of a broad range of mathematical tools and concepts (Van den Heuvel-Panhuizen, 2000), the teacher finds that she has developed a greater appreciation of the connections between the various topics in mathematics. Previously, she tended to teach topics in isolation, according to the way they are set out in the traditional English textbooks, but now she spends more time highlighting the connections between the different topics in mathematics.

Overall, there was clear evidence of the RME 'guidance' principle (Van den Heuvel-Panhuizen, 2000) in the lesson observed, through the combination of rich contextual problems and a learning environment which allowed students room to develop ideas and insights related to area themselves.

Benefits

The teacher feels that it is too early to judge whether there has been a significant increase in student achievement as a result of using the MSM textbooks. She believes that students would need to be exposed to the RME approach from a younger age if there is to be a positive impact on exam results. However, she feels that there are many other notable benefits of this approach. She has observed a significant increase in student confidence, claiming that students are no longer frightened to participate in mathematics lessons. Since students can "access the context", they are not "put off by the fact that they can't do the maths straight away." For her, teaching mathematics is much more rewarding now because "pupils are learning things or realising things for themselves." She finds that students are more willing to tackle new problems and do not depend upon having been taught a set rule or procedure in the way they used to: "They don't need to only have seen things and have done similar examples before to be able to attempt things." She also notices an improvement in student engagement: "I think their interest levels are easier to maintain, like proper interest, rather than just superficial good behaviour. They're actually much more interested in what's happening in the lessons." Another reported benefit, she feels, is that students are more aware of mathematics in the world: "It's not something that just happens in the classroom; it actually happens out there in all sorts of areas of life." Given that mathematics topics are inter-related throughout the MSM textbooks, the teacher feels that students are also able to see the connections between the different areas of mathematics more clearly now.

Challenges

The implementation of the new approach to mathematics education has not been without a number of challenges. Despite the many benefits outlined, the teacher states that the RME approach is "not an easy fix at all." She admits that this approach is very demanding both inside and outside the classroom. Even though the MSM textbooks are very clearly set out and provide detailed guidance for teachers, she suggests that RME lessons need more preparation than traditional lessons. Behaviour management can also be more demanding at times, given the increased emphasis on whole class interaction and discussion. In the teacher's words, "Anybody that goes into it thinking that it's going to be an easier life is actually going to be quite surprised." She adds that it is not something that a teacher can partially commit to: "It's all or nothing in many respects in that you can't really dip in and out of it."

There was some initial concern that the new approach might have a negative impact on student achievement. Given that the pace of work within the MSM textbooks is much slower, the teacher worried that the more able students "weren't getting through the work fast enough … or covering the topics." However, as time went on, she realised that the various mathematical topics were still being addressed; it's just that the teaching approach and the order in which they were taught had changed. She recalls that students in the higher ability classes also found the new approach quite frustrating and felt that what they were being asked to do was quite basic: "They wanted the quick fixes that they were used to … they wanted the prescribed methods for things and they wanted to be able to get quick answers and that doesn't happen with RME." However, she believes that the work is actually much more challenging in terms of what the students are expected to do.

The teacher admits that, in the early stages of the project, she was wary about introducing an approach which incorporated a broader range of topics within one lesson. She was concerned that students might become confused if they were exposed to different topics and experiences at the same time. However, she observes, "Through this I've realised that the more experience they can get, the less confused they become, the more they can make sense of things and order things themselves and make links between things."

The use of visual models to support the progression from informal methods to formal methods is an important characteristic of the RME approach. However, there was an ongoing tension between drawing neat and accurate models in student exercise books and drawing rough sketches of a visual model which is essentially meant to be a 'thinking tool.' Perhaps this is not an issue in the Netherlands, but student exercise books may be examined as part of the process of government inspection of schools in England. Ensuring that students record their work neatly was an important focus within the school visited. When implementing the RME approach within her classroom practice, the teacher found that student exercise books became more "notebook like" and they were "a bit messier." For example, in the lesson observed, students did not rely on rulers to draw the various shapes; instead, they sketched a 'model' of each tiling pattern and then added lines where appropriate as they partitioned each shape in order to determine the overall area.

The teacher highlighted the need for adequate support and training in introducing the MSM textbooks. She found the training days and twilight sessions at MMU extremely beneficial as she implemented the new approach in her classroom practice. They provided a forum for the discussion of progress and, more importantly, an opportunity for teachers to find help and reassurance: "Teachers need reassurance that the problems they're encountering are the problems everybody encounters and the tensions they feel are the tensions that everybody has." She said it was important to be reassured that they were "doing the right thing" and that "pupils will get there in the end."

CONCLUSION

Curriculum materials which reflect the key principles of RME are seen as important vehicles in improving the landscape of mathematics education, not just in the Netherlands where the theory was developed but also in England. The study of the impact of introducing RME textbooks in one post-primary school in Manchester highlights a number of important benefits for both teacher and students and demonstrates that cross-national curriculum collaboration is possible. The recently revised NI primary curriculum (CCEA, 2007), with its emphasis on the use and application of mathematics in a range of contexts, particularly in real-life situations, echoes Freudenthal's (1968, p. 3) view that mathematics should be taught 'so as to be useful.' Further, the importance the NI primary curriculum (CCEA, 2007) places upon introducing mathematics through meaningful contexts is consistent with the RME 'reality principle' which relates also to Freudenthal's belief that learning mathematics should originate in reality (Van den Heuvel-Panhuizen, 2000). Given that Gravemeijer (1994) points out the need for appropriate instruction materials when implementing new ideas and approaches, a pilot of RME textbooks in NI primary schools is considered worthwhile. However, it is worth emphasising that improvement in mathematics education does not depend on textbooks alone (Gravemeijer, 1994). The lessons learned from the Dutch-English collaboration highlight the need for teacher support and training throughout such a pilot project. In the teacher's words, initially it may feel like "hitting your head against a brick wall" but with hard work and perseverance the benefits are worth pursuing. With appropriate materials and an effective support system in place, a reform of mathematics education in NI is clearly achievable.

THANKS AND ACKNOWLEDGEMENTS

The visit to MMU was made possible by the generous travel scholarship awarded by the Universities Council for the Education of Teachers (UCET). Thanks are also due to Anna Jordan from Blessed Thomas Holford Catholic College in Altrincham, Greater Manchester, for permitting an observation of one of her mathematics lessons and for agreeing to participate in an interview.

REFERENCES

Anghileri, J. (2009). *Gatsby Project: Mathematics in Context – Summary evaluation report.*

Council for the Curriculum, Examinations and Assessment (CCEA). (2007). *The Northern Ireland curriculum: Primary.* Belfast: CCEA.

Eade, F., Dickinson, P., Hough, S. & Gough, S. (2006). *Developing mathematics through context: Year 2. Annual report.* Retrieved July, 7, 2008 http://www.partnership.mmu.ac.uk/cme/DMtC/Updates/ReportAnnual2006.html

Freudenthal H. (1968). Why to teach mathematics so as to be useful. *Educational Studies in Mathematics, 1,* 3-8.

Gravemeijer, K.P.E. (1994). *Developing realistic mathematics dducation.* Utrecht: CD-β Press / Freudenthal Institute, Utrecht University.

National Center for Research in Mathematical Sciences Education & Freudenthal Institute (Eds.). (1997-1998). *Mathematics in Context.* Chicago: Encyclopaedia Britannica.

Romberg, T.A. (2001). *Designing middle-school mathematics materials using problems set in context to help students progress from informal to formal mathematical reasoning.* Retrieved July, 7, 2008 http://www.partnership.mmu.ac.uk/cme/DMtC/Updates/ReportAnnual2006.html

Van den Heuvel-Panhuizen, M. 1996). *Assessment and realistic mathematics education.* Utrecht: CD-β Press / Freudenthal Institute, Utrecht University.

Van den Heuvel-Panhuizen, M. (2000). *Mathematics education in the Netherlands: A guided tour. Freudenthal Institute Cd-rom for ICME9.* Utrecht: Utrecht University.

Van den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics, 54* (1), 9-35.

ASSESSING THE LEVEL OF SUITABLY QUALIFIED TEACHERS TEACHING MATHEMATICS AT POST-PRIMARY EDUCATION IN IRELAND

Máire Ní Ríordáin — NCE-MSTL, UL
Ailish Hannigan — University of Limerick

Widespread coverage in the national media has highlighted the underperformance of Irish second level students in mathematics. However, little research has been undertaken to investigate issues of causality in relation to this decline. Smith (2004) emphasises that an adequate supply of suitably qualified mathematics teachers is an essential prerequisite for delivering long term improvements at post-primary mathematics education. Thus, the aim of this study is to investigate the level of out-of-field teaching occurring in Irish second level mathematics classrooms and to assess the type of second level schools in which this is dominant. The sampling frame consisted of a list of all 731 post-primary schools in Ireland (Dept. of Education website, November 2008). 12.5% of these schools are community/ comprehensive schools, 34.5% are vocational schools and the remaining 53.2% are secondary schools. A stratified random sample of 60 schools was selected so that the sample of 60 has approximately the same proportions of the different types of schools as the population. There are 30,000 students in the schools selected. 51 schools responded with 324 teachers of mathematics completing the questionnaire. There are 26,634 students in the schools who responded. Some preliminary findings are presented and discussed.

INTRODUCTION

Widespread coverage in the national media has highlighted the underperformance of second level students in mathematics and the low uptake of Higher Level mathematics at Senior Cycle education (EGFSN, 2008). In particular, performance in the Leaving Certificate examinations has been subjected to scrutiny, with growing concerns for the number of students failing Ordinary level mathematics, and thus restricting their opportunities for further education and training. However, little research has been undertaken to investigate issues of causality in relation to the decline in mathematics in Irish post-primary education. The authors propose that out-of-field teaching may be one of the key influences on students' poor performance in mathematics and the low uptake of the subject at Higher Level. Smith (2004) emphasises that an adequate supply of suitably qualified mathematics teachers is an essential prerequisite for delivering long term improvements at second level mathematics education. This research seeks to establish if there is a need to address and improve teacher qualifications in order to improve mathematics education at post-primary level.

OUT-OF-FIELD TEACHING

The definition of out-of-field teaching employed in this study is that of 'teachers assigned by school administrators to teach subjects which do not match their training or education' (Ingersoll, 2002, p.5). These teachers generally possess a teaching qualification but will have little or no training or education in the area of mathematics education. The Teaching Council of Ireland has been established since 2006 in order to promote teaching as a profession and to regulate standards within the profession. In order to teach mathematics in a post-primary school in Ireland, they stipulate that teachers must:

- Have studied Mathematics as a major subject in the degree extending over at least three years and of the order of 30% at a minimum of that period
- Provide details of the degree course content to show that the breadth and depth of the syllabi undertaken are such as to ensure competence to teach Mathematics to the highest level in post-primary education
- Provide explicit evidence of standards achieved in degree studies in Mathematics with at least an overall Pass result in the examinations in Mathematics

(Teaching Council, 2009)

However, this may not be enforced by school principals. For example in the Irish context many qualified science teachers are employed to teach mathematics to Junior Cycle level but their degree/post-graduate studies does not contain sufficient mathematics, any mathematics pedagogy, or explicit instruction and training on how to teach mathematics. Current international research advocates that one of the causes of inadequate student achievement in mathematics is the failure of schools to assign suitably qualified teachers to appropriate subject areas relevant to their undergraduate and post-graduate qualifications (Darling-Hammond, 1999; Elmore & Fuhrman, 1995; Haycock, 1998; National Commission on Excellence in Education, 1983). The Chief Inspector's report in the UK (2001/'02) found that the quality of mathematics teaching is suffering in many schools due to the limited number of specialist mathematics teachers whose expertise is usually assigned to A-level courses (equivalent to Senior Cycle in Ireland). As a consequence, non-specialist mathematics teachers are assigned to Key Stages 3 and 4 (equivalent to Junior Cycle in Ireland), where they often fail to respond to students' mathematical learning needs. Consequently it is having an adverse effect on students' performance in mathematics.

METHODOLOGY

This investigation is quantitative in nature and a preliminary statistical analysis of the data has been undertaken by the investigators. A questionnaire was designed to assess each teacher's undergraduate and post-graduate qualifications, number of years teaching experience of mathematics and other subject areas, the year group(s) being taught mathematics by the teacher and level of mathematics (Higher, Ordinary, Foundation), and the number of students in each of the teacher's mathematics classes. A number of open ended questions were also included in order to examine the training/qualification needs of all teachers engaged in mathematics teaching in second level schools. The sampling frame for this study was a list of all 731 post primary schools in Ireland (Dept. of Education website, November 2008). 12.5% of these schools are community schools, 34.3% are vocational schools and the remaining 53.2% are secondary schools. The targeted sample size was approximately 400 mathematics teachers giving a margin of error for the estimate of the percentage of unqualified mathematics teachers of ±5%, with a 95% confidence level. Using an estimate of an average of seven mathematics teachers in each school, a stratified random sample of 60 schools was selected so that the sample of 60 has approximately the same proportions of the different types of schools as the population. There are 30,000 students in the schools selected.

To date, 51 schools (85% of the targeted sample) have responded, with 324 questionnaires returned from teachers teaching mathematics in these schools. There are 26,634 students in the schools who responded with a median of 463 students and a range of 69 to 1230 students in each school. The number of teachers who returned the questionnaire in each school ranged from 2 to 14 teachers with a median of 6 teachers. 51% of the teachers taught in secondary schools, 35% in vocational schools and the remaining 14% in community schools. Two thirds of the teachers were full time teachers, a quarter of the teachers were full-time but only employed during the school year and 10% of the teachers worked part-time. 53% of the teachers were female. A bar chart of the age of the teachers is given in Figure 1. 71% of the teachers are aged 40 or under.

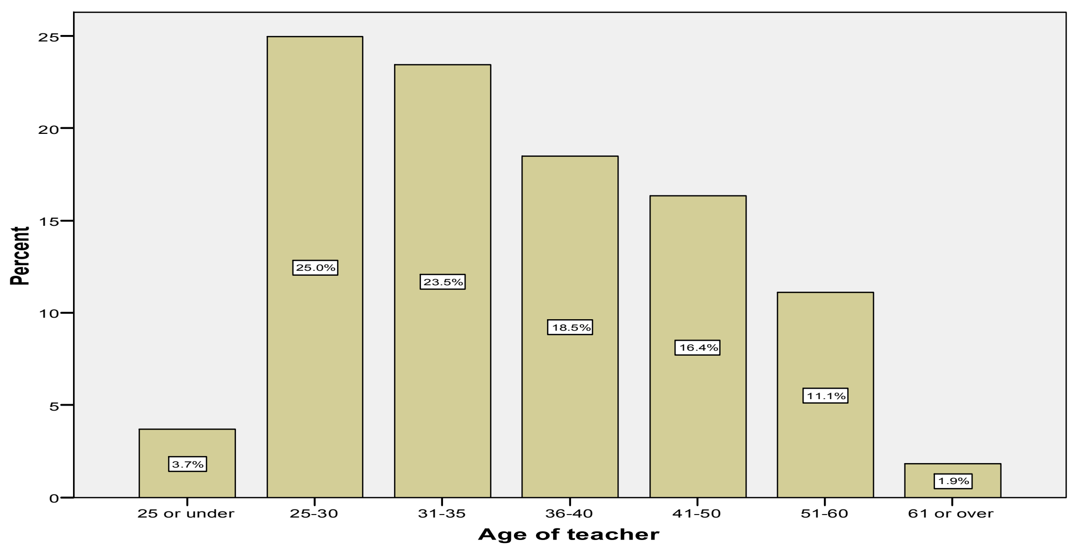


Figure 1: Percentage of teachers in each age category (n=324)

FINDINGS

A preliminary analysis of the data was undertaken and the findings will be discussed in the following subsections.

Experience of Teaching Mathematics at Post-Primary Education

62% of the teachers had been teaching at second level for 10 or more years. A similar percentage had been teaching mathematics at second level for 10 or more years. A bar chart of the categories of years of experience teaching mathematics at post-primary is provided in Figure 2.

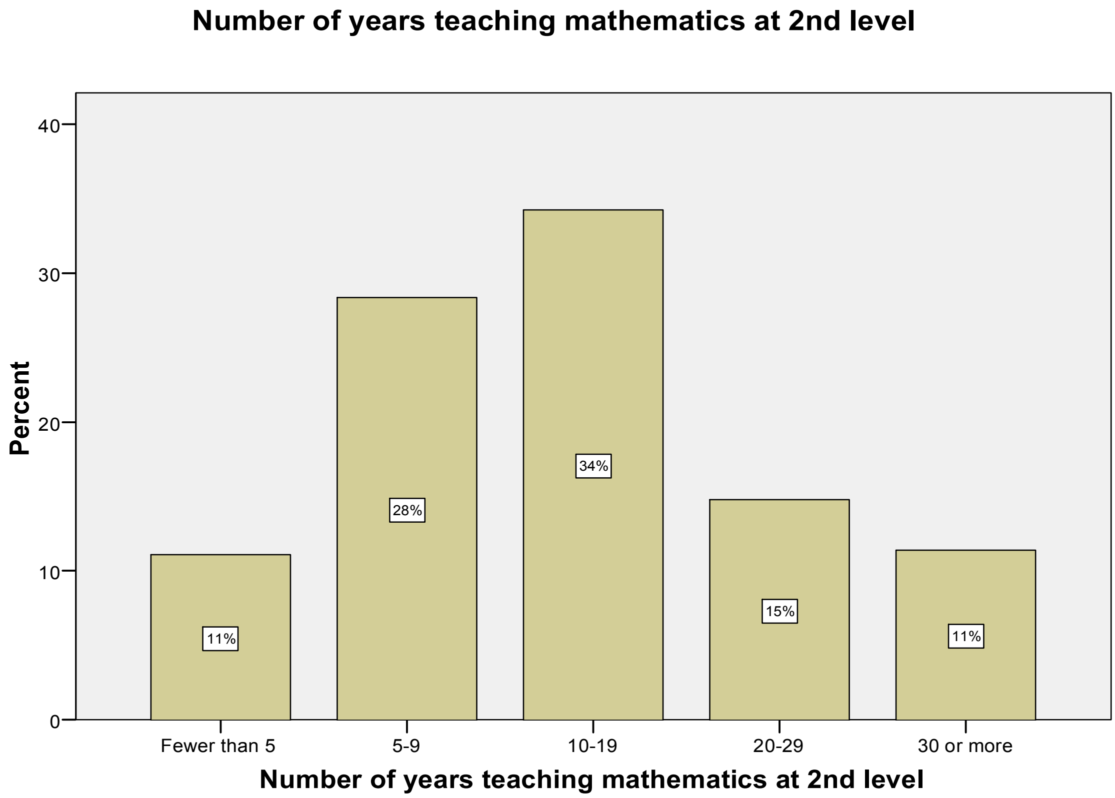


Figure 2: Bar chart of categories of years of experience teaching mathematics (n=324)

The teachers taught an average of 10 hours of mathematics a week with a range of 1 to 22 hours. 25% of teacher taught less than 7 hours of mathematics a week. The average number of mathematics classes taught by the teachers was 15 classes with a range of 2 to 36. 25% of teachers taught 10 classes or less of mathematics each week. The most popular subjects for the teachers to teach with mathematics were science (33%), Business Studies (18%), Biology (15%), Resource (14%), Chemistry (13%), CSPE or SPHE

(13%), ICT (12%), Physics (11%) and Accounting (11%). All other subjects were taught by less than 10% of the teachers in the sample. 90% of the teachers said they enjoyed teaching mathematics at second level. Of those who gave further information on why they enjoyed teaching mathematics (n=153, 47% of the sample), 44% said they loved the subject and helping students, 27% said they enjoy when classes are motivated and 19% said it was challenging.

Teacher Qualification

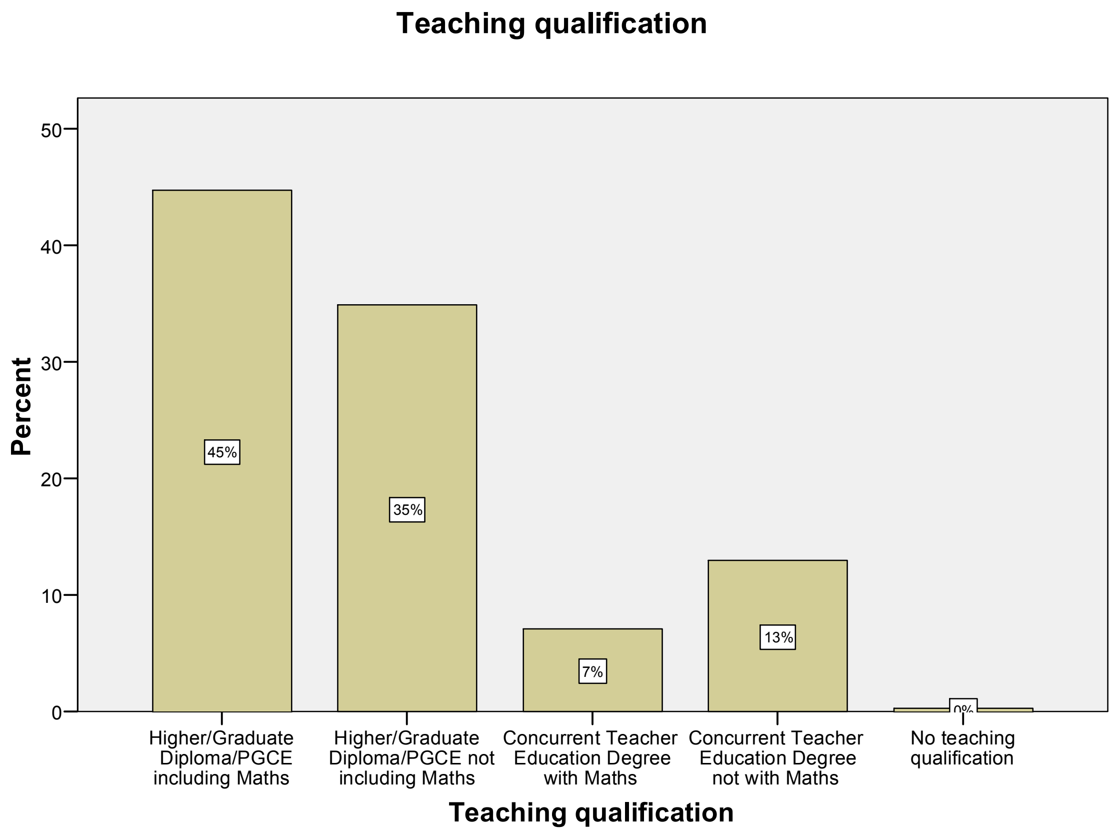


Figure 3: Bar chart of types of teaching qualification (n=324)

Only 1 of the 324 teachers did not have a teaching qualification though 48% of the teachers did not have a mathematics teaching qualification. A bar chart of the type of teaching qualification obtained by the teachers is given in Figure 3. Of the 156 (48%) of teachers without a mathematics teaching qualification, 35% had a BSc. primary degree, 34% had a B. Commerce /Business primary degree and 27% had a concurrent teacher education degree without mathematics. Of the 168 teachers with a mathematics teaching qualification, 73% had a BA/BSc. with maths primary degree, 14% had a concurrent teacher education degree with maths and 11% had a BSc. primary degree.

The highest qualification obtained by 18% of the teachers was a degree (58 of the 59 teachers in this category had a degree in Education). Almost two-thirds of the teachers had a Higher Diploma as well as their primary degree. The remaining 16% of teachers had a Grad Dip/Masters/PhD. Only 4% of the teachers were currently undertaking a further qualification. 45% of the teachers had a significant mathematics content in their primary degree (BSc. with mathematics, BA with mathematics or a mathematics education degree). Overall, 78% of teachers felt that their qualifications were adequate for preparing them to teach mathematics at second level. All the teachers with a concurrent teacher education degree with maths felt their qualifications were adequate for preparing them to teach mathematics at second level compared to 46% of those with a concurrent teacher education degree without maths. Table 1 summarises the percentage of those who felt their qualifications were adequate by type of teaching qualification.

	Type of teaching qualification				
Qualification adequate?	H Dip/ Grad Dip/ PGCE with maths	H Dip/ Grad Dip/ PGCE without maths	Concurrent teacher education degree with maths	Concurrent teacher education degree without maths	None
Yes	131 (90%)	79 (70%)	23 (100%)	19 (46%)	0 (0%)
No	14 (10%)	34 (30%)	0 (0%)	22 (54%)	1 (100%)

Table 1: Type of teaching qualification by whether the teachers felt it was adequate in preparing them to teach mathematics at second level

Table 2 summarises the years taught by the teachers in second level by whether or not they had a teaching qualification in mathematics. Teachers with a teaching qualification in mathematics are more likely to be teaching students in the examination year of the Junior Cycle (Third year) and the Senior Cycle. Teachers without a teaching qualification in mathematics are more likely to teach students in the Junior Cycle.

Teaching Qual. in maths	Year of study					
	First	**Second**	**Third**	**Fourth**	**Fifth**	**Sixth**
Yes (n=168)	85 (51%)	100 (60%)	134 (80%)	95 (57%)	133 (79%)	131 (78%)
No (n=156)	81 (52%)	94 (60%)	79 (51%)	18 (12%)	45 (29%)	38 (24%)

Table 2: Numbers teaching in each year by whether or not they had a teaching qualification in mathematics (% of total in teaching qualification category).

Teacher and School Characteristics and Type of Teaching Qualification Obtained

The percentage of male and female teachers with a teaching qualification in mathematics was similar i.e. 54% of male teachers and 50% of female teachers. Oder teachers tended to be more likely to have a teaching qualification in mathematics. Only 40% of the teachers aged 35 or under had a teaching qualification in mathematics compared to 65% of the teachers aged over 35. Table 3 examines the relationship between age and whether or not the teacher had a teaching qualification in mathematics.

	Teaching qualification in mathematics?	
Age group	Yes	No
Under 25 (n=12)	4 (33%)	8 (67%)
25-30 (n=81)	31 (38%)	50 (62%)
31-35 (n=76)	33 (43%)	43 (57%)
36-40 (n=60)	34 (57%)	26 (43%)
41-50 (n=53)	41 (77%)	12 (23%)
51-60 (n=36)	22 (61%)	14 (39%)
61 or over (n=6)	3 (50%)	3 (50%)

Table 3: Numbers in each group by whether or not they had a teaching qualification in mathematics (% of total in age group).

There is no significant relationship between type of school and whether or not the teacher had a teaching qualification in maths though vocational schools tender to be slightly more likely to have qualified teachers. Care should be taken when interpreting that result because 5 (24%) of the 21 vocational schools, selected in the sample, did not respond compared to 3 (9%) of the 32 secondary schools selected and 1 (14%) of the 7 community or comprehensive schools selected. Table 4 examines the relationship between type of school and whether or not the teacher had a teaching qualification in mathematics.

	Teaching qualification in mathematics?	
Type of school	No	Yes
Secondary (n=165)	83 (50%)	82 (50%)
Community or comprehensive (n=44)	23 (52%)	21 (48%)
Vocational (n=115)	50 (44%)	65 (56%)

Table 3: Numbers in each type of school by whether or not they had a teaching qualification in mathematics (% of total in school type).

DISCUSSION OF FINDINGS

The findings presented here are intended to provide a possible explanation for the poor performance in mathematics of second level students in Ireland. Naturally, there are many other factors at play (e.g. attitudes, motivation, learning difficulties, etc.) but the authors strongly feel that specifically qualified mathematics teachers play a significant role in improving the quality of mathematics learning at second level education and thus is one of the key reasons for undertaking this study. One of the significant findings emerging from this research is that nearly half (48%) of the teachers teaching mathematics in our post-primary schools are not qualified mathematics teachers. The majority of these teachers are qualified science and business studies teachers and appear to be assigned mathematics classes, perhaps due to timetabling and staffing issues. The number of unqualified mathematics teachers is considerably lower than the estimated 80% as suggested by the Royal Irish Academy (2008), so this is positive in terms of the numbers requiring further training and staffing considerations.

However, there is no doubt that a significant number of our second level students are being taught mathematics by out-of-field teachers. This is particularly prominent at the Junior Cycle level, during the first and second years of study. Concern also lies in the fact that a considerable number (63%) of these out-of-field teachers feel that they are suitably 'qualified' to teach mathematics even though their degrees and postgraduate qualifications do not contain sufficient mathematics nor a qualification to teach the subject. The out-of-field teachers who had pursued a concurrent teacher education degree (e.g. science education) were more conscious of their inadequacies to teach mathematics, which may be reflective of the type of degree they undertook and the explicit instruction they received on the teaching of their specific subject area. The qualified mathematics teachers are predominantly assigned the exam years (3rd and 6th year) and the Senior Cycle mathematics classes. Clearly this demonstrates that school principals are acutely aware that mathematical qualifications and experience are important in the teaching of mathematics but this importance is placed on exam years/Senior Cycle. This, unsurprisingly, is reflective of our exam orientated post-primary system and the norm of teaching towards the exam (NCCA, 2006). The irony lays in the reality that specialist mathematics teachers should be employed at the Junior Cycle classes so to ensure sufficient mathematical skills/concepts are developed at the early stages and to lay the foundation for further study in mathematics at second and third level education (Smith, 2004).

Another consideration is that the qualified mathematics teachers in this study are older and more experienced than the out-of-field teachers. This is perhaps reflected in that mathematics teaching was valued as a career option after undertaking a third level qualification in mathematics, up to the 1990's. Whereas now, students who pursue mathematics or courses with a significant mathematical content at third level education have more 'appealing' career opportunities available to them e.g. finance, accounting, computing, engineering, etc. Therefore, concerns for the future of securing suitably qualified mathematics teachers at second level education is an issue we must address in our education system and entice suitably qualified mathematics graduates into the field.

CONCLUSION

This is the first time a study of this type into the nature of out-of-field teaching at second level education has been undertaken in the Irish context. This paper presents some of the insights emerging from the initial data analysis in order to generate awareness amongst practitioners in the Irish context. This study was undertaken not to draw attention to the negatives but rather to assess what needs addressing most urgently within teaching of mathematics at second level education. Out-of-field teaching in mathematics is significant and clearly needs to be dealt with in the Irish context. Many of these teachers enjoy teaching mathematics and this needs to be fostered. Also, school logistics relies on these teachers undertaking mathematics teaching in order to facilitate timetabling and staffing issues. Thus concern now lies with addressing the training needs of these out-of-field teachers (e.g. introducing training and qualification courses/CPD). What these teachers lack is sufficient mathematical knowledge and pedagogical knowledge of how best to teach the subject. New directives need to be targeted towards this cohort of mathematics teachers. In turn, improved student learning and achievement in mathematics may be achieved.

REFERENCES

Expert Group on Future Skills Needs (EGFSN, 2008) *Statement on Raising National Mathematical Achievement*, Dublin: EGFSN.

Darling-Hammond, L. (1999). *Teacher quality and student achievement: A review of state policy evidence.* Center for the Study of Teaching and Policy, Seattle, WA: University of Washington.

Department for Education and Skills, (2002). *Chief Inspector's Report for 2001/2*, HMI 0-10-292032.

Department of Education and Science (2008). *List of Post-Primary Schools.* Retrieved November 16, 2008 http://www.education.ie/servlet/blobservlet/ppschools.xls

Elmore, R. & Fuhrman, S. (1995). Opportunity to learn and the state role in education. *Teachers' College Record*, 96(3), 433-458.

Haycock, K. (1998). Good teaching matters….a lot. *Thinking k-16: A Publication of the Education Trust*, 3(2), 3B14.

Ingersoll, R. M. (2002). *Out-of-Field Teaching, Educational Inequality and the Organisation of Schools: An Exploratory Analysis.* Center for the Study of Teaching and Policy, Seattle, WA: University of Washington.

NCCA (2006), *Review of Mathematics in Post-Primary Education – Report on Consultation*, Dublin: NCCA.

National Commission on Excellence in Education (1983). *A Nation at risk: The imperative for educational reform.* Washington, DC: Government Printing Office.

Royal Irish Academy Committee of Mathematical Sciences and Chemical & Physical Sciences (2008), *Response to the Proposal to offer Bonus Points for Maths*, Dublin: RIA.

Smith, A. (2004). *Making Mathematics Count: The Report of Professor Adrian Smith's Inquiry into Post-14 Mathematics Education,* London.

The Teaching Council (2009). *General and Special Requirements for Teachers of Recognised Subjects in Mainstream Post-Primary Education,* Retrieved April 23, 2009 http://www.teachingcouncil.ie/_fileupload/TC_ProcDocs/General_And_Special_Requirements_19282059.doc

ASSESSING THE EFFECT OF MATHEMATICS TEXTBOOK CONTENT STRUCTURE ON STUDENT COMPREHENSION AND MOTIVATION

Lisa O'Keeffe and John O'Donoghue

NCE-MSTL, University of Limerick

Much research on the influence of mathematical textbooks has been carried out internationally, with Irish textbooks only playing a minor role in the TIMSS analysis (Valverde et al, 2002). As part of her doctoral studies the author is investigating the impact of Irish Junior Cycle Mathematical Textbooks on mathematics learning and understanding. This research includes an analysis of four textbook series from a number of different perspectives. One such perspective focuses on the structure of the content of each textbook and this is examined in this paper. The Rivers Matrix (Rivers, 1990) method combined with the content structure grids used by the TIMSS analysis (Valverde et al, 2002), were employed by the author to examine each of the four textbook series in order to identify the key areas that can enhance students mathematical comprehension and motivation (Rivers 1990.) . Through an analysis and comparison of current Junior Cycle mathematics textbooks this paper reports on the effectiveness of their content structure for student comprehension and motivation to engage in mathematical learning and understanding. The research presented here is the first of its kind specifically addressing the influence of mathematics textbooks on teaching and student learning. It is expected the findings will provide significant insights into to the teaching of mathematics at Junior Cycle.

INTRODUCTION

Textbooks are widely accepted as a commonly used mathematical resource. O'Keeffe, (2007) noted that over 75% of Irish secondary school teachers use a textbook on a daily basis. Yet despite a minor inclusion in the TIMSS Report (Valverde et al, 2002) no Irish research has been carried out to evaluate the effectiveness of the textbooks currently in use. This lack of research from an Irish perspective justifies an analysis of the current mathematical textbooks employed at second level. When the current Leaving Certificate curriculum was introduced in 1992 it specified an aim of increasing the uptake of Higher Level students to 25%, however in 2008 only 17% of Leaving Certificate students attempted the Higher Level examination. This trend is echoed at Junior Cycle; whereby the recommended uptake of Higher Level is 60% but unfortunately only 43% of students in 2008 sat the Higher Level paper. In order to maximise the uptake of Higher Level Mathematics at Leaving Certificate it is vital and obvious that the initial problem to be tackled is increasing uptake in the Higher paper at Junior Certificate (EGFSN, 2008).

Over reliance on inappropriate textbooks is thought to contribute to this problem and needs to be addressed at this level.

The NCCA consulted with a number of interested parties, such as 2nd and 3rd level educators to identify and discuss recommendations for enhancing the proficiency of mathematics in Ireland (NCCA, 2006). As part of this report the NCCA put forward a number of suggested improvements, one of these being to improve the textbooks and available resources. 51% of those involved felt that this would be very effective at enhancing mathematics proficiency, 42% felt it would be effective with 7% stating it would be ineffective. Over all 90% of those involved felt strongly about the effective role an improved textbook could play in pupil learning (NCCA, 2006). With the aim of improving Junior Cycle mathematics teaching and learning, the author has analysed four Junior Cycle Mathematics textbook series. The initial framework for analysis is based on the TIMSS Report (2002) with further more in-depth and specific frameworks being incorporated throughout the various aspects of the review. The aim of this review is to identify elements of the textbook which can be improved in order to enhance both teaching and learning.

RESEARCH METHODOLOGY

The four major Mathematics textbook series currently used in Irish second level schools were indentified and subjected to detailed analysis by the author; eight textbooks were included in total. In order to analyse the effectiveness of these textbooks for pupil motivation and comprehension, the Rivers Matrix (1990) was applied, together with the TIMSS framework (Valverde et al, 2002). The River's framework examines the structures in place in the current textbooks based on the following factors:

- Motivational factor - historical notes, scientist and mathematician biographies, career info, applications and photographs
- Comprehension cues - colour and graphics
- Technical Aids - inclusion of material related to calculators and computers
- Philosophical Position - emphasis and predominant philosophy

The aim of this research is to assess the effect of these textbooks on student motivation and comprehension. As evident above, motivation and comprehension both appear as measureable strands. However the role of technical aids and the textbook philosophy both play an equally important role in the overall affect on pupil motivation and comprehension.

The author analysed each textbook on a page by page basis. Prior to commencing the analysis a number of definitions were required in order to avoid ambiguity:

- Firstly a problem is defined as a "situation in which a goal is to be attained and a direct route to the goal is blocked, [which] usually requires the presence of a person who has the problem" (Kilpatrick, 1985:2)

- A number of diagrams assisting one question all count as only one diagram as they have only one purpose.

- Theorems were counted as being examples.

FRAMEWORK FOR THE TEXTBOOK ANALYSIS

The TIMSS framework for the analysis of textbooks has three main features (TIMMS, 2002): **Structure**, **Content,** and **Expectation**. **Language & Readability** has also emerged as a significant research area for mathematics education and mathematics textbook research.

This paper examines how one element of the overall research project, the structure of 'Textbook Content', can influence the selection of Mathematics content and emphasis adopted by mathematics teachers and students, and consequently the impact on mathematical learning outcomes. The structure of the content is analysed based on the Rivers Matrix (Rivers, 1990) which has previously been outlined and will now be discussed in detail. The matrix identifies four main elements of analysis:

- Motivational factors
- Comprehension cues
- Technical Aids
- Philosophical Position

Motivation:

This matrix has been used by Mikk (2000:129-131) in his work when he combines and examines a number of methods for textbook analysis. Mikk notes how previous frameworks developed for content structure analysis are similar to that created by Rivers or at the very least echo the same ideals, such as that put forward as far back as 1969 by Gerbner. According to Mikk (2000) new information evokes pupil interest. However, over use of new information can intimidate and evoke fear in pupils thus causing a decline in interest. Mikk (2000) suggests that textbooks should include approx 30-40% new information and no more if they wish to engage and motivate pupils. The reality is that the majority of textbooks currently comprise of approx 70-80% new information (Mikk, 2000). On the other hand, motivation can be increased by including:

- Historical Data: make reference to mathematical discoveries and applications.

- Practical Implications: provide concrete examples as opposed to theoretical explanations.

- Inclusion of Problems: such as problematic situations which motivate students; the textbook should include questions aimed at reproducing, analysing and applying information, and unanswered questions, and provide options before solutions.
- Humour: Directly address the reader, use proverbs, riddles, jokes etc.
- Figurative Representations: provide illustrations and mental images.
- Narrate about People: (interlinked with historical data) provide information on mathematics, talk about modern people for realistic problems.

Comprehension can be greatly influenced by the colour, layout and inclusion of graphics within a text (Dowling, 1996). The combination of colour and graphics with the motivational factors can greatly enhance pupil comprehension.

Technical Aids:

By encouraging and evoking the use of technical aids e.g. computers and calculators; the textbook can help to broaden the learning experience of pupils. The rapid evolution of technology and its potential for use in Mathematics teaching and learning suggest a strong case for incorporating the use of ICT into mathematics teaching and learning (Mackie & Scott, 1988). Yet few mathematics teachers take this opportunity to maximise learning outcomes. Duffy and O'Donoghue (1992) suggested that many Irish Mathematics teacher neglect technology, as the novelty of computer based learning has worn off.

Philosophical Orientation:

The Philosophical orientation of a textbook is concerned with the aims, and the nature of the learning and teaching of mathematics. Ernest (1994) discusses the new directions in mathematical philosophy, the key elements of modern mathematics which influence this and the areas which are often omitted by mathematical philosophers. However, simply put, Ernest's (1994) work can be summarised as follows; the philosophy of mathematics considers the language of a text, the concepts present, the prior knowledge assumed, the methodology used and the conversation engaged in. These ideas combine with those of (Rivers, 1990). She provides a list of possible emphases and philosophies evident in textbooks. The predominant emphasis deals with the aims and nature of teaching and learning mathematics and this in turn provides a basis for identifying the predominant philosophy of a textbook.

MAIN FINDINGS:

The author will now discuss the main findings emerging from the analysis of Mathematics 'Textbook' Content at Junior Cycle using the framework described above.

Motivation:

The following table details the findings for motivational and comprehension factors for each of the eight textbooks involved:

Table 1: Comparison of Motivation Factors across Junior cycle Mathematics Textbooks

Textbook:	Historical Notes:	Biographies:	Career Information:	Photos:	% of Exercises which are Problems:
Text and Tests 1	5	0	0	3	20.08%
Text and Tests 2	0	0	0	1	18.27%
Concise Maths 1	0	0	0	0	15.08%
Concise Maths 2	0	0	0	0	13.78%
Edco Maths 1	0	0	0	0	17.58%
Edco Maths 2	0	0	0	0	13.93%
Discovery Maths 1	12	22	0	22	23.58%
Discovery Maths 2	14	12	0	14	21.29%

The presence of historical notes, biographies, career information and photographs is almost non-existent across most of the textbooks. The percentage of exercises which are problems is particularly low in each case. Given that Pisa (2006) highlights how only 10% of Irish students fall into the highest level of proficiency with regard to mathematical knowledge (the OECD average is 13%), the lack of emphasis on problem solving is not surprising.

Comprehension:

The use of attractive colours, identified by Rivers (1990) as being significant to student comprehension, is limited throughout all textbooks and there is a lack of consistency across colour use (Table 2). For all textbooks the colour schemes tend to depend on a grouping of chapters as opposed to common topics and themes throughout.

Table 2: Comparison of Comprehension Factors across Junior cycle Mathematics Textbooks

Textbook:	Page Background Colour	Font Colour:	Graph-line Colour:
Text and Tests 1	White, Yellow	Black	Black
Text and Tests 2	White, Blue	Black	Black
Concise Maths 1	White, Blue, Yellow	Black, Blue	Red
Concise Maths 2	White, Blue, Yellow	Black, Blue	Red
Edco Maths 1	White, Green	Black	Black
Edco Maths 2	White, Orange	Black	Black
Discovery Maths 1	White, Blue, Red, Orange	Black, White, Red, Blue, Orange	Black, Red, Blue Orange
Discovery Maths 2	White, Green, Purple	Black, White, Green, Purple	Black, Green, Purple

The following two graphs highlight the differences in page background and the use of colour throughout each textbook.

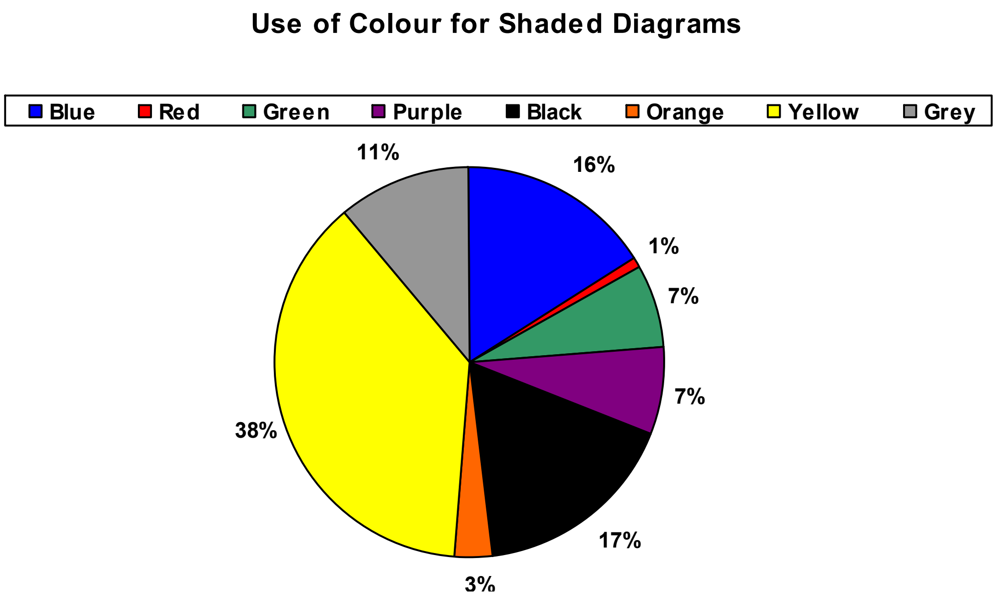


Figure 1: Use of Colour for Shaded Diagrams

Yellow is the most common colour evident through all textbook series, however the shade of yellow used is extremely dull and faded. Following yellow, black, blue and grey are the next most commonly used colours.

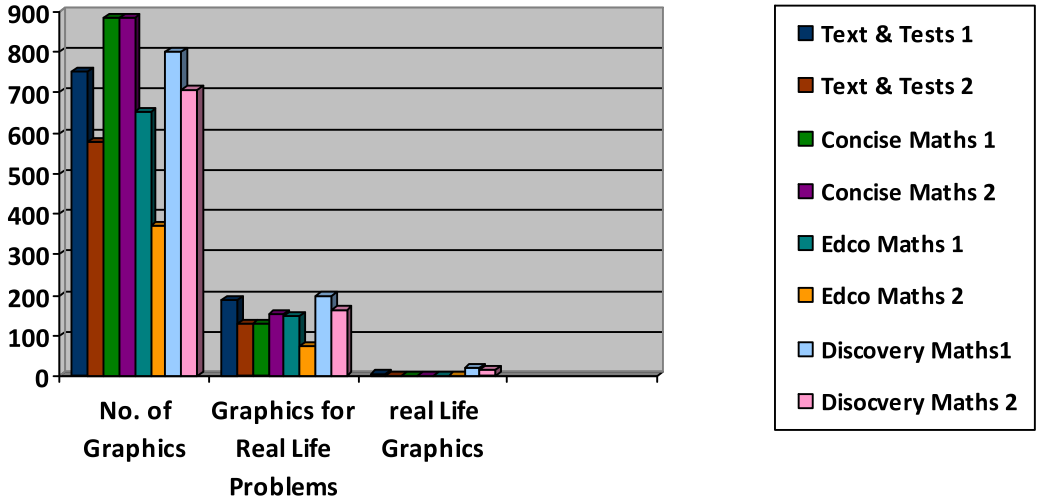


Figure 2: Distribution of Graphics throughout the Junior Cycle Mathematics Textbooks

The above graph demonstrates the presence of graphics throughout each textbook. As is evident there is a multitude of graphics present in all textbooks, however the number of graphics assisting real life problems or indeed real life graphics is worryingly low in comparison to the number of figures used through-out the textbooks. This raises questions about the purpose of the graphics throughout the textbooks.

Technical Aids:

The following table indicates the number of chapters in each textbook which make reference to calculator or computer based work. (Calculator and computers are the most basic of the available technical aids)

Table 3: Technical Aids

Textbook:	Calculator Reference:	Computer Reference:
Text and Tests 1	5	0
Text and Tests 2	2	0
Concise Maths 1	5	0
Concise Maths 2	3	0
Edco Maths 1	0	0
Edco Maths 2	0	0
Discovery Maths 1	3	3
Discovery Maths 2	4	0

The use or encouraged use of technical aids is not evident in any of the textbook series analysed. There is an abundance of research outlining how ICT can enhance the learning experience of students such as Duffy & O'Donoghue (1992), however none of these textbook series took this into consideration. The use of ICT is not reflected in Junior Cycle mathematics examinations and it appears that textbooks may be reflected this inclusion.

Philosophical Orientation:

A textbook can have any number of emphases evident throughout but only one predominant philosophy. Emphasis number 5 (retention and depth of understanding) is not relevant in the study as the author felt that as a combined emphasis it was not present. This is due to the lack of educational weight on understanding throughout each textbook despite an obvious focus on retention.

Table 4: Outline of possible Emphasis & Philosophy

Emphasis:	**Philosophy:**
1. Utility and motivation 2. Career Goals & Immediate Reinforcement 3. Proficiency & Logic 4. Not stated - Inference: Reduction of threat 5. Retention & Depth of Understanding 6. Language of Mathematics & Active Participation 7. Critical thinking & Communication.	A. Reconstructionism - Focus on society and change B. Pragmatism - Practical approach to problems, problem solving central focus. C. Perennialism - Focuses on the intellect and importance of understanding. D. Realism - Focus on observable facts and information, no fantasy only actual experience. E. Existentialism - emphasis is entirely on the individual very much student based learning. (Rivers, 1990)

Identifying the presence of emphasis and philosophy was difficult for all the textbooks, but one common theme did emerge. All the textbooks highlighted retention and practice, with little focus on active learning. In the last two textbook series, as outlined in table 5, the presence of a 'Language and Active Learning Participation' was identified as an emphasis. However, it is important to note that the focus on language was the emphasis which came across and active learning was minimal to non-existent in this textbook series. With regard the predominant philosophy of each textbook, none of the identified philosophies sat comfortably with the textbooks however with the emphases are best linked to perennialism. It is important to note that the textbooks did not foster an environment of understanding.

Table 5: Emphasis & Predominant Philosophy

Textbook:	Emphasis:	Predominant Philosophy:
Text and Tests 1	3	C
Text and Tests 2	3	C
Concise Maths 1	3	C
Concise Maths 2	3	C
Edco Maths 1	3 + 6	C
Edco Maths 2	3 + 6	C
Discovery Maths 1	3 + 6	C
Discovery Maths 2	3 + 6	C

DISCUSSION:

As evident in Table 1, the current Mathematics textbooks employed at Junior Cycle are lacking in a number of key areas. According to Rivers (1990) textbooks should offer innovative resources. None of the textbooks examined were innovative in this sense. All four series of the textbooks studied seem to be falling short with regard to the features identified by Rivers as significant for student motivation. There are no references to careers evident across all four textbook series. Only three textbooks have any historical references, with two of these including biographies. Four textbooks have photographs; however two of these four textbooks only have 4 photographs in total. As regards problem solving, less than one quarter of all exercises present in all the textbooks could be classified as problems with the highest presence of problems evident in the Discovery Maths textbook series. In general it appears that the Discovery Maths Series offers the greatest potential for pupil motivation.

Colour, and the presence and correct use of diagrams play a vital role in pupil comprehension. The use of colour for background, font and graphical demonstrations appears to be limited across all textbooks. However, it is worth remembering in this context that Dowling (1996) states that excessive and unnecessary use of colour can hinder comprehension. Each textbook series does take this into consideration but colour consistencies across a textbook are minimal. For example, graph lines appear as black in the majority of textbooks. However in the books that use more than one colour for graphical lines there is no distinction made between red and black graph lines, for example.

As is evident in Figure 1, yellow, blue and black are the most commonly used textbook colours. However, only pale dull shades of each colour, as opposed to bright vibrant colours, are used in these textbook. Many researchers have highlighted the need for relevant, bright, attractive illustrations (Dowling, 1996; Mikk, 2000; Noonan, 1990; Rivers, 1990). Not only do graphics assist with pupils' understanding of the mathematical topic/problem, they also assist in grabbing and holding a pupils' attention (Mikk, 2000). While there is no shortage of graphics throughout any of the textbooks, the presence of real life graphics is extremely low with four textbooks having no real life graphics. This trend follows through for technical aids with only one textbook making any reference to ICT and calculator references being minimal throughout each series. As previously stated a textbook can have any number of emphases evident throughout but only one predominant philosophy. The main emphasis which came across in all textbooks examined was 'Proficiency and Logic'; all books were focused on being able to complete a number of questions. . The textbook analysis determined that none of River's (1990) philosophies sat comfortably with any of the textbooks. However the closest was perennialism, where the focus is on the intellect. This is true for the Irish context, but understanding is not given the same importance as procedure and method throughout each of the textbooks (Rivers, 1990; Valverde et al, 2002).

CONSEQUENCES FOR JUNIOR CYLE MATHEMATICS STUDENT LEARNING:

Despite the fact that O'Keeffe (2007) noted that over 50% of teachers in her study rely on only one textbook for classroom planning and teaching – Text and Tests, it appears that the Discovery Maths series proves itself to be more beneficial to learning. It provides the greatest opportunity for motivation; they have a greater number of graphics present, they incorporate a number of graphics assisting real life problems and real life graphics. Also Discovery Maths 1 is the only textbook to make reference to ICT. A variance in emphasis and philosophy is non-existent across all eight textbooks. However, the Discovery Maths series is one of two series which encourage active participation and knowledge of the language of mathematics. However, this paper is not suggesting that this textbook is the most beneficial for student learning, but rather it is the best available in the current situation. This paper is highlighting the need for this type of research on textbooks to be conducted and most importantly applied. What this research is advocating is a need for standards and/or a checklist for textbooks in order to enhance the learning at Junior Cycle.

Currently in Ireland, Project Maths is being piloted in 24 second level schools. This curriculum initiative aims to enhance the uptake of Leaving Certificate Higher Level mathematics. Project Maths is approaching this aim from a number of directions, one of which is improving the standard of Junior Cycle mathematics. In order to achieve this aim they will need to introduce and encourage the use of effective resources and textbooks. The failure of textbooks in this study to allow for motivation and comprehension highlights the need for improved textbooks. It is hoped that research of this kind will lead to standards being developed in mathematics textbook design at Junior cycle which if employed will lead to better teaching and learning resources, and will have a knock on effect of contributing to an increase in the take-up of Higher Level Junior Cycle mathematics.

Almost 80% of teachers currently teaching mathematics at second level do not have a mathematics specific qualification (EGFSN, 2008) and the majority of qualified mathematics teachers in second level schools are teaching at Leaving Certificate level. This indicates that Junior cycle Mathematics is poorly populated with qualified Mathematics teachers and this reality underlines the need for improved textbooks. Unqualified teachers may lack sufficient subject matter knowledge and reliance on substandard textbooks may be harmful to students learning experience. Howson (1995) suggests that the role of the textbook is the most dominant and powerful in mathematics education; he believes that it is associated with almost all aspects of teaching and learning mathematics.

In order to facilitate and complement changes in the Irish Mathematics curricula at both Junior and Senior Cycle it is then necessary to review the available textbooks. This paper highlights the failure of the current Junior Cycle Mathematics textbooks to motivate pupils, or to provide for the comprehension and processing of the information provided. The textbooks are detrimental to pupil learning; however teachers are slow to change pedagogies. Project Math will bring change into Irish classrooms in 2010, improved textbooks can help facilitate such change and enhance pupil learning. The authors' future work will be concerned with creating a checklist for use in the design of effective Mathematics textbooks, and validating the significance and role of well designed Mathematics textbooks in teaching and learning.

REFERENCES

Dowling, P. (1996), 'A Sociological Analysis of School Mathematics Texts, *Educational Studies in Mathematics',* 31(4): 389-485.

Duffy, D. and J. O'Donoghue, (1992), 'Use of the Complier as a Mathematics Teaching Resource', *International Journal of Mathematics Education in Science and Technology,* 23(5): 637-650.

EGFSN (Expert Group on Future Skills Needs) (2008), Statement on raising National Mathematical Achievement, EGFSN: Dublin

Ernest, P. (1994), Mathematics, Education and Philosophy, London: The Falmer Press.

Howson, G. (1995), Mathematics Textbooks: A Comparative Study of Grade 8 Texts, Vancover: Pacific Education Press

Kilpatrick, J. (1985), Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives, Chapter A: Retrospective Account of the Past 25 Years of Research on Teaching Mathematical Problem Solving, 1-15, Lawrence Erlbaum Associates.

Mikk, J. (2000), Textbook: Research and Writing. Lang: Oxford.

NCCA, (National Council for Curriculum and Assessment) (2006), Review of Mathematics in Post Primary Education, Report on the Consultation: NCCA: Dublin

Noonan, J. (1990), 'Readability problems presented by mathematics texts' *Early Child Development and Care*, 54: 57-81.

O'Keeffe, L. (2007), 'An Analysis of the Junior Cycle Mathematical Textbooks', unpublished final year dissertation, University of Limerick, (submitted March, 2007).

Rivers, J. (1990), Contextual Analysis of Problems in Algebra 1 Textbooks, University of South Carolina, presented at the annual meeting of the American Educational Research Association, Boston: Massachusetts.

Valverde, G., Bianchi, L., Wolfe, R., Schmidt, W. and Houang, R. (2002), 'According to the Book: Using the TIMSS to Investigate the Translation of Policy into Practice through the World of Textbooks' London: Kluwer Academic Publishers.

SOLVING PROBLEMS IN MATHEMATICS EDUCATION: CHALLENGES FOR PROJECT MATHS

Elizabeth Oldham — Trinity College, University of Dublin

Sean Close — Educational Research Centre, St Patrick's College

In recent decades, problem solving has been a major focus in many mathematics curricula. However, often such curricula have been incompletely implemented; this may reflect, among other things, a divergence between the philosophies of mathematics underpinning the problem-oriented curricula and those espoused by teachers of mathematics. This paper outlines research on problem solving and on philosophies of mathematics education. It aims to promote discussion on these issues and to identify challenges facing the implementation of a revised curriculum in Irish second level mathematics via 'Project Maths.'

INTRODUCTION

Problem solving has always been an important part of mathematics; for some mathematicians, the activity of solving problems is the defining aspect of the subject. Similarly, and especially since 1980, there is a history of problem solving as a part of mathematics education. Many modern curricula highlight the importance of higher-order learning outcomes such as identifying and generalising mathematical relationships, modelling real-world situations and solving unfamiliar problems ('problem-oriented' curricula); however, it has been found that, all too often, implementations lack fidelity to the specified aims (see for example Stein, Remillard & Smith, 2007). This may reflect – among other things – a divergence between, on the one hand, the philosophies of mathematics and of mathematics education that underpin problem-oriented curricula and, on the other hand, the philosophies espoused (perhaps unconsciously) by mathematics teachers who implement the curricula.

In recent years in Ireland, the role of problem solving in mathematics education has been the subject of considerable discussion, at any rate at the level of curriculum design. The revised Primary Curriculum in Mathematics (DES/NCCA, 1999) places more emphasis on problem solving, especially in real-life contexts, than did its predecessor (Department of Education, 1971). The current second level curriculum documents specify applications and problem solving among their objectives (Department of Education, n.d. [1992]; DES/NCCA, 2000). However, Chief Examiners' Reports on performance in the State examinations indicate that students' achievement of higher-order objectives is poor (see for example DES, 2001), and a revised curriculum with increased focus on problem solving and applications is now being phased in for both Junior Cycle and Senior Cycle. The process of revision constitutes 'Project Maths' (NCCA, 2008). In a highly innovative approach in the Irish context, the draft syllabuses are being piloted in a small number of schools, and their final form and implementation are being shaped in part by the teachers in these schools, notably through the development of 'teaching and learning plans' that promote the active involvement of the learners. This work to date has focused chiefly on the development of students' understanding. It remains to be seen to what extent the problem-solving aspects will be addressed, both by the teachers and via the promised restructuring of the State examinations.

Successful implementation of Project Maths poses many challenges. This paper aims to promote reflection on some of the issues involved, in particular those mentioned above: the introduction of problem-oriented curricula and their implementation by teachers who may not fully understand or endorse their underlying philosophy. It therefore provides a historical overview and a summary of key research in the areas. First, problem solving in mathematics curricula is addressed; then, an account is given of philosophies of mathematics and mathematics education. Finally, implications are considered for implementing problem-oriented curricula, in particular through Project Maths.

HISTORY OF PROBLEM SOLVING IN MATHEMATICS EDUCATION

Problem solving is considered here under four headings. First, general features of problem solving are outlined; secondly, an account is given of research and development with regard to problem solving in mathematics education. The third subsection considers the assessment of problem-solving skills, while the fourth addresses curriculum implementation by teachers.

Nature of Problem Solving

The psychologist William Brownell defined problem solving as the process of responding to '...perceptual and conceptual tasks, the nature of which the subject, by reason of original nature, of previous learning or of organisation of the task, is able to understand, but for which, at the time he knows no direct means of satisfaction. The subject experiences perplexity in the problem situation, but he does not experience utter confusion' (Brownell, 1972, p. 132). Today, we can find many definitions in the literature that more or less correspond to Brownell's. There is a lack of agreement among researchers about the precise nature of problem solving, although all seem to agree that initial uncertainty about a solution procedure is an essential ingredient. A complication is that a problem to some may not be a problem to others, or what may be a problem to an individual at one point in time may not be a problem for him or her at a later stage. If a 'problem' is completely incomprehensible to an individual, then he or she cannot engage in problem solving with it. On the other hand, if individuals become familiar with a particular class or type of problem, and by reasoning and experience develop a procedure or algorithm for dealing with this type of problem, then, arguably, it is no longer a problem for them. The word 'routine' is often used as a descriptor for this type of problem, where little or no uncertainty is involved. For this reason it is probably useful to think of problems as being on a continuum, ranging from non-routine and complex problems towards one extreme to routine and familiar problems towards the other extreme.

One kind of problem solving can be viewed as a broad-based or generic competence or skill set (including for example analytical reasoning, critical thinking, creative thinking, metacognition) which is used in dealing with cross-curricular problems in schooling, or problems which arise in non-domain-specific situations in the real world or computer-based simulations of them. This kind of problem solving is more often associated with the world of work rather than schooling, and is prized by employers who often refer to the lack of it in the workforce (for example EGFSN, 2008). The other kind of problem solving is domain-specific and involves solving problems in domains of knowledge such as mathematics, science, engineering, economics, finance, and so forth. The former kind of problem solving has been considered important enough, and different enough, to be assessed separately from key subject domains in international studies such as PISA and ALLS (Klieme, 2004; Murray, Clermont & Binkley, 2005; OECD, 2004a).

Problem Solving in Mathematics Education

In the field of school mathematics education, the 1960s were considered to be the decade of ‘Modern Mathematics’ or ‘New Maths,’ the 1970s to be the decade of ‘Back to Basics’ and the 1980s to be the decade of ‘Problem Solving.’ Though the 1990s might be considered to be the decade of ‘Constructivism’ in mathematics education, problem solving has retained a high standing in mathematics education research and development since the 1980s.

Prior to the 1980s, when mathematics was viewed mainly as sets of ‘unpacked’ and isolated concepts and skills, and learning and teaching mathematics was dominated by behaviourist approaches including use of teaching examples, repetition and reinforcement, students were assigned problems (usually routine word problems) for the purpose of providing them with the opportunity to apply the mathematical concepts and skills they had been taught. Any uncertainty or challenge in these problems was generally minimal as they came at the end of a lesson or chapter on a specific topic, and so students would probably know what concept or procedure to apply. Occasionally, students would be presented with puzzle-type problems, often of a purely mathematical or contrived nature, which were generally aimed at challenging the more able students. As problem solving became the focus of attention of research, the initial studies investigated a number of aspects of problem solving, both as a product and as a process. These included:

- factors that make problems difficult to solve (for example reading demands, number of steps in the solution procedure, difference from preceding problems, information not in the order in which it is to be used, size/type of numbers, redundant or distracting information, difficult mathematical content);

- strategies used in problem solving (for example Polya's heuristic for problem solving, draw a diagram, construct a model, make a table/chart, look for a pattern, work backwards, guess and test, reduce to a simpler version);
- characteristics of successful problem-solvers (for example mathematical understanding, above average general ability, perseverance, tolerance for ambiguity, good spatial ability, ability to classify problems, ability to identify critical vs. irrelevant information, ability to estimate, ability to switch strategies easily, ability to plan, monitor and check solutions, confidence in ability to solve problems).

A criticism of the research was that most of it was done with standard word problems (Lester, 1995).

More recently, developments in problem solving in mathematics include efforts to use problem situations to enable children to develop concepts and skills and to motivate learning, i.e. *problem-based learning*, with a particular emphasis on use of realistic mathematics problems as exemplified by the Realistic Mathematics Education (RME) approach (Freudenthal, 1981). In this approach the teacher begins with a problem, often practical, that makes sense to the learners. The learner is expected to extract the mathematics from the problem context and then use it to solve the problem. The mathematical knowledge gained can then be used to solve other similar or more complex problems. A range of problems is selected so that they cover key understandings and skills. Further development, clarification and varied practice of some these skills and concepts is often carried out as part of the teaching/learning sequence (for example Romberg, 2001).

Achievement in Problem Solving

Traditionally, problem-solving achievement has been assessed using word problems requiring application of one or two specific concepts and/or skills in simple contexts that were often somewhat contrived or unrealistic. In keeping with this approach, standardised mathematics tests usually had three sections; (i) Computation (recall facts and rules, implement learned procedures); (ii) Concepts (identify, classify, interpret); and, (iii) Problems (apply, solve, analyse, generalise). Early international mathematics surveys more or less followed suit. However, with the increased focus on problem solving in the 1980s, efforts were made to assess achievement on problems requiring a broader range of mathematical knowledge and processes, including higher-order thinking and reasoning, in a wide variety of contexts. An early example of this trend is reflected in the focus on mathematical investigations in the Cockcroft Report (Cockcroft, 1982). The Third International Study of Mathematics and Science (TIMSS), carried out in some 40 countries in 1994-95, incorporated this development into its mathematics framework by aiming to test the cognitive processes of knowing, using routine procedures, investigating and problem solving, reasoning, and communicating (Martin, Mullis, Gregory, Hoyle & Shen, 2000). To complement the written tests, TIMSS also carried out a performance test that consisted of a set of practical tasks in mathematics and science. Given the additional expense involved, only 21 countries participated in the performance testing (Harmon, Smith, Martin, Kelly, Beaton, Mullis, Gonzalez & Orpwood, 1997).

With the increasing emphasis on problem-based learning and teaching, efforts were made to develop improved methods of testing the outcomes of problem-based learning in mathematics. The most notable manifestation of this work at the international level is the 2003 PISA study of the mathematical literacy of 15 year-olds (OECD, 2003). The PISA mathematics test is almost entirely composed of problems – ranging from routine familiar problems to complex non-routine problems. As well as involving the traditional assessment dimensions relating to mathematical content and cognitive process, PISA introduced a third dimension – the type of context or situation in which the mathematics is embedded: personal, social, occupational or scientific. The cognitive processes addressed included reasoning; argumentation; communication; modelling; problem posing and solving; representation; using symbolic, formal and technical language and operations; and, use of aids and tools. A PISA problem may involve one or more of these processes at three levels of complexity – reproduction, connections, and reflection. Ireland achieved an overall mean score of 503 in PISA 2003, which was about the international mean, and ranked 17th of 29 OECD countries and 20th of 40 participating countries (OECD, 2004b). This performance compared unfavourably with Ireland's performance on TIMSS 1995, which was significantly above the international mean and placed Ireland 12th among 41 countries (Martin et al., 2000). The less than satisfactory performance of Ireland on the PISA test has been a factor in promoting review of the second level curriculum and initiating Project Maths.

From Design to Achievement: Curriculum Implementation

The curriculum, as decreed by a State Department of Education or a similar agency, is not the only factor that influences student achievement. Among the other factors, and of particular importance for this discussion, is the way in which teachers implement the curriculum. According to Hiebert & Grouws (2007), '[the] nature of classroom mathematics teaching significantly affects the nature and level of students' learning' (p. 371) and 'different teaching methods might be *effective* for different learning goals' (p. 374, emphasis added).

It is pertinent to ask: effective for *whose* goals? For curriculum designers, effective teaching would entail practices that achieve the declared aims and learning outcomes of the curriculum; thus, for problem-oriented curricula, effective teaching would develop students' problem-solving disposition and competence. For students, however, effective teaching may be instruction perceived as helping them to do well in the State or other examinations that act as gatekeepers to further study and employment. Teachers may be caught between these two perspectives. Teachers' views on effective teaching have been the subject of recent attention (Cai, Kaiser, Perry & Wong, 2009). Their goals with regard to curriculum implementation may be affected by – among other factors – the teachers' own differing philosophies of mathematics education. The latter aspect is the subject of the next section of the paper.

PHILOSOPHIES OF MATHEMATICS AND MATHEMATICS EDUCATION

This section is divided into three subsections. First, the context is set by examining aspects of the Irish State curricula, and by posing questions for consideration by teachers of mathematics as to which aspects of the curricula they themselves enjoyed most in their schooldays. This raises issues about the essential nature of mathematics, and hence leads in to the second subsection: an outline of well-established philosophies of mathematics. Thirdly, the related but perhaps more nuanced issue of the philosophy of mathematics education is investigated.

Aspects of the Irish State Curricula

The discussion is started by considering two contrasting areas in the Irish Junior Cycle Mathematics syllabuses introduced in 2000 (DES/NCCA, 2000). One content area is 'Geometry.' The Higher course syllabus, in particular, contains a section on 'synthetic' geometry in the style of Euclid's Elements, with specified axioms [1] allowing for the establishment – by logical reasoning – of a set of theorems. The *Guidelines for Teachers* (DES/NCCA, 2002) provides the following rationale, in which phrases in italics are taken from, or closely echo, the general aims set out in the syllabus (DES/NCCA, 2000, p. 2):

> Synthetic geometry is traditionally intended to promote *students' ability to recognise and present logical arguments.*
>
> - [Higher course] students address one of the greatest of mathematical concepts, that of proof, and hopefully … appreciate the *abstractions and generalisations* involved.
> - Other students may not consider formal proof, but should be able to draw appropriate conclusions from given geometrical data….
> - Tackling 'cuts' and other exercises based on the geometrical system presented in the syllabus allows students to *develop their problem-solving skills*.
> - Moreover, in studying synthetic geometry, students are encountering one of the great monuments to intellectual endeavour: a very special part of Western culture (DES/NCCA, 2002, pp. 20-21).

The content area 'Applied Arithmetic and Measure' has a contrasting rationale:

This topic is perhaps one of the easiest to justify in terms of *providing mathematics needed for life, work and leisure.*

- Students are likely to use the skills developed here in 'everyday' applications, for example in looking after their personal finances and in structuring the immediate environment in which they will live. For many, therefore, this may be a key section in *enabling students to develop a positive attitude towards mathematics as a valuable subject of study*.
- There are many opportunities for *problem-solving*, hopefully in contexts that the students recognise as relevant (DES/NCCA, 2002, p. 20).

Thus, the topic of Applied Arithmetic and Measure is justified on highly pragmatic grounds. By contrast, synthetic geometry is justified, not for its immediate and practical usefulness, but rather in terms of abstract beauty, cultural significance, and development of reasoning powers. (The latter aspect may indeed be practical and useful, but may not readily be perceived as such by a typical Junior Cycle student.) The two rationales reflect the dual nature of mathematics, encapsulated in the well-known catch-phrase 'the queen and servant of the sciences' – or perhaps the gender-neutral 'monarch and servant' would be more apposite.

A similar dichotomy can be found in the revised Primary Curriculum for Mathematics (DES/NCCA, 1999). The strands on Number and Measures have obvious real-life applications, while some content in the strand on Shape and Space is notable perhaps chiefly for its aesthetic value which can serve to initiate the learners into the beauty and intrinsic fascination of mathematics. Even in the Number strand, some of the work for senior classes – that on prime and composite numbers, for instance – is perhaps more likely to appeal through its capacity to fascinate children than for the applications that they may encounter later in their school careers and their everyday lives.

Pertinent questions could be posed to teachers at this point, and it is suggested that readers provide their own answers before considering the remaining argument.

- In your schooldays, which aspects of mathematics appealed to you most, and why?
- For second-level teachers specialising in mathematics, which aspects drew you to the study of mathematics and to a career in teaching the subject? In particular, with regard to the aspects contrasted above, did your choice owe more to the abstract beauty or the all-permeating applications and occurrences in the world? Or did neither aspect appeal, and did you choose mathematics in order to enjoy the rapid execution of unambiguous procedures and the reinforcement of getting unique right answers?

Teachers' answers to these questions probably have a bearing on the way in which they present the subject in the classroom – a claim discussed below. Other factors are likely to contribute also: for example, students' reactions and their developmental readiness to engage with the curricular aspects, as well as the constraints imposed by assessment. However, the discussion here focuses on *research on teachers' conceptions of mathematics and their personal beliefs with regard to the value of the subject* [2]. In the context of this paper, it is important to emphasise that – as highlighted in the quotations from the Junior Cycle *Guidelines* – both the 'beauty' and 'applicability' conceptions can be seen as supporting problem-solving approaches. Bearing in mind the research on problem solving outlined above, however, the 'unambiguous procedures' conception may not.

Philosophies of mathematics

The differing approaches to synthetic geometry and to arithmetic – evidence of a 'Great Schism' in the philosophy of mathematics – have their roots in history. Traditional synthetic geometry reflects the Greeks' passionate pursuit of truth and of the manner in which truth can be established or preserved by rigorous reasoning. In fact, the Greeks brought the same approach to the study of number, which they related to geometry in that number was considered in terms of length or extent; but in this case the legacy for modern-day mathematics was more restricted. It was the Indian and Arabic influence that freed number from its association with geometry and introduced a theory of arithmetic that was based less on a search for abstract truth and more on the provision of a workable calculus (Kneebone, 1963; Rouse Ball, 1922).

Later, the development of algebra, and eventually of co-ordinate geometry and analysis, led to a massively creative period in mathematics. However, such a burst of creativity can lead to a demand for critique. In the nineteenth century, questions were being asked as to whether or not mathematics was a secure discipline, one in which the theorems were really 'true' and the methods used were defensible. Around the beginning of the twentieth century, this 'crisis in the foundations' led to the emergence of three important philosophies of mathematics – Logicism, Formalism and Intuitionism – designed to address the difficulties.

- For Logicism, mathematics is envisaged as a body of absolute truth derived by impeccable reasoning from obviously true starting points (the view of Bertrand Russell and others)
- In Formalism, associated in particular with Hilbert, it as a set of consistent structures in which results are derived by irrefutably rigorous methods
- Intuitionism, notably in the work of Brouwer, endeavoured to build mathematics up constructively from the natural numbers.

Fuller accounts are provided for example by Kneebone (1963) and Dossey (1992). All three philosophies were ambitious and profound, but all ultimately failed to achieve the objective of showing that traditional mathematics (or at any rate most of it) has secure foundations. The chief aspect of relevance to this paper is that the mathematics in question was regarded as a body of knowledge, pre-existing and eternal. In terms of the 'Great Schism' identified above, the approach was Greek rather than Hindu-Arabic in spirit. This view of mathematics can be described as *absolutist*.

A contrasting approach emerged, notably in the 1960s through the work of Lakatos. In the spirit of Popper's 'critical fallibilism,' he claimed that 'informal, quasi-empirical mathematics does not grow through a monotonous increase in the number of indubitably established theorems, but through the incessant improvement of guesses by speculation and criticism' (Lakatos, 1963, p. 6). The introduction of this *fallibilist* view perhaps marks the end of a chapter in the development of the philosophy of mathematics. It can also be seen as introducing a new chapter in mathematics education, in which the focus is on the 'philosophy of mathematics education' as discussed below.

Philosophies of mathematics education

The claim that the philosophies of mathematics espoused by teachers have implications for their teaching and for students' learning was mentioned earlier. The relationship between *conceptions of mathematics* and *teaching approaches* has been the subject of theoretical and empirical study since the early 1980s (Philipp, 2007; Prediger, Vigiani-Bicudo & Ernest, 2008; Thompson, 1992). Ernest (1985) provided a threefold categorisation of conceptions of mathematics that has been a basis for much of the work in the area. The categories are labelled and described here as follows (Dossey, 1992; Ernest, 1985; Thompson, 1992):

- Platonist: a body of knowledge, already existing and summarised in *true* theorems that can be discovered rather than created;
- Instrumentalist: a bag of tools, such as routine procedures and 'tricks,' useful in obtaining answers to exercises and standard problems;
- Problem-solving: a continuously expanding field of human creation, reflecting ongoing activity rather than a static product.

As the name suggests, Platonism sits on the Greek side of the 'Great Schism.' In terms of the philosophies described above, Logicism is firmly Platonist, while both Formalism and Intuitionism stemmed from Platonist concerns. The Instrumentalist approach can be seen as an inheritor of the Hindu-Arabic tradition. As formulated here – with mathematics seen as a collection of procedural rules divorced from meaning and logic – it is far removed from a philosophy of mathematics that a mathematician might recognise. However, if the 'tools' are interpreted as powerful and coherent methods of analysis and computation, it perhaps relates to a philosophy of applied mathematics, rather than of the pure mathematics that is the subject of most of the discussion so far. The Problem-solving approach is Lakatosian in spirit. In terms of the absolutist/fallibilist dichotomy, both Platonism and Instrumentalism are absolutist, whereas the Problem-solving approach reflects fallibilism.

For some time, there was a tendency for the mathematics education literature to associate absolutist views with behaviourist or instructivist approaches ('bad'), while fallibilist views were associated with constructivist approaches ('good'); but it has now been acknowledged that the connection is not so straightforward (Prediger et al., 2008). While the relationship between fallibilism and constructivism appears natural, it is over-simplistic to assume that teachers with absolutist views would necessarily present mathematics to their classes in dogmatic form. Platonists, in particular, may well wish to emphasise meaning, understanding and some form of problem solving. However, for absolutists espousing simplistic Instrumentalist views, according to which mathematics is a set of arbitrary rules unrelated to meaning or context – rules without reasons – the criticisms may have more substance.

Another way of viewing the situation is in terms of the source of authority in mathematics teaching. For holders of Platonist views, the natural source of authority is the subject itself and the logic inherent in it. For those with Problem-solving views, the natural source may be the classroom community, jointly and as individuals forging their understanding of the subject. For Instrumentalists – at least those holding the simplistic views outlined above – the source of authority may be the text-book, or the teacher, or the teacher's own teachers: 'why' questions are answered by 'because that's the rule.'

IMPLICATIONS FOR IMPLEMENTING PROBLEM-ORIENTEDCURRICULA

It was indicated above that views on what might constitute effective teaching vary, between as well as within countries (Cai et al., 2009). It would be interesting, albeit challenging, to investigate the views of a large group of Irish teachers. Work with small numbers of teachers has been done by Lyons, Lynch, Close, Sheerin and Boland (2003) and Brosnan (2008). Their classroom video material and interviews with the teachers captured much procedure-dominated teaching, in which authority lay with the teacher or perhaps the subject, with little reference to context or applications, and – especially for Brosnan's teachers of third year classes – a conspicuous focus on teaching towards the examinations. Such approaches are consistent with rather narrow Instrumentalist views of mathematics. Of course, as indicated above, the approaches may have been driven by constraints: time, curricular material, student interest, and school and parent pressure, to mention only the obvious possibilities. Moreover, a larger sample of teachers, or of lessons per teacher, might have captured a wider range of approaches. Nonetheless, it might be inferred that the teachers' philosophies, if *not* Instrumentalist, were insufficiently strongly held to over-ride the constraints.

While the authors of this paper have encountered many good teachers of mathematics, whose approaches are consistent with Platonist or (less often) Problem-solving views of the subject, or who have an inspiring focus on applications, it has to be said that they have also encountered teachers whose classroom performance can be seen as narrowly Instrumentalist. In terms of the questions posed above to teachers with regard to aspects of the Irish State curricula, choice of a mathematics teaching career based only on enjoyment of rapid execution of unambiguous procedures may indicate Instrumentalist views that do not bode well for faithful implementation of problem-oriented curricula – or, indeed, of curricula that emphasise understanding and/or applicability. Choice of a mathematics teaching career based on views of mathematics as pre-existing and eternal may also necessitate some degree of teacher adjustment if problem-oriented curricula are to be implemented with fidelity.

In conclusion, it is hoped that the foregoing discussion will help to provide some background against which mathematics educators and teachers can reflect on their present philosophies and approaches to teaching mathematics, and can consider how they relate to a more Problem-solving philosophy and approach such as that outlined for Project Maths. If teachers are to engage seriously and substantially with this approach, they will need time to experiment with it and discuss it in an atmosphere which is relatively free from fear of loss of status or of poor performance on the part of their students on high-stakes examinations. This time will also facilitate the development of suitable teaching materials and resources envisaged as part of the implementation process.

NOTES

1. To cater for students still at Piagetian concrete operational level, the axioms were labelled as 'facts.' The syllabus for Project Maths outlines a different presentation but also exemplifies the aspects discussed above.

2. Writers have distinguished between the concepts of 'conception', 'perception,' 'belief,' and so forth (see Philipp, 2007). Here, such terms are used as synonyms.

REFERENCES

Brownell, W.A. (1972). Problem solving. In J.F.Weaver & J. Kilpatrick (Eds.), *Studies in Mathematics Vol. XXI: The place of meaning in mathematics instruction: Selected theoretical papers of William Brownell* (pp 129-159). New Haven: Yale University.

Brosnan, A. (2008). *The teaching of mathematics at post-primary level in Ireland: A review of traditional patterns and an exploration of future practices*. Unpublished Ph. D. thesis, National University of Ireland Maynooth.

Cai, J., Kaiser, G., Perry, B. & Wong, N.G. (Eds.) (2009). *Effective mathematics teaching from teachers' perspectives: National and cross-national studies*. Rotterdam: Sense Publishers.

Cockcroft, W. H. (1982). *Mathematics counts*. Report of the Committee of Inquiry into the Teaching of Mathematics in Schools under the chairmanship of of Dr. W. H. Cockcroft [the Cockcroft Report]. London: HMSO.

Department of Education (1971). *Primary School Curriculum: Teacher's handbook, part 1*. Dublin: Stationery Office.

Department of Education (n.d. [1992]). *The Leaving Certificate: Mathematics syllabus.* Dublin: Stationery Office.

DES (Department of Education and Science). (2001). *Leaving Certificate Examination 2001: Mathematics – Chief Examiner's report.* Retrieved July 28, 2009, from http://www.examinations.ie/archive/examiners_reports/cer_2001/lcmaths_ol_01_er.pdf.

DES/NCCA (Department of Education and Science / National Council for Curriculum and Assessment) (2000). *Junior Certificate Mathematics: Higher, Ordinary and Foundation level.* Dublin: Stationery Office.

DES/NCCA (Department of Education and Science / National Council for Curriculum and Assessment) (1999). *Primary Curriculum: Mathematics*. Dublin: Stationery Office.

DES/NCCA (Department of Education and Science / National Council for Curriculum and Assessment) (2002). *Mathematics Junior Certificate: Guidelines for teachers.* Dublin: Stationery Office.

Dossey, J. (1992). The nature of mathematics: its role and influence. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 39-48). New York: Macmillan.

Ernest, P. (1985). The philosophy of mathematics and mathematics education. *International Journal for Mathematical Education in Science and Technology,* 16 (5), 603-612.

EGFSN (Expert Group on Future Skills Needs) (2008). *Statement on raising national mathe-matical achievement*. Report to the Tánaiste & Minister for Enterprise, Trade and Em-ployment and the Minister for Education and Science; November 2008. Dublin: The Nat-ional Policy & Advisory Board for Enterprise, Trade, Science, Technology and Innovation.

Freudenthal, H. (1981). Major problems in mathematics education. *Educational Studies in Mathematics,* 12, 133-150.

Harmon, M., Smith, T.A., Martin, M.O. Kelly, D. L., Beaton A.E. Mullis, Gonzalez, E.J. & Orpwood, G. (2000). *Performance assessment in IEA's Third International Mathematics and Science Study* (TIMSS). Boston: TIMSS International Study Center, Boston College.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371-404). Charlotte, NC: Information Age Publishing.

Klieme, E. (2004). Assessment of cross-curricular problem-solving competencies. In J.H. Moskowitz & M. Stephens (Eds.), *Comparing learning outcomes: International assessment and educational policy* (pp. 81-107). London: Routledge-Falmer.

Kneebone, G. T. (1963). *Mathematical logic and the foundations of mathematics*. London: D. Van Nostrand.

Lakatos, I. (1963). Proofs and refutations. *British Journal for the Philosophy of Science*, 14, 1-25.

Lester, F. K. (1995). Musings about mathematical problem-solving research: 1970-1994. *Journal for Research in Mathematics Education,* 4, 660-675.

Lyons, M., Lynch, K., Close, S., Sheerin, E. & Boland, P. (2003). *Inside classrooms: The teaching and learning of mathematics in social context*. Dublin: Institute of Public Administration.

Martin, M.O., Mullis, I.V., Gregory, K.D., Hoyle, C., & Shen, C. (2000). *IEA's Third International Mathematics and Science Study: Effective schools in science and mathematics*. Boston: TIMSS International Study Center, Boston College.

Murray, T. S., Clermont, Y. & Binkley, M. (2005). *Measuring adult literacy and life skills: New frameworks for assessment*. Ottawa: Statistics Canada.

NCCA (National Council for Curriculum and Assessment). (2008). *Project Maths: Information for schools*. Dublin: Author. Retrieved July 28, 2009, from http://www.ncca.ie/uploadedfiles/mathsreview/PMaths_En.pdf.

OECD (Organisation for Economic Co-operation and Development). (2003). *The PISA 2003 assessment framework: Mathematics, reading, science and problem solving knowledge and skills*. Paris: Author.

OECD (Organisation for Economic Co-operation and Development). (2004a). *Problem solving for tomorrow's world. First measures of cross-curricular skills from PISA 2003*. Paris: Author.

OECD (Organisation for Economic Co-operation and Development). (2004b). *Learning for tomorrow's world:- First results from PISA 2003.* Paris: Author.

Philipp, R. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257-315). Charlotte, NC: Information Age Publishing.

Prediger, S., Vigiani-Bicudo, M. & Ernest, P. (2008). DG4: Philosophy of mathematics education. In M. Niss (Ed.), *ICME-10 Proceedings: Proceedings of the 10th International Congress on Mathematical Education, 4-11 July, 2004* (pp. 438-442). Roskilde, Denmark: IMFUFA, Department of Science, Systems and Models, Roskilde University.

Romberg, T. A. (2001). Designing middle-School mathematics materials using problems set in context to help students progress from informal to formal mathematical reasoning. In: *Designing Middle-School Mathematics Materials.* (pp 1-19). National Center For Improving Student Learning & Achievement In Mathematics & Science, University of Wisconsin–Madison. Retrieved July 28, 2009, from: http://ncisla.wceruw.org/publications/articles/MiCChapter.pdf

Rouse Ball, W. W. (1922). *A short account of the history of mathematics.* London: Macmillan.

Stein, M. K., Remillard, J. & Smith, M. (2007). How curriculum influences student learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 319-369). Charlotte, NC: Information Age Publishing.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 127-146). New York: Macmillan.

THE LADDER OF KNOWLEDGE: KNOWLEDGE FOR EFFECTIVE TEACHING

Niamh O'Meara Professor John O'Donoghue

University of Limerick NCE –MSTL

Recent years have witnessed a decline in the mathematical standards of students entering third level (Gill & O'Donoghue, 2006). A contributing factor to this problem is the existence of the under-preparedness cycle whereby poor teaching is leading to under prepared students entering third level and some of these students are graduating as teachers, and are returning to the classroom with the same low levels of mathematical knowledge and poor attitudes towards mathematics - thus continuing the cycle. The authors believe that this problem must be targeted at its root and teachers must develop a better knowledge of mathematical content in order to improve the standards of teaching and learning in Ireland. Researchers such as Shulman (1986), Ernest (1989), Fennema & Franke (1992) and Rowland (2007) have devised models of the packages of knowledge which they believe contribute to effective teaching. Following a thorough analysis of these, the authors have developed a model of teacher knowledge in an attempt to resolve some of the problems currently facing mathematics education in Ireland. A detailed description of this 'Ladder of Knowledge' will be presented in this paper, while demonstrating the need for sufficient knowledge (mathematical, pedagogical, etc.) for effective mathematics teaching.

INTRODUCTION

Furinghetti (2000) has proposed the idea of the cycle of under preparedness. This cycle clearly demonstrated how poor teaching leads to underprepared students entering third level and many of these then graduate, as teachers, and return to the classroom with the same low levels of mathematical understanding and poor attitudes towards mathematics that they first entered university with, thus continuing the vicious cycle. This project initially set out to help resolve the 'Mathematics Problem' by breaking this cycle. After a thorough investigation into the impact that teachers have on student learning, it was decided to attack it from a teacher's perspective.

> "If we want to improve student learning, we must find a way to improve teaching in the average classroom. Even slight improvements in the average can affect millions of students" (Stigler & Hiebert, 2004, p.12)

The impact that teachers have on students is undeniable as this quote indicates while others have also found teachers impact upon students' affective and cognitive development (An et al., 2004; Klinger, 2005; Osborne et al., 1997). As a result of this, it is essential that we first address the 'Mathematics Problem' from the point of view of teachers and only when this is achieved will we begin to see improvements in the other areas discussed in the cycle.

This paper will first look at the current situation in Ireland in relation to mathematics education before discussing the importance of subject matter knowledge for effective teaching. Following on from an analysis of previously proposed models of teachers' knowledge the author will then discuss a framework which she believes will help teachers develop the knowledge required for teaching before finally looking at the impact that such improvements could have on mathematics teaching and learning in Ireland.

MATHEMATICS EDUCATION IN AN IRISH CONTEXT

Mathematics education in Ireland has been seriously affected by the 'Mathematics Problem' in recent years. Research suggests that two of the main problems facing Irish mathematics education are the poor uptake of Higher Level mathematics and the levels of attainment currently evident in mathematics state examinations (Chief Examiners Report, 2001; Healy, 2003; Donnelly, 2007) .

Uptake of Mathematics in Ireland

Mathematics is one of two compulsory subjects for the Leaving Certificate (Established) and as a result of this policy, research indicates that the number of students studying mathematics in Ireland in upper second level education is above the international average (NCCA, 2005). However when such findings are investigated further it is obvious that an underlying problem still exists in this regard. Researchers, internationally, have found that students tend to opt for lower level mathematics at senior cycle (Bagnall, 2002) and unfortunately Ireland has not been able to avoid this problem. In 2002 only 17.5% of the 55,496 students who sat Leaving Certificate mathematics chose the higher level paper, while in 2007 this figure fell to 15.6% (Donnelly, 2007; NCCA, 2005). In 2009 the figure is expected to again be well below 20% and this has resulted in people raising questions about the Irish government's ability to deliver a 'knowledge economy' (Flynn, 2009).

Despite such findings, the new 'Project Maths' initiative, which was launched in September 2008 aims to raise this figure to 30% in the near future. Such an increase would be welcomed wholeheartedly. However if this goal were to be achieved a worrying problem would still exist, when the numbers taking higher level mathematics are compared with similar findings for the subject of English. Like mathematics, the majority of students in Senior Cycle study English. However according to the NCCA (2005) in 2004 over 60% of students sat the higher level paper. The reasons for such a difference in uptake range from poor attitudes towards mathematics to the perception of mathematics as a time consuming and difficult subject (English et al. 1991; NCCA, 2005). However these negative perceptions need to be overcome in order to ensure a high number of students graduate from second level with high levels of mathematical skills which isn't currently the case.

Attainment Levels in Mathematics

> "…poor performance in Maths can exclude students from almost all third level courses in universities and institutes of technology"
> (Donnelly, 2007)

This quote underlines the importance of achieving good grades in Senior Cycle mathematics. The attainment of good grades can play a decisive role in deciding the future career choices of many students. Since the turn of the century levels of attainment both at higher and ordinary level appear to be improving. In relation to higher level, Kennedy (2007) found that of the cohort of students who sat the higher level paper in 2007 80% obtained an honour (Grade A/B/C) an increase of 2.7% from 2004 (NCCA, 2005). The Department of Education and Science (2008) also found that 97% of students who sat the higher level paper in 2008 passed (obtained a D3 or higher) which is a slight increase from the 96.2% who passed in 2007. The results in the ordinary level paper show that in 2000, 65.9% of students achieved an honour in ordinary level mathematics while this figure rose to 69.2% in 2004. Despite a slight decrease to 67.4% in 2008 these figures still appear to offer much hope for the future of mathematics education in Ireland (DES, 2008).

Despite this positive picture further investigation suggests that problems do still exist in relation to mathematical attainment. Firstly when compared to international studies it appears that the levels of achievement in Ireland are lagging behind those being attained internationally. According to the Irish Mathematical Society (2006) Ireland's top mathematical students are failing to reach the same levels of mathematical competency as students of the same age in different countries. Secondly, in the Chief Examiner's Report on Ordinary Level Geography (2002) it was found that in that year 73.3% of students sitting the ordinary level paper obtained an honour and this is compared with 62.6% in mathematics. As a result, despite recent improvements, mathematical attainment still appears to be lower than achievement levels in other subjects and in other countries. Finally there exists a problem in relation to the high number of students who fail to achieve a D grade at ordinary level. Of the 2003 cohort, 4,000 failed to attain a grade higher than a D in ordinary level mathematics. This figure, in conjunction with the 5,702 students who sat the Foundation Level paper that year meant that 9,702 (17.3%) students were, as a result of their performance in Leaving Certificate mathematics, prohibited from entering the majority of Universities and Institutes of Technology. This figure rose to 20.4% in 2008 (DES, 2008).

These problems in relation to attainment and uptake provide us with a very challenging picture of the standards of mathematics teaching and learning in Ireland and their consequences are far reaching.

The Impact of Poor Uptake and Attainment Levels on Third Level and Beyond

These two problems are contributing to the worsening problem of under preparedness among students entering third level.

> Evidence is accumulating that the incoming level of mathematical expertise [to higher education institutions] – hence, the expertise of students who achieved a D grade or higher in the Leaving Certificate Ordinary level examination – is insufficient, and does not match expectations (NCCA, 2005, p.16)

Overall it is clear that the gap that currently exists between school mathematics and 3rd level mathematics is having a detrimental effect both on students, the higher level institutes and the Irish economy. In England it is feared that if the gap between second and third level continues to exist then the knock on effect will result in Britain being unable to keep abreast of international developments in science, engineering and technology (LMS, 2005). The LMS (1995: 6) believes that Britain will become even more dependent on foreign nations for '…inventions, specialists and products' and the entire economy will suffer. In order for the already faltering Irish economy to avoid a similar fate in the near future the problem of under preparedness in Ireland needs to be rectified. The author will now look at the role which teacher subject knowledge may play in such remedies.

IMPORTANCE OF SUBJECT MATTER KNOWLEDGE

According to Askew & Williams (1995, p. 42) 'many aspects of mathematics teaching are under researched'. One such area appears to be the knowledge base required for mathematics teaching and the impact of such knowledge on student learning (Shulman, 1986; Stigler & Hiebert, 2004). In the next section the authors will look at different knowledge types that they and previous researchers feel are necessary in order to develop knowledge for effective teaching. Before any model of knowledge is developed it is essential that teachers first develop a strong knowledge of mathematical content. The work of Ball et al (2005) indicates that subject matter knowledge includes the common knowledge of mathematics that the majority of adults have access to as well as specialised knowledge for the purpose of teaching. Boero et al., (1996, p. 1099) state that 'He who knows mathematics knows how to teach it'. The significance of subject matter knowledge for the profession of teaching is undeniable. Shulman (1986) suggests that it is impossible for teachers to prepare to successfully teach mathematical content that they themselves have never had the opportunity to learn. Many aspects of effective teaching such as the use of resources, a teacher's ability to link mathematics to everyday life as well other subjects, the need for clear explanations and effectively involving students in classroom proceedings are all affected by a teacher's level of content knowledge (Ernest, 1989; Watson, 2008). Levels of subject matter knowledge have also been seen to impact upon student learning. Hill et al. (2005) found that there was a strong link between the score teachers attained on a test of their mathematical knowledge and academic gains made by their students. While Fennema & Franke (1992, p. 148) state:

> …one of the most widely offered explanations of why students do not learn mathematics is the inadequacy of their teachers' knowledge of mathematics

From such findings it is evident that teachers' subject matter knowledge has a big impact on the attainment levels of their students, which the authors have shown to be an area of serious concern in Ireland. Levels of subject matter knowledge among teachers have been shown to be deficient in the United States, Australia, Great Britain and Spain (Ma, 1999; Bagnall, 2002; Blanco, 2003; Smith, 2004). The author intends to assess the current levels of knowledge among Senior Cycle teachers in the near future but currently it is assumed that Ireland is following international trends in relation to low levels of knowledge among teachers. As a result and due to the importance of content knowledge as well as the influence which we know teachers have on students it is essential that we look to develop teachers' knowledge base. This in turn will help to resolve the mathematics problem.

MODELS OF TEACHERS' KNOWLEDGE BASE: A THEORETICAL FRAMEWORK

In order to identify a 'package of knowledge' which the author considers to be essential for the effective teaching of mathematics an analysis of other key theoretical frameworks was required. In the past many models of teacher knowledge have been proposed including those drawn up by Shulman (1986), Ernest (1989), Fennema & Franke (1992) and Rowland (2007) and it is these four that the authors will first analyse before discussing a model which they designed, one which they believe is better equipped to dealing with the problems we face in the Irish mathematics system.

Analysis of Models of Teachers' Knowledge

The idea of teachers requiring a number of different knowledge types was first introduced by Shulman (1986). His work suggested that there were three main domains in which teachers were required to be competent in, notably, subject matter content knowledge, pedagogical content knowledge and curricular knowledge. The first and most important component which Shulman (1986) alluded to was subject matter content knowledge, which he referred to as a knowledge of content (in our case mathematics), as well as a knowledge of the structure of the subject. It is clear from his work that subject matter knowledge takes precedence over all other knowledge types. Only when teachers have mastered content knowledge can they begin to focus on pedagogical knowledge. This refers to a teacher's ability to use mathematical representations and applications which the students understand. Shulmans' work would suggest that content knowledge feeds into pedagogical knowledge and it is only when one has mastered content knowledge can he/she become concerned with identifying and using appropriate representations, analogies and demonstrations. Shulman (1986) accepts that pedagogical content knowledge plays a secondary role to subject matter knowledge yet he feels it is still an important element of a teacher's knowledge base and therefore needs to be afforded more attention during teacher training and throughout ones professional career. He also believes the third element of his model needs to be given more attention. Shulman (1986) believes that curricular knowledge will allow teachers to enhance their pedagogical capabilities. However as with pedagogical knowledge this knowledge has little use without profound subject matter knowledge. Overall Shulman (1986) believes that the three types of knowledge discussed, when combined, provide teachers with sufficient knowledge for teaching and each domain contributes significantly to a teacher's ability to teach effectively.

In 1989 Ernest also looked at this issue and he too designed a model of the knowledge and beliefs which he believed to be essential for teaching but unlike Shulman's (1987) his was actually mathematics specific. The areas included in Ernest's model are

- Knowledge of Mathematics
- Knowledge of Other Subject Matter
- Knowledge of Teaching Mathematics
- Classroom Organisation and Management for Mathematics Teaching
- Knowledge of the Context of Teaching Mathematics
- Knowledge of Education

Despite its complex appearance many of the components are the exact same as those included by Shulman (1986). For example we again see content knowledge at the top of the list. Like Shulman, Ernest too, believes that this type of knowledge provides teachers with an essential foundation for mathematics teaching and nothing can be achieved prior to competency in mathematical content being accomplished. Furthermore, Ernest (1989) also included a knowledge of the curriculum and pedagogical knowledge under the heading Knowledge of Teaching Mathematics. Some additional components that Ernest (1989) saw fit to include were a knowledge of other subject matters so as to be able to make connections between mathematics and other subjects being taught in the schools and to gain an insight into the pedagogical principles adopted in other classrooms that may be used for mathematics. Ernest (1989) also found it important for teachers to develop a knowledge of students both as individuals and as members of the larger class group. Overall the comprehensive model proposed by Ernest (1989) contains many of the components discussed by Shulman (1986). Despite the detailed nature of this model, Ernest (1989) presents a strong argument for the inclusion of each knowledge type and although the latter few are not dealt with in too much detail he still feels it is important that they are recognised as important elements of a teacher's repertoire of knowledge.

Fennema & Franke (1992) were the next in the field to put forward a model. They adopted a similar approach to previous researchers but from the outset they stated their belief that all components were interlinked and each served to enhance the next. The types of knowledge proposed in their model are: content knowledge, knowledge of learning, knowledge of mathematical representations and pedagogical knowledge. As with the previous two models Fennema & Franke (1992) believe content knowledge to again be the most important knowledge for teaching. They believe that in order for teachers to help students learn mathematics they themselves need a strong foundation in the subject. They acknowledge that there is little evidence to suggest a strong link between teacher subject knowledge and student attainment but they still refuse to dismiss the importance of subject matter knowledge because like other researchers they believe it to provide a foundation for all other knowledge types. The next knowledge discussed by Fennema & Franke (1992) is a knowledge of student learning. They believe it is essential for teachers to understand individual student learning in order to deal with mixed ability and diversity in the classroom. Again this knowledge domain appears to have little impact when analysed in isolation but serves to enhance the standard of teaching and in turn learning when combined with other types of knowledge included in the model. The final two components of this model have again featured in previously discussed models and involve the transformation of knowledge into material students can understand and the decisions made by teachers during instruction as well as those made in the planning phase. Overall the package of knowledge proposed by Fennema & Franke (1992) incorporates many of the same types of knowledge that Shulman (1986) and Ernest (1989) included. Throughout their work, however, Fennema & Franke (1992) highlighted the interrelated nature of a teacher's knowledge base and it is essential that future researchers also take this into consideration when designing new or updated models.

The most recent research carried out in this field was reported by Tim Rowland in 2007. He labelled his model as the 'Knowledge Quartet'. The four components of this model were categorised as foundation, transformation, connection and contingency. To date we have seen all researchers in the field attribute huge importance to subject matter knowledge and Rowland (2007) continues this trend. Foundation knowledge, which he deems most important, refers to content knowledge. As the name suggests, he too believes this to be the foundation on which all other knowledges can be built. The second component in this model, transformation knowledge, is what would have been traditionally called pedagogical knowledge. Despite this knowledge not being considered the most important knowledge Rowland (2007) accepts that it is this component that distinguishes a mathematics teacher's knowledge from that of a mathematician. The third element of Rowland's quartet relates to teachers' knowledge of students and their learning. This relates to teachers knowing how to arrange or sequence topics in an order that students understand. It also involves having sufficient knowledge of mathematics to make connections between topics or a knowledge of other subjects to highlight the relevance and importance of mathematics. The importance of this type of knowledge was highlighted in the work of Brown et al. (1999) whereby they found five out of six teachers, deemed effective, demonstrated this type of knowledge. The final component of the quartet, contingency, relates to a teacher having the knowledge and flexibility to deal with the unexpected. This knowledge involves the teacher being aware of how to respond to unexpected questions or answers from students as well as having the awareness to know when and the flexibility to know how to alter an intended lesson plan due to unforeseen circumstances.

The Ladder of Knowledge

Overall the models proposed in the past do not differ greatly from each other and many of the same elements/characteristics will also be evident in the authors' own model. However due to their own mathematical philosophies and experiences the authors felt it necessary to design a unique model that would meet the needs and address the problems facing mathematics education in Ireland. The model which they propose to serve this purpose is the Ladder of Knowledge.

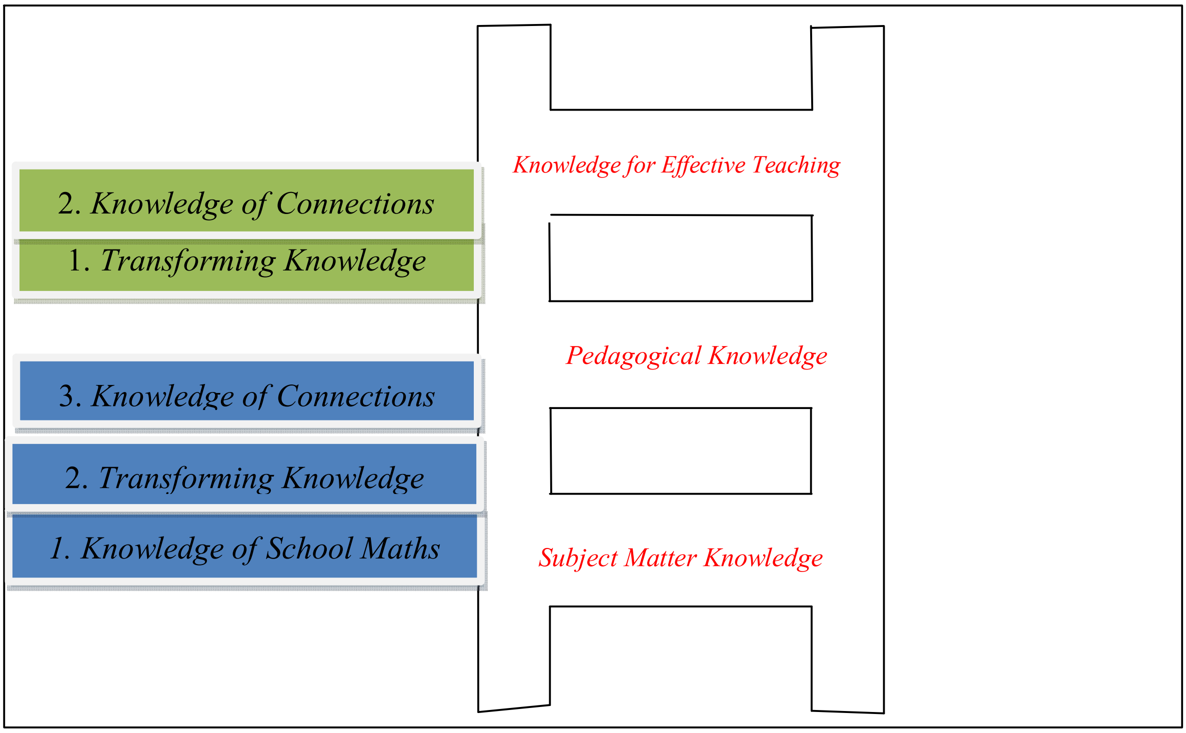


Figure 1: The Ladder of Knowledge

The rungs on the ladder analogy highlight the three types of knowledge which the authors believe teachers must develop in order to demonstrate sufficient knowledge for teaching. It also demonstrates the importance she attributes to each as well as the associated knowledge she considers essential to get from one rung to the next. The importance of content knowledge has already been discussed in great detail and the authors believe no teacher can attempt to get on the ladder with out a deep understanding of mathematical content. Once this deep understanding has been acquired teachers may then concern themselves with reaching the second rung

> …it does not make good sense…to make teachers believe that they can make a full scale assault on pedagogical content knowledge without first acquiring a strong content knowledge. If we ask them to run before they can walk, they will surely fall flat on their faces (Wu, 2005, p. 2).

In order to reach this second rung they must first develop a knowledge of school mathematics. They must become familiar with what school mathematics entails as well as the relevance and more importantly the applications of such content. Secondly, they must know how to transform the knowledge they have acquired into representations and analogies that students will understand and appreciate. The final element which teachers need to develop before progressing to the second rung is a knowledge of connections. They must know how to connect particular mathematics strands to others previously taught as well as connecting mathematics to other subjects, an aspect that others often referred to as curricular knowledge. They must also be aware of how school mathematics connects with historical milestones in mathematics to again show the relevance of school mathematics and to develop an appreciation of the subject among students. Once the second rung has been reached and teachers develop pedagogical knowledge they can then strive to reach the final rung which the authors believe to be the elusive knowledge that distinguishes mathematics teachers from mathematicians. In order to be proficient in this area a teacher must be capable of connecting or combining the knowledge they would have acquired to date and they must then be capable of transforming these different types of knowledge in order to transfer their extensive knowledge to students in an effective manner.

CONCLUSIONS & FUTURE WORK

This model clearly highlights the knowledge domains that are critical for effective teaching and the authors intend to help mathematics teachers' progress on this ladder through Continued Professional Development. With a model now in place the author will seek to help in service mathematics teachers to progress through this model through the use of continued professional development. The authors hope to refine this model in the future through fieldwork and additional desk research and ultimately to develop a viable model that will help teachers become aware of and develop appropriate knowledge for mathematics teaching. These improvements in teaching will have a knock on effect on students, the education system; third level institutes, industry in Ireland and the Irish economy and mathematics education in Ireland will finally be seen to be moving in the right direction.

REFERENCES

An, S. Kulm, G. & Wu, Z. (2004) The Pedagogical Content Knowledge of Middle School, Mathematics Teachers in China and the U.S., Journal of Mathematics Teacher Education, 7, 145 – 172.

Askew, M. and William, D. (1995) Recent Research in Mathematics Education 5 – 16, London: Her Majesty's Stationary Office.

Bagnall, D. (2002) *The Not So Clever Country,* The Bulletin – October 2002 [online]. Available: http://www.austms.org.au/Jobs/Library3.html [Accessed: 15th April 2008].

Ball, D.L., Hill, H.C., Bass, H. (2005) *Knowing Mathematics for Teaching: Who Knows Mathematics Well Enough to Teach Third Grade, and How Can We Decide,* paper presented at the Annual Meeting of the American Educational Research Association 2005, Montreal, Canada.

Blanco, L.J. (2003) *The Mathematical Education of Primary Teachers in Spain,* International Journal for Mathematics Teaching and Learning [online] April 18th 2003, pp: 1 – 10, Available: http://www.cimt.plymouth.ac.uk/journal/blanco1.pdf [Accessed: 8th April 2008].

Boero, P., Dapueto, C. and Parenti, L. (1996) *Didactics of Mathematics and the Professional Knowledge of Teachers* in A.J. Bishop et al. (eds.), International Handbook of Mathematics Education: Part 2, Holland: Kluwer Academic Publications, 1097 – 1121.

Brown, T., McNamara, O., Hanley, U. & Jones, L. (1999) *Primary Student Teachers' Understanding of Mathematics and its Teaching,* British Educational Research Journal, 25(3), 299 – 322.

Department of Education and Science (2008) *Percentage Breakdown of Candidate by Grade Awarded in Each Subject,* [online], Available: http://www.examinations.ie/statistics/statistics_2008/nat_stats_2008n.pdf, [Accessed: 8th December, 2008]

Donnelly, K. (2007) *High Failure Rates in Maths and Science Worries Employers,* Irish Independent, 15th August 2007.

English, J., O'Donoghue, J. & Bajpai, A.C. (1991) *What Irish Pupils Say About Mathematics,* International Journal of Mathematics Education in Science and Technology, 23(5), 749 – 763.

Ernest, P. (1989) The Knowledge, Beliefs and Attitudes of the Mathematics Teacher: A Model, Journal of Education for Teaching, 15(1), 13 – 33.

Fennema, E. & Franke, M.L. (1992) Teachers' Knowledge and Its Impact in D.A Grouws (ed.) Handbook of Research on Mathematics Teaching and Learning, New York: Macmillan Publishing Company, 147 – 164.

Furinghetti, F. (2000) The History of Mathematics as a Coupling Link between Secondary and University Teaching, International Journal of Mathematics Education in Science and Technology, 31(1), 43 - 51

Gill, O. & O'Donoghue, J. (2006) Exploring Mathematics Under – Preparedness at Third Level in Ireland, in Moore, S., Carr, P., Crehan, M., Crowley, J., Flynn, J., Morgan, P. & Shannon, P. (eds.) Keeping Students at University: The Retention Debate at Third Level, Dublin: Intersource Group Publishing.

Hill, H.C., Rowan, B., & Ball, D.A. (2005) *Effects of Teachers' Mathematical Knowledge for Teaching on Student Achievement,* American Educational Research Journal, 42(2), 371 – 406.

Irish Mathematical Society (2006) NCCA Review of Mathematics in Post Primary Education: Response of the Royal Irish Academy Committee for Mathematical Sciences, Bulletin 57, p. 11 – 20.

Kennedy, J. (2007) *Leaving Cert Maths Numbers Don't Add Up,* [online], Available: http://www.siliconrepublic.com/news/news.nv?storyid=single9004, [Accessed: 15th June 2008]

Klinger, C. (2005) Challenging Negative Attitudes, Low Self – efficacy Beliefs and Maths Anxiety in Pre – Tertiary Learners, Proceedings of the 12th International Confernece on Adults Learning Mathematics, 3rd – 7th July 2005, Melbourne, Australia.

London Mathematical Society (1995) *Tackling the Mathematics Problem*, Southend – on – Sea: LMS.

Ma, L. (1999) Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States, New Jersey: Laurence Erlbaum Associates.

National Council for Curriculum and Assessment (2005) *Review of Mathematics in Post Primary Education: a discussion paper,* Department of Education and Science, Dublin: The Stationary Office.

Osborne, J.P., Black, P., Boaler, J., Brown, M., Driver, R. & Murray, R. (1997) attitudes to Science, Mathematics and Teachnology: A Review of Research, London: King's College.

Rowland, T. (2007) Developing Knowledge for Mathematics teaching: A Theoretical Loop, paper presented at Second National Conference on Research in Mathematics Education in Ireland: MEI 2, St. Patricks College, Dublin, 14th – 15th September 2007.

Shulman, L.S. (1986) Those Who Understand: Knowledge Growth in Teaching, Educational Researcher, 15(2), 4 – 14.

Smith, A. (2004) *Making Mathematics Count,* The Report of Professor Adrian Smith's Inquiry into Post – 14 Mathematics Education.

Stigler, J.W. & Hibbert J. (2004), Improving Mathematics Teaching, Educational Leadership (Association for Supervision and Curriculum Development), February 2004, 12 – 17.

CONTINUOUS PROFESSIONAL DEVELOPMENT IN MATHEMATICS EDUCATION: CONSIDERATIONS FOR THE IRISH CONTEXT

Mark Prendergast, University of Limerick

John O'Donoghue, NCE – MSTL,

There has been much focus of late concerning effective teaching methodologies in Irish classrooms, particularly in relation to second level mathematics. Dwindling numbers of students opting to pursue the subject at higher level in state examinations is just one of the factors motivating this concern. Traditionalists continue to voice the advantages of a behaviourist approach focusing on 'drill and practice'. Reformists on the other hand argue about the benefits of a more constructivist centred classroom with a focus on group work and collaborative learning. However, simply identifying successful practices and methodologies will not lead to immediately effective mathematics teaching. The next step must be to train and promote mathematics teachers to use these new methodologies along with other advances in resources and technology. This is where the issue of Continuous Professional Development (CPD) comes to the fore. Professional development is the ticket to reform (Wilson and Berne, 1998) and is undoubtedly a central component to effective mathematics teaching. This paper will examine the relevant literature regarding CPD, and a comprehensive review will be provided. Conclusions will also be reached about developing a CPD program which will have the potential to successfully bring effective teaching methodologies to the Irish mathematics teacher and their classrooms.

INTRODUCTION

The provision of high quality Continuous Professional Development (CPD) is a central component to effective teaching (Smith, 2004). Teaching is a complex practice that can be learned and continually improved (Ball, 2001), starting with initial training and continuing until retirement. It is an active process. However, development does not happen merely as a result of years of teaching. Dean (1991) makes the point that teachers' must actually work to develop. Continuous professional development involves many possible forms of support for teachers to update and enhance their subject knowledge and pedagogy and to sustain their enthusiasm and commitment to teaching (Day, 1999). Wide ranging research in education has led to changes in content, applications and assessment of mathematics. Advances in technology have also impacted both on the subject matter and on possible modes of teaching and learning the subject. Thus, each day teachers are met with new challenges in their classrooms. This is hard work. The support through effective CPD is essential. Teachers must be afforded ongoing opportunities to develop their knowledge of mathematics, both from subject matter and pedagogical perspectives (Ball, 2001). Such opportunities are not readily available and this is one of the main challenges facing mathematics education at present (Smith, 2004).

PURPOSE OF CPD

Tomlinson (1997: 28) states that *'the ultimate aim of professional development in schools is to improve the quality of learning and teaching'*. The main focus of recent research in mathematics education has been directed towards finding ways of increasing student learning. However, the focus of CPD lies with increasing teacher learning. It is perceived that the knock on effect of this will thus increase student learning. Hence, the main purpose of CPD, is improving teacher learning with the hope of initiating change.

IMPORTANCE OF CPD

'Teacher professional development is essential to efforts to improve our schools'

(Borko, 2004:3)

As highlighted in the Introduction there have been many advancements and developments in the teaching and learning of the mathematics. However, these cannot impact on the educational lives of students unless their teachers have the knowledge, know-how and motivation to do so (Borko, 2004). This is where the importance of CPD comes to the fore. In order to foster students' understanding of a subject, teachers must have a rich and flexible knowledge of the material they teach (Smith, 2004). In the case of mathematics teachers, they must understand the central facts and concepts of the discipline, how these theories are connected and the process used to establish new knowledge (Anderson, 1989; Ball, 1990 as cited in Borko, 2004). This is certainly not an easy endeavour and can pose problems for teachers whose understanding is limited. This is a particular concern in Ireland at present as there are a high proportion of teachers, teaching the subject whose degree does not contain a significant, if any mathematics component (NCCA, 2006). However, the importance of CPD is also important for experienced mathematics teachers. Teachers today remain in their positions for long periods of time. It is essential they are given the opportunity to refresh their skills and to renew their enthusiasm for the subject. They must also be kept up to date in new content and curricular changes such as those proposed by *Project Maths*. *'Teachers need the opportunity to develop their understanding of mathematics and their teaching throughout their careers'* (MET Report, 2001 as cited in Cuoco, 2003: 785). Smith (2004) ascertained that this is essential in order to develop and challenge the full range of students whom they teach.

FORMS OF CPD

Borko (2004) points out that for teachers, learning can occur in many different aspects of practice. The daily experience of the classroom is one such practice (Cohen and Ball, 1999). Each day teachers are met with new challenges in their classrooms and Borko

(2004) considers these as having possibilities for their learning. In addition to the classroom, the school community is also an influential source of teacher learning. Traditionally the majority of teacher learning occurred in the community of practice (Alder, 2000). Such learning consisted of conversations with colleagues, observing another teacher's classroom or advice offered in the staff room. However, the classroom has been dramatically altered, necessitating different skills and competencies even on the part of the more experienced teachers (Cawelti, 1993 as cited in Basinger, 2003). Hence other forms of CPD are needed.

The OECD (1998) give a description of the forms of Irish in – service programmes which they describe as a variety of models ranging on a continuum from short one day courses to post graduate programmes leading to diplomas, MAs or PhDs. However, the numbers of teachers involved are small. The forms of CPD undertaken by the majority of teachers, involve in – school developments or short or once off courses. Such courses have come under much criticism of late, particularly because of their short duration and the lack of any sufficient follow up. Research conducted in Ireland by Finucane (2004) determines that the average amount of time spent on CPD by respondents in her study is 2.5 days a year. This figure suggests that a mere 0.7 per cent of a year is spent by teachers on their development. This is just one of the many problems facing CPD at present.

PROBLEMS WITH CURRENT CPD

In the U.S., as far back as 1967, Davies offered a strong condemnation of CPD in his testimony before the Senate Subcommittee on Education. He concluded, '*In-service education is the slum of American education-disadvantaged, poverty stricken, neglected, psychologically isolated, riddled with exploitation, broken promises, and conflict*' (as cited in Guskey, 1986:5). Nearly thirty years later Sykes (1996:465) characterises the failure of professional development as '*the most serious unsolved problem for policy and practice in American education today'*. Such problems affecting CPD are not just confined to the U.S. Research from an Irish perspective is sparse but evidence from Finucane (2004) paints a bleak picture regarding CPD in this country. Problems are also widely reported in the U.K. Respondents to the Smith Inquiry (2004) have noted with concern that in contrast to other professions, there is not a strong tradition of CPD among teachers in England, Northern Ireland and Wales. Responses from England and Northern Ireland in particular indicate clear needs for both subject matter and pedagogy CPD. Teachers in Northern Ireland also expressed the view that more mathematics subject specific CPD would be desirable. Furthermore in the U.K., the Advisory Committee on Mathematics Education (ACME) Report (2002) makes it clear that there is an urgent need to provide infrastructure to support the enhancement of existing teachers of mathematics. In this section the authors will examine why there is such an urgent need and why existing CPD infrastructure is in such a bad condition. Problems such as the short duration and lack of follow up to courses were already mentioned. However, additional problems which may not be so apparent will now be examined.

Table 1: Influences on participation at in – service programs

Participation in In – service was influenced by:	%
School based decision influenced by govt policies/ initiatives	59.5
Solely external priorities	1.4
Personal choice influenced by govt policies/ initiatives	18.9
Personal choice based on self development needs	20.3

(Source: Finucane, 2004:88)

Borko (2004) concedes that current CPD is severely fragmented. Wilson and Berne (1998) go on to reinforce this by acknowledging that while some teachers pursue any opportunity to learn with passion, others simply attend workshops when mandates arrive in their school mailbox or to get a day off school. This was evidenced in Ireland in research carried out by Finucane (2004). As illustrated in Table 1, almost 60 per cent of the respondents in her study reported that their participation at in – service was decided upon by their schools, based on Government initiatives. Just over 20 per cent of teachers made a personal choice to attend based on self development needs. Such a figure is in stark contrast to the findings of Osterman and Kottkamp (1993), who argued that the motivating force behind CPD is a personal desire to improve one's teaching and professional capacity.

Many additional problems concerning CPD were evident throughout the literature. Newly qualified teachers require appropriate induction and support (Smith, 2004). However, the ACME (2002) report mentions that in the U.K there is currently a shortage of individuals with appropriate experience and expertise to offer such support and guidance to teachers of mathematics. The prospect of utilising experienced teachers in CPD programs must be investigated in more detail. As John Dewey, the famous American psychologist, philosopher and educational reformer pointed out, '*one of the saddest things about education is that the wisdom of our most successful teachers is lost to the profession when they retire'* (Stigler and Hiebert, 2004:1).

In addition to a shortage of experienced individuals, the ACME (2002) report also notes that there is a shortage of money for professional development generally. The report concludes that current levels of resources are inadequate to even begin to address current concerns relating to mathematics CPD needs. Undoubtedly, the mathematics teaching profession will not develop a culture of CPD unless continuous and improved funding is made available.

It was noted previously that one of the main purposes of CPD was to bring about change. However, this change also brings a certain amount of anxiety and can be very threatening to teachers (Guskey, 1986). To change or to try something new, means to risk failure. Students might learn less well than they do under current practices. Hence, like practitioners in many other professions, teachers are reluctant to adopt new practices or procedures unless they feel sure they can make them work (Lortie, 1975 as cited in Guskey, 1986). Therefore, even when presented with evidence from the most prestigious research, teachers do not easily alter or discard the practices which they have developed and become familiar with over the years.

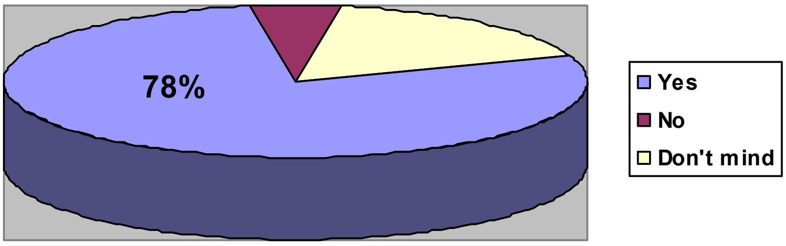
The advantages of having a strong community of learning amongst teachers were already outlined. However, central to this community was a deep sense of trust. This may take time to develop, and its absence is often an inhibiting factor to effective CPD. Teachers are often so concerned with presenting themselves as ‘good teachers’ that this compromises their ability to share their problems (Wilson and Berne, 1998). Admitting that one's practice is less than perfect is an act of exposure that depends on group trust and mutual respect (Wilson and Berne, 1998). Such trust and respect, in a profession often besieged by consistent criticism, takes time to develop.

Accordingly, the many problems and inhibitors to CPD have been identified and examined. With these in mind, the next section will propose a number of possible improvements to CPD mathematics programs with the aim of increasing its overall quality from an Irish perspective.

POSSIBLE IMPROVEMENTS WITH IRISH CPD

One of the most prominent difficulties facing CPD as evidenced from the previous section was how to encourage the active participation of teachers in such programs. In the U.K. the ACME (2002) report envisages that it may be necessary to reward teachers of mathematics who engage in CPD in some way - for example by salary increments on completion of accredited components of CPD. The report also suggests that building up a CPD portfolio should become an important part of career progression, and the key to higher salaries and promotion. The debate of whether or not CPD should be accredited in such a way is rife throughout the literature. More than three quarters of respondents to Finucane’s (2004) Irish study agreed that CPD should be accredited (see Figure 1).

Figure 1: Should CPD be accredited?



(Finucane, 2004: 98)

One respondent to Finucane's study believed that accreditation would be a way of increasing teacher professionalism; others believed that accreditation would be an incentive to motivate people to develop themselves. A number of respondents believed that accreditation should be used as a means of promotion and salary increase (Finucane, 2004). Hence, the case for teacher accreditation in Ireland must be considered. Based on Finucane's (2004) research, accreditation of some form would undoubtedly lead to higher rates of teacher participation.

However, teacher participation is not the only problem. The next difficulty is how to encourage teachers to implement new practices. As established in the previous section many teachers hold concerns about how these new practices would affect their classrooms. Guskey (1986) points out it is important to remember that very few teachers will leave a staff development initiative thoroughly convinced that a new program or innovation will work for them. Teachers must be encouraged to try the new practices, at least on a trial basis. Guskey (1986) determines that these new practices will be accepted and retained when teachers perceive them as having increased their success with students. Therefore, plans for implementing a new program or innovation should also include some procedures by which teachers can measure the effectiveness of their efforts. These procedures may be in the form of informal, classroom based tests. Such assessment provides teachers with direct evidence of the results of their efforts and show clearly and accurately the improvements made in students' achievement.

Other problems mentioned in relation to CPD programs included the lack of any follow – up activities. Follow up of professional development is extremely important (Wilson and Berne, 1998). Fullan (1982) reminds us that no matter how much effective staff development occurs, it is when teachers actually try to implement a new approach that they have the most specific concerns and doubts. Hence providing continued support following the initial training is crucially important and must not be overlooked.

There are many other possible improvements to CPD offered through the literature, touching on a wide range of issues. These improvements have originated from other countries but could certainly be adapted to the Irish system.

For example;

~ Research carried out in New Zealand by Irwin (1994) highlights the benefits of a peer mentoring system. Irwin feels that teachers have a great deal to learn from observing colleagues and skilled practitioners in their own and other schools.

~ In the U.K. Smith (2004) ideally feels that every teacher should have a personal professional development plan, to which both teacher and school commit in writing, placing obligation for continuing CPD on them both.

~ The ACME (2002) report in the U.K. recognises the need to create both a national centre and local centre's to support and deliver CPD for mathematics teachers. Without such a local network, respondents are agreed that the majority of teachers will feel too remote from a national centre to become involved in developments. However, the national centre is also seen as essential to provide strategic direction and coordination of expertise in all aspects of the support of the teaching and learning of mathematics, as well as to provide a focus for close working with national stakeholders.

~ In the U.K. Cordingley et al. (2003) calls for teachers to take ownership of their own professional development. This is reinforced by Basinger (2003) who concedes that teachers must be involved in planning and implementing their own professional development activities. *'Teachers should determine the shape and course of their own development'* (Ball, 1996: 502).

~ In the U.S. Marx et al. (1998) calls for a collaborative effort between researchers and teachers in developing effective CPD. He feels such collaboration can contribute to the learning of both groups. Researchers bring new knowledge to inform teachers of new practices and concepts. Teachers bring experience with students and contexts, knowledge of the limits imposed by curriculum frameworks and knowledge of the everyday life and happenings in schools.

Many of these improvements make perfect sense and one wonders why they have not been implemented. Perhaps the limitations imposed by time and resources (mentioned previously) have a direct effect here. Or perhaps these improvements are just too idealistic and taken totally out of the school context. When it comes to implementation, there may be too many barriers. The authors feel that it may be more beneficial to take a look at successful initiatives already in place in other countries which have already breached any barriers to their implementation. For example, our Scottish neighbours currently employ a most successful CPD program. One of their initiatives is similar to the proposal previously mentioned by Smith (2004). It involves incorporating CPD responsibilities and entitlements into a written agreement between local authorities and teachers. This was put in place in Scotland following the report of The McCrone Inquiry (2001). The aims of the agreement are to enhance opportunities available to all teachers. Every teacher has an annual CPD plan and every teacher is required to maintain an individual CPD record.

In a similar initiative, the recognised professional status of teachers in the Netherlands brings with it a requirement that 10 per cent of the working year (171 hours) must be devoted to the individuals CPD (Le Metais, 1997). Up to 50 hours may be spent meeting schools' needs (for example the school CPD plan). However, the remainder is at the discretion of the individual teacher who must account for the way in which the time is spent. In addition the problem of losing experienced teachers has also been taken into account by the Dutch. Their Government has introduced a 'package' of shorter hours and less demanding, non contact duties, to retain 'seniors' who are retiring prematurely due to the demands of the job and the lack of salary increases (Le Metais, 1997). Their new duties involve mentoring and induction to newly qualified teachers.

In France, staff are entitled to pay leave (at 85 per cent of salary) to undertake personal or professional training (Le Metais, 1997). This may be to prepare for competitive examinations within or outside the teaching profession, or to undertake other long term studies leading to university credits or diplomas.

Wilson and Berne (1998) confirm that for teachers in countries such as Japan and China developing professional knowledge and skills are part of their work. Not only do they have time to learn and improve their practice as part of the regular work week, but what they work on is practice –curriculum, mathematical content, students' learning- together with other professionals. Their learning is ongoing, systemic and systemically connected to their professional career (Ball, 2001).

Finally, although the many problems facing CPD in the U.S have been well documented, the work being carried out towards effective staff development in the country is worthy of a mention. Recent policy such as the No Child Left Behind Act (2001) appropriately calls for highly qualified teachers and high quality professional development in all academic subjects and at all grade levels (http://www.ed.gov). Similarly, a report released by The Teaching Commission (2004), calls for continued support for teachers. Hence, there can be no doubting that the U.S. recognise the need for CPD. Hopefully, like Ireland they too can turn such policy into practice similar to the efforts already in place in countries such as France, Scotland and Netherlands.

CONCLUSION

The main focus of the authors' current research is concerned with the dwindling numbers of students opting to pursue mathematics at higher level in state examinations. For instance in 2008, figures show that only 17 per cent of the Leaving Cycle cohort took the Leaving Certificate higher level mathematics course (EGFSN, 2008). Among the many cause for such low numbers of students taking the higher level paper, poor teaching has been singled out as a major contributing factor (NCCA, 2005). Subsequently the lack of any purposeful CPD in Ireland is a contributing factor to poor teaching. The era of Irish mathematics teachers attending half day in – services, twice a year must come to an end. Such effort is no longer acceptable in the pursuit of effective teaching.

Many possible improvements to professional development which are in place in other countries have been highlighted. These improvements must be adapted to the Irish education system. Initiatives such as the setting up of national and local centers, the assignment of individual teacher development plans and the collaboration between teachers and researchers must be built upon. These initiatives are the way forward. Matters such as salary increases and promotional prospects on the basis of CPD must also be looked at. The Irish government must follow the lead of countries such as Scotland, China and Japan and put in place a CPD 'system' which works. There is no point continuing to spend millions of euro developing effective teaching methodologies for teachers, if structures are not in place to train and promote teachers to use them. Effective professional development programs provide the access to such structures. As denoted by Shulman (1983: 495), they '*evoke images of the possible'*.

REFERENCES

Adler, J. (2000). Social practice theory and mathematics teacher education: A conversation between theory and practice. *Nordic Mathematics Education Journal,* 8(3), 31–53.

Ball, D. L. (1996). Teacher learning and the mathematics reforms: What do we think we know and what do we need to learn? *Phi Delta Kappan*, 77, 500-508.

Ball, D.L. (2001). 'Developing a mathematically proficient public: What are the problems, what do we know about them, and what would it take to solve them?' Paper prepared for the Aspen Institute congressional conference on '*Promoting Excellence in the New Economy: The Challenges to National Policy*', St, Petersburg, Florida.

Basinger, D. (2003). 'Teachers Integrating Technology: Case Studies', *Essays in Education, 8*. [Online], available: http://www.usca.edu/essays/archives.html [accessed April 2008].

Borko, H. (2004). 'Professional Development and Teacher Learning: Mapping the Terrain', *Educational Researcher,* 33 (8), 3 – 15 [online], available: http://www.aera.net/uploadedFiles/Journals_and_Publications/Journals/Educational_Researcher/Volume_33_No_8/02_ERv33n8_Borko.pdf [accessed May 2008].

Cohen, D. K., and Ball, D. L. (1999). *Instruction, capacity, and improvement.* CPRE Research Report Series RR-43. Philadelphia, PA: Consortium for Policy Research in Education, University of Pennsylvania.

Cordingley, P., Bell, M., Rundell, B. and Evans, D. (2003). *The impact of collaborative Continuing Professional Development (CPD) on classroom teaching and learning,* [online] available http://eppi.ioe.ac.uk/EPPIWeb/home.aspx?&page=/reel/reviews.htm [accessed May 2008].

Cuoco, A. (2003). 'Teaching Mathematics in the United States', *Notices of the AMS*, 50 (7), 777 – 787.

Day, C. (1994). 'Professional Development and reflective practice: purposes, processes and partnerships', *Culture and Society,* 7 (2), 221 – 233.

Dean, J. (1991). *Professional Development in School – Developing Teachers and Teaching*, Philadelphia: Open University Press.

Expert Group on Future Skills Needs (EGFSN) (2008). *Statement on Raising National Mathematical Achievement* [online], available http://www.skillsireland.ie/media/egfsn081215_raising_mathematical_achievement.pdf [accessed April 2009].

Finucane, P. (2004). *Continuing Professional Development: The Experiences and Perceptions of a Sample of Irish Post Primary Teachers,* unpublished thesis, (M.A.), University of Limerick.

Fullan, M. (1982). *The meaning of educational change*, New York: Teachers College Press.

Guskey, T.R. (1986). 'Staff Development and the Process of Teacher Change', *Educational Researcher,* 15 (5), 5 – 12 [online], available: http://www.jstor.org/stable/1174780 [accessed May 2008]

Irwin, K. (1994). 'Ongoing Development as a Teacher of Mathematics'. In J. Neyland (ed), *Mathematics Education: A Handbook for Teachers,* 1, 367 – 374, New Zealand: The Wellington College of Education.

Le Metais, J. (1997). 'Continuing Professional Development: the European Experience'. In Tomlinson, H. (1997) *Managing Continuing Professional Development in Schools,* London: Paul Chapman Publishing Ltd.

Marx, R.W., Blumenfeld, P.C., Krajcik, J.S. and Soloway, E. (1998). 'New Technologies for Teacher Professional Development', *Teaching and Teacher Education,* 14 (1), 35 – 52.

National Council for Curriculum and Assessment (2005) *Review of Mathematics in Post – Primary Education*, Department of Education and Science, Dublin: The Stationary Office.

National Council for Curriculum and Assessment (NCCA) (2006). 'Review of Mathematics in Post – Primary Education: Response of the Royal Irish Academy Committee for Mathematical Sciences', *Irish Mathematics Society Bulletin,* 11 -20.

Organisation for Economic Co – operation and Development (OECD) (1998). *Staying ahead: In – Service Training and Teacher professional Development,* Paris: OECD.

Osterman, K.F. and Kottkamp, R.B. (1993). *Reflective Practice for Educators – Improving Schooling Through Professional Development,* Newbury Park, California: Corwin Press, Inc.

Shulman, L. S. (1983). 'Autonomy and obligation: The remote control of teaching', in L. S. Shulman and G. Sykes (Eds.), *Handbook of teaching and policy* (pp. 484–504). New York: Longman.

Smith, A. (2004). *Making Mathematics Count,* The Report of Professor Adrian Smith's Inquiry into Post – 14 Mathematics Education [online], available: http://www.dfes.gov.uk/mathsinquiry/Maths_Final.pdf [accessed March 2008].

Stigler, J.W. and Hiebert, J. (2004). 'Improving Mathematics Teaching', *Improving Achievement in Math and Science* [online], available: http://www.udel.edu/education/mathed/downloads/Hiebert2004b.pdf [accessed February 2009].

Sykes, G. (1996). 'Reform of and as professional development' *Phi Delta Kappan*, *77*, 465–467.

The McCrone Inquiry (2001). *A Teaching Profession for the 21st Century* [online], available: http://www.snct.org.uk/library/278/2001%20Teachers%20Agreement.pdf [accessed May 2008].

The Teaching Commission (2004). '*Teaching at Risk: A Call to Action'* [online], available: https://uteach.utexas.edu/download.cfm?DownloadFile=3FF40BFA-E6B4-2333-8F9ADEAA8CDDD781 [accessed May 2008].

Tomlinson, H. (1997). *Managing Continuing Professional Development in Schools,* London: Paul Chapman Publishing Ltd.

Wilson, S.M. and Berne, J. (1998). 'Teacher Learning and the Acquisition of Professional Knowledge: An Examination of Research on Contemporary Professional Development', *Review of Research in Education,* 24, 173 – 209 [online], available: www.jstor.org/stable/1167270 [accessed May 2008].

http://www.ed.gov/policy/elsec/guid/states/index.html#nclb [accessed January 2009].

CONSTRUCTING AND VALIDATING AN INSTRUMENT TO MEASURE STUDENTS' ATTITUDES AND BELIEFS ABOUT LEARNING MATHEMATICS

Sinead Breen | Joan Cleary | Ann O'Shea

CASTeL & St Patrick's College | Institute of Technology, Tralee | NUI Maynooth.

When embarking on a study of attitudes and opinions, it is important to design an instrument of inquiry that will provide valid, reliable and interpretable information that addresses the specific question of interest. Reported here are attempts made to evaluate the reliability and validity of a survey instrument using Rasch analysis, factor analysis, and Cronbach's alpha. The instrument aimed to measure students' confidence and anxiety in relation to learning mathematics, and to examine their goal orientation and beliefs on the nature of intelligence, by recording their responses to a number of statements using a Likert-style format. A previous study conducted by the authors failed to endorse Dweck's (1986) claims that a student's theory of intelligence and confidence in his/her present ability combine to determine the student's behaviour when presented with an unfamiliar and/or challenging task. The present study was initiated in order to provide the opportunity to investigate this claim more comprehensively. The techniques (of Rasch and factor analysis) used to investigate the validity of the instrument designed, also facilitate the computation of a quantitative measure of the level of each trait possessed by a student.

INTRODUCTION

Motivation for Study

Dweck (1986) maintains that a student's theory of intelligence (as to whether intelligence is fixed or malleable) and confidence in his/her present ability combine to influence the student's behaviour when presented with an unfamiliar task. As part of a study carried out in 2006 and 2007, first year students in three third level institutions in Ireland were asked whether they believed that mathematical ability could be improved and were asked to rate their confidence in approaching mathematics. However, in that study it was found that an individual's level of confidence but not his/her view of the nature of intelligence played an important role in how he/she approached, persevered with and performed on unfamiliar tasks (Breen, Cleary & O'Shea, 2007). A more comprehensive survey on third-level students' beliefs about the nature of mathematical intelligence and the level of confidence felt in approaching mathematical tasks and courses was undertaken to investigate these issues further. Explicitly included in the questionnaire were items addressing goal orientation, tendency to seek out or avoid challenges, and persistence in the face of difficulty (or unfamiliarity) as Dweck maintains that a child's theory of intelligence orients him towards a certain class of goal (learning or performance) and that this is turn influences his pattern of behaviour (adaptive or maladaptive) which can be identified by the child's attitude towards challenges and the levels of persistence or perseverance he displays (Dweck, 1986; Elliot & Dweck, 1988).

Traditionally, factor analysis has been used to validate rating-scale instruments by seeking to establish the underlying factorial structure of the data, uncovering the dimensions inherent in the dataset. For instance, in an investigation of the process by which theories of intelligence and performance affect student learning, Stipek and Gralinski (1996) used factor analysis to identify the dimensions underlying their items assessing beliefs, and verified the reliability of the two factors found by computation of the associated alpha coefficients. Mulhern and Rae (1998) evaluated the reliability of the Fennema-Sherman Mathematics Attitudes scales for a sample of 196 secondary schoolchildren using Cronbach's alphas and then also employed factor analysis to investigate the validity of these scales. Their findings lead them to develop a shorter instrument involving only 51 of the original 108 items, while retaining a comparable level of reliability.

In recent times, Rasch analysis has also been used for the purpose of instrument validation. A systematic evaluation of the impact of technology on the teaching and learning of mathematics was reported by Galbraith and Haines (1998). They used Rasch analysis to judge the effectiveness of the rating-scales they developed, including reliability and goodness-of-fit. Apparent associations between their attitude scales (e.g. mathematics confidence, computer confidence) were further explored using factor analysis. Bradley, Sampson & Royal (2006) emphasised the fundamental role the quality of a measurement tool plays in the analysis of the data it seeks to produce and lamented the lack of attention sometimes suffered by this phase of a research study. They employed Rasch analysis to address this concern for their own survey instrument. Meanwhile, Grimbeek and Nisbet (2006) explained how they had previously used factor analysis to examine a dataset of primary teachers' beliefs about compulsory numeracy testing in Queensland, but decided to improve on the outcome by utilising a combination of Rasch analysis and an iterative sequence of factor analyses. They reported this approach to have produced an instrument with a factor structure statistically and conceptually more elegant than their initial model. The fact that Rasch analysis explicitly takes into account the categorical and ordinal nature of Likert-scale items motivated its use by Grimbeek and Nisbet.

METHOD

Description of Study - Design & Administration of Questionnaire

The questionnaire used collected personal information (including gender, age, level of mathematics achievement at post-primary school) from the participants as well as recording responses to sets of rating-scale items relating to Confidence, Anxiety, Theory of Intelligence, Goal Orientation (Learning/Mastery and Performance) and Persistence. All rating-scale items were presented using a 5-point Likert scale where 1 represented 'disagree strongly', 2 'disagree', 3 'not sure', 4 'agree' and 5 'agree strongly'. In developing the questionnaire, the authors were mindful of including sufficiently many items to enable a statistically significant analysis of the data collected, while taking into account the time it would take respondents to complete it. The Confidence and Anxiety items used appear in tables 1 and 2 in the Results section, while the remaining rating-scale items are included in the appendix. (Note that negatively worded questions were coded in reverse order before analysis.)

Most of the items on the rating-scales exploring Goal Orientation (Learning/Performance Goals) and Theory of Intelligence were those used by Stipek & Gralinski (1996), reworded slightly to use terms appropriate to third-level students in Ireland. Theory of Intelligence 2 was modified from Schoenfeld (1989). Items on the Confidence and Anxiety rating scales were similarly adapted from the Fennema-Sherman scales (see Mulhern & Rae, 1998) and those used in the PISA 2003 student questionnaire (OECD, 2003), with the exception of Confidence 4 which was taken from Pietsch, Walker and Chapman (2003). The remainder of the items were constructed for this study, including the Persistence items which were based on hypotheses put forward by Dweck and Elliott as to the behaviour of, and strategies employed by, students with differing goal orientations (Dweck, 1986; Dweck & Elliott, 1988).

186 students from three third level institutions voluntarily participated in this study that was conducted in the second semester of the 2007/2008 academic year. All participants were in the first year of their respective programmes, namely BEd or BA (Humanities) at St Patrick's College, Drumcondra, BA or BA (Finance) at the National University of Ireland, Maynooth and Higher Certificate in Engineering or BSc at Institute of Technology, Tralee. Participants were given approximately 20 minutes to respond to the questionnaire.

Introduction to Rasch Analysis

Rasch analysis is a means of constructing an objective fundamental measurement scale from a set of observations of ordered categorical responses (to assessment items or rating-scale items). In the case of rating-scale items, some will be easier to endorse than others and this is reflected in the measurement scale produced so that the items that are easiest to endorse appear at the bottom of the scale while those that are most difficult to endorse are at the top. For instance, it seems reasonable that the item 'I am so afraid of maths that I avoid going to maths classes' indicates a higher level of math-anxiety than 'I am sometimes afraid that I will make mistakes when doing maths problems', and so the former should appear at a higher level on the resulting measurement scale than the latter. The scale produced is an interval one, and so, a one-unit difference between the positions of a pair of items (that is, between the item difficulties) at any point on the scale reflects the same level of difference in the underlying trait being measured. Moreover, the Rasch model places the respondents on the same scale as the items and relates the two via a probabilistic function of the difference between their positions. Following the assumption that useful measurement involves the consideration of a single trait or construct at a time, the Rasch model incorporates a quality control mechanism, by means of error estimates and fit statistics, to verify that all items are contributing to the measurement of the trait of interest.

RESULTS

Rasch Analysis

In order to usefully employ measures of the affective traits that we mean to study in this project, we need to be confident that the measures are constructed in a scientifically valid manner. To this end, we used Rasch analysis to validate our scales. Item fit statistics as well as reliability statistics were used to check that all items on a scale were addressing the same construct and that the measures were reliable.

We used two specialist software packages to analyse our data. They were Conquest and Winsteps (Linacre, 2009). Both packages gave similar results and we will report the results from the Winsteps analysis here. As the items to be analysed were of Likert-scale type (with responses 1-5, and higher numbers representing higher levels of agreement), it was appropriate to apply a Partial Credit Model (Masters, 1982) to the data. The software estimates the items' difficulty levels and error estimates in logits. It also computes both weighted and un-weighted mean-squared residuals for each item. The residuals represent the differences between the model's theoretical expectation of item performance and the performance actually encountered. Each raw residual is standardised using its variance before inclusion in the un-weighted mean square. In the weighted mean-square the standardised residuals are also 'information-weighted' using the variance of the observed performance: this reflects the fact that better targeted observations provide more information for the analysis of the data. Therefore, the un-weighted mean-square is more sensitive to outliers. The un-weighted mean-square is usually called the outfit statistic and the weighted mean-square is called the infit statistic. The mean squares are chi-squared statistics divided by their degrees of freedom and thus have expected values of 1. Bond and Fox (2007, p.243) report that a reasonable range of infit and outfit statistics for Likert scale items is 0.6 to 1.4. Any items whose infit or outfit does not lie in this range should be investigated.

The Rasch model also provides person and item reliability indices. The item reliability index indicates how stable the item estimates are, that is it tells us how likely it is that the estimates would remain the same if the items were given to a different group of the same size and similar behaviour patterns. The person reliability index tells us how likely it is that the person ordering would remain the same if the sample were given a new test with similar items. Rasch person reliability measures are comparable to traditional Cronbach's alphas with the proviso that the former tend to underestimate the true reliability while the latter tend to overestimate it (Linacre, 2009). Thus, the same criteria recommended for Cronbach's alphas will be employed here for Rasch person reliabilities: Kline (1993, p.11) recommends that ideally Cronbach's alphas should be high, in the region of 0.9, and that they certainly shouldn't fall beneath 0.7. The Cronbach's alpha statistic for all 38 questions was 0.918, comfortably satisfying Kline's criterion. Cronbach's alphas were also computed for the individual scales and are reported below. Low item reliability indicates that the sample size may be too small for stable comparison of items (Linacre, 2009): in what follows, it can be seen that the item reliabilities were all greater than 0.9 and also always greater than the associated person reliabilities, and so they were deemed to indicate acceptable reliability of items.

Confidence Scale

Table 1 contains the measures and infit and outfit item statistics (as calculated using Winsteps) for the Confidence items. It is clear that all items are behaving well in this scale. The person reliability index is 0.82 and the item reliability index is 0.94. Thus we can be confident that the items are measuring the same trait, and that our person measures will be reliable. The Cronbach's alpha value for this scale was 0.855.

Item	Estimate	Error	Infit MNSQ	Outfit MNSQ
Confidence 1: I learn mathematics quickly.	0.46	0.11	1.01	0.98
Confidence 2: I feel confident in approaching mathematics.	0.65	0.11	0.81	0.73
Confidence 3: I can get good marks in mathematics.	-0.18	0.13	1.11	1.10
Confidence 4: I have trouble understanding anything with mathematics in it.	-0.77	0.12	1.06	1.00
Confidence 5: Mathematics is one of my worst subjects.	0.06	0.10	0.83	0.91
Confidence 6: I am just not good at mathematics.	-0.23	0.11	1.15	1.28

Table 1: Item fit for the confidence scale

The item appearing at the bottom of the scale (with estimate –0.77) is Confidence 4, 'I have trouble understanding anything with mathematics in it'. Taking into account the negative wording, this indicates that the participants were able to disagree with this item most easily. At the top of the scale (with estimate 0.65) is Confidence 2, 'I feel confident in approaching mathematics', indicating that this item was the most difficult one for the participants to endorse.

Anxiety Scale

There were six anxiety questions in our questionnaire. We can see from Table 2 (overleaf) that one of the anxiety items does not perform well. Anxiety 5 has very large infit and outfit values. On reflection, this item, 'I like contributing and answering questions in maths class', seemed to have a slightly different focus to the other scale items. As a result, we decided to drop this question from the scale. When we did this all remaining items had infit and outfit values in the allowable range and the person reliability index (0.81) and item reliability index (0.98) for the reduced scale were good. The Cronbach's alpha value for the full scale was 0.825 but this increased to 0.847 when Anxiety 5 was deleted.

Item	Estimate	Error	Infit MNSQ	Outfit MNSQ
Anxiety 1: I get very nervous during maths classes.	-1.38	0.10	1.13	1.04
Anxiety 2: I often worry that it will be difficult for me in maths classes.	0.42	0.09	0.70	0.67
Anxiety 3: I often feel helpless when doing a maths problem.	0.38	0.09	1.09	1.17
Anxiety 4: Mathematics makes me feel uneasy and confused.	-0.54	0.10	0.71	0.66
Anxiety 5: I like contributing and answering questions in maths class.	0.97	0.10	1.64	1.76
Anxiety 6: I usually feel at ease doing mathematics problems.	0.16	0.10	0.76	0.75

Table 2: Item fit for the anxiety scale

Theory of Intelligence Scale

There were seven items on the theory of intelligence scale. An examination of the fit statistics computed for this scale showed Theory of Intelligence 6 to have a high outfit value (1.41) and so it was decided to drop item 6 from this scale. The new scale behaved well when the analysis was run again. The new scale has person reliability index of 0.72

and item reliability index of 0.98. The original scale had a Cronbach's alpha of 0.756 and this increased slightly to 0.757 for the reduced scale.

Persistence Scale

The persistence scale contained seven items. The items mostly concern students' strategies when faced with difficult problems. In this study, persistence will also be measured using the students' performance on the difficult items on a mathematical literacy test. The fit statistics computed in this case showed all items to be behaving well. The person reliability index is 0.69 and the item reliability index is 0.97. Thus no adjustments need to be made to this scale. The Cronbach's alpha value for this scale is 0.737.

Learning Goal Scale

There were five questions on the learning goal scale. Once again all items were seen to have infit and outfit values in the acceptable range. The person reliability index was 0.8 and the item reliability index was 0.96. Thus we can be confident that our scale is uni-dimensional and the measures derived from it can be trusted. The Cronbach's alpha value for this scale was 0.829.

Performance Goal Scale

This scale had seven items. Again all items had acceptable infit and outfit values. The person reliability index for this scale was 0.67, the item reliability index was 0.96 and Cronbach's alpha statistic was 0.752. Thus this scale is behaving well.

Factor Analysis

In order to further confirm that the questionnaire administered was measuring the traits intended, the entire dataset was subjected to an exploratory factor analysis, with the number of factors sought set to six following an examination of the scree plot (see Kline, 1994; Thompson, 2004). Using Principal Components Analysis and oblique rotation, items were identified with factors according to the principle of allowing as many items as possible to be identified with one factor while minimizing the number identified with more than one (minimum saliency criterion of |0.51|) (following Mulhern & Rae, 1998, p.299). In general, the factors thus obtained matched the traits under investigation with the first factor combining Confidence and Anxiety, the second factor representing Performance Goals, the third predominantly Learning Goals and the fourth Theory of Intelligence. The final two factors contained Persistence items 2, 4, 5, 6, 7. There was further agreement with the Rasch analysis results in terms of identifying items that did not behave as expected: for instance, Theory of Intelligence 6 was not identified with any factor, while Anxiety 5 appeared with the Learning Goals in the third factor – Rasch analysis indicated that neither of these items fit well with their respective scales. Although, this factor analysis also suggested a small number of other anomalies in the dataset (for instance, it failed to associate Persistence 1 or Performance Goal 2 with any of the six factors), it strongly corroborated the results of the Rasch analysis. Further factor analyses using different methods of factor extraction and rotation and/or various criteria for the retention of factors produced very similar results.

Measures for the Confidence Scale

Following validation of the scales, we computed a measure of each student's confidence using Winsteps. The measures ranged from -5.39 logits to 5.45 logits. For instance, a student with the pattern of responses 5,5,5,1,1,2 (where 1 represents 'disagree strongly' and 5 represents 'agree strongly') to Confidence 1-6 in order was allocated 5.45 on this scale, whereas another with responses 2,2,2,2,4,2 obtained –0.92. The mean confidence measure was 0.9381 logits. Moreover, using Winsteps' facility to produce an item-person map, which shows the positions of the items (in terms of their difficulties) and the participants (in terms of their levels of confidence) on the same scale, indicated that the Confidence items used were well matched to the participants. That is, there was a suitable range of Confidence items to cater for the range of levels of confidence possessed by the participants.

Using these measures of confidence we investigated some hypotheses. We found a significant difference between the mean confidence measures of students who had taken Higher Level mathematics at secondary school and those who had taken Ordinary Level (t-test, $p<0.001$). Higher Level students had much higher confidence measures than the Ordinary Level students. When we looked at the group as a whole we did not find a significant difference between the confidence levels of males and females. For the Higher Level students we again found no significant gender difference, however when the Ordinary Level students were considered, there was a significant difference between the male and female students (t-test, $p=0.034$), with the male students displaying higher levels of confidence.

CONCLUSION

The survey instrument used, following removal of just two of the original items, was found to be both valid and reliable by Rasch analysis. Factor analysis and the computation of Cronbach's alphas provided support for these findings. Despite the self-selecting nature of the sample of participants, the range of trait levels observed matched well with the range of participants (as evidenced in the Item-Person maps produced by Winsteps), further endorsing the appropriateness of the instrument for the sample. This provides a strong foundation on which to lay further exploration of Dweck's assertion of the relationships between confidence, goal orientation and beliefs about the nature of intelligence, through the construction of numerical measures of the levels of these traits possessed by the participants. The existence of such measures on the same measurement scale as the difficulty of the items themselves (an attractive feature of the Rasch model) enhances this exploration. Some work has been undertaken on the construction of estimates of a participant's confidence and its relationship to gender and prior mathematical achievement. A similar approach will be taken in the measurement of students' goal orientation and beliefs in relation to the nature of intelligence, before investigating the interplay between all three.

REFERENCES

Bond, T. G. & Fox, C. M. (2007). *Applying the Rasch model – Fudamental measurement in the human sciences* (Second edition). Lawrence Erlbaum Associates, New Jersey.

Bradley, K., Sampson, S. & Royal K. (2006). Applying the Rasch rating scale model to gain insights into students' conceptualisation of quality mathematics instruction. *Mathematics Education Research Journal*, 18(2), 11-26.

Breen, S., Cleary, J. & O'Shea, A. (2007). A study of third level students' beliefs about mathematics. In S. Close, T. Dooley, D. Corcoran (Ed.s), *Proceedings of the Second Conference on Research in Mathematics Education*, Dublin.

Dweck, C., (1986). Motivational processes affecting learning. *American Psychologist*, 41, 1040-1048.

Elliott, E. & Dweck, C. (1988). Goals: an approach to motivation and achievement. *Journal of Personality and Social Psychology*, 54 (1), 50-12.

Galbraith, P. & Haines, C. (1998). Disentangling the nexus: Attitudes to mathematics and technology in a computer learning environment. *Educational Studies in Mathematics*, 36, 275-290.

Grimbeek, P. & Nisbet, S. (2006). Surveying primary teachers about compulsory numeracy testing: Combining factor analysis with Rasch analysis. *Mathematics Education Research Journal*, 18(2), 27-39.

Kline, P. (1993). *Handbook of psychological testing*. Routledge, London.

Kline, P. (1994). *An easy guide to factor analysis.* Routledge, London (& New York).

Linacre, J. M. (2009). *Winsteps (Version 3.68)* [Computer Software]. Chicago: Winsteps.com.

Masters, G. N. (1982). A Rasch model for partial credit scoring. *Psychometrika*, 47, 149-174.

Mulhern F. & Rae, G. (1998). Development of a shortened form of the Fennema-Sherman mathematics attitudes scales. *Educational and Psychological Measurement*, 58 (2), 295-306.

OECD (2003). *PISA 2003 Student Questionnaire*. Retrieved June 26, 2009 http://www.pisa.oecd.org/dataoecd/34/7/37617728.pdf

Pietsch, J., Walker, R. & Chapman, E. (2003) The relationship among self-concept, self-efficacy, and performance in mathematics during secondary school. *Journal of Educational Psychology*, 95 (3), 589-603.

Schoenfeld, A. (1989) Explorations of students' mathematical beliefs and behaviour. *Journal for Research in Mathematics Education*, 20 (4), 338-355.

Stipek, D. & Gralinski, J. (1996). Children's beliefs about intelligence and school performance. *Journal of Educational Psychology*, 88 (3), 397-407.

Thompson, B. (2004), *Exploratory and confirmatory factor analysis*. American Psychological Association, Washington.

APPENDIX

Rating Scale Items from Questionnaire
Theory of Intelligence 1. You have to be smart to do well in maths. 2. People are either good at maths or they are not. 3. Some people will never do well in maths no matter how hard they try. 4. You can succeed at anything if you put your mind to it. 5. You can succeed at maths if you put your mind to it. 6. It is possible to improve your mathematical skills. 7. Everyone can do well in maths if they work at it.

Persistence
1. I will risk showing that I don't know something in order to acquire new mathematical knowledge.
2. I am most proud of my mathematical performance when I feel I have done my best.
3. When presented with a choice of mathematical tasks, my preference is for a challenging task.
4. When presented with a mathematical task I cannot immediately complete, I increase my efforts.
5. When presented with a mathematical task I cannot immediately complete, I persist by changing strategy.
6. When presented with a mathematical task I cannot immediately complete, I give up.
7. When presented with a choice of tasks, my preference is for one I know I can complete.
Learning Goals
1. I work at maths because I like finding new ways of doing things.
2. I work at maths because I like learning new things.
3. I work at maths because I like figuring things out.
4. I work at maths because I want to learn as much as possible.
5. I work at maths because it is important for me that I understand the ideas.
Performance Goals
1. I work at maths because I want other people to think I'm clever.
2. I work at maths because it is important to me that the lecturer/tutor thinks I do a good job.
3. I work at maths because I don't want people to think I'm stupid.
4. I work at maths because it is important for me to do better than the other students.
5. I work at maths because I don't want to do worse than the other students in the class.
6. I will sacrifice acquiring new mathematical knowledge in order to avoid looking stupid.
7. I am most proud of my mathematical performance when I feel my performance made me look good.

DETERMINING THE VALIDITY OF MATHEMATICAL STATEMENTS IN A THIRD-LEVEL ANALYSIS COURSE

Nuala Curley Maria Meehan

University College Dublin

Our study is based on data collected from interviews with seven students from a second-year Analysis course in University College Dublin. During the interviews, students were given statements about concepts from Analysis and were asked to say whether the statements were "True" or "False" and to verify their answers. One aim of the study was to determine whether the students took a semantic or syntactic approach when determining the validity of the statements, and to observe whether they could switch flexibly between approaches if necessary. Our work is based on the studies of Alcock and Weber where they explore in detail the notions of semantic and syntactic approaches to proving.

INTRODUCTION

In an advanced mathematics course, the authors believe that giving students a statement about concepts from the course and asking them to decide if is "True" or "False" and verify their answer, can provide excellent learning opportunities. In coming to a decision about the truth or falsity of the statement the student may need to engage with examples, counterexamples, theorems from the course, and formal definitions of the concepts involved. He or she may start out believing that a statement is false and attempting to generate a counterexample, only to change his or her mind and then attempting to prove the result is true by using theorems from the course and/or formal definitions. This type of flexibility in approach however may be very difficult for a student just embarking on the study of advanced mathematics. In this paper, we analyse the approaches taken by seven students from a second-year Analysis course in University College Dublin when presented with three True/False tasks. We will describe the most common approaches taken by our students, and the main difficulties they encounter.

THEORETICAL FRAMEWORK AND LITERATURE REVIEW

Weber and Alcock (2004) distinguish between taking a *syntactic* approach to proof production versus a *semantic* (or *referential*) approach. They define a syntactic proof production "as one which is written solely by manipulating correctly stated definitions and other facts in a logically permissible way", while in a semantic proof production the "prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws" (Weber & Alcock, 2004, p. 210).

By presenting case studies of Brad and Carla - two undergraduates from a transition-to-proof course in a US university - Alcock and Weber (2008) elaborate on the strengths and weaknesses associated with each approach. Carla, who consistently took a syntactic approach, was able to use mathematical syntax and form to construct her arguments, but admitted that she lacked a sense of understanding. Brad, who consistently took a semantic approach, had the ability to use examples to explore concepts or statements, but was unable to translate his ideas into formal mathematical arguments. The authors observe that both Brad and Carla "seemed to have an underdeveloped notion of how to use examples and syntax together to construct a proof" (Alcock & Weber, 2008, p. 122).

Brad and Carla were just two of eleven undergraduates interviewed from the transition-to-proof course. Alcock and Weber (2008) found that six of the students consistently took a semantic approach, four consistently took a syntactic approach, and one student (incidentally, the most successful of the eleven on the interview tasks) couldn't be classified as taking one approach over the other. This led Weber, Alcock & Radu (2007) to say that ten of the students could be classified as having either a semantic or syntactic *proving style.* They note that if undergraduates have a tendency towards a particular proving style, then given the strengths and weaknesses associated with each, an important pedagogical challenge is to help students become proficient in each approach and to develop the flexibility to let the mathematical situation guide which approach they take.

Interestingly, Alcock and Simpson (2009) admit that at the undergraduate level at least, a student taking a syntactic approach may have more chance of being successful in passing a mathematics course, than one taking a semantic approach. A student taking a syntactic approach could, "provided that they recognise the structures at the appropriate level, pass many undergraduate exams with minimal need to engage with a wide range of examples for every concept they meet" (Alcock & Simpson, 2009, p. 93). On the other hand, a student taking a semantic approach may fail to produce proofs, despite having access to a wide range of examples of the concepts and statements involved.

During the one-to-one interviews conducted for their study, Alcock and Weber (2008) required each of the participating undergraduates to complete a number of mathematical tasks. However, they deliberately avoided including tasks where students were asked to determine the validity of mathematical statements, or tasks that required the production of counterexamples. Their aim was to choose interview questions that could reasonably be approached either semantically or syntactically, and they believed that such True/False tasks would perhaps lead the students to generate examples. However they note that they are interested in students' responses to such tasks and in the role they have to play in teaching and learning mathematics. Thus one aim of our study, and the focus of this paper, is exactly this:

> What approaches do students take in determining the validity of a mathematical statement?

In discussing our students' responses we found it useful to think in terms of Selden and Selden's (1995) notion of *statement image.* They note that, given a mathematical statement, a person may attach to it a "rich mental structure" that includes "all of the alternative statements, examples, nonexamples, visualizations, properties, concepts, consequences, etc., that are associated with a statement" (Selden and Selden, 1995, p. 133). Many will recognise this notion as an extension of Tall and Vinner's (1991) *concept image*.

RESEARCH CONTEXT AND METHODOLOGY

In Spring Semester 2007, the second author taught a second-year *Analysis* module in University College Dublin. The module was a formal definition-and-proof based course that covered the main concepts and results in sequences and series of real numbers, such as convergence, completeness and countability. While the mathematical background of the 65 students registered to the course varied, all had taken third-level modules in Calculus, Linear Algebra, and Calculus of Several Variables.

To encourage students to generate examples of the concepts involved in Analysis, students were required to keep a *Portfolio of Examples.* This involved completing several example-generation exercises of the type described by Watson and Mason (2005). It was hoped that these exercises might enable and encourage students to take a semantic approach to proving. On the other hand, to enable and encourage students to take a syntactic approach to proving, *In-class Exercises*, which were designed with the aim of emphasising the role of definitions and syntax in constructing arguments, were also incorporated into the module. In addition, throughout the module the lecturer took care, when possible, to illustrate the role of both examples and formal definitions in Analysis. We note that it is extremely difficult to determine whether either of these exercises had the intended effect, but we mention them because we feel that their inclusion in the course may be relevant to this study. We believe we have some evidence that the students benefited from the completing the *Portfolio of Examples,* however this will not be discussed further in this paper.

Towards the end of the semester, the first author interviewed seven student-volunteers from the module. All seven students had completed and submitted their *Portfolio of Examples* and had completed at least five of the seven *In-class Exercises*. In addition, all had taken an in-class test two weeks previously. In this test, students were given ten statements about concepts from Analysis and were asked to decide if the statements were "True" or "False" and to verify their answers. Ranked in order of their overall performance in the module: Adrienne, Anna, and Aoife all achieved As; Brendan achieved a B; Catherine and Colm achieved Cs; and David was awarded a D.

The interviews were one-to-one and each lasted approximately half-an-hour. None of the participants had previously known the interviewer. At the start of the interview, the students were asked to describe the following: what they thought about the subject of Analysis; how they studied the subject; aspects of the course they found difficult; and, to rate their understanding of Analysis on a scale from one to five. It was hoped that these questions would give the student time to relax. During the second part of the interview, the students were given three statements about concepts from Analysis and were asked to decide the truth or falsity of the statements and to verify their answers. This paper will focus specifically on the students' responses to these three tasks, and therefore we will describe them in more detail below. The third part of the interview explored the student's experience of completing the *Portfolio of Examples*, while in the final part of the interview each student was asked to explain what was meant by a "bounded sequence". The purpose of this final task was to see whether the student would refer to examples and/or the formal definition in his or her description.

We now discuss the second part of the interview in more detail. Each participant was systematically presented with three statements from the in-class test along with the following instructions: *Decide whether the statement is 'True' or 'False' and verify your answer. You must clearly state any theorem or result that you use.* The three statements are as follows:

Statement 1. If (a_n) has a convergent subsequence, then (a_n) is bounded.

Statement 2. If there exists $M > 0$ and $N \in \mathrm{N}$ such that $|a_n| \leq M$ for all $n \geq N$, then (a_n) is bounded.

Statement 3. If (a_n) and (b_n) are positive and increasing, then $(a_n b_n)$ is increasing.

The first statement is "False" and the other two statements are "True". While the aim of including this task in the study was to examine the approaches taken by students when determining the validity of mathematical statements, we had particular reasons for including each statement. The first statement was chosen to see if the student would be able to take a semantic approach and produce a counterexample. The second statement was chosen to see if the presence of mathematical syntax in the statement would encourage the student to take a syntactic approach and provide a proof. The third statement was chosen as we conjectured that a person might initially take a semantic approach, but clearly the proof requires engagement with the formal definition of positive, increasing sequence. The students had encountered all three statements in the in-class test about two weeks previously. The reason we chose statements from the test was that we wanted to ask students what their thought processes had been during the test. However, in most cases the student seemed to "figure-out" the problem anew during the interview, and many of the answers differed from those given on the test. While the students had received their final score on the test, their papers had not yet been returned nor the solutions discussed in class.

At the start of this part of the interview, a list of formal definitions of concepts was also provided. If, for Statement 3, the student hadn't attempted to introduce the definition of increasing sequence and became stuck, the interviewer then prompted the student to look at the list of formal definitions. Otherwise, no direct reference was made by the interviewer to the list. While working on the statements, each participant was encouraged to "think-aloud" and to write down anything he or she wished. In addition, when the student had completed work on each task, he or she was asked to rate his or her confidence in the answer out of two, and also to rate what he or she believed the lecturer would give the answer out of two.

We now discuss how the data was analysed. Each author independently examined the students' responses to each of the three statements with the following objectives in mind: note if the student initially takes a syntactic or semantic approach; note the progress made and difficulties encountered; note if he/she switches approach; and, note the progress made and difficulties encountered; repeat. On meeting to discuss the findings, we found that we more-or-less agreed on when a student was taking a semantic approach, however we had more difficulty deciding on when a student was taking a syntactic approach. If the student introduced the formal definition of a concept and attempted to use it, we were both agreed that this student was taking a syntactic approach. However, many students took the approach of stating a theorem or result from the course and attempting to use it. While this is considered to be a syntactic approach, we felt the skills required for this were different to those needed to manipulate the syntax of a formal definition. A detailed discussion of our findings and initial analysis can be found in Curley (2007).

Based on our findings from the first analysis of the responses, we decided to analyse them again, this time with the following codes in mind:

A: The student generates/attempts to generate a counterexample.

B: The student generates/attempts to generate an example.

C: The student states and/or attempts to use a theorem or result.

D: The student uses or attempts to use a formal definition.

We consider the students in A and B to be taking a semantic approach, while the students in C and D are taking a syntactic approach. In analysing the responses, we also attempted to note when the student decided the statement was "True" or "False". We both assigned the codes to all seven students' responses independently, and then we met to discuss our findings. There was considerable agreement on the coding, and after some further discussion and debate, we reached agreement on all responses. We now present our results.

RESULTS

The following table outlines the approaches taken by each student in attempting to validate the three statements. The codes A, B, C, and D are as described above. The subscripts "*t*" and "*f*" give an indication as to when the student decided that the statement was "True" or "False". They are only a rough indication since a number of times the student only said aloud that they believed the statement was "True" or "False" when specifically prompted by the interviewer.

Name	Statement 1	Statement 2	Statement 3
Adrienne	$_t$C A $_f$	D$_f$	$_t$C
Anna	$_t$C	$_f$D	$_t$A
Aoife	$_f$A	$_t$B	$_f$A $_t$B
Brendan	$_f$A C	B$_f$	A $_t$C
Catherine	$_f$A $_t$C B	B$_t$	$_t$C
Colm	$_f$A D	$_t$B	$_t$C D
David	$_t$C D$_f$ C $_t$B	D$_t$	B$_t$

For Statement 1, four of the students initially generated or attempted to generate a counterexample, while the other three initially stated and/or attempted to use a theorem or result. Five of the students altered their initial approach while attempting the task. Of the five students who attempted at some point to generate a counterexample, all but Catherine were successful. Adrienne, Aoife, Brendan, and Colm decided the statement was false, while Anna, Catherine and David believed it to be true.

To give the reader some insight into how we coded the responses, we give Adrienne as an example. Her initial response was that Statement 1 was actually the statement of the Bolzano-Weierstrass Theorem so we coded that as a "C":

Adrienne: Okay, so I would say that is true because, ehm, that theorem the Bolzano-Weierstrass theorem, that states that if, ehm, that a bounded sequence has at least one convergent subsequence, so eh, okay … [*pause*] but then I think I'm the opposite way around, it's the other way.

She pauses and then switches her approach to what we have coded as "A", that is, she generates a counterexample, while very clearly describing her thought-processes.

Adrienne: Okay, I'm just trying to think of a counterexample of where something could have a convergence subsequence but it isn't bounded so then if you take ... [*pause*] Okay [*writing*: $(-1/n)^n = -1/1, 1/2, -1/3$] I was just thinking of an oscillating sequence just to see what that is, … would have a convergence subsequence and also is bounded. My instincts tell me that the answer is false but I'm just trying to find an example … [*writing*: 1/n =1,1,1/2,1,1/3,1, 1/4] ... and if you did something like that where you have a decreasing sequence but you have like a constant every second term. That would have a convergent subsequence and would also be bounded as well. [*pause*] And then if you took, ehm, like, set of natural numbers, ehm, that would be one, two, three, four, off to infinity, then if you have a constant every second term [*writing*:1,1,2,1,3,1] … that would have a convergent subsequence cause one, one, one is convergent to one but the sequence itself isn't bounded above, it's only bounded below by one. [*pause*] So then I'd say that the statement is false.

In attempting to validate Statement 2, four of the students initially adopted the approach which we have coded "B", that is they graphed or attempted to graph what the statement was saying. The other three students initially introduced the formal definition of bounded sequence and thus we have coded their attempts as "D". Incidentally, Adrienne and Anna simply compared the statement to the formal definition of bounded sequence, noted that they were different, and thus said the statement was "False". No student switched his or her initial approach as all were happy that they had made a good attempt. Adrienne, Anna and Brendan believed the statement to be false, while the other four students believed it to be true, but didn't give a correct proof.

Three students initially approached Statement 3 by trying to find a counterexample, coded "A", three stated and/or attempted to use a theorem or result, coded "C", and one student attempted to generate examples, coded "B". Three students altered their initial approaches. All seven students believed the statement to be true, but none could prove it.

Again, to give the reader some insight into how we coded the responses, we look at Colm's approach to Statement 3. He first believes that the statement is a theorem from the course, so we code his approach as a "C":

Colm: I remember this was definitely one of the proofs and theorems that I was writing down on the rough work before I started. [*pause*] I consider the statement to be true [*pause*]. Because if all of a_n is positive and

> increasing it is usually tending off to infinity or if b_n is increasing and positive it will tend to plus infinity. Then by multiplying the two together they should both tend to infinity they should both follow the pattern…of their parent sequence…As I said that's a theorem I would regurgitate back out again.

He then attempts to use the formal definition – coded "D":

> Colm: I am saying that the next value in the a_nb_n sequence is greater than the value before it as well as in a_n and b_n. So which, it means it is increasing it is getting bigger with each value. So next value is getting bigger and bigger. Therefore it is increasing all the time.

Then Colm wrote down the following: $a_n \geq a_{n+1}$; $b_n \geq b_{n+1}$; $a_nb_n \geq a_nb_{n+1}$.

We will now discuss the results in more detail.

DISCUSSION OF RESULTS

We return to our research question: What approaches do students take in determining the validity of a mathematical statement? Our first comment is that in attempting to address this question, rather than trying to determine whether a student took a semantic or syntactic approach, we found it more useful to break these two approaches down further. Our first analysis of the data suggested we examine the responses to determine whether a student generated or attempted to generate a counterexample or an example; stated and/or attempted to use a theorem or result; and, used or attempted to use a formal definition.

Our second comment is that our data set is rather small and thus coming to any conclusions about the research question is difficult, but we believe our results provide a starting point for further research on this topic. Statement 2 was chosen because it contained mathematical notation, and we were interested to see if this would encourage the students to introduce the formal definition of bounded sequence and engage with the syntax. Three of the seven students did introduce the formal definition, while the remaining four attempted to graph a picture of what the notation in the statement was saying. We cannot say much more than this, but still feel that the idea of including mathematical notation in a statement and asking students to determine its validity may encourage them to engage with the formal definitions.

Statements 1 and 3 were written "in English" - without using any mathematical notation. By examining all fourteen responses here, we can say a bit more about the students' approaches to determining the validity of these two statements. The initial approach taken by all students to Statement 1 was either to attempt to generate a counterexample, or attempt to state and use a previously established theorem or result. With the exception of David, who initially generated some examples, the same was true of the other six students' initial approaches to Statement 3. These seemed to be the preferred initial approaches of our students.

In attempting to determine the validity of Statements 1 and 3, eight students “switched” from their initial approach if they believed they weren’t getting anywhere. In four of these cases, the student switched from attempting to generate a counterexample to trying to state and/or use a theorem from the course or vice-versa. We see this with Adrienne, Brendan and Catherine for Statement 1, and Brendan again for Statement 3. Only Catherine and David on Statement 1 and Aoife on Statement 3, went on to generate examples of the statements.

Despite the fact that a list of definitions had been provided, Colm and David were the only two students to introduce formal definitions without prompting, in determining the validity of Statements 1 and 3. (We believe we may be able to account for Colm’s behaviour and will discuss his approach below.) Anna, Aoife, Brendan and David were all prompted by the interviewer to consult the list of formal definitions when they became stuck on Statement 3. Even with prompting, only Aoife and David considered the formal definition of increasing sequence, and only David made any progress. (We note that he had consulted the lecturer about this problem a few days earlier.) Few attempts were made by the students to use the definitions, despite the fact that at various stages throughout the interviews, a number made comments that suggested they were aware of the pivotal role that formal definitions play in producing a proof. The following is a comment made by Brendan when asked what mark he thought the lecturer would give them out of two for his response to Statement 2:

> Brendan: Em, I mean she would give a one out of two because there are specific … What I said is, while I think it is true, it is not as concise and complete as it would be if I stated the relevant definitions and used them appropriately, I don’t think. I have kinda answered the questions intuitively by my, by the image I have of this in my head, whereas she would be expecting me, after all the talking she has done about these definitions and all the writing she has done up on the board to be able to remember the relevant definitions and theorems…

We conjecture that to introduce and use a formal definition in producing a proof, is still a step-too-far for these students. As Brendan explains when awarding himself a four out of five for his understanding of Analysis in general:

> Brendan: A four because I see a five as someone whose intuitive understanding is almost exactly what the reality is, and who would get it right, you know, 99% of the time. I am not quite there yet. I think I’ve a fairly good intuitive understanding of [*inaudible*] – I can manipulate these things in my head well, but I don’t quite have the X-factor – that extra bit.

Thus we summarise our findings using Selden and Selden’s (1995) notion of *statement image.* When given a mathematical statement (written “in English”, without mathematical notation), what are the most common approaches taken by students to enhance their statement image? Our students' most common approaches when working on Statements 1 and 3 were to attempt to generate counterexamples and to attempt to use other statements and theorems to make sense of the given statement.

We finish this discussion with a note of caution. The reason that a student may take a particular approach in tasks such as those described above may not be that they have either a syntactic or semantic proving style. It may instead be the case that they are being *strategic,* particularly when the task is given in an examination. We conjecture that some of our students may believe that an answer which involves the statement and use of a theorem or result may get more marks in an examination, than one which involves a counterexample - even if it is correct! Of the four students who correctly generated a counterexample to show that Statement 1 is false, only Aoife rated her confidence in her answer as being two out of two. Catherine, Brendan and Colm all rated their confidence as being one out of two. Explaining his mark Colm states:

Colm: Ehm, because I am not entirely sure about it. Ehm, as I said, I was using in a test, I would have studied this. I would have known exactly what definition and proof I would sort of need for this, but as they are not fresh in my head at the moment, I am not 100% sure of it.

Brendan initially generated a correct counterexample but then went off on a tangent. He rated his confidence on his answer as being a zero, before changing it to a one:

Brendan: Out of two, ehm, I would give myself a zero because there is definitely much more … in asking a question, like that a person is asking a question like that, the person has something special in mind, a theorem, a definition in mind, that they want or that should be written down here. And I think I have taken a very airy-fairy and not concrete way of looking at it.

Thus, in addition to semantic and syntactic approaches to proving, we believe we have identified a third approach – a *strategic approach*. As mentioned above, Colm voluntarily introduced formal definitions when working on both Statements 1 and 3. However we believe his approach was more strategic than syntactic. During his attempt at Statement 1 he was quite explicit about how he approaches such tasks in an examination situation.

Colm: But generally what I do for class tests or whatever, I learn the theorems, most of the theorems. I would write them down in a book and I'd learn them before I go in. As soon as I open the maths paper, then I would immediately write down all the theorems I remember in the rough work column. I wouldn't even look at the questions. I would read the questions and then first of all decide myself if it was true or false, so I'd say whichever it is here in this case. Then I would go back and look at all the theorems I have down and see if one of those theorems were this I felt. Otherwise, I would write down to the best of my ability trying to use as much maths terms as I can, to make it look like I know what I'm doing. That would be the way I would probably answer this question.

Alcock and Simpson (2009) noted that an undergraduate who has a syntactic proving style may achieve higher grades than one who has a semantic proving style. Colm's comments lead us to wonder whether some students who have been labelled as having a syntactic proving style, might actually have a *strategic proving style*.

CONCLUSIONS

In advanced mathematics courses at university, students often struggle with proof production for various reasons (Alcock & Simpson, 2009). This is not surprising as it *is* difficult to write proofs and it is not a skill that we believe can even be developed in just one introductory course. Our students encountered difficulties with proof production, and specifically, with using the formal definitions to frame and write the proof. However we need to focus on what they *could* do. They were able to generate examples and counterexamples, and recognise and use theorems from the course to gain an understanding of the given statement. The tasks required them to make decisions, to be creative, and would have afforded the opportunity for debate and discussion had they been posed in the classroom setting. Even though none of our students could actually prove that Statement 3 was true, all were able to engage with it, and gain insight into what it was saying. Had we simply asked them to prove it we may have just left them sitting there demoralised. For all of these reasons, we are advocates of the use of True/False tasks in the teaching and learning of advanced mathematics.

REFERENCES

Alcock, L. & Simpson, A. (2009). *Ideas from mathematics education: An introduction for mathematicians.* Birmingham, UK: The Higher Education Academy: Maths, Stats & OR Network.

Alcock, L. & Weber, K. (2008). Referential and syntactic approaches to proving: Case studies from a transition-to-proof course. In F. Hitt, D. A. Holton, & P. Thompson (Eds.), *Research in Collegiate Mathematics VII* (pp. 101-123). Providence, RI: American Mathematical Society.

Curley, N. (2007). *Concept development and usage in an advanced mathematics course.* Unpublished masters thesis, University College Dublin.

Selden, J. & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics,* 29, 123-151.

Tall, D. & Vinner, S. (1981). Concept image and concept definition with particular reference to limits and continuity. *Educational Studies in Mathematics,* 12(2), 151-169.

Watson, A. & Mason, J. (2005). *Mathematics a constructive activity – learners generating examples*. Mahwah, NJ: Lawrence Erlbaum Publishers.

Weber, K. & Alcock, L. (2004). Semantic and syntactic approaches to proving. *Educational Studies in Mathematics,* 56, 209-234.

Weber, K., Alcock, L. & Radu, I. (2007). Proving styles in advanced mathematics. In *Proceedings of the 10th Conference on Research in Undergraduate Mathematics*, California, USA. Retrieved June 19, 2009, from http://cresmet.asu.edu/crume2007/papers/weber-alcock-radu.pdf.

THE CHANGING PROFILE OF THIRD LEVEL SERVICE MATHEMATICS STUDENTS (1997-2008)

Fiona Faulkner   Ailish Hannigan   Olivia Gill

University of Limerick   (NCE-MSTL)

In 1997 diagnostic testing was introduced in the University of Limerick (UL) to identify and notify students who would, more than likely, require help to complete first year service mathematics courses successfully and proceed through to second year. A database of diagnostic test and end of semester results was initiated in 1997 and is updated annually by the authors. It has been observed by the authors that the student profile and their needs have changed in the 12 years since the initiation of the database. In this paper, the authors describe the profile of the current student cohort in first year service mathematics courses in UL and how it has changed in the last 12 years. Furthermore the progressions of the Mathematics Learning Centre and facilities it currently provides to address such issues will be discussed.

Supported by MACSI - the Mathematics Applications Consortium for Science and Industry (www.macsi.ie), centred at the University of Limerick.

BACKGROUND AND METHODOLOGY

Diagnostic testing in the University of Limerick

Diagnostic testing has been used in the University of Limerick (UL) since 1997 to help identify students who may be at risk of failing end of semester mathematics examinations (O'Donoghue, 1999; Gill, 2006). This form of testing has become an increasingly popular tool in third level education, both in Ireland and abroad, to help identify weaknesses in basic mathematical skills (Tall & Razali, 1993; Malcolm & McCoy, 2007).

The UL diagnostic test, which is paper-based, was developed by O'Donoghue, a professor of mathematics education in the University of Limerick, in 1997. The two largest service mathematics groups in UL are Science Mathematics and Technological Mathematics. It is these two groups that are targeted annually (Technological Mathematics were the only group tested in 1997). The total number of students registered for Science Mathematics within the period 1997-2008 is 2,530 and for Technological Mathematics students is 3,676. The diagnostic test consists of 40 questions covering nine topics. The vast majority of questions on the test are aimed at Ordinary Level Leaving Certificate Mathematics standard or below with the exception of six questions, which are only covered on the Higher Level syllabus(Leaving Certificate Mathematics can be taken at three levels in Ireland; Higher, Ordinary and Foundation Level. The Higher Level curriculum, as its name suggests, is the most advanced level and includes more topics than the Ordinary Level curriculum. The minimum mathematics entry requirement for direct entry to third level education in Ireland is a grade C or higher in Ordinary Level Leaving Certificate Mathematics). The diagnostic test has remained the same since it was first implemented in 1997.

The test is distributed, without prior warning, to students in their first mathematics lecture of the academic year and is in adherence with the ethical checklists of Burgess (1989). Students are advised that if they receive 19 out of 40 or below in the test that they should avail of the support services provided by the Mathematics Learning Centre (MLC). These students are categorised as being 'at risk' (O'Donoghue, 1999,p.). The reason for 19 being the cut off point is that these students were considered to be seriously insufficient in basic mathematical knowledge necessary for third level mathematics and therefore were categorised as being at risk and in need of support. There is however "a strong argument for extending the 'at risk' category to students with higher scores" (O'Donoghue, 1999, p. 15).

Research Methodology

The analyses carried out in this paper, which is concerned with the changing profile of service mathematics students in the last ten years, will be carried out through the use of the statistical package SPSS (Statistical Package for the Social Sciences).The investigation into the changes in support services over the time period 1998-2008 will be carried out simply by detailing the chronological order in which new mathematics support services were introduced to UL.

MAIN FINDINGS-THE CHANGING STUDENT PROFILE (1998-2008)

Many aspects of the student profile of Science and Technological Mathematics students in UL have changed between the years 1998 and 2008, as detailed in Table 1 on the next page. A changing profiling of service mathematics students such as this, which has impacted on the teaching and learning of mathematics in third level education, has also been documented internationally(Kitchen,1999; Taylor & Mander,2002).

Gender balance

Within Technological Mathematics the male dominance has remained largely unchanged between 1998 and 2008. However the Science Mathematics cohorts have seen a considerable change in gender balance from 1998 when 36.6% of the group were male, to 2008 when the males made up more than half of the cohort, 53.1%. This increase in male students in Science Mathematics may be explained by the addition of degree programs such as Health and Safety and Physical Science to this mathematics module.

Degree Programmes

There has been an increase in the number of degree programmes within Technological and Science Mathematics between 1998 and 2008. Within Technological Mathematics there had been an increase from 8 to 14 degree programmes with the addition of programmes of study such as Music, Media and Performance Technology and Digital Media and Design to name but a few. Science Mathematics has seen an increase from 8 degree programmes in 1998 to 11 in 2008 with courses such as Health and Safety now being required to take Science Mathematics.

Percentage of 'at risk' and 'no-show' students

The percentage of students turning up to their first lecture to take the diagnostic test has declined for both Technological and Science cohorts since 1998 when 100% of students registered for each module sat the test. In 2008 19% of Technological Mathematics students and 20.5% of Science Mathematics students failed to show up at their first lecture when the test is administered. The reasons for these students not turning up to their first lecture to take the test has not yet been examined however upon analyses of their end of semester results the vast majority of 'no-show' students perform well below the average when compared to the rest of the cohort.

There has been an increase in the percentage of 'at risk' students, which as previously highlighted are students who receive 19 out of 40 or below in the diagnostic test. 13.6% more Technological students were labelled 'at risk' in 2008 than in 1998 and a massive increase of 24.7% occurred for Science students. A decrease in mathematical standards such as this may be due to, as suggested by Hunt and Lawson (1996), a shift in emphasis of A Level Mathematics (or, in this case, Leaving Certificate Mathematics) towards topics not covered on the diagnostic test. Although the suggestions of Hunt and Lawson (1996) may hold in an Irish context also, it is thought that the increase in at risk students may be partially explained by the decrease in students taking Higher Level Leaving Certificate Mathematics, a decrease of 6.8% and 17.4% within Technological and Science Mathematics cohorts respectively between 1998 and 2008. The increase of non-standard students is also shown to contribute to the decline in diagnostic test performance over time. A non-standard student is defined as a student who did not enter UL through the CAO system i.e. through obtaining a sufficient mathematics Leaving Certificate grade to meet the third level course requirements. They therefore consist of mature students (i.e. anyone over the age of 23), non-national students and those who have completed previous degree/diplomas/certificates and have used these as an entry qualification to UL. This will be discussed in greater detail later on in section 3.

Note: In Table 1 over, all percentages refer to the total number of students registered for the mathematics programme in question with the exception of the % 'at risk' category which refers to the percentage of the number taking the diagnostic test only.

Year	1998	2008		1998	2008
Technological Mathematics			Science Mathematics		
Males	85.2%	87.4%		36.6%	53.1%
Females	14.8%	11.8%		63.4%	46.2%
% of total taking the test	100%	81%		100%	79.5%
% 'at risk'	32.8%	46.4%		21.3%	46%
% doing HL	41%	33.2%		55.4%	38%
% doing OL	58.7%	57.5%		43.1%	54.8%
Standard students	99.7%	90.6%		98.5%	93.1%
Non-standard students	.3%	9.4%		1.5%	6.9%

Table 1: Profile of students in Science and Technological Mathematics

CHANGES IN MATHEMATICAL BACKGROUND OF STUDENTS ENTERING UL (1998 & 2008)

From table 1 it is evident that within the Science and Technological Mathematics student cohorts the percentage of students taking Higher Level Leaving Certificate Mathematics has declined over the period (1998-2008). This is an important finding as performance in mathematics in third level education has been shown to be better when students have higher level mathematics as pre-requisite knowledge (Barry & Chapman, 2007). The distribution of grades within Higher and Ordinary Level will now be considered.

Technological students' mathematical background

Table 2 shows the changes in frequencies of students' entry grades to Technological Mathematics (Note: Tables 2 and 3 only include the grades for which there was a substantial frequency increase/decrease between 1998 and 2008). The decrease in Higher Level grades, particularly Higher Level A1, A2 and B1 grades is quite substantial between 1998 and 2008. There are no Higher Level A1 students in Technological Mathematics in 2008. There is also a noticeable decrease in the number of students achieving Ordinary Level A1 grades over time. These Ordinary Level students also follow the pattern of declining frequencies that are seen among the top Higher Level students.

An increase in Ordinary Level Leaving Certificate grades occurred over the ten year period particularly noticeable are the increases in the percentage of students getting Ordinary Level A2-B3 grades.

Science students' mathematical background

Table 3 paints a similar picture of decreasing frequencies in the top Higher Level grades and Ordinary Level A1 grades and increases in frequencies in Ordinary Level A2-C1 grades over time.

Analyses of Higher Level A1-B3 grades in 1998 shows 76 (37.6% of total cohort) people falling into this category compared to 36(11.9%) in 2008, a drop of 25.7% of the total cohort of Science Mathematics students entering UL with these mathematics grades.

The increasing frequencies in the Ordinary Level grades OA2-C1 are just as startling over time. In the case of Science Mathematics students in 1998 25.3% of the total cohort achieved these grades compared to 47.1% in 2008.

The tendency for students entering Science and Technological Mathematics in 2008 to hover around the minimum entry grade, which is an Ordinary Level B3 for Science Mathematics and an Ordinary Level C1 for Technological Mathematics, creates a very different cohort to that which was present ten years previous. Students' entry grades have a huge bearing on their performance in the diagnostic test therefore the lower the Leaving Certificate entry grade the more likely a student is to perform poorly in the diagnostic test and find themselves in the 'at risk' category.

Increase in non-standard students

A 9.1% and 5.4% increase of non-standard students occurred, in Technological and Science Mathematics respectively between 1998 and 2008, which is highlighted in table 2. A shift in student intake like this has been shown to affect overall mathematical performance in UL as well as in other institutions such as Coventry University (Lawson, 2003).The previous knowledge of the students can no longer be assumed to be Leaving Certificate material. This opens up many new challenges to lecturers of these modules and calls for adaptations in teaching styles so as to cater for the current cohort and not that which sat in front of mathematics lecturers ten years previous.

Key: OA1-OrdinaryLevel A1 grade HB1- Higher Level B1 grade NS- Non-standard student

Increase in frequency over time Decrease in frequency over time

Year	Grade	Frequency (%)	Year	Grade	Frequency (%)
1998	HA1	8 (2.6)	2008	HA1	-
	HA2	7(2.3)		HA2	3(.8)
	HB1	16(5.2)		HB1	6(1.6)
	OA1	56(18.4)		OA1	40(10.7)
	OA2	45(14.8)		OA2	58(15.5)
	OB1	30(9.8)		OB1	49(13.1)
	OB2	30(9.8)		OB2	42(11.2)
	OB3	11(3.6)		OB3	20(5.3)
	NS	1(.3)		NS	35(9.4)

Table 2: Frequency changes of entry grades between 1998 and 2008-Technological Mathematics

Year	Grade	Frequency (%)	Year	Grade	Frequency (%)
1998	HA1	9(4.5)	2008	HA1	1(.3)
	HA2	13(6.4)		HA2	2(.7)
	HB1	13(6.4)		HB1	10(3.3)
	HB2	20(9.9)		HB2	6(2)
	HB3	21(10.4)		HB3	17(5.6)
	OA1	36(17.8)		OA1	23(7.6)
	OA2	21(10.4)		OA2	41(13.5)
	OB1	18(8.9)		OB1	38(12.5)
	OB2	8(4)		OB2	29(9.6)
	OB3	4(2)		OB3	27(8.9)
	OC1	-		OC1	8(2.6)
	NS	3(1.5)		NS	21(6.9)

Table 3: Frequency changes of entry grades 1998 and 2008- Science Mathematics

PROFILE OF NON-STANDARD STUDENTS IN 2008

Breakdown of non-standard students

Within the 2008 non-standard group the following breakdown, which can be computed from Table 2 and 3 occurs(see section 2.3 for definition of non-standard students):

- 70% of the students are mature students all of whom have no previous qualifications in UL, are over 23 and have not completed the Leaving Certificate in at least 5 years. Upon passing access course examinations and/ or on the basis of an interview some mature students gain places in degree programmes.
- 19% are students who have gained entry to UL in 2008 based on their completion of a previous degree/diploma/certificate in a variety of universities around Ireland. The previous qualifications of these non-standard students include Engineering in Industrial Automation, Automation Technology, Environmental Science and Archaeology
- the remaining 11% are made up of non-national students who have completed an alternative school leaving qualification to the Leaving Certificate.

Performance of non-standard students in the diagnostic test

The statistics mentioned above highlight that mature students make up the majority of the non-standard cohort in UL, therefore the majority of non-standard students have not engaged in mathematics for a number of years. This is evident in their diagnostic test results which can be seen in figure 1 below. Non-standard students have mean diagnostic test scores below that of the standard student coming directly from Leaving Certificate. In the case of both Science and Technological Mathematics the majority of non-standard students are at risk of failing their end of semester examinations. It is very clear to see from the box plots in figure 1 that non-standard students are mathematically less prepared entering UL than standard students are.

Performance of non-standard students in end of semester examination

In contrast to the diagnostic test result the non-standard students outperform the standard students in the end of semester examination for both Science and Technological cohorts. The non-standard students within Technological Mathematics outperform the standard students by a median value of 12% and the non-standard Science students outperform the standard students by a median value of 8% in the end of semester examination..

Comparison of attendance at support tutorials of standard and non-standard students

It is evident from tables 4&5 and figures 1&2 that the non-standard students improve hugely in their mathematical performance between the beginning of the semester and the end. So what is happening in between? The proportion of non-standard students attending support tutorials, run by the Mathematics Learning Centre (MLC), is much larger than that of the standard students. Technological and Science Mathematics students are offered a weekly support tutorial in addition to their regular weekly tutorial. Within Technological Mathematics the best attendance from standard students to these support tutorials occurred when 3.6% of students (12 students) turned up in comparison to the highest attendance within the non-standard support tutorial which was 40 %(14 students). Similar findings can be seen for Science Mathematics students. The findings shown in figures 1 & 2 and tables 4 & 5 reveal positive indicators for engaging in support services such as support tutorials and one-to-one consultations in the drop-in centre. These findings could also be attributed to the fact that non-standard students may be more motivated than standard students (Hirst, 1999).

Diagnostic Test	**Mean value out of 40(SD)**	**Median**	**% of students at risk**
Standard students (S)	21.24 (6.3) n=273	21	43% n=117
Non-standard students(NS)	13.0 (8.4) n=30	12	76.7% n=23
Semester 1 examination	**Mean value out of 40(SD)**	**Median**	
S	53.9(22.5) n=315	52	
NS	55.8(29.5) n=33	64	

Table 4 Technological Mathematics students' diagnostic tests and end of semester results in 2008

Diagnostic Test	**Mean value out of 40(SD)**	**Median**	**% of students at risk**
Standard students (S)	20.9 (6.6) n= 224	21	43.2% (96)
Non-standard students(NS)	10.9(8.4) n= 17	8	86.7% (13)
Semester 1 examination	**Mean value out of 40(SD)**	**Median**	
S	53.9 (22.2) n=267	**55**	
NS	59(17.2) n= 17	**63**	

Table 5: Science Mathematics students' diagnostic test and end of semester results in 2008

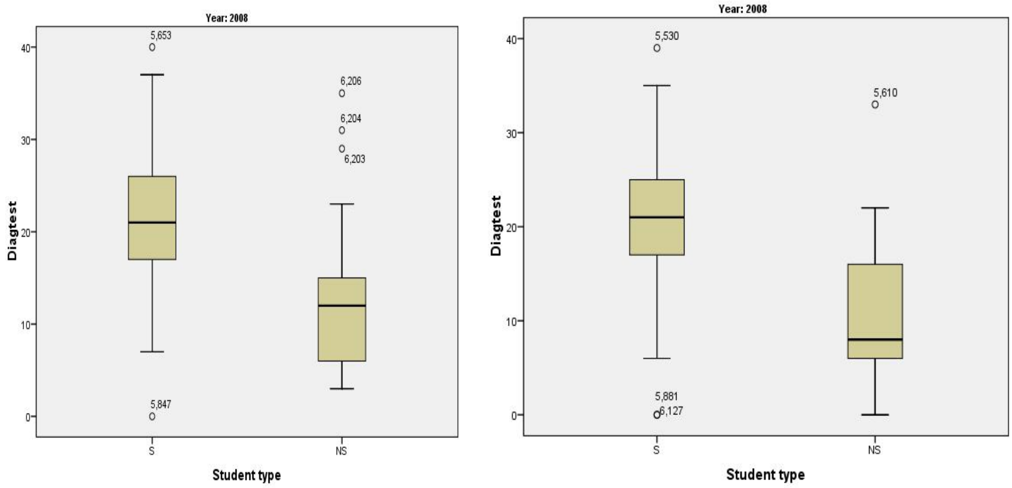


Figure 1: Comparison of diagnostic test performance for standard and non-standard students- Science (left) & Technological (right) Mathematics

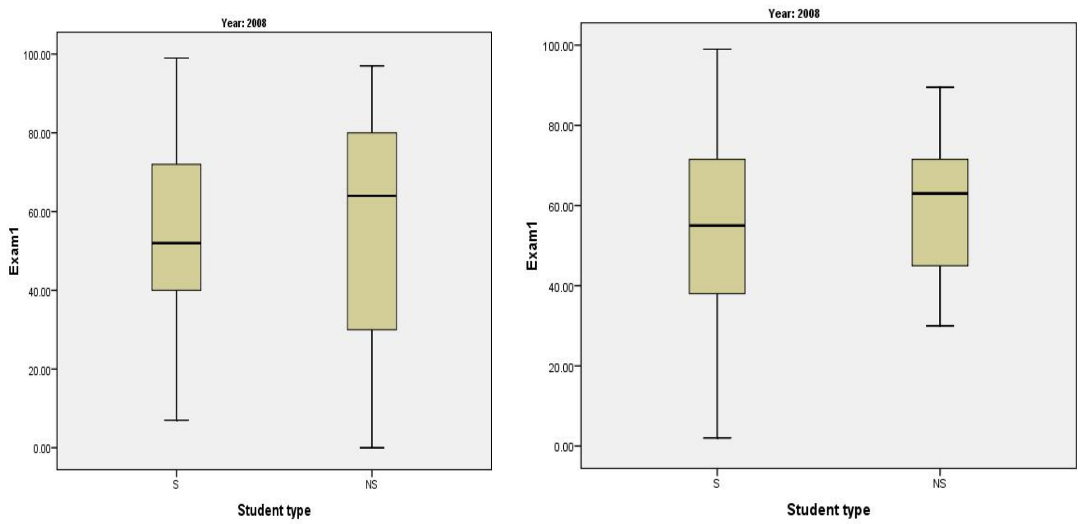


Figure 2: Comparison of End of semester 1 exam performance for Standard and Non-standard students- Science (left) & Technological (right) Mathematics

INCREASE IN SUPPORT SERVICES

The change in student profile and decline in mathematical standards on entry to UL are further evidenced in the provision and uptake of support structures in place in UL.

In 1997, when O' Donoghue carried out his pilot study to measure the extent of the errors and gaps in students' mathematical knowledge, support was initiated in two forms. 'Front-end' tutorials were set up in the first two weeks (one each week) of the first term to help students revise fundamental mathematics skills in arithmetic and algebra. Additionally, students were strongly encouraged to attend a 'support tutorial' on a weekly basis throughout the term. Support tutorials run parallel to regular tutorials and cover the same material but at a slower pace and in a smaller group. The number of support tutorials has increased significantly since 1997, when two support tutorials (one for traditional age students and one for mature students in Technology Mathematics) ran on a weekly basis. In 2008/09, a total of 206 support tutorials in 21 modules were provided by the MLC with a total of 2,395 contacts/attendances recorded at these classes.

It was acknowledged by the Department of Mathematics and Statistics that further support was warranted and in 2001, the Mathematics Learning Centre (MLC) opened. A fully supervised drop-in centre where students could study and/or receive 1-1 attention was provided for 20 hours a week. In the first year of operation, 1516 visits to the drop-in centre were recorded. In 2008/09, the total number of visits to the drop-in centre was 1,177 with a further 3,632 visits made the weeks prior to examinations. Between the drop-in centre, examination revision and support tutorials, over 7,200 contacts were made with the MLC.

With the view to increasing contact time with students who needed mathematics support, online support was set up by the MLC. A series of fact sheets tailored specifically for UL service mathematics courses were designed and peer reviewed by colleagues from the Department of Mathematics and Statistics, and made available online for all students. Past examination papers and sample solutions are made available on the site as are links to useful websites such as the University of Loughborough's Engineering Mathematics website.

One further progression made by the MLC to cope with growing numbers of non-standard students was the introduction of a one week refresher mathematics course, entitled 'Head Start Mathematics', designed to help students who had been away from formal mathematics education for a long time catch up on essential mathematics skills they would need for their service mathematics courses. This course, which is run in August before students start in UL, is free of charge and covers topics such as number systems, algebra and graphing to name but a few. The feedback from participants thus far has been very positive.

DISCUSSION

Implications of the changing student profile

Gill (2006) highlighted, through the use of diagnostic testing, the declining mathematical standards of third level students entering Technological and Science Mathematics courses in UL. This paper offers a discussion on a probable contributor to this decline i.e. the changing profile of students within these service mathematics courses. The main changes in student profile between 1998 and 2008 can be summarised as follows:

- Large increase in males within Science Mathematics over time

- Increase in the number of degree programmes within the service mathematics courses
- An increase of over 10% of Technological Mathematics students being labelled at risk and an increase of over 20% of Science Mathematics students as being at risk
- The percentages of students entering UL with Higher Level Mathematics has declined along with an increase in the frequency of entering students with Ordinary Level Leaving Certificate grades near to the minimum entry requirements
- The percentage of non-standard students has gone from approximately 1% of the entire cohort to almost 10% for Science and Technological Mathematics combined

Upon further investigation of the non-standard students' examination performance in 2008, the figures tell us that these students perform below average in the diagnostic test when compared to the standard students and above average in the end of semester examination. They are also reported to avail more of the support service put in place by the Mathematics Learning Centre. A major implication of the changing student profile was the need for increased mathematics support services within the period 1997 and 2008, which the Mathematics Learning Centre have carried out with great success.

The increase in support services is but one of the changes that needs to occur in mathematics education in order to attempt to cater for the varying backgrounds of this new profile of students. Lecturers and tutors of mathematics need to take note of the changes which have occurred over time and plan their teaching content and style accordingly. A lack of willingness to change to suit the needs of the current student profile would be likely to lead to "deterioration in the effectiveness of the learning" in UL and indeed in other third level institutions which are experiencing similar changes in student profile (Hunt & Lawson, 1996, p. 171).

REFERENCES

Barry, S.I., & Chapman, J. (2007). Predicting university performance. *ANZIAM J,* 49, 36-50.

Burgess, R.G. (1989). *The Ethics of Educational Research*. London: Routledge Falmer.

Gill, O. (2006). What Counts as Service Mathematics? An Investigation into the 'Mathematics Problem' in Ireland. unpublished thesis (PhD), Department of Mathematics and Statistics, University of Limerick.

Gill, O., O'Donoghue, J., Hannigan, A., & Faulkner, F. (2009). Trends in Performance of Science and Technology Students (1997-2008) in Ireland'. Submitted to *International Journal of Mathematical Education in Science and Technology,* July, 2009.

Hirst, K. (1999). Mature students studying mathematics. *International Journal of Mathematical Education in Science and Technology,*30(2),207-213.

Hunt, D.N., & Lawson, D.A. (1996). Trends in the Mathematical Competency of A Level Students on Entry to University. *Teaching mathematics and its applications*, 15, 167-173.

Kitchen, A. (1999). The Changing Profile of Entrants to Mathematics at A Level and to Mathematical Subjects in Higher Education. *British Educational Research Journal,* 25(1),57-74.

Lawson, D. (1997). What can we expect from A-Level students?. *Teaching Mathematics and its Applications,* 16(4),151-156.

Lawson, D. (2003). Changes in student entry competencies 1991-2001. *Teaching mathematics and its applications,* 22(4), 171-175.

Malcolm, R.K., & McCoy, P. (2007). Evaluation of numeracy skills in first year pharmacy undergraduates 1999-2005. *Pharmacy Education: An International Journal for Pharmaceutical Education,* 7,53-59.

O'Donoghue, J. (1999). *An intervention to assist at risk students in service mathematics courses at the University of Limerick*: *University of Limerick teaching fellowship scheme*. University of Limerick, Limerick.

Tall, D., & Razali, M. R. (1993). Diagnosing students' difficulties in learning mathematics. *International Journal for mathematics education in Science and Technology,* 24,209-222.

Taylor, J.A., & Mander, D.(2002). Managing diversity in first year service mathematics courses: integration of effective numeracy teaching principles and management theory. *In: Proceedings of Changing Agendas "Te Ao Hurihuri": Sixth Pacific Rim - First Year in Higher Education Conference*, Christchurch, New Zealand, 8-10 July 2002 [online], available: http://eprints.usq.edu.au/995/1/Taylor_Mander_2002.pdf [accessed Sept,2009].

DIAGNOSTIC TESTING IN DCU: A FIVE-YEAR REVIEW

Eabhnat Ní Fhloinn

Dublin City University

Diagnostic testing in mathematics is in widespread use across many third-level institutions in order to identify, and hopefully redress, areas of particular mathematical weakness at an early stage in a student's university career. Since September 2004, first-year service-mathematics students in Dublin City University have taken such a test upon entry. The test consists of fifteen, multiple-choice questions on a range of basic mathematics topics. Students who score 50% or less are deemed to be at-risk of failing their mathematics module and are advised to attend "refresher sessions" held during the first two weeks of semester. During the course of the past five years, it has been considered necessary to introduce certain modifications to the test; we discuss in detail the rationale behind these changes as well as the likely impacts on overall test results. In addition, we highlight the questions which were most poorly answered and look at possible relationships between examination grades and performance in the diagnostic test. Finally, we discuss the relative merits of module-specific diagnostic tests, where the difficulty of the test is related to the level of mathematics the student will encounter in their module.

INTRODUCTION

The poor core mathematical skills of a large number of students entering third-level education continue to be a cause of growing concern for many mathematics educators. This concern has been expressed in numerous journal articles and conference proceedings, and inquiries have been undertaken to ascertain the mathematical accomplishment of these students. In Ireland, studies were being undertaken as early as 1985, when Cork Regional Technical College concluded that their incoming undergraduates were deficient in basic mathematics (Cork Regional Technical College, 1985). Numerous other universities and institutes were soon reporting similar findings (Hurley and Stynes, 1986; Brennan, 1997; O'Donoghue, 1999). By 1995, in the United Kingdom, the London Mathematical Society (LMS), in collaboration with the Institute of Mathematics and its Applications (IMA) and the Royal Statistical Society (RSS), had produced a report entitled "Tackling the Mathematics Problem" (LMS, 1995), which investigated concerns amongst mathematicians, scientists and engineers in third-level education about the mathematical preparedness of new undergraduates. This was followed up by a report by the UK Engineering Council which showed strong evidence of a "steady decline" in basic mathematical skills and "increasing inhomogeneity in mathematical attainment and knowledge" (Savage, Kitchen, Sutherland & Porkess, 2000). One of the main recommendations of this report was that "students embarking on mathematics-based degree courses should have a diagnostic test on entry.” However, the report was also at pains to point out that diagnostic testing is a means to an end:

> Diagnostic testing should be seen as part of a two-stage process. Prompt and effective follow-up is essential to deal with both individual weaknesses and those of the whole cohort. (Savage et al, 2000, p. iii)

Without some category of support structure in place for students identified as being at-risk of struggling substantially with their mathematics module, there is a risk that diagnostic testing may not be of major benefit to these students:

> In situations where students are simply told their test result and advised to revise certain topics on their own, there is little evidence that this happens. (Lawson et al, 2003, p. 8).

DIAGNOSTIC TESTING

In 2003, the U.K. Learning and Teaching Support Network (LTSN) MathsTEAM project produced a detailed collection of case studies of diagnostic testing throughout the UK, which highlighted the range of testing being undertaken, as well as the results obtained, possible barriers to test execution and general recommendations (LTSN MathsTEAM, 2003). They concluded that:

> Diagnostic testing provides a positive approach to a situation. For the student it provides a constructive method, which leads to ongoing support, and for the academic it is an indication of “what is needed” in terms of teaching and curriculum changes. As the number of institutions implementing these tests increases it is becoming an integral part of mathematical education for first year students. (LTSN MathsTEAM, 2003, p. 7)

Diagnostic tests fall into two main delivery types: paper-based and computer-based. The optimal choice is generally dependent upon internal resources within each university, with a lack of sufficiently reliable computing facilities in some locations (including Dublin City University) meaning that paper-based is the only sensible choice, particularly at the start of the academic year when many students may not yet be set up on the computing system. Although a large number of these tests are multiple-choice, to enable speedy return of marks to the students involved, some universities, such as the University of Limerick, have opted instead for open-ended questions, as this provides them with greater information about the mathematical deficiencies in question:

> The test was designed for marking by hand so that one could investigate the specific errors that students make and identify where the gaps in student knowledge lie…As the students were provided with rough work areas, it was possible to determine why students were making the type of mistakes they were. (Gill & O’Donoghue, 2007, p. 228).

The majority of diagnostic tests are given during orientation week or in the first couple of weeks of the academic year (LTSN MathsTEAM, 2003, p. 4); in some cases, such as the Institute of Technology in Tralee, the tests are repeated several weeks later to assess students' improvements (Cleary, 2007).

DIAGNOSTIC TESTING IN DUBLIN CITY UNIVERSITY

The Maths Learning Centre in Dublin City University (DCU) opened in February 2004, and before the start of the 2004/2005 academic year, developed a diagnostic test for incoming first-year service mathematics students. The test consists of fifteen, multiple-choice questions on a range of basic mathematical skills, including percentages, fractions, numerical and algebraic manipulation, and solving linear and quadratic equations. It is paper-based and was initially conducted during the first mathematics class of the year, but in the past two years has been done during orientation week instead. There are two versions of the test, to prevent students who take the test earlier in the week passing it on to those taking it later in the week, as the solutions are given to students once they have completed the test. Both copies of this test can be found in Appendix A.

Student who receive below a certain grade on the diagnostic test are deemed to be at risk of failing their mathematics module and are advised to attend refresher sessions which take place during the first two weeks of semester, revising basic mathematics, and to make frequent use of the Maths Learning Centre during the year.

Over the past five years, approximately 700 students per year have taken the diagnostic test upon entry to the university (with the exception of 2006, when about 500 students took the test, due to a once-off alteration in the orientation programme for the business faculty). This amounts to roughly 70% of the first-year service mathematics cohort who have sat the test. The students involved are from six different service mathematics modules: Mathematics for Computing, Mathematics for Scientists, Mathematics for Physicists, Business Mathematics, Accounting Mathematics, and Engineering Mathematics.

MODIFICATIONS MADE

Introducing any modifications to a diagnostic test means that a full comparative study of results with previous years is no longer possible, and as such, the benefits of any such changes must be carefully weighed up against this loss of comparative data. However, bearing in mind the principal aim of the test, which is to identify those most at risk, occasionally changes are necessary in order to improve the accuracy of the information being obtained.

One example of this in the DCU diagnostic test was in regard to the most poorly answered question on the test, based on fractional indices (see Appendix A, question 3). Last year, a discrepancy was noticed between the two versions of the test: in one test, students were asked to simplify $\left(\frac{16}{9}\right)^{\frac{5}{2}}$, a question involving only fractional indices, whereas the corresponding question in the second test asked students to simplify $\left(\frac{100}{9}\right)^{-\frac{3}{2}}$, involving both fractional and negative indices. To rectify this oversight, the second test was altered to include only fractional indices (students were asked to simplify $\left(\frac{9}{100}\right)^{\frac{3}{2}}$ instead), and the results for the question showed a substantial improvement, with 48% of students answering correctly compared with 26% the previous year. Clearly, this is still an area of extreme difficulty for students, but it would seem likely that the inclusion of negative indices skewed the results for this question, given that they only appeared on one version of the test.

Another modification introduced two years ago was in regard to the marking scheme. For the first three years, negative marking was not used, as it was felt that this might be overly intimidating for students at the start of their university experience; however, in 2007, it was decided that the introduction of negative marking was necessary for more accurate results and to avoid random guessing of answers. In addition, negative marking provides a greater level of knowledge about the students' perceptions of their own learning gaps, as it would seem likely that, on the whole, students who leave questions blank are at least aware that they are unsure of that area, while students who answer incorrectly think they know how to solve the problem and as such, may be unlikely to seek further help for that topic. Therefore, as there were five options given for each question, students received 4 marks for a correct answer, and lost 1 mark for an incorrect answer, while receiving no marks for an unanswered question. Students were made aware of this fact and advised to leave blank any questions about which they were unsure.

MOST COMMON AREAS OF DIFFICULTY

Among the fifteen questions on the test, there are several questions which are invariably poorly-answered, with little variation shown on these from year-to-year. Figure 1 below shows the results for all six modules for each question of the diagnostic test in 2008.

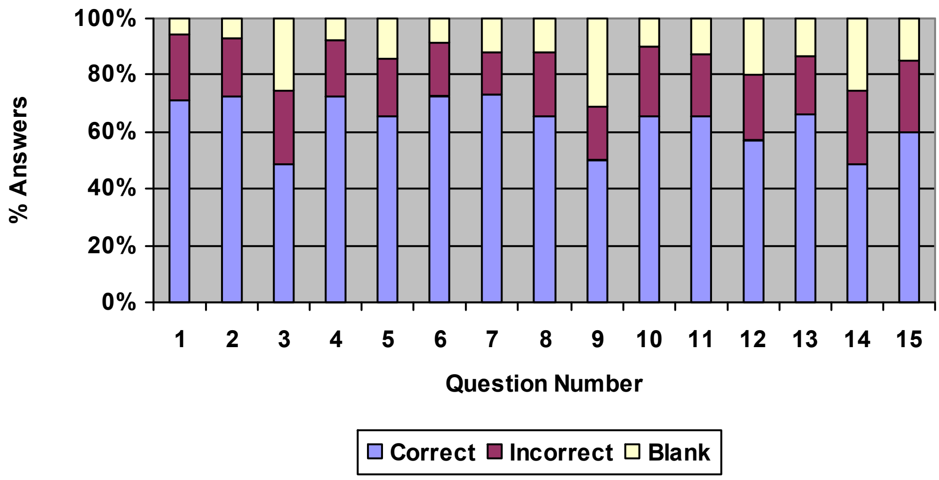


Figure 1: Results from all modules for each question of the diagnostic test in 2008.

The question which causes the most difficulty for students every year is based on fractional indices (see Appendix A, question 3). This is closely followed by questions on algebraic indices (question 9), solving a partially-factored cubic (question 12) and inequalities (question 14). In some modules, such as Mathematics for Computing, the percentage of correct answers for indices questions such as 3 and 9 can drop lower than 25%. It is also worth noting that a large number of students leave these particular questions unanswered.

DIAGNOSTIC RESULTS

As a result of the introduction of negative marking in 2007, students now obtain a percentage mark for the test, whereas pre-2007, they received a mark out of 15. The pass mark for the test is now 50%, whereas previously, it was 8 out of 15 (53.3%). Table 1 below shows the mean mark and standard deviation for 2007 and 2008, once negative marking was introduced.

Year	**2008**	**2007**
# Student Responses	726	730
Mean Mark (%)	58.22	62.52
Standard Deviation (%)	23.63	22.08

Table 1: Number of student responses to diagnostic test, along with mean mark and standard deviation for 2007 and 2008. Negative marking was used in these years.

Table 2 shows the results before negative marking was introduced. However, for greater clarity, all marks are given in percentages, instead of as marks out of 15.

Year	**2006**	**2005**	**2004**
# Student Responses	474	694	728
Mean Mark (%)	64.07	69.93	68.60
Standard Deviation (%)	21.73	28.47	19.67

Table 2: Number of student responses to diagnostic test, along with mean mark and standard deviation for 2004-2006. Negative marking was not used in these years.

If these results are compared with students' Leaving Certificate mathematics grades on a student-by-student basis, the resulting correlation is consistently low. There are several possible reasons for this: one is that students are given no prior warning about the test, and as such, have no opportunity to revise any material, and it has been at least three months since they last looked at any mathematics. Some students have naturally better recall than others. In addition, many students are trained to answer questions in state examinations and as such, do not cope as well with a new format of test. This theory is supported in countless instances of classroom observation (Lyons et al, 2003), as well as in the comments of the State Examination Commission:

> (E)xaminers have been commenting on a noticeable decline in the capacity of candidates to engage with problems that are not of a routine and well-rehearsed type. (SEC, 2005, p. 73)

Also, the structure of the marking scheme for Leaving Certificate mathematics means that it is possible for students to make small errors in basic components of questions and still obtain high "attempt marks" for that question, as they lose one mark for numerical slips, or three marks for mathematical errors or omissions (SEC, 2009 b, p. 2), whereas any such error would result in an incorrect answer in the diagnostic test. For all of these reasons, the diagnostic test identifies students whose grasp of some basic mathematical concepts is not as good as is needed to cope in their course, and as such, is a valuable add-on to their Leaving Certificate grade in identifying at-risk students as early as possible.

FUTURE WORK

The Leaving Certificate Mathematics examination can be taken at three different levels: Foundation (F), Ordinary (O) and Higher (H). Within each level, fourteen possible grades can be awarded, as laid out in Table 3 below.

Result, r (%)	Grade	Result, r (%)	Grade	Result, r (%)	Grade
$90 \leq r \leq 100$	A1	$65 \leq r < 70$	C1	$40 \leq r < 45$	D3
$85 \leq r < 90$	A2	$60 \leq r < 65$	C2	$25 \leq r < 40$	E
$80 \leq r < 85$	B1	$55 \leq r < 60$	C3	$10 \leq r < 25$	F
$75 \leq r < 80$	B2	$50 \leq r < 55$	D1	$r < 10$	NG
$70 \leq r < 75$	B3	$45 \leq r < 50$	D2		

Table 3: Percentage range for each grade awarded at Leaving Certificate (SEC, 2009).

The standard required for first-year service mathematics modules in DCU varies considerably, depending on the course involved. Indeed, the minimum mathematics requirement for Engineering Mathematics is a HC3 in Leaving Certificate mathematics (namely, 55% or greater in the Higher Level paper), whereas for many of the other modules, it is a HD3 or OC3. Even apart from this, however, the level that students are expected to reach by the end of their module is very different. If we then look at the diagnostic results for 2008, in which the mean mark was 58.22 ± 23.63 when all modules were taken together, we see that the mean for Mathematics for Computing, for example, was as low as 50.83 ± 22.99, while the mean for Engineering Mathematics was 74.74 ± 17.15. As a result, only 8.5% of the Engineering students who took the diagnostic test were identified as being “at risk” of failing their module, while 51.9% of the Computing students were similarly classified. However, based on anecdotal evidence, continuous assessment results and observations within the Maths Learning Centre, a significant proportion of the students in Engineering Mathematics will struggle with their mathematics module every year. Therefore, there is a real danger that the diagnostic test currently being used is not suitable for some of the more demanding modules.

However, the advantage of having all service mathematics students take the same diagnostic test is that it allows a direct comparison between students in different modules. Therefore, the optimum response is to produce two different tests, one a subset of the other, and give the more demanding test to students in the more difficult modules. As a result, a modified test is being designed which will hopefully be introduced in the coming year for three of the modules (Mathematics for Physicists, Accounting Mathematics and Engineering Mathematics), based on the original test, but with a number of additional questions, in order to better identify those at risk in such modules.

REFERENCES

Brennan, M. (1997). The 2nd-3rd level mathematics interface. In Cawley, S., (Ed.), *A Mathematics Review,* pp. 3-15, Dublin: Blackhall Publishing.

Cleary, J. (2007). Diagnostic testing – An evaluation 1998-2007. In S. Close, D. Corcoran & T. Dooley (Eds.), *Proceedings of Second National Conference on Research in Mathematics Education (MEI2)* (pp. 215-227), St. Patrick's College, Dublin.

Cork Regional Technical College, Department of Mathematics and Computing (1985). Report on the basic mathematical skills test of 1st year students in Cork R.T.C. in 1984. *I.M.S. Newsletter,* 14, 33-43.

Gill, O. & O'Donoghue, J. (2007). The mathematical deficiencies of students entering third level: An item by item analysis of student diagnostic tests. In S. Close, D. Corcoran & T. Dooley (Eds.), *Proceedings of Second National Conference on Research in Mathematics Education (MEI2)* (pp. 228-239), St. Patrick's College, Dublin.

Hurley, D. & Stynes, M. (1986). Basic mathematical skills of U.C.C. students, *I.M.T.A. Bulletin,* 17, 68-79.

Lawson, D., Croft, T. and Halpin, M. (2003). *Good practice in the provision of mathematics support centres*. Retrieved September 1, 2009. http://www.mathcentre.ac.uk/staff.php/all_subjects/guides/maths_support_centres/resources/resources/81

London Mathematics Society (LMS) (1995). *Tackling the Mathematics Problem.* Southend-on-Sea: LMS.

LTSN MathsTEAM (2003). *Diagnostic Testing for Mathematics.* Retrieved June 10, 2009 http://www.mathstore.ac.uk/mathsteam/packs/diagnostic_test.pdf.

Lyons, M., Lynch, K., Close, S., Sheerin, E. and Boland, P. (2003). *Inside classrooms: The teaching and learning of mathematics in social context.* Dublin: Institute of Public Administration.

O'Donoghue, J. (1999). *An intervention to assist at risk students in service mathematics courses at the University of Limerick*, University of Limerick teaching fellowship scheme, University of Limerick, Limerick.

Savage, M., Kitchen, A., Sutherland, R. & Porkess, R. (2000) In Hawkes, T. & Savage, S. (Eds.), *Measuring the Mathematics Problem*. London: Engineering Council

State Examination Commission (SEC) (2005). *Leaving Certificate Examination 2005, Mathematics, Chief Examiner's Report.* Retrieved September 1, 2009. http://www.examinations.ie/archive/examiners_reports/cer_2005/LCMathematics.pdf

State Examination Commission (SEC) (2009). *The Leaving Certificate (Established) Programme*. Retrieved September 1, 2009. http://www.examinations.ie/index.php?l=en&mc=ca&sc=sb

State Examination Commission (SEC) (2009 b). *Leaving Certificate 2009 Marking Scheme Mathematics Ordinary Level*. Retrieved September 1, 2009. http://www.examinations.ie/archive/markingschemes/2009/LC003GLP000EV.pdf

APPENDIX A: DCU DIAGNOSTIC TEST

Question 1

$\frac{2}{3}+\frac{2}{5}=?$

A) $\frac{4}{8}$ B) $\frac{2}{8}$ C) $\frac{16}{15}$ D) $\frac{2}{15}$ E) None of these

$\frac{3}{4}-\frac{1}{3}=?$

A) $\frac{2}{1}$ B) $\frac{2}{12}$ C) $\frac{5}{12}$ D) $\frac{2}{4}$ E) None of these

Question 2

$\frac{2}{3}\div\frac{1}{5}=?$

A) $\frac{2}{15}$ B) $\frac{10}{3}$ C) $\frac{10}{15}$ D) $\frac{3}{15}$ E) None of these

$\frac{3}{7}\div\frac{2}{3}=?$

A) $\frac{9}{14}$ B) $\frac{2}{7}$ C) $\frac{2}{21}$ D) $\frac{6}{21}$ E) None of these

Question 3

Simplify $\left(\frac{9}{100}\right)^{\frac{3}{2}}$

A) $\frac{1000}{27}$ B) $\frac{300}{18}$ C) $\frac{81}{1000}$

Simplify $\left(\frac{16}{9}\right)^{\frac{5}{2}}$

A) $\frac{40}{9}$ B) $\frac{40}{22.5}$ C) $\frac{1024}{9}$

D) $\frac{1000}{90}$ E) None of these

D) $\frac{1024}{243}$ E) None of these

Question 4

What is 1.5 expressed as a percentage of 2?

A) 75% B) 1.5% C) 15% D) 150% E) None of these

What is 6 expressed as a percentage of 15?

A) 37% B) 40% C) 75% D) 250% E) None of these

Question 5

A car bought for €12,500 loses 15% of its value in one year. What is the car worth at the end of this year?

A) €12,485 B) €11,000

C) €10,625 D) €1,875

E) None of these

A house increases it value by 8% each year. If a house cost €325,000 last year, how much will it cost this year?

A) €333,000 B) €26,000

C) €585,000 D) €348,000

E) None of these

Question 6

$-(2x+4y)-2(-x-2y)=?$

A) x B) $-4x+8y$

C) $-4x-8y$ D) 0 E) None of these

$(3x-6y)-5(-x+3y)=?$

A) $8x+9y$ B) $-2x+9y$

C) $8x-21y$ D) $8x-9y$ E) None of these

Question 7

Expand & simplify $(x-3)(2x+1)$

A) $2x^2-6x-3$ B) x^2-5x+3
C) $2x^2+x-6$ D) $2x^2-5x-3$

E) None of these

Expand & simplify $(4+x)(-3x+1)$

A) $-12x^2+x+1$ B) $-3x^2-11x+4$
C) $5-2x$ D) $-12x^2+4$

E) None of these

Question 8

Write $\frac{2}{x}-\frac{1}{1+x}$ as a single fraction.

A) $\frac{2+x}{x(1+x)}$ B) $\frac{2}{x(1+x)}$

Write $\frac{1-k}{2k}+\frac{k}{1-k}$ as a single fraction.

A) $\frac{1}{1+k}$ B) $\frac{1}{2k(1-k)}$ C) $\frac{1}{2}$

C) $\frac{2+2x}{x+x^2}$ D) $\frac{1}{-1}$ E) None of these

D) $\frac{1-2k+3k^2}{2k(1-k)}$ E) None of these

Question 9

Simplify $a^4\left(\frac{a^5}{a^3}\right)$

A) a^{17} B) a^8 C) a^6 D) a^{12} E) None of these

Simplify $b^{-2}\left(\frac{b^3}{b^4}\right)$

A) b^{-3} B) b C) b^3 D) b^{-2} E) None of these

Question 10

Find the value of s if $\frac{2}{s}-1=3$.

A) $s = 2$ B) $s = 8$ C) $s = 4$ D) $s = -8$ E) None of these

Find the value of Q if $-\frac{6}{Q}+5=8$.

A) $Q=2$ B) $Q=\frac{1}{2}$ C) $Q=-2$ D) $Q=-\frac{1}{2}$ E) None of these

Question 11

Find all values of x such that

$2x^2-x-3=0$

A) $x=\frac{3}{2}$, $x=1$ B) $x=\frac{3}{2}$, $x=-1$

C) $x=-\frac{3}{2}$, $x=1$ D) $x=-1$, $x=3$ E) None of these

Find all values of x such that

$3x^2+5x+2=0$

A) $x=\frac{2}{3}$, $x=1$ B) $x=\frac{2}{3}$, $x=-1$

C) $x=-\frac{2}{3}$, $x=-1$ D) $x=-2$, $x=-3$ E) None of these

Question 12

Find all values of t such that

$t(t^2-1)=0$

Find all values of r such that $r(r^2-9)=0$

A) $t=1$ B) $t=0,\ t=1$ C) $t=0,\ t=1,\ t=-1$	A) $r=0,\ r=3,\ r=-3$ B) $r=0,\ r=3$ C) $r=3$
D) $t=0$ E) None of these	D) $r=0$ E) None of these

Question 13

Let $y=(2x-1)^2+\sqrt{-8zx^3}$. If $x=-1$ and $z=2$, what is the value of *y*?	What is the value of $a-(b\sqrt{a+c})$ when $a=10$, $b=12$ and $c=15$?
A) $y=-5$ B) $y=5$	A) –10 B) 50
C) $y=3$ D) $y=13$ E) None of these	C) 70 D) 10 E) None of these

Question 14

Find all values of *x* for which $-2x+7>3$	Find all values of *x* for which $2x+4<-2$
A) $x<2$ B) $x=2$	A) $x=-3$ B) $x>-3$
C) $x>2$ D) $x<8$ E) None of these	C) $x>-6$ D) $x<-6$ E) None of these

Question 15

At what point do the lines $y=2x+4$ and $y=-4x+1$ intersect?	At what point do the lines $y=3x-2$ and $y=-x+6$ intersect?
A) (–0.5, 3) B) (1.5, –5)	A) (–2, 4) B) (2, 4)
C) (0.5, 3) D) (–3, 3) E) None of these	C) (2, 2) D) (0, 6) E) None of these

EXTENDING THE MATHEMATICAL CAPACITY OF GAEILGEOIRÍ: ASSESSING THE EFFECTIVENESS OF BILINGUAL MATHEMATICS INSTRUCTION IN FIRST YEAR UNDERGRADUATE EDUCATION

Máire Ní Ríordáin	Aisling McCluskey

NCE-MSTL, UL	NUI Galway

This project investigates the transition from learning mathematics through Gaeilge (Irish) at second level education to learning mathematics through English at tertiary level. It explores how such a transition may be affected by bilingual instruction for Gaeilgeoirí (students who learn through the medium of Gaeilge) in their first year of undergraduate mathematics education. In particular, the project quantitatively assesses whether the level of language proficiency in Gaeilge and in English is related to mathematical performance and whether the bilingual year assists in developing additive bilingualism. More importantly, the research evaluates if differences in mathematical performance exist between Gaeilgeoirí who opt to take bilingual instruction (lectures through the medium of Gaeilge; workshops through the medium of English) and those Gaeilgeoirí who opt for all English instruction at third level education at NUI Galway during the academic year 2008/2009. The authors have identified a gap in the literature specific to addressing how to support bilingual students in the transition between language mediums of learning. This research is the first of its type to be undertaken in an Irish context and provides significant insights into advantages for the learning of mathematics in a bilingual context.

INTRODUCTION

Examination of linguistic influences on mathematics education within the Irish context is a relatively new phenomenon. This is surprising given that two mediums of instruction – Gaeilge (Irish) and English – exist at primary and second level education in Ireland. Within the realm of Gaeilge-medium education, two types exist: Maintenance Heritage Language (Gaeltacht schools) and Immersion education (Gaelscoileanna and Gaelchloláistí). Immersion education has experienced substantial growth in the past decade with the numbers receiving this type of education increasing by more than sixty percent, and the number of schools increasing by more than fifty percent (Gaeilscoileanna Teo., 2006). The introduction of Acht na dTeangacha Oifigiúla (2003) has increased the prominence and use of our native language at a national and international level. Naturally, these are notable developments for the Irish language, but what is lacking is follow-up research on the development and use of Gaeilge as a language of learning across all levels of education and in particular at third level. The majority of Gaeilgeoirí face an imminent transition to English-medium tertiary education and this is what is of concern to the authors. Transitioning to a new language of learning for mathematics can be a source of difficulty for some Gaeilgeoirí (Ní Ríordáin & O' Donoghue, 2009), as Gaeilgeoirí submerged in the transition will be required not only to learn mathematics, but also to learn mathematics through the medium of English (Barwell, 2003). This is not just a localised problem; such issues are prevalent in international literature. Mathematics is not 'language free' and due to its particular vocabulary, syntax and discourse it can cause problems for students learning it in a second language (Barton & Neville-Barton, 2003). Research investigating the cognitive effect of bilingualism on mathematical learning began in the early eighties. Both Dawe (1983) and Clarkson (1992) concluded that bilingual mathematics students proficient in both their languages performed significantly better in mathematics than bilingual students dominant in only one language, and better than their monolingual peers. They also found that mathematics students who were weak in both their languages performed poorly mathematically also. More recent research carried out at second and third level education in New Zealand (Neville-Barton & Barton, 2005) with students for whom English is a second language concluded that these students experience a disadvantage of between 10 and 15 percent in mathematics as a result of language difficulties, which again reinforces the notion of the necessity of language proficiency in both languages. Overall, conflicting views exists on the benefits/disadvantages of bilingualism on mathematics learning where positive findings have been found by some (e.g. Barwell, 2003; Swain, 1996; Williams, 2002), while others view it as negative for mathematical learning (e.g. Adetula, 1990; Adler & Setati, 2000; Gorgorió & Planas, 2001). However, the significance of the authors' work lies in assessing how a bilingual (Gaeilge and English) first year undergraduate mathematics programme may help *facilitate* this key transition from Gaeilge medium to English medium education, and accordingly improve mathematical learning for Gaeilgeoirí at third level education.

BACKGROUND TO THE STUDY

NUI Galway has a special role in the protection and promotion of the Irish language (Gaeilge). Accordingly, it provides the subject Honours Mathematics at first year undergraduate level to both Arts and Science students in a choice of either entirely English medium or bilingual (Gaeilge/English) medium. The subject is year-long and is taught as four lectures per week together with a compulsory weekly workshop. Additionally, weekly tutorials are provided. The bilingual option offers lectures through Gaeilge while these students then integrate with their English counterparts in the weekly workshops. For a variety of reasons, uptake of the bilingual option is small (typically in the range 3 – 10 students) and there is certainly scope for improvement on that alone. A second subject, Pass Mathematics, is also offered in first year with a similar outline to that for Honours Mathematics but with the key exception that no bilingual option is available. This bilingual option is only available to ***first*** year undergraduate students at NUI Galway. All mathematics instruction is through the medium of English from second year through to the reminder of the degree. Therefore, this bilingual year provides Gaeilgeoirí with a unique opportunity to help facilitate the transition to all-English medium education.

LIMITATIONS OF THE STUDY

There are a number of limitations to this research which the authors would like to address for the readers of this study, and which are of importance when interpreting the findings emerging from the work.

1. The number of participants in this study is low. Seven Gaeilgeoirí in total took part: four were participating in the bilingual first year undergraduate programme; three were completing their study entirely through the medium of English.
2. For the students participating in the bilingual programme, they receive their mathematics lectures through the medium of Gaeilge and are then integrated with all first year mathematics students for their weekly workshops through the medium of English. Clearly, there is a low number of students in the lecture situation (4 in total) and thus they receive significant attention and instruction from the lecturer in comparison to those who opt to take the subject through the medium of English (typically, around 110 students are in this class). However, in the workshop context, no distinction between the groups is observed.
3. In this study, the Gaeilgeoirí opting to take mathematics through the medium of English at third level are taking Pass Mathematics. This subject is seen as a natural progression from Ordinary level mathematics at second level education. However, all Gaeilgeoirí participating in this study sat their Leaving Certificate examination at Higher Level and all received comparable marks (grade B average). The mathematics tests administered in this study are designed for Senior Cycle second level mathematics students and thus this eliminates discrepancies between the levels of mathematics taken at third level education for the participants in this study.

The purpose of the study undertaken is to give an insight into any advantages or disadvantages experienced by Gaeilgeoirí who opt to take mathematics through a bilingual medium of Gaeilge/English at NUI Galway during their first year of study.

METHODOLOGY

Description of Test Instruments

In subsequent sections of this paper we will discuss participants' performance on a variety of linguistic and mathematical tests, and the relationship(s) between these tests. A description of the tests employed follows:

(a) *Short Algebra Test* (Mestre, 1986). This was a locally designed to test Gaeilgeoirí's ability with algebraic manipulations. The problems were symbolic as opposed to verbal and included topics such as solving two simultaneous equations and factoring quadratic polynomials. The students had to complete twenty algebraic questions in total. The material in this test is consistent with the Leaving Certificate – Ordinary Level syllabus and thus was deemed to be appropriate for all participants in this study.

(b) *Mathematics Word Problem Test* (Newman's Research Method, 1977). The English mathematics word problem test consisted of nineteen word problems, with a number of subparts in some of the questions. Sixteen of the word problems were constructed using the PISA mathematical literacy framework (OECD, 2006). The remaining three questions consisted of cloze type questions (see Hater & Kane, 1975). Several words were deleted at random from each of these questions and the participants were required to fill in the missing word in each of the blank spaces provided.

(c) *English Proficiency Test* (Cambridge Examinations Publishing, 2002). A standard English cloze proficiency test was sourced through the Cambridge Certificate of Proficiency in English. In total, sixteen words were required to be filled in by the participants in order to complete the proficiency test.

(d) *Gaeilge Proficiency Test* (Aonad na Gaeilge, UL). Permission was granted by Aonad na Gaeilge, UL to use their internal Gaeilge proficiency test for this research project. This proficiency test consisted of sixty five multiple choice cloze questions.

(e) *Language Use Survey* (Clarkson, 2007). This survey enabled us to distinguish how much the participants employ English only, English and Gaeilge, or Gaeilge only when answering each algebra question and mathematics word problem question.

Description of the Language Proficiency Groups

All participants in this study completed all of the above tests (February, 2009). In total, seven Gaeilgeoirí participated in the study. Four were completing first year undergraduate mathematics education through the medium of Gaeilge and English, and three were taking mathematics through the medium of English. This is to facilitate comparison and to assess the effectiveness of participating in a bilingual (Gaeilge and English) year in third level mathematics education. A modification of the technique employed by Clarkson (2007) was used to segregate the participants into language proficiency groups. The median scores for the proficiency tests in Gaeilge and in English were used in order to divide Gaeilgeoirí into comparatively high or low proficiency groups in Gaeilge and in English (Clarkson originally divided English language proficiency into three groups – high, medium, low but the number of participants is too low in this study). Gaeilgeoirí were then categorised as having relatively high proficiency in both languages, dominance in one language (combination of high/low) or relatively low proficiency in both languages (combination of low/low). Each student was assigned to only one of these language proficiency groups (see Table 1).

	Categorisation	No. of Students	Medium of Learning - 3^{rd} Level
High Proficiency	**High Gaeilge & High English**	**3**	**Bilingual**
Dominant Proficiency	**High Gaeilge & Low English** ***Or*** **Low Gaeilge & High English**	**2** **1**	**English** **Bilingual**
Low Proficiency	**Low Gaeilge & Low English**	**1**	**English**

Table 1: Description of the Language Proficiency Groups.

OBSERVATIONS

All data was coded and imported into SPSS (Version 16) for analysis. The findings described in subsequent sections are intended to suggest or emphasize any advantages/disadvantages of undertaking a first year bilingual (Gaeilge & English) undergraduate mathematics course for Gaeilgeoirí in the transition to long term English-medium mathematics education.

Performance on the Algebra Test

The first concept explored in the analysis of the data gathered is to establish if a relationship exists between Gaeilgeoirí's performance on the algebra test and their overall language proficiency, as well their chosen medium of learning mathematics at third level education.

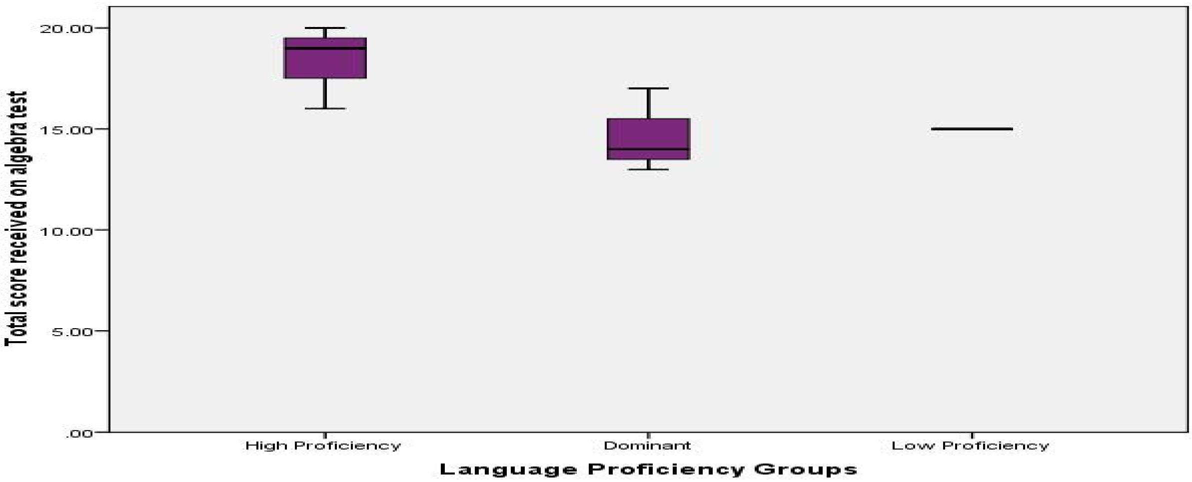


Figure 1: Comparison of language proficiency groups with performance on algebra test.

From Figure 1 it is clear that Gaeilgeoirí with relatively high proficiency in both languages performed better mathematically on the algebra test than Gaeilgeoirí dominant in one language and the Gaeilgeoir with low proficiency in both languages. Given that the high proficiency group dominates the course group learning through the medium of Gaeilge, it is not surprising that the bilingual group learning through the medium of Gaeilge and English at third level did better than the Gaeilgeoirí who opted to study mathematics entirely through the medium of English (Figure 2).

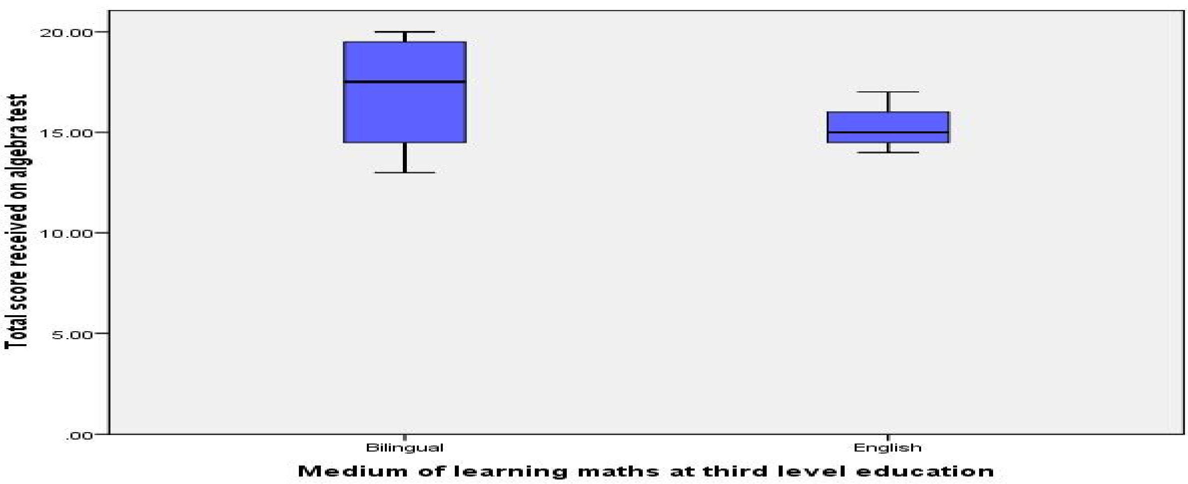


Figure 2: Comparison of language medium of learning with performance on algebra test.

No significant difference was found between the means of the various groups in the above boxplot diagrams (Mann-Whitney U, $p<0.05$). However, this may be attributed to the low number of participants in the study and it is anticipated a significant difference would exist between the groups if a larger cohort were involved in the study.

Performance on the Mathematics Word Problem Test

Given the previous observations reported on the algebra test, the authors were keen to see if these findings are replicated in the mathematics word problem test (in English). From Figure 3, it is evident that the high language proficiency group outperformed the dominant group and the student with low language proficiency. However, the differences in means was not found to be statistically significant (Mann-Whitney U, $p<0.05$), which again may be attributed to the low numbers participating in the study.

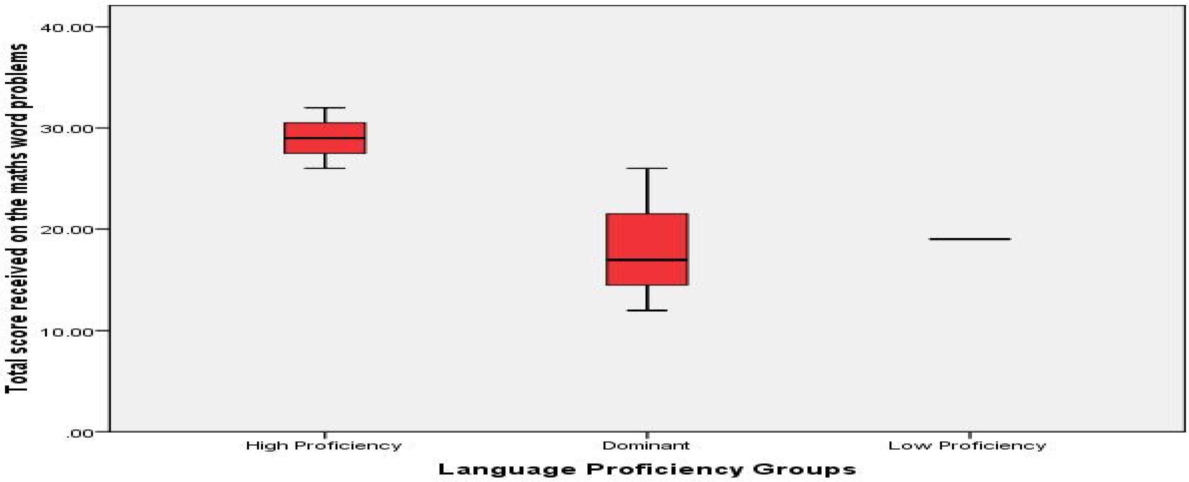


Figure 3: Comparison of language proficiency groups with performance on mathematics word problem test.

When segregating the Gaeilgeoirí according to the medium of mathematics education at third level, visibly the bilingual instruction group outperform the Gaeilgeoirí learning through the medium of English only (Figure 4). The difference was found to be significant (Mann-Whitney U, $p<0.05$). This suggests that Gaeilgeoirí pursuing mathematics through the bilingual approach are reaping the cognitive benefits from being bilingual, evident for those with high proficiency in both languages (Ní Ríordáin & O' Donoghue, 2009).

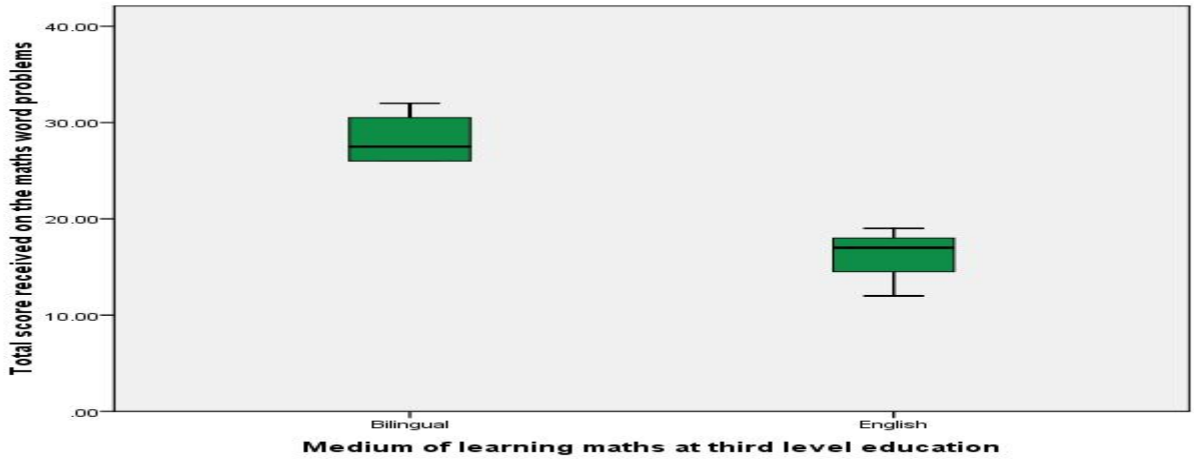


Figure 4: Comparison of medium of learning with performance on mathematics word problem test.

Language Use Survey

Gaeilge is still of importance for mathematical learning for the participants in this study. From analysis of the language use survey, it is evident that the Gaeilgeoirí drew on their first language of learning (Gaeilge) primarily when answering the algebra questions, even though all of the questions were presented in English (Table 2).

Language Group	**Algebra Questions** *(average language used overall)*	**Word Problem Questions** *(average language used overall)*
High Proficiency	**Gaeilge**	**Gaeilge & English**
Dominant	**Gaeilge & English**	**English**
Low Proficiency	**Gaeilge & English**	**English**

Table 2: Findings of the language use survey

In particular the High Proficiency language group employed Gaeilge when answering both sets of questions, demonstrating its importance to them for undertaking problem solving and their ability to accomplish mathematical thinking in ***two*** languages. This may explain why the High Proficiency group (who all are enrolled in the bilingual 1st year mathematics course) outperformed the other students mathematically in this study.

DISCUSSION

Three of the four students pursuing mathematics through the medium of Gaeilge and English demonstrate high proficiency in both languages, whereas the students opting for instruction entirely through the medium of English demonstrate dominant proficiency in one of their languages or low proficiency in both (only one student). From the findings presented in the previous section, it is obvious that the high language proficiency group perform better mathematically on the algebra and word problem tests than all other language proficiency groups. This demonstrates that bilingualism and bilingual instruction is a positive outcome for mathematics at third level education in NUI Galway. It suggests that bilingual instruction at NUI Galway may be facilitating the development of additive bilingualism, and accordingly the cognitive advantages associated with this type of bilingualism (Dawe, 1983).

All of the participants in this study were transitioning from Gaeilge-medium second level education to English-medium third level education. Research on the transition between language mediums for mathematics learning has shown that it can be difficult and may affect mathematical learning for those experiencing subtractive bilingualism (Neville-Barton & Barton, 2005; Thomas & Collier, 2002). Some of the participants opted for bilingual instruction, while others opted for all English instruction. Given that the bilingual mathematics group in this study performed mathematically better than the Gaeilgeoirí who opted for all English instruction it implies that this bilingual year may assist the transition to a new language of learning mathematics (English) for Gaeilgeoirí, as well providing Gaeilgeoirí with the opportunity of developing additive bilingualism.

Nationally, little research has been undertaken on mathematics education and the effects of bilingualism, even though Gaeilge-medium education has been well established with a number of decades. This research suggests/exposes positive consequences for mathematical learning for Gaeilgeoirí and reinforces the positive impact of bilingualism provided appropriate language thresholds have been reached in each of the languages. Internationally very little research has been carried out in relation to bilingualism and mathematics education at tertiary level (Neville-Barton & Barton, 2005). Therefore, this research provides a significant contribution to this area of research, as well as providing a foundation for future work to be undertaken.

RECOMMENDATIONS FOR FUTURE DIRECTIONS

This research project was conducted in order to give some insights into advantages or disadvantages that Gaeilgeoirí may experience from undertaking bilingual (Gaeilge & English) mathematics education during their first year of undergraduate education at NUI Galway. Despite the obvious deficit in terms of a low number participating in this study, initial suggested outcomes offer substantial insights into the potential benefits of this type of course in developing additive bilingualism and in easing the transition to all-English medium mathematics education subsequent to completing their first year at NUI Galway. A number of recommendations can be suggested from the work undertaken. These include:

- There is an obvious need to conduct repeat and further research into this bilingual mathematics course with larger numbers in order to assess its effectiveness in developing additive bilingualism and for facilitating the transition to learning mathematics through the medium of English.
- The results emerging from the research undertaken should be utilised to encourage Gaeilgeoirí entering NUI Galway and who wish to study mathematics to enrol in the bilingual mathematics option.
- The data suggests that Gaeilgeoirí with high proficiency in both languages experience a cognitive advantage in mathematics. Adopting a longitudinal perspective, the Gaeilgeoirí who participated in this study should be monitored and further testing undertaken throughout their mathematical studies at NUI Galway.
- Gaeilgeoirí entering first year undergraduate education at NUI Galway also have the option of taking some science subjects through this bilingual stream also. Research should be conducted into how effective these science courses are in facilitating the transition and to compare and contrast with those findings emerging from the mathematics stream.
- Other third level institutes should be made aware of these results and encouraged to adopt a similar opportunity in their mathematics programmes in order to cater for the Gaeilgeoirí in their institutions, and to foster additive bilingualism.

REFERENCES

Adetula, L. O. (1990) ‘Language factor: Does it affect children's performance on word problems?’, *Educational Studies in Mathematics*, **21**(4), pp.351-365.

Adler, J. & Setati, M (2000) ‘Between languages and discourses: Language practices in primary multilingual mathematics classrooms in South Africa’, *Educational Studies in Mathematics,* **43**(3), pp.243-269.

Barwell, R. (2003). Patterns of attention in the interaction of a primary school mathematics student with English as an additional language. *Educational Studies in Mathematics*, 53(1), 35-59.

Barton, B. and Neville-Barton, P. (2003) ‘Language issues in undergraduate mathematics: A report of two studies’, *New Zealand Journal of Mathematics (Supplementary Issue),* **32**, pp.19-28.

Cambridge Examinations Publishing (2002). *Cambridge Certificate of Proficiency in English.* Cambridge: Cambridge University Press.

Clarkson, P. C. (2007). Australian Vietnamese students learning mathematics: High ability bilinguals and their use of their languages. *Educational Studies in Mathematics,* 64(2), 191-215.

Clarkson, P. C. (1992) 'Language and mathematics: A comparison of bilingual and monolingual students of mathematics', *Educational Studies in Mathematics*, **23**(4), pp.417-429.

Dawe, L. (1983). Bilingualism and mathematical reasoning in English as a second language. *Educational Studies in Mathematics*, 14(4), 325-353.

Gaeilscoileanna Teo (2006). *Fás ar an nGaelscolaíocht sa Ghalltacht.* Retrieved January 10th, 2008 http://www.gaelscoileanna.ie/index.php?page=aims&lang=en.

Gorgorió, N. and Planas, N. (2001) 'Teaching mathematics in multilingual classrooms', *Educational Studies in Mathematics*, **47**(1), pp.7-33.

Hater, M.A. & Kane, R.B. (1975), 'The cloze procedure as a measure of mathematical English', *Journal for Research in Mathematics Education*, **6**(2), 121-127.

Mestre, J.P. (1986). Teaching problem-solving strategies to bilingual students: what do research results tell us? *International Journal of Mathematical Education in Science and Technology,* 17(4), 393-401.

Neville-Barton, P. and Barton, B. (2005). *The Relationship between English Language and Mathematics Learning for Non-native Speakers: A TLRI research report for NZCER,* Wellington: NZCER. Retrieved October 22, 2005 http://www.tlri.org.nz/index.html.

Newman, M. A. (1977). An analysis of sixth-grade pupils' errors on written mathematical tasks. *Victorian Institute for Educational Research Bulletin,* 39, 31-43.

Ní Ríordáin, M. & O' Donoghue, J. (2009). The relationship between performance on mathematical word problems and language proficiency for students learning through the medium of Irish. *Educational Studies in Mathematics*, 71(1), 43-64.

OECD (2006). *Assessing Scientific, Reading and Mathematical Literacy: A Framework for PISA 2006*. Retrieved February 2, 2007 http://www.oecd.org/dataoecd/63/35/37464175.pdf

Roinn Gnothaí Pobail, Tuaithe & Gaeltachta (2003). *Acht na dTeangacha Oifigiúla.* Retrieved April 23, 2006 http://www.achtanna.ie/ga.act.2003.0032.1.html.

Swain, M. (1996) 'Discovering successful second language teaching strategies and practices: From programme evaluation to classroom experimentation', *Journal of Multilingual and Multicultural Development*, **17**(2), pp.89-113.

Thomas, W. and Collier, V. (2002). *A National Study of School Effectiveness for Language Minority Students' Long-Term Academic Achievement.* University of California, Santa Cruz: Center for Research on Education, Diversity and Excellence.

Williams, C. (2002) *A Language Gained: A Study of Language immersion at 11-16 Years of Age,* University of Wales, Bangor: Education Transactions.

THE POWER OF THE SHORT STORY AND THE BIG PICTURE IN MATHEMATICS EDUCATION IN SCHOOLS, UNIVERSITIES AND FOR THE GENERAL PUBLIC

Fiacre Ó Cairbre

NUI, Maynooth

In this paper I will present a case for how the 'short story and the big picture' can play a significant role in enhancing the understanding, awareness and appreciation of mathematics in schools, universities and among the general public. The big picture of mathematics includes its history, motivation, beauty, practical applications, Irish connections, tricks, word origins, stories, famous characters, outdoor activities, puzzles and more. Evidence for the case above comes from many years of university teaching (including a history of mathematics course), giving school talks and Maths Week events and also conducting a wide variety of public events including the annual Hamilton walk, radio and TV shows, newspaper articles and public talks in many different locations. I will outline the positive feedback and some interesting and unexpected consequences of the 'short story and the big picture' approach. The general public plays an important role in mathematics education in schools and universities because parents, decision makers and the media are all members of the general public and can exert great influence on the attitude of young people and society at large towards mathematics.

INTRODUCTION

This paper discusses the positive role played by the short story and the big picture in mathematics education. In this paper the big picture of mathematics includes, in no particular order, (a) history of mathematics, (b) stories, (c) human element and famous characters, (d) beauty, (e) practical power and applications, (f) motivation, (g) Irish connection, (h) research, (i) word origins, (j) humour, (k) outdoor activities, (l) cultural connections, (m) tricks/magic, (n) puzzles, (o) drama (p) deductive reasoning and/or abstraction. See Ó Cairbre, McKeon & Watson (2006) for more on each of the above 'big picture' items (a) to (p) and other 'big picture' items. For a wide variety of topics in mathematics, many of the 'big picture' items above can be wrapped up in a short story, or a longer story if appropriate, like in transition year etc. I find from experience that the 'short story and the big picture' approach enhances the understanding, awareness and appreciation of mathematics among students, teachers and the general public.

I will now elaborate a little on items (a) and (d) above, which are two of the most important items. NCCA (2005) states, regarding mathematics in the junior cycle, that students should develop an appreciation of mathematics including its history. There are many reasons why students should hear about the history of mathematics. For example, it shows the humanity of mathematics and that mathematics is created by people motivated by pure imagination or problems in the physical world. Mathematics in School, (2003) and Swetz, Fauvel, Bekken, Johansson & Katz (1995) discuss how the history of mathematics can improve teaching. Two of the general objectives in mathematics education, in Department of Education and Science (2000), are that the students should appreciate mathematics as a result of being able to acknowledge the beauty and that students be aware of the history of mathematics. From my experience in the promotion and teaching of mathematics, I have found that the concept of beauty in mathematics shocks many people. This is good in one sense because it arouses curiosity. Then, after a quick example (like 'Big sum for a little boy' below) and a little chat, the very same people have changed their perception of mathematics and agree that beauty is a feature of mathematics. The beauty in mathematics typically lies in the beauty of ideas because mathematics essentially comprises a myriad ideas. For example, number is an idea and cannot be sensed with our five senses, even though many people think they see it on the board. A sheet of music is important and useful, but it's nowhere near as interesting or powerful as the music itself. Similarly, mathematical notation is important and useful, but nowhere near as interesting or powerful as the ideas it represents. Ó Cairbre (2009) gives many examples of beauty in mathematics and presents a case for why I consider beauty to be arguably the most important feature in mathematics. The three main reasons in my case are:

(i) The quest for beauty has typically been the motivation for why the great mathematicians do research in mathematics. Historically, research has been very vibrant with mathematicians trying to solve unsolved problems and develop new theories.
(ii) The practical power of mathematics is often an offspring of the search for beauty.
(iii) Mathematics, as we know it today, was essentially born out of a pursuit for aesthetic pleasure and beauty by the Classical Greeks around 600 BC. This is because the two main pillars of mathematics are deductive reasoning and abstraction and both of these appealed to the Greeks for aesthetic reasons.

The Classical Greeks reckoned there were three ingredients in beauty and these were lucidity, simplicity and restraint. Note that 'simplicity' here means simplicity in hindsight, because it may not be easy to come up with the idea initially. On the contrary, it may require much creativity and imagination to come up with the idea initially. These three ingredients may not be a complete recipe for beauty for everybody, or maybe a recipe for beauty doesn't even exist. However, it can be interesting to have these ingredients in the back of your mind when you encounter beauty in mathematics. Also, for the Classical Greeks, the three ingredients applied to beauty, not just in mathematics, but for many of their interests like literature, art, architecture, sculpture, music etc.

I will now illustrate the power of the 'short story and big picture approach' by bringing many 'big picture' items into seven examples below. All examples have received very positive feedback from a wide variety of students, teachers and the general public. The examples provide a taste because many of the 'big picture' items can be brought into any topic in mathematics. The word 'student' in the following examples may also include 'member of the general public' in the appropriate places. The 'big picture' items typically merge together naturally in a topic and so it's not like one has to tick the boxes to ensure certain items are included. In each example the teacher can decide what level of student is suitable. There is at least one example, called Magic, which is appropriate for all levels from primary to third level.

SIMULTANEOUS EQUATIONS AND THE KING'S CROWN OF GOLD

In the 'Implementation Experience' in Ó Cairbre et al (2006) the secondary school teacher, John McKeon, wrote:

> I found 'Archimedes and the Crown of Gold' to be useful for the motivation of the teaching of simultaneous equations. It is also a story and pupils like stories.

Archimedes (287-212 BC) lived in the Greek city-state of Syracuse. King Hiero of Syracuse received a crown of gold but he suspected that the interior contained the lesser metal silver. He asked Archimedes to find out if the crown was made of pure gold without damaging the crown. Later on, while in the bath, Archimedes had his famous 'Eureka' moment and solved the problem. How did he solve this practical problem? First note that a body immersed in water displaces water equal in volume to the space the body occupies. What is less obvious is that bodies of equal weight are not necessarily of equal volume. For example, 1kg of silver occupies more volume than 1kg of gold. Consequently, he realised that 1kg of silver would displace more water than 1kg of gold. He then created two objects, one made entirely of silver and the other entirely of gold, but both equal in weight to the crown. He immersed each in a tub full of water and, as expected, more water overflowed when the silver object was immersed than when the gold object was put in. King Hiero's crown was then immersed in the tub and displaced more than the gold object and less than the silver one. Thus, he knew that the crown was not made of pure gold and had solved the King's problem. Apparently, he jumped out of the bath and ran naked down the street shouting 'Eureka' meaning 'I have found it'. The following clerihew captures the event nicely:

> Archimedes of Syracuse
> To get into the news
> Called out Eureka
> And became the first streaker

Archimedes was curious to do more than just solve the problem he was given. He wanted to know how much gold and silver was in the crown. This is where simultaneous equations appear. He measured the water displacement of each of the three objects above. Suppose the 5kg crown displaced 21cc of water, the 5kg of silver displaced 30cc and the 5kg of gold displaced 15cc. Now suppose the crown contains x kg of silver and y kg of gold. We wish to find x and y. Since 5kg of silver displaced 30cc of water, then x kg of silver will displace 6x cc. Similarly, y kg of gold will displace 3y cc. So, the crown will displace (6x+3y)cc and so 6x+3y=21. We also have that x+y=5 and so we have two simultaneous equations:

$$6x+3y=21$$
$$x+y=5$$

Solving these equations gives us x=2 and y=3. He now knows that the crown contains 2kg of silver and 3kg of gold. His determination of the amount of gold in the crown, by solving simultaneous equations, is quite elegant. Recall the three Greek ingredients of beauty mentioned earlier. It's also kind of magical that he was able to do it without damaging the crown This example has all the big picture items except (g) and (k) unless a student has a Eureka moment in an outdoor hot tub for (k)!

PYTHAGORAS' THEOREM AND THE MUSICMAKER

Pythagoras (585-500 BC) was born on the Greek island, Samos, in the Aegean Sea. There is a town there now called Pythagorea and it has an impressive sculpture commemorating him. The students can be asked to find out what the sculpture is. Yes, not surprisingly, it's a huge right angle triangle. It's interesting that the original statement of Pythagoras' theorem was not about numbers but rather about a relationship between areas. The original statement said that:

> The area of the square on the hypotenuse is equal to the sum of the areas of the two squares on the other two sides of the right angle triangle.

There have been several hundred different proofs of Pythagoras' Theorem. Some proofs are aesthetically pleasing. The proof with the greatest element of surprise is probably the proof by a former US President. The students can be asked to find out which one and to compare some of the other proofs. It was President Garfield. In fact a mathematician president is not so unusual for us when you realise that De Valera was a mathematician. The students can be asked to find out which Irish president was a mathematician. Pythagoras was the leader of a cult, called the Pythagoreans, in Crotona, which was a Greek city in what is now southern Italy. He made many significant contributions to mathematics and is also regarded as the founder of the theory of music. The story goes that one day Pythagoras entered a blacksmith's forge and found the sounds of the hammers quite harmonious (or hammeronious even!). He was puzzled and on investigation, he realised that the weights of the hammers were in small whole number ratios, like 1 to 2, 2 to 3 etc. This led to the observation that harmonious sounds were made by plucking two strings whose lengths were in small whole number ratios. Indeed, the relative lengths in every harmonious combination of plucked strings could be expressed as ratios of small whole numbers, i.e. rational numbers like $\frac{2}{3}$ etc. The word rational is used because it relates to the ratio of two numbers. His mathematics led to the theory of harmonics and hence the music of today. Disney has produced a DVD of a highly acclaimed award winning film/cartoon called ‘Donald Duck in Mathmagicland’. The film is very funny and entertaining and covers the above story and many other features of mathematics. This example has all the big picture items except (k) unless one organises a school trip to Pythagorea in Samos!

THE IRRATIONAL MURDER OF HIPPASUS

Hippasus (c. 500 BC) was a Pythagorean. The motto of the Pythagoreans was ‘All is number’, meaning everything in the universe could be explained by rational numbers. Hippasus proved that the $\sqrt{2}$ was not a rational number, i.e. it was irrational, and so shattered Pythagoras' theory of the universe. His proof is a very elegant piece of mathematics that can be shown to students. This led to the first crisis in the history of mathematics, because all the proofs that assumed all numbers were rational had to be checked and potentially redone if possible, with that assumption deleted. Pythagoras was so annoyed that he went overboard and literally threw Hippasus overboard into the sea where he drowned! In this way, Hippasus became the first known martyrmatician!

HAMILTON AND THE ANNUAL WALK

This example will have all the big picture items. William Rowan Hamilton (1805-1865) is Ireland's greatest mathematician and one of the world's most outstanding mathematicians and scientists ever. He was born in Dominick St. in Dublin and spent his early youth on the Banks of the Boyne in Trim across from the spectacular ruins of Trim Castle. He then lived in Dunsink Observatory for the rest of his life. Like most great mathematicians, his motivation for doing mathematics was the quest for beauty. He was successful in finding much beauty in mathematics. As is frequently the case in mathematics, practical power is the offspring of the quest for beauty. His mathematics has turned out to be incredibly powerful when applied to science, engineering, and now also to special effects in movies and computer games etc. Ó Cairbre (2000) and Ó Cairbre (2006) have more on Hamilton’s life and works.

I organise the annual Hamilton walk on October 16 which commemorates his famous creation of a strange new system of four dimensional numbers called Quaternions, in a flash of inspiration, on the banks of the Royal Canal in Broombridge, Cabra, in Dublin in 1843. The walk retraces his steps from Dunsink Observatory to Broombridge and takes about forty-five minutes. There is a plaque at Broombridge with the Quaternion formulas which Hamilton scratched on the bridge in a 19th century act of graffiti! The plaque was unveiled by De Valera, who was a big fan of Hamilton.

The annual walk will celebrate its 20th anniversary this year in 2009 since Anthony G. O' Farrell of Maynooth initiated it in 1989. So, I suppose you could say 'Fiche bliain ag siúl'! The walk typically attracts over 200 people from diverse backgrounds including staff and students from universities and secondary schools and many from the general public. The walk is ideal for a mathematics outing for transition year students and teachers have said that the walk and the Hamilton story have had a positive impact on students' perception of mathematics. Many famous people have come on the walk. Andrew Wiles, of Fermat's Last Theorem fame, launched the walk in 2003 and the walk appeared on the 6 o'clock news on RTE I television. The Fields Medallist, Timothy Gowers, and several Nobel Prize winners in Physics including Murray Gell-Mann, Steven Weinberg and Frank Wilczek have participated in the walk in recent times. Also, in 2005 Hamilton's great-great grandson, Mike O'Regan, came on the walk. Cabra Community Council have made the event into a very festive affair in the last few years with a large banner about Hamilton draped across the bridge and stalls along the canal. Aodhan Perry, of Cabra Community Council, said:

> The walk has had a huge impact on the local community. In fact it has gone way beyond just being a walk because all the local school children and the community are extremely proud of Hamilton and their local connection with him. It really has touched the local people in a big way. The fact that famous mathematicians and Nobel Prize winners mingle with school children and the local community on the walk and at the bridge is a great experience. Also, not one but two local artists have been commissioned in recent times to do portraits of Hamilton which are then publicly displayed at the bridge during the walk.

I am delighted to be involved in an event that has had this impact on a local community. To me, this is a very rewarding and significant part of Mathematics education. I find, from experience, that there is an appetite for mathematics among school children and general public when the 'big picture approach' is taken. Broombridge has become a world famous site in the history of mathematics and science because of Hamilton's creation of Quaternions. The large number of participants in the annual walk also shows the massive interest of the general public in Hamilton and the Broombridge site. The word Broomsday is now sometimes used in mathematical and scientific circles to indicate October 16 and the walk, and the word plays the same role as Bloomsday for literary groups.

Quaternions liberated algebra from the shackles of arithmetic and Hamilton has been called the Liberator of Algebra. The mathematical community was stunned by his audacity in creating a useful system of numbers that did not satisfy the usual commutative rule (ab = ba) of multiplication that exists in arithmetic. In Quaternions the order in which the numbers appear is important in multiplication and this did not bother the creative Hamilton because this is what usually happens in nature. For example, consider an empty swimming pool and the two operations of diving head first into the pool and turning the water on. The order in which the operations take place is quite important! Of course, students are advised not to try this at home, or wherever there is a swimming pool! The word Liberator above often leads on to the notion of freedom in mathematics. The idea of freedom in mathematics surprises many people. However, as Cantor (1845-1918) once said: 'Freedom is the essence of mathematics'. The reason freedom is a significant feature of mathematics is because one is free, just like Hamilton, to conceive of any ideas one wants in mathematics. Whether or not these ideas will lead to anything interesting or useful is another matter. Historically, the major breakthroughs in mathematics have typically happened because the great mathematicians were free to conceive of any ideas they wanted even if they broke with conventions and seemed bizarre to other mathematicians and the general public. Two examples, of many, are Quaternions which liberated algebra and the discovery of Non-Euclidean geometry which liberated geometry in the 19th century. Students can be shown the Quaternion formulas and encouraged to work with Quaternions a little bit.

Hamilton's mathematics has been, and still is, crucial for many important applications to Science and many other areas. Quaternions play a significant role in Computer Games. One example of this, which always appeals to journalists, radio hosts and students of course, is the fact that Lara Croft in Tombraider was created using Quaternions! Continuing with the theme of entertainment, Quaternions now play a prominent role in special effects in movies. For example, an Irish company called Havok used Quaternions in the creation of the acclaimed new special effects in the movie 'The Matrix Reloaded' and also in the movie Poseidon which was nominated for an Oscar for its visual effects in 2007. Havok won an Emmy award in the US in 2008 for pioneering new levels of realism and interactivity in movies and games. Also, the dramatic visual effects in the new James Bond movie, Quantum of Solace, were created by Havok. Students always give a positive reaction when I show a movie character like Shrek etc. and then tell them its creation depends heavily on mathematics. Quaternions played an important role in Maxwell's mathematical theory and prediction of electromagnetic waves in 1864. Thus, the inventions of radio, television, radar, X-rays and countless other significant products of our time are directly related to Hamilton. Maxwell's work illustrates the magical power of mathematics because his pure mathematics made the invisible visible since radio waves are invisible to our five senses. Maybe mathematics has this magical power because it comprises many ideas which are not limited to our five senses. Hamilton's fundamental theory of dynamics was indispensable for the creation of Quantum Mechanics, which is how we now understand the physical world at the microscopic level. His famous Hamiltonian function is fundamental to many aspects of Physics. Vector Analysis, which is indispensable in Physics, is an offspring of Quaternions. Also, Hamilton created the Icosian game, which was produced as a commercial puzzle board.

The Government designated the bicentenary year of his birth, 2005, as 'Hamilton year – Celebrating Irish Science' and many events were held all over Ireland. In that year RTE 1 television showed a Léargas documentary on Hamilton that covered the walk too. Also, inspired by the walk in 2003, Cabra resident, Jack Gannon, wrote a song about Hamilton and June Robinson has also written a poem about him. These are two, or many, interesting and unexpected consequences of the walk. Jack has said:

> On account of the walk, Hamilton is in the folk consciousness of the local people.

From an early age Mick Kelly had an interest in science and technology but had 'built a glass wall around himself that prevented him from learning mathematics'. Then he participated in the walk in 2005 and wrote:

> The walk had a profound effect on me. Hearing not only a Nobel laureate and a professor of mathematics sing Hamilton's praises, but also local poets, school children, balladeers and the Cabra community council, spurred me to turn my desire to celebrate Ireland's science heritage into action. That action turned out to be a family run business called Science Heritage Ireland selling placemats and coasters celebrating Hamilton.

The glass wall above was cracking. Regarding the later walk in 2007, he wrote:

> There was a sense of magic at Broombridge on that fine Tuesday, October 16, 2007 when the canal bank was alive with children playing all kinds of mathematics games. I couldn't help but wonder how many bridges to the future the organisers of this walk and maths week had created for our children.

Maths Week Ireland was initiated in 2006 by Eoin Gill of Calmast and it now occurs annually around mid October so that it includes October 16 and the walk. The aim of Maths Week is to popularise mathematics among school children and the general public. I organise some events for Maths Week, involving many of the big picture items, and the feedback from people has been very positive. Typically around fifty events take place during the week all over the country. Also, it's very heartening to see that, separate from the fifty or so events above, many schools now organise their own Maths Week events.

BIG SUM FOR A LITTLE BOY

Here is a simple example of what I consider to be beauty in mathematics. For those people who are shocked by the notion of beauty in mathematics, this example usually changes their perception of mathematics very quickly for the better and they then agree that beauty can be a feature of mathematics. A German boy, Karl Friedrich Gauss (1777-1855), was in his first arithmetic class in the late eighteenth century and the teacher had to leave for about 15 minutes. The teacher asked the pupils to add up all the numbers from 1 to 100 assuming that would keep them busy while he was gone, Gauss put up his hand before the teacher left the room. He had the answer and his solution exhibits both beauty and practical power. He observed that

$$1+100=101,\ \ 2+99=101,\ \ 3+98=101,....\ \ 50+51=101$$

and so the sum of all the numbers from 1 to 100 is 50 times 101 which is 5050. Notice how his solution exploits the symmetry in the problem and flows very smoothly. Compare it to the direct brute force approach of 1+2+3+... which is very cumbersome and would take a long time. Both approaches will give the same answer but his solution has beauty and the other is tedious. Also, his approach can be generalised for more complicated problems and is much more powerful than 1+2+3+…

MAGIC

I will give an example of magic in the form of a trick below. Tricks can often be a good way to stimulate students. They can also provide an intriguing setting for the discussion of mathematics. The trick below has many important applications to science. For example, the trick relates to why students can listen to their favourite music on a CD (and why a CD supposedly has no flaws/scratches etc. like the old LPs) and also relates to why we can view images from Mars!

Here is the trick: Create an audience of students. Ask a volunteer to set up a square with five rows and five columns of cards (or anything that has a front and a back side that are different), with a random number of cards face up and face down. Ask the volunteer to turn one of the cards over while you are not looking. The trick is that you will be able to say which card was turned over. However, just before the student turns the card over, you suggest adding in one card to each of the five original rows and one card to each of the five original columns in order to make your problem more difficult. This action is crucial to the trick but you don't let the audience know this. You carefully, yet seemingly carelessly, append a new card to each row and a new card to each column such that the number of cards face up in each of the five new rows is even and the number of cards face up in each of the five new columns is even. You then look away and let the volunteer turn one card over. You look back at the cards (and wave your magic wand!) and simply silently count the number of cards face up in each row and column and note where you get an odd answer. This will tell you where the overturned card lies. It will seem like magic. The above trick can also be performed by using zeros and ones on the blackboard instead of cards face up and face down. I have performed this trick many times in my public promotion of mathematics in schools and the general public and the trick definitely makes a big impression on people. The idea behind the trick also has a certain beauty.

How does the above trick relate to applications in science? Well, in the trick you are using a basic version of a technique that is fundamental in the powerful practical area of 'error correction in codes'. This is the technique where information is appended to the code (message) by the transmitter, in order that the receiver of the message will be able to detect a possible error, due to physical interference etc, and hopefully correct the error. The analogue of the error in the above trick is the overturned card and you were able to detect where the error lies essentially by appending extra information before the card was turned over. Error correction in codes is crucial in the performance of compact discs. Sound is digitally stored on a CD. This digital information can be thought of as a code (message) consisting of zeros and ones, just like the face up cards and face down cards in the trick above. Extra information is also appended to the CD as in the trick above to give the total code on the CD. A laser beam in your CD player transmits this total code to a decoder. The decoder receives the total code and attempts to detect any errors which may have been caused by dirt or a scratch etc. This detection process is an advanced version of the method used in the trick above. When an error is detected it can then be corrected so that the sound emanating from your CD player is correct. This is a far cry from the needle on the turntable! Error correction in codes is also fundamental in analysing data transmitted from spacecraft. For example, when a spacecraft takes photos of Mars, the data is digitally stored like in the CD above. This data (and the extra appended information like above) is transmitted to earth. Any errors caused along the way, like radio interference etc, can be detected and corrected like above. We can then see the correct images of Mars.

GEOMETRY

The word geometry is derived from the Greek words 'geo' meaning earth and 'metron' meaning measure. Thus, geometry was probably originally mainly concerned with measuring land. Now, of course, geometry encompasses much more, but the notion of measurement, in a very broad sense, is still a fundamental part of geometry. The Greek historian, Herodotus (484 - 425 BC), wrote the following about the Egyptian pharaoh, Sesostris, who lived in the fourteenth century BC.

> Sesostris divided the land among all the eligible Egyptians. Each person received a rectangle of the same area and was taxed accordingly. If part of the land was flooded during the annual overflow of the Nile, then the owner was obliged to report the loss to the Pharaoh, who would then dispatch an inspector to measure the area of the flooded portion of land and grant the appropriate reduction in tax.

The inspector needed to know some geometry to accomplish the above task. One can imagine that inspectors/geometers might have been tempted with a few nixers by the pyramid makers as they sailed down the Nile. The art of geometry would have played a crucial role in the building of the pyramids. Geometry is exceptionally powerful in science, engineering, navigation, meteorology, astronomy and many other areas. The following extract of a poem, which appears in the Welsh mathematician Robert Recorde's (1510-1558) 'Pathway to Knowledge', captures the diverse applications of geometry:

Sit merchants by ships great riches do win,
I may with good right at their seat begin.
The ships on the sea with sail and with oar,
Were first found and still made by geometries lore.
Their compass, their card, their pullies, their anchors,
Were found by the skill of witty geometers.
To set forth the capstock and each other part,
Would make a great show of geometry's art
Carpenters, carvers, joiners and masons,
Painters and limners with such occupations,
Broderers, goldsmiths, if they be cunning,
Must yield to geometry thanks for their learning.
The cart and the plough, who do them well mark,
Are made by good geometry. And so in the work
Of tailors and shoemakers, in all shapes and fashion,
The work is not praised if it want proportion.

Note that back then the word witty meant skilful or clever. The list of important applications of geometry is practically endless. The students can be asked to compose a modern version of Recorde's poem with their favourite items like ipods, DVDs, movies and their special effects, TV, radio, CDs, mobiles, computer games, computer animation, planes, trains and automobiles etc.

CONCLUSIONS

The short story and the big picture can have a significant positive role to play in mathematics education for students, teachers and the general public. The time involved in class for teachers can range from a few minutes to much longer if appropriate, like in transition year etc. In the future, I hope to be involved with colleagues in the creation of more resources, that include many 'big picture' items, for teachers at all levels.

REFERENCES

Department of Education and Science (2000). Junior Certificate Mathematics Syllabus, www.education.ie

Mathematics in School, Special Issue on the History of Mathematics. (2003). 32 (1).

NCCA (National Council for Curriculum and Assessment) (2005). Review of mathematics in Post-Primary education. Discussion paper. Dublin: NCCA.

Ó Cairbre, F. (2000). William Rowan Hamilton (1805-1865), Ireland's greatest mathematician. *Ríocht na Midhe*, XI, 124-150.

Ó Cairbre, F. (2006). William Rowan Hamilton and the Quaternion walk. *Irish Mathematics Teachers' Association Newsletter*, 105, 3-14.

Ó Cairbre, F., McKeon, J. &Watson, R. (2006). *A resource for Transition year mathematics teachers.* Dublin: Department of Education and Science.

Ó Cairbre, F. (2009). The importance of being beautiful in mathematics. *Irish Mathematics Teachers' Association Newsletter*. (in press).

Swetz, F., Fauvel, J., Bekken, O., Johansson, B. & Katz, V. (1995). *Learn from the Masters*. Mathematical Association of America.

THE ROLE OF PROOF VALIDATION IN STUDENTS' MATHEMATICAL LEARNING

Kirsten Pfeiffer

NUI Galway

The study of proofs is a major obstacle in the transition from school mathematics to university mathematics. Given the importance of argumentation and proof in the spectrum of mathematical activities, the incoming students' understanding, appreciation and knowledge of the nature and role of proof must be considered. I describe the results of an exploratory study of first year mathematics undergraduates' criteria and learning process when validating mathematical arguments or proofs. The study is based on a series of written tasks and interviews conducted with first year honours mathematics students at NUI Galway. I presented the students with numerous (correct and incorrect) proposed proofs of mathematical statements, and asked them to evaluate and criticize those. The first year students' comments on different and partly incorrect 'proofs' of mathematical statements gave some understanding about their criteria when validating mathematical arguments. In recently held interviews with eight randomly chosen students I focussed on the learning experience during the process of proof validation. Considering the observed learning effect and its large potential extent during the process of proof validation I propose its practice in the teaching of mathematics.

INTRODUCTION

In the first part of this article I explain what I mean by *proof validation* and consider various aspects of this activity, including how and why mathematicians validate mathematical arguments. I will then describe our experiments and findings, and consider the undergraduates' validation skills and practices in relation to previously characterized aspects of proof validation. I emphasize the learning effect during the process of proof validation and finally argue for explicit inclusion of its practice in the teaching of mathematics, as the development of validation skills not only improves the practice of validation itself, but also the ability to construct proofs, the understanding of mathematical context, the knowledge of proving strategies and the links between different areas of mathematics.

ON PROOF VALIDATION.

Before considering how students validate proofs I discuss the nature of proof validation; I further distinguish it from other types of reading and from construction of proof. Selden and Selden (1995) call the reading and the mental processes associated with reflections on mathematical proofs to determine their correctness *"validations of proof"*. I extend this description of proof validation considering that mathematicians, advanced students or maths teachers validate not only to determine the correctness of an argument. Discussions with a number of experienced mathematicians suggest that they also wish to reach an understanding of why a mathematical statement is true and often to understand the content and the position of the proved statement in a wider context.

Therefore when comparing proofs they would prefer an argument which helps to understand content and context to another one which might equally be correct but does not enhance one's understanding. (That might be a reason why mathematicians often are not really satisfied with very long or computer generated proofs. It is common practice in the mathematical community to seek more illuminating proofs even though correct proofs already exist.) The **providing with understanding** aspect is one criterion for valuable proofs that we use when validating mathematical arguments.

Alcock and Weber (2004) describe how mathematicians determine whether an argument is a valid proof. Their results are based on interviews with eight mathematicians who are presented with six to eight arguments from number theory. Their study is focused on the process of validating. One of Alcock and Weber's results is about the chronological order of the mathematicians' proceedings. They state that the participants first examined the proof methods employed. If those were judged to be reasonable the participants would proceed to the second stage of line-by-line verifications. These findings and our own discussions with mathematicians indicate that they consider **proof idea** and **strategy** substantial in a valuable argument.

Another important aspect when validating mathematical explanations is its **structure**. The structure of an argument proving a statement like *'When you add any two even numbers, your answer is always even'* could be

1. State properties of the objects (e.g. even numbers). These properties might be previously learnt definitions or simply properties that the student or mathematician believe are true.
2. Justify the statement from the chosen properties.
3. State the conclusions.

Other types of mathematical theorems or statements require different proof structures. I consider clarity of the structure of the argument significant in a valuable proof.

After checking proof idea and structure I regard **correct and sufficient reasoning** as a major criterion in argument validation. This includes clear and accurate use of language and notation and explanation of what is needed without excess.

A mathematical argument is valuable if it is **convincing.** This aspect plays a major role for mathematics educators. David Tall (1989) claims that a mathematical proof must be accepted by the mathematical community; Mason, Burton and Stacey (1982) define a proof as an argument that convinces an enemy; Hersh (1993) as a convincing argument, as judged by qualified judges.

This concludes my discussion of the factors that experienced mathematicians might consider while validating proposed proofs. In Section 2 below I report on my investigation of the criteria that students consider as essential for a valuable mathematical argument.

The activity 'proof validation'.

Validation, in comparison to the reading of non-mathematical texts, requires the reader to put some additional effort into understanding of the reasoning. Validation usually takes more time, the validator might consider the whole proof or parts of it (certain logical steps) several times, is more inclined to write a few notes checking deductions, verifying justifications, etc. According to Selden and Selden (2003) the mental process when validating proofs can include for example asking/answering questions, constructing subproofs or remembering other theorems and definitions.

It is well documented that construction of proof is a major obstacle for students. Selden and Selden (2003) describe how the ability to validate proofs relates to the ability to construct them. On the one hand proof construction and proof validation are **different.** Proof construction requires 'the right idea' at the 'right time'. The validation process can usually be managed in a linear order, which appears unlikely in construction of proof. On the other hand proof construction and proof validation **entail each other** as one considers during the process of proof construction how that proof would be validated, and as validation of a proof is likely to require the construction of **subproofs**. I summarize this relationship in the following diagram.

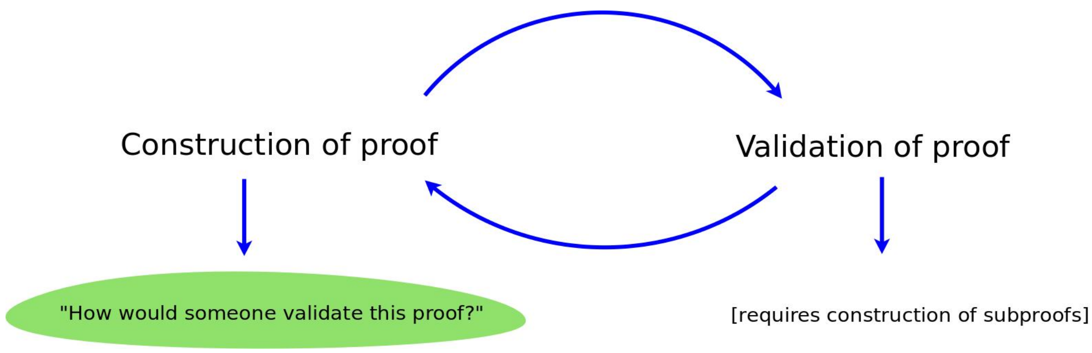


Fig 1: Construction versus Validation

Considering my reflections above on the nature of proof validation, I extend this diagram to highlight the extent of the learning effect through the process of proof validation.

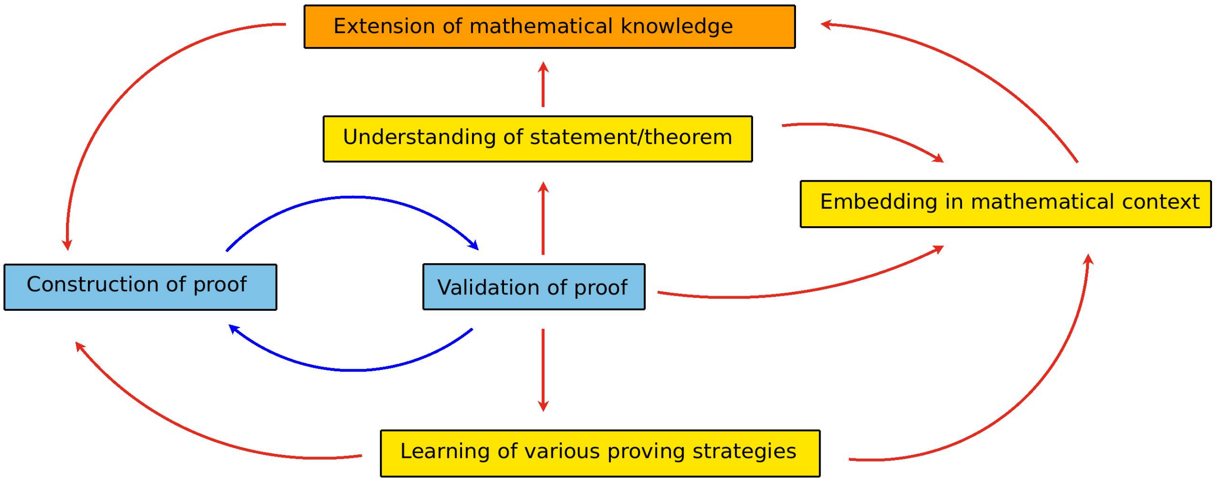


Fig 2: Proof Validation in the process of learning about mathematical proof.

I summarize my **hypothesis**: *The ability to validate proofs can improve the ability to construct proofs, develop deeper understanding of the meaning and significance of the proved theorem and develop knowledge of proving methods or strategies.*

THE EXPERIMENT.

The study is based on several tests and interviews conducted with first year honours mathematics students at NUI Galway. The students' post-secondary experience in mathematics at this stage consists of two foundation courses in Algebra and Analysis and a weekly workshop designed to introduce essential skills of advanced mathematical thinking through specific team-oriented tasks. During the year I worked intensively in small groups with the students. One of my main impressions at the end of the students' first year at University was that many of them were not able to recognize a satisfying mathematical proof.

The students' way to validate mathematical proofs first caught my interest when I was analysing their responses to a written exercise that I held in May 2008, at the end of their first year at University. My research has focussed on that topic since. Therefore the design of further research instruments concentrated on proof validation. The test for the new incoming students (D-test08), held in September 2008, included tasks designed to give me some insights into their proof validation skills. Based on the findings of the analysis of the written exercises I designed interviews to be held with a smaller number of students. The aim of these oral exercises was to get a deeper insight in the students' validation processes. The interviews were held with eight undergraduates in March 2009.

I report below on findings arising to date from all three of these exercises. Analysis of the data is ongoing.

Some of the questions had been adapted (with permission) from the Longitudinal Proof Project which ran 1999 until 2003 in the U.K. (http://www.mathsmed.co.uk/ioe-proof).

Considering the previously described aspects of the process of proof validations I investigate

- how our first year students validate mathematical proofs,
- what criteria they use to decide whether an argument is correct or not and
- the learning process during the exercise of validating and comparing different mathematical arguments.

Written exercises.

The test held in May 2008 was attended by 37 participants. [1] The students were presented with six attempts to determine, with proof, whether the statement

"When you add any two even numbers, your answer is always even"

is true or false. For each of the following attempts, I asked the students to give a mark out of five and a line of advice.

Aoife's answer:

a is any whole number.
b is any whole number.
2a and 2b are any two even numbers.
2a+2b=2(a+b).
So the statement is true.

Barry's answer:

2+2=4 4+2=6
2+4=6 4+4=8
2+6=8 4+6=10
So it's true.

Cathy's answer:

Even numbers are numbers that can be divided by 2.
When you add two numbers with a common factor,
2 in this case, the answer will have the same common factor.
So the statement is true.

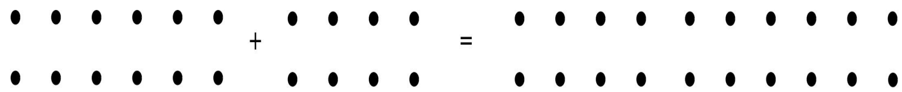
Dan's answer:

Even numbers end in 0,2,4,6 or 8. When you add any two of these the answer will still end in 0,2,4,6 or 8. So it's true.

Eve's answer:

Let x= any whole number, y= any whole number.
x+y=z
z-x=y
z-y=x
z+z-(x+y)=x+y=2z. So the statement is true.

Finn's answer:



▪

▪

So the statement is true.

Observations from written exercises: students' criteria when validating proofs

Discussions with mathematicians and students indicate that first year university students are typically not very experienced in the skill of proving. [2] I was surprised by some comments indicating what the students found essential in a good proof. The prevalence of certain expressions like *"proof by example"* or *"not mathematical enough"* raised my curiosity. What do students think is a *"mathematical proof"*? Do they all believe that *"mathematical"* means *"includes formulas"* or is that a small minority? I list below a number of themes that caught my interest, either because they appeared quite often or because they surprised me. I will analyse the students' comments to the given different answers in relation to these themes.

The role of examples

When confronted with a few examples to show the truth of the statement, I found that most students recognized the necessity of rigour and commented on that. *"He has just given examples, this is not a proof"* or *"Not a general solution"* are typical remarks. Though this certainly is an encouraging result I have to keep in mind that I don't know how the students would have attempted their own answers without getting the chance to compare to other, more 'general' proofs. [3]

Some of the students' comments indicate that *"proof by example"* might be seen as another type of proof, just not as good as one including a general formula: *"It's not bad. But it's only proved by example."*, *"Although this does prove the statement, it only does so for a few egs..."*.

Students do give examples a surprisingly high if not essential value within a proof. They often deduct marks on the basis of absence of examples. Some students comment on the absence of examples to explain the answers (*"Give an example to show ...")* but most criticize the lack of examples as an essential part of the reasoning. Some students request examples in (correct) answers to *"back up proof"* or to *"finish the proof"*. One student criticizes a correct answer because the statement is *"not proven by numerical example"*. Without any examples students don't seem to be satisfied by an argumentation. [4]

"Not mathematical enough"

The following are a few selected comments on a correct answer that was expressed purely in the form of text: *"The proof makes sense but she could have used a more mathematical approach"*, *"Good intuitive answer but needs a mathematical proof"*, *"Correct answer but show mathematically"*, *"The proof should be shown mathematically as well as in words"*. The argument formulated as text was *"not mathematical enough"* to most of the undergraduates. In comparison to another more algebraic looking approach one student comments: *"Although Cathy's answer is true there is too much English and does not mathematically prove it unlike Aoife"*. These comments raise the question of what the students associate with the term *"mathematical"*. Schoenfeld noted that mathematicians and students seem to have different perceptions of "thinking mathematically".

> For the students, thinking mathematically involves algebraic tricks and formal language (Schoenfeld, 1985) .

Our students' comments on the different answers indicate a similar impression: *"mathematical"* to them appears to mean including formulas (*"Try to come up with formula"),* algebraic equations (*"Give clear equation to support your answer", "Would like to see this expanded with a general equation"*) and mathematical notation (*"Use mathematical notation to show this", "Cathy's answer is well written and ... although she should sum her answer up .. using formal notation"*). Consequently Eve's answer, which is fundamentally and irreparably incorrect but includes algebraic equations and mathematical notation, seemed basically correct to more than half of the students. Less than 30% of the students recognized that this answer was wrong.

A visual approach to prove the statement is generally not accepted by the students. Finn's answer consisted of a diagram showing how an even number can be represented by two rows of dots, and how addition of even numbers can be interpreted as concatenation of two such representations. Most students interpreted the answer as just one example, visualized in a diagram. *"Again Finn's answer only covers 1 solution. He needs to give a general statement."*, *"This proves that it works for 12+8. It doesn't prove for all cases."*, *"Not a proof, just an example"* are a few typical answers. 27% recognized the idea behind the illustration, but didn't acknowledge a graphical representation of the correct idea as a mathematical proof: *"Good visual proof but use mathematics"*, *"Good visual representation but needs notational explanation"*. A proof without numbers and words can't be sufficient: *"There are no words in this proof"*, *"Proof is illustrated using graphics rather than numbers", "This does not prove anything, words and numbers are needed"*, *"There are no words in this proof"*.

"No definitions".

Many of the students regard a proof as good, if it is written in a certain structure, beginning with a definition. *"Define an even number before* [using] *them in the proof"*, *"I believe Aoife's answer would be more acceptable had she defined an even number"*, *"State definition of even no at start"* are some examples. [5]

"it doesn't explain..."

As mentioned in the previous section '*On Proof Validation*' some mathematical educators argue that whether or not an argument is accepted as a proof depends not only on its logical structure, but also on how convincing the argument is. That aspect seems to play a role in the students' proof evaluation as well. *"Nice pictures, you could have written a line explaining it though", "Intuitively correct but needs to explain why the answer means the statement is true", "Need more explanation"* or *"She should explain what she is doing"*. The positive comments on the highly marked answers often include a note about the good explanations: *"Aoife has a very clear and straightforward answer"*, *"Well explained answer"* or *"Aoife is using clear and simple language to get her answer across..."* are a few comments on the students' favourite answer. Those comments indicate that just having a good idea to prove a statement is not sufficient for the students. The skills of convincing and explaining ideas to others matter to them.

I summarize that after their first year in university most students are aware that checking the truth of a statement for a few examples is not sufficient to prove the statement. On the other hand examples play an important role in mathematical argumentation to students. Even after accepting the correctness of an argument they are not convinced until it is shown with a few examples. Overall the students' picture of a proof seems to be vague. I didn't get the impression that the students had been taught explicitly certain properties of a mathematical proof; their view must have been developed by the amount of proofs they have seen so far, probably mostly in textbooks. [2] To the students a valuable proof should have a certain structure, starting with a definition, followed by some algebraic equations or formulas, and finishing with a few examples. Structure and formalism seem to be more important to the students than the idea behind the proof. If these requirements are met most of the students give at least a few marks regardless of the correctness of the particular steps or whether the whole idea makes sense to them or not. Ergo, a good idea to prove a statement is not being valued as highly as the structure and formalism of a proof. Beside structure and formalism the quality of explanations played a role in the students' proof evaluations. It seemed that if an answer didn't show attempts to convince the reader of an argument, most of the students would deduct marks.

Oral exercises

In March 2009 I held interviews with eight randomly chosen students who had attended the written exercise in September as well. The aim was to get a deeper insight into

- students' opinions about valuable proofs. What do students mean when they use the term “mathematical”? Do they ask for examples in order to understand the given reasoning or because they consider them as essential part of proofs? etc.
- students' validation process. How do they attempt the task of validation? Do they read the proposed proofs line by line? Do they write notes, verify the arguments? etc.

- the learning effect during the validation process.

To facilitate comparison of the results one of the statements chosen for consideration in the interviews was similar to the one in the written exercise, but a bit more difficult.

The squares of even numbers are even, and the squares of odd numbers are odd.

The other statement was different from the exercises the students had performed so far.

Let f be a quadratic function, $f(x) = ax^2 + bx + c$ with $a,b,c \in R$ and $a>0$.

Show: f can't have more than two common values with its derivative f'.

Again the students were confronted with five or six different arguments, some incorrect or partly incorrect, and asked to comment on them and finally rank them. Some of the proposed proofs were algebraic, some visual, some written in text, others wrong but expressed using “typical” mathematical notation.

Observations from the oral exercises: the learning process when validating and comparing different mathematical arguments.

A detailed analysis of the interviews is in progress. In structuring and partly transcribing them a remarkable pattern caught my attention. The students were very quick in deciding whether they liked an answer or not and their first opinions and comments were similar to those in the written exercises. During the meeting though I could observe a process of understanding when spending more time with the task, comparing ideas with some proofs they have seen somewhere else, and sometimes even questioning their own criteria. Especially when ranking the different answers, the students reconsidered their opinions and sometimes changed their minds about certain answers.

These findings are in line with Selden & Selden and Alcock & Weber:

- Selden and Selden describe that students' performance in distinguishing valid from invalid arguments improved dramatically through the reflection and reconsideration during the interview. (Selden & Selden, 2003)
- “Our results suggest that many of the students in our study could perform this task competently, but did not do so without prompting.” (Alcock & Weber, 2004)

In my interviews I tried to avoid prompting. Therefore I believe it is the fact that the students were forced by the ranking task to spend some time on their reflections which encouraged the learning process. I conclude that the understanding of mathematical concepts can improve considerably during the process of careful proof validation.

CONCLUSION

The findings show on the one hand that undergraduates have a vague yet inflexible picture of valid proofs. Structure and 'mathematical' looking formalism seem more important to them than the idea behind its appearance. On the other hand I discovered that reflection during the process of proof validation hugely encourages a learning process about the nature of mathematical proofs. Recalling the discussion in the first part of this article about the nature of proof validation, in particular the relation between construction and validation of mathematical proof and the attainment of understanding through validation, I conclude that practice of proof validation can not only improve students' validation skills but can also lead them to a better understanding of mathematical content and to improved appreciation of deductive reasoning. Therefore I suggest explicit inclusion of practice of proof validation in the curriculum, not only to develop the ability to construct proofs, but also to improve the students' overall understanding and knowledge of mathematics and its coherence.

NOTES

1. In September 2008 we asked new incoming students to perform similar exercises. This time 103 students attended. The analysis of this second test (D-test08) is in progress.

2. A survey about first year students' background and secondary school experience regarding proof is in preparation.

4. In D-test08 the students were asked to attempt the tasks themselves before they got to see the prepared answers.

5. K. Hemmi describes in her doctoral thesis that incoming students in her survey (held in Sweden) like our first years students used examples to convince themselves of the correctness of a formula, but (different to our students) they felt that *"that was something private and not accepted in real mathematics."* (Hemmi, 2006, p.157). That raises the question why so many Irish beginner students regard examples as crucial in a mathematical proof.

6. The second questionnaire (D-test08) show that definitions didn't play a similar role for new incoming students. Definitions was one of the topics in our workshops and it seemed to be new to the students to think about the role of definitions in mathematics. Obviously those exercises had an influence on the students' validation habits.

REFERENCES

Alcock, L. and Weber, K. (2004). How do mathematicians validate proofs? *Paper presented at the annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Toronto, Canada.

Alcock, L. and Weber, K. (2005). Proof validation in real analysis: inferring and checking warrants. *Journal of Mathematical Behaviour* 24, 125-134.

Hemmi, K. (2006) Approaching proof in a community of mathematical practice. Doctoral Dissertation. Department of Mathematics, Stockholm University.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, 24(4), 389-399.

Küchemann, D. and Hoyles, C. (1999-2003). The Longitudinal Proof Project. *A Longitudinal Study of Mathematical Reasoning: Student Development and School Influences*, London.

Mason, J., Burton, L. and Stacey, K. (1982). *Thinking mathematically*. Addison Wesley, London.

Schoenfeld, A. (1985). *Mathematical problem solving.* Orlando, Academic Press.

Selden, A. and Selden, J. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, 29(2), 123-151.

Selden, A. and Selden, J. (2003). Validations of proofs written as texts: can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, 34(1), 4-36.

Tall, D. (1989). The nature of mathematical proof. *Mathematics Teaching*, 128, 28-32.

WHAT MAKES MATHEMATICS ATTRACTIVE AT UNIVERSITY?

Rachel Quinlan
NUI Galway

We discuss the findings arising from a survey of students entering NUI Galway in September 2008, who had either enrolled on degree programmes with high mathematical content or had selected the specialized mathematics option within the undenominated Arts and Science programmes. Students were invited to write a few lines in response to each of the questions "What do you enjoy about studying mathematics?" and "What do you dislike about studying mathematics?". The 54 responses received reveal a number of identifiable and distinct themes in the case of each question. We present the issues that emerge most prominently and discuss them in the context of the students' embarkation or progress on the transition to modes of conceptual mathematical thinking that are characterized by emphasis on such matters as abstraction, generalization and proof. We include some reflections on the implications for our practice as university mathematics educators, in accordance with the conference theme of extending mathematical capacity.

INTRODUCTION AND CONTEXT

> ". . . we go through the motions of saying for the record what we think the students 'ought' to learn, while the students are grappling with the more fundamental issues of learning our language and guessing at our mental models. Books compensate by giving samples of how to solve every type of homework problem. Professors compensate by giving homework and tests that are much easier than the material 'covered' in class, and then grading the homework and tests on a scale that requires little understanding. We assume that the problem is with the students rather than with communication : that the students either don't have what it takes, or else just don't care. Outsiders are amazed at this phenomenon, but within the mathematical community, we dismiss it with shrugs."

These words are taken from William Thurston's provocative, critical, and famous essay on communication in the academic mathematical community (Thurston, 1994). It is likely that many academic mathematicians, despite their own sustained and energetic efforts, can (however reluctantly) recognize more than a grain of truth in Thurston's damning description of our collective teaching practice. Naturally we would like to do far better than this. Yet the task of articulating our practice in a manner that is consistent with the reality of our experience and also with our aspirations for our students and for our communication of our subject remains a major challenge.

The difficulties associated with the transition from school to university mathematics are widely documented and seem to be geographically and historically ubiquitous in their themes. Hoyles, Newman and Noss (2001) give an account of the issues surrounding this transition in the UK, in the current context of the changing roles and objectives of university education and surrounding policy. Tall (1991) presents us with a detailed discussion of the cognitive challenges that are likely to be faced by students embarking on the specialized study of mathematics at university. He characterizes the transition from elementary mathematical thinking to *advanced mathematical thinking* (a cognitive transition that we tacitly or explicitly expect specialist mathematics students to make in the early undergraduate years) in the following terms :

> ". . . from *describing* to *defining*, from *convincing* to *proving* in a logical manner based on those definitions. This transition requires a cognitive reconstruction which is seen during the university students' initial struggle with formal abstractions as they tackle the first year of university."

If we wish to avoid the scenario alleged by Thurston, then surely one of our goals as lecturers must be to try to find ways of attending to the "cognitive reconstruction" that our students must achieve in order to make progress in our subject. There appears to be a consensus that for most students this reconstruction, if it is achieved at all, is likely to involve a protracted process rather than a quick transformation. If we are to have some hope of assisting students through this journey, it seems reasonable to suggest that we need to consider the road that they travel as well as their destination. As practitioners who view this journey from the transformed perspective of having successfully completed it some time ago, it is not easy for us to identify the hazards that our students might expect to encounter (see (Meyer & Land, 2003) for a discussion of this point). However, a first step might be to consider what in our students' experience has led them to the point of embarkation on a journey that they may not be aware they are expected to undertake. That is the theme of this paper. Every student has studied mathematics for a number of years upon entry to university. If a student chooses to study mathematics at advanced level at university, it seems reasonable to suppose that this choice is informed by an experience of mathematics that has been affirming in some way. The body of this paper consists of a report on incoming mathematics students' comments on what they like and dislike about the study of mathematics, and an accompanying commentary on possible implications for their prospects for transition to advanced mathematical thinking.

QUESTIONS FOR STUDENTS ON EMBARKATION

Students entering the Colleges of Arts and Science at NUI Galway have the opportunity to study Mathematics either at general or honours level (or not at all). The general courses are primarily intended for students who do not plan to become mathematical specialists, but require or wish to study some mathematics, for example in order to support their learning in other subjects. Students who are considering the specialized study of mathematics to degree level take the honours course. A questionnaire was distributed in September 2008 to students of Arts and Science, who had either opted to take Mathematics at honours level, or who had enrolled in programmes requiring honours mathematics in First Year (all such programmes have a high mathematical content throughout). Students were invited to return their completed questionnaires on the next day, and the number of students who did so was 54. The purpose of the questionnaire was to invite students to reflect on their expectations of mathematics at university and on their mathematical experience to date. The questionnaire consisted of the following three questions :

- What do you enjoy about studying mathematics?
- What do you dislike about studying mathematics?
- Why did you decide to take the First Year Honours Mathematics course?
  (If you are required to take it as part of your programme, why did you decide to enrol in a programme with high mathematical content?)

Our analysis here is concerned with responses to the first two questions only. It is our opinion that these data furnish us with some insight into the mathematical experience of our students, their beliefs about mathematics, and the typical positions from which they embark on the study of mathematics at university level. We categorized the responses to Questions 1 and 2 according to the prominent themes which emerged. These themes were not identified in advance but became apparent from inspection of the students' responses. The two questions were handled separately and independently.

1. What do you enjoy about studying mathematics?

The assumption that mathematics has *some* appeal for the students in this class is a reasonable one, as each of them had either chosen to take mathematics at honours level in first year, or had chosen a degree programme with mathematics as a major subject. We hoped that the students' answers to this question would help us to determine what aspects of their mathematical experience makes further study of the subject attractive to school leavers. Unlike subjects such as philosophy or computer science which may be encountered for the first time at university, mathematics has a prominent place in the Irish second level curriculum. Students enter third level with patterns of mathematical thinking that have been developed through long experience and may not be immediately amenable to the sort of "cognitive reconstruction" described by Tall (1991). We remark also that students participating in this survey are likely to have enjoyed high levels of success in mathematics at second level, where (in Ireland at least) assessment in mathematics tends to be dominated, in practice if not in intent, by tasks involving the implementation of methods and procedures. This point was made by in the 2005 Chief Examiner's report on Leaving Certificate Mathematics (Higher):

> examiners have been commenting on a noticeable decline in the capacity of candidates to engage with problems that are not of a routine and well-rehearsed type . . . more prevalent is the inclination of quite good candidates to not even attempt parts of questions that do not resemble well rehearsed examples from class.

The themes arising in the responses to Question 1 are reported below.

1. "Getting the Answer": 24 students (coded as GA1)

Responses concerned with satisfaction arising from being able to complete an exercise and get the right answer were included in this category, which accounts for over 46% of participants. Many of the students in this category cited the belief that answers in mathematics are either right or wrong and not subject to opinion as a favourable aspect of the subject. Responses in this category are distinguished by apparently being primarily concerned with satisfaction at successfully implementing mathematical procedures rather than success at advancing mathematical understanding or other aspects of the subject.

The following are examples of responses categorized as (GA1):

> "Like practical side, I enjoy working out the problems as it is very rewarding. Dislike writing out pages like language subjects, rather work out the problems . . . satisfaction in working hard on a particular problem and it working out in the end".
>
> "I enjoy working with numbers, working out problems – the satisfaction of knowing I figured it out -- knowing there is either a right or wrong answer"
>
> "I like studying mathematics and enjoy working sums out"
>
> "I find it rewarding when I find a solution to a problem."
>
> "I like . . . that we need to find a method to arrive at that right answer"
>
> "I like solving problems and being able to get the answer. I like the way the answer is either right or wrong"
>
> "I get a feeling of satisfaction when I get the right answer."

We acknowledge that being able to master difficult procedural skills in mathematics is indeed satisfying, particularly when instantaneous positive feedback is available through checking of "the answer" which will typically consist of a number or a concise statement. We acknowledge also that procedural skills are very important in mathematics. However, we are concerned that students who experience pleasure and success in mathematics purely through positive reinforcement arising from consistently "getting the right answer" are at risk of experiencing severe difficulty in the transition to university mathematics. This concern is not based on doubts about ability. It is based rather on the danger that a student who has been successful at mathematics in the past and whose motivation and confidence have been affirmed by instantly verifiable "right answers" may find herself or himself on rockier ground when the task is one of reflection and exploration and the answer may be neither completely right nor completely wrong. This concern seems to be universal; it is highlighted in the Irish context by Meehan and Paolucci (2007) and in the USA and UK by Boaler and Greeno (2000) who comment that

> Students who choose mathematics as their main field of study, based on the idea that the subject is structured, certain, and non-negotiable, may encounter significant problems as the mathematics they learn at university becomes more advanced.

We conclude the discussion of the category (GA1) by quoting the response of another student in our own group. The limited (if not inaccurate) view of what mathematics is about that may be held by some mathematically successful school leavers is succinctly encapsulated by this student's comment.

> "I like maths because its not a subject that you have to learn off and memorize as such. It's about working out things and having particular procedures to follow. It's a practical subject."

2. "Advancing Understanding" : 12 students (coded as AU1)
Any response to Question 1 that was concerned, even tenuously, with feelings of pleasure and satisfaction arising from advancing understanding of mathematical concepts was included in this category. For example any response indicating an interest in *why* mathematical techniques work rather than simply *how* they are implemented was included here. Any response explicitly referring to styles of thinking was also included here. Responses in the (AU1) category were fewer than those in (GA1), but they were also less homogeneous in terms of their content. A selection from these 12 responses is given below.

> "Coming to grips with the real meaning and concepts behind maths is what I enjoy because I think once you understand something fully it will stay with you for life"

> "I enjoy the fact that I can make sense of it."

> "You have to develop a certain way of thinking instead of learning things off."

> "I enjoy studying mathematics because it is a subject that involves thinking for yourself as opposed to learning information off-by-heart."

> "At second level, Mathematics was the only subject that I found difficult - subjects that involved mere memory provided little challenge and were therefore dull and uninspiring. But Maths was different - it depended more on you + your mind."

> "That 'ping' of realization when you realize how/why you do something . . . The frustration you feel when you're trying to work out a problem."

It is perhaps interesting to note that the last two students quoted above were the only ones who explicitly identified difficulty or frustration as a positive feature of mathematical learning. We will comment further on this below in our discussion of the responses to the second question.

It is our opinion that the responses in the (AU1) category are indicative of a more comprehensive conception of the nature of mathematics and advanced mathematical learning than those discussed above under the (GA1) label. Since the students in the (AU1) category indicate that their enjoyment of studying mathematics arises from the pleasure of mental engagement and "making sense of it" rather than the more external assurance of correct answers, we feel that they may be better equipped to navigate the transition to advanced mathematical thinking than those in the (GA1) category. We also feel that students in this category, who appear already to have reflected somewhat on the cognitive processes involved in mathematical learning, may be less likely to be daunted by obstacles encountered along the way than those who are used to appreciating the quick and frequent reassurance of "right answers". We will comment further on this when we look at the responses to Question 2 of students in the (AU1) category on Question 1.

3. "Challenge" : 12 students (coded as CH1)

Students in this category stated that they enjoy the challenge of learning mathematics, but did not elaborate on the nature of this challenge or on how they find it enjoyable. We will not comment further on these responses as any interpretation that we could offer would be speculative.

4. "Other" : 6 students (coded as O1)

None of the miscellaneous responses in this category contained information contributing significantly to our goal of understanding what makes mathematics attractive to beginning university students.

We now proceed to a discussion of students' responses to the second question.

2. What do you dislike about studying mathematics?

After reviewing our students' responses to this question, we were able to divide them into six categories based on the major themes arising. Some responses included comments on more than one of these themes, so the sum of the numbers of students in the various categories exceeds 54. In addition to these six categories there were twelve responses classified as "Other"; some of these were quite interesting isolated comments. For each category on Question 2, we note the numbers of students whose responses on Question 1 were classified as (GA1), (AU1), (CH1) and (O1).

1. "Not Getting the Answer" : 15 students, 8(GA1), 4(AU1), 2(CH1), 1(O1) (coded as NA2)

Fifteen students cited frustration at being unable reach the answer to a problem in their responses to Question 2. These fifteen students included one third of the students from each of the categories (GA1) and (AU1) (8 students and 4 students respectively).

Some typical responses in the (NA2) category are listed below.

"Not getting solutions"

"The frustration when I can't get something right."

"You can't sit down like other subjects and learn things off, which can be frustrating sometimes. You have to logically think through things which can be annoying when you can't come to the right answer."

"When a problem doesn't work out it can be very frustrating, especially when it's hard to find out where you went wrong."

"Sometimes not having the logical train of thought to find the solution to the problem."

"I dislike the way I sometimes get frustrated with maths when I have trouble finding the solution to a problem."

"Not being able to do the sums."

One of the (NA2) responses to Question 2 is worthy of special mention in our opinion as it seems to acknowledge the fact that it is through the potentially frustrating experience of struggling with a problem or concept that genuine progress is made. The same student's response to Question 1 was in the (AU1) group and included a reference to frustration as an enjoyable aspect of learning mathematics:

"I also hate that frustrated feeling when you're trying to work on a problem, and it just ain't working (depending on my mood frustration can be good or bad)."

2. "Memorizing": 6 students, 4(GA1), 1(CH1), 1(AU1) (coded as M2)

Six students disliked the fact that for them the study of mathematics involved a significant amount of memorization. This is an interesting finding as seven students expressed an appreciation of the fact that mathematics *does not* involve much memorization in their responses to Question 1. There seems to be a lack of consensus in the class about the amount of memory work involved in the learning of mathematics. Most mathematicians would take the view that mathematics if fully understood does not need to be memorized, though some would acknowledge that memorization might be a useful step along the path to that complete understanding. It is noteworthy in our view that only one of the twelve students in category (AU1) (advancing understanding) cited memorization as a problem in responding to Question 2. Typical responses in this category included the following.

"I don't like learning off theorems, it can be boring and time-consuming"

"Trying to remember formulas, theorems and the names for certain things"

"Too many formulas to learn."

"There are many formulas and theorems to be learned"

Another striking feature of these comments is the view of theorems and formulae as objects "to be learned". This language may suggest that some students are used to relying on memorizing such material rather than constructing accessible understanding of it, which would be a more demanding but ultimately more valuable and reliable manner of attaining knowledge.

3. "Abstraction" : 5 students, 2(GA1), 1(O1), 2(AU1) (coded as A2)

Five students expressed dislike of mathematical abstraction in response to Question 2. This is not so surprising to us as we would consider that an advanced state of mathematical maturity is needed in order to truly recognize the value and power of abstraction in mathematical reasoning. However, students in this category may be at risk of reinforcing or developing a habit of trying to avoid engagement with abstract ideas in favour of a more procedural focus.

The following comments were included under this heading.

"I dislike the very abstract part of Mathematics."

"I hate the way the numbers and equations mean nothing, and that it's all philosophical and hypothetical."

"That most of the more difficult maths seemed to lack practical applications. It did not really relate to everyday life."

4. "Time Consuming" : 5 students, 3(GA1), 2(CH1) (coded as TC2)

Five students reported that mathematics is time-consuming in their responses to Question 2. Examples of responses in this category include :

"The amount of study that you have to put in to get a good result."

"You have to spend a lot of time studying. More than other subjects."

There is no suggestion that these students expect this situation to change at university.

5. "Particular Topics" : 6 students, 3(GA1), 2(CH1), 1(AU1) (coded as PT2)

Six students answered Question 2 by citing a particular mathematical topic that they did not enjoy. Since the six topics mentioned were all different there is probably not much information to be gleaned from these responses. However the following answer in this category did stand out because of its optimistic tone and suggestion of personal responsibility:

> "Geometry - it doesn't make sense to me yet."

This comment was from a student whose response to Question 1 was in the (AU1) category.

6. "Hard" : 6 students, 4(GA1), 1(AU1), 1(CH1) (coded as H2)

Six students responded to Question 2 by saying that they find the study of mathematics difficult. Examples of responses in this category include :

> "Needs 100% concentration."

> "I find it difficult at times."

Amongst these six students, only one (whose answer was in category (AU1) on Question 1) also cited the difficulty as a reason to like mathematics. The following is this student's response to Question 2 :

> "Being a challenge, Mathematics can also be very frustrating for me. It places a demand on the human intellect which can be exciting but at the same times immensely taxing. It can be very abstract and therefore difficult to understand and grasp."

7. "Other": 12 students, 3(GA1), 4(AU1), 2(CH1), 3(O1) (coded as O2)

Twelve students' responses to Question 2 do not fit into any of the six categories described above. Some of these 12 students did not respond to Question 2 and some responded saying that there is nothing that they dislike about studying mathematics. Two responses in this section are of particular interest to us. Both are from students whose answers to Question 1 placed them in the (AU1) category, and both are isolated in terms of the sentiments expressed. One student responded to Question 2 with the following comment :

> "I'd say the calculating part, the calculation of lists and the endless tipping (?) to get numbers. And if you finally have them, you realise that you made a mistake with one of them and you've to start all over again."

This remark is starkly at odds with the dominance of "satisfaction at getting the right answer" amongst the responses to Question 1. It is interesting from our point of view to know that a small number of students (at least one!) recognize that "the calculating part" is only a part of the study of mathematics, and not always a particularly interesting one at that. On the other hand it is perhaps a little disappointing that comments of this nature were not more numerous.

The other comment which we consider worthy of note in this section is the following.

> "When you're given a formula/sum/definition and we are to accept it as fact, I'd prefer to know the origin and the meaning behind these theories, so as to have a better understanding of what I'm doing with it."

This comment indicates an appetite for conceptual understanding on the part of the student, and an awareness that expositions that present facts or methods without justification for them are incomplete. In the overall response to this survey, sentiments of this nature were conspicuous by their absence. We do not think that this is because most of the students in the group do know the origin and meaning behind the mathematical theories that they use; this opinion is based on our experience and it is backed up by the responses in categories such as (GA1) and (M2) here.

CONCLUDING REMARKS

As lecturers on the 1st year Honours Mathematics course, our hope of course is to encourage our students to develop their abilities in exploring mathematical concepts, theories and techniques, both through procedural tasks and through investigation of ideas. Our hope is that our students will be enthusiastic and curious to broaden their conceptual understanding, and not satisfied to merely confirm that they are correctly implementing procedures. The responses to this survey strongly suggest that curiosity is not a dominant feature of the styles of mathematical thinking typically employed by our beginning undergraduate students. The survey also suggests that many of our students have developed study habits in mathematics that are almost entirely procedure driven, and that some have relied upon memorization to cope with those parts of the curriculum that are not concerned with procedures. There is little evidence to suggest that the desire to identify and remedy weaknesses in their own conceptual understanding is a driving force in most of our beginning students' mathematical activity. As they proceed through the transition to university mathematics, we would like our students to focus not only on the practice of procedures (in which progress is easily confirmed by correct answers) but also on the more tentative business of constructing and fortifying their own personal knowledge by being alert to unexpected connections and by avidly seeking satisfying explanations. At the time of the survey, these students were at (or near) the beginning of a mathematical journey which, to quote from Tall (1991) again, can be described as follows :

> "What is essential [for students] is an approach to mathematical knowledge that grows as they grow: a cognitive approach that takes account of the development of their knowledge structure and thinking processes. To become mature mathematicians at an advanced level, they must ultimately gain insight into the ways of advanced mathematicians but, en route, they may find a stony path that will require a fundamental transition in their thinking processes."

As lecturers we are charged with the task of trying to provide some guidance along this stony path to those students who choose to travel it. It behoves us then to try to explore the path ourselves from the position of having already navigated it. This means trying to identify the rougher stretches and trying to devise strategies that might help a novice to negotiate the likely hazards and overcome the many obstacles. As evidenced by the intensity of research activity and vast body of literature on this subject, as well as the informal but impassioned conversations taking place in coffee rooms in mathematics departments everywhere, this is no easy task. However a reasonable starting position might be to hear what our students have to say at the beginning of the journey, bearing in mind that it is their experience of mathematics to date that has led them to this point. It is hoped that in the Irish context at least, this short paper might make a small contribution in this direction.

REFERENCES

Boaler, J & Greeno, J. (2000). Identity, agency and knowing in mathematics worlds, in J. Boaler (ed.) *Multiple Perspectives on Mathematics Teaching and Learning* (pp.171-200). Westport, CT : Ablex Publishing.

Hoyles, C., Newman, K. & Noss, R. (2001). Changing patterns of transition from school to university mathematics. *International Journal of Mathematics Education in Science and Technology,* 32(6), 829-845.

Meehan, M. & Paolucci, C. (2007). Reflections of university students on the Leaving Certificate Mathematics experience, in Close, Corcoran & Dooley (eds) *Proceedings of the Second National Conference on Research in Mathematics Education* (pp.255-265). St. Patrick's College.

Meyer, J. & Land, R. (2003). Threshold concepts and troublesome knowledge: linkages to ways of thinking and practising within the disciplines. ETL project, University of Edinburgh.

State Examinations Commission (2005). Chief Examiners Report : Higher, Ordinary and Foundation Levels – Mathematics Leaving Certificate Examination 2005.

Tall, D (1991). The psychology of advanced mathematical thinking, In D. Tall (Ed.) *Advanced Mathematical Thinking* (pp.3-21). Dordrecht, Kluwer.

Thurston, W.P. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society,* 30 (2), 161-167

PROOFS, WRANGLERS AND VIRTUAL CONVERSATIONS

Tim Rowland

University of Cambridge, UK

This paper is overtly mathematical, being about the different solutions that several individuals devised for a particular problem in Euclidean geometry, and the communication of these solutions in email messages. Apart from the ingenuity and intrinsic interest of the solutions themselves, the focus of the paper is on creativity, generosity and camaraderie in mathematical endeavour. The paper also acts as advocate for Euclidean geometry in the school mathematics curriculum, and its potential for inventiveness and connectedness in the search for multiple solutions to problems.

Introduction

The geometry problem at the heart of this chapter came to my attention at a seminar given by Orit Zaslavsky in Cambridge early in 2006. The problem features in Zaslavsky (2005), a paper about the design and implementation of mathematical tasks that evoke uncertainty for the learner. Zaslavsky presents it in the form shown in Figure 1, although the original textbook problem is "Prove that α=60^{o}".

The point is that not revealing [the measure of] the angle α provoked a debate between students to resolve their competing claims. One student, Bob, claims that α=60 (angles will be in degrees throughout), and presents a proof that he found in a textbook, based on what looks like an ingenious construction. However, this fails to convince Ruth, another student, who believed at first that α=55. Zaslavsky goes on to write about ways in which this task was developed over time to provoke uncertainty and debate.

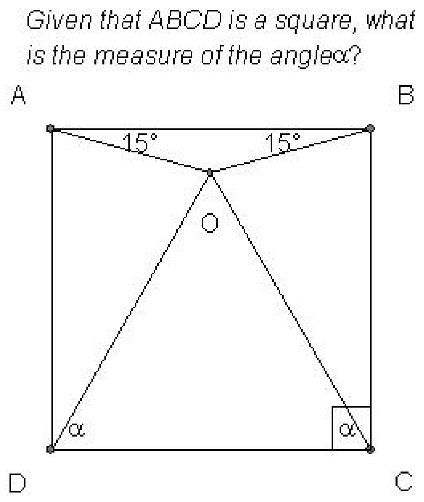


Figure 1

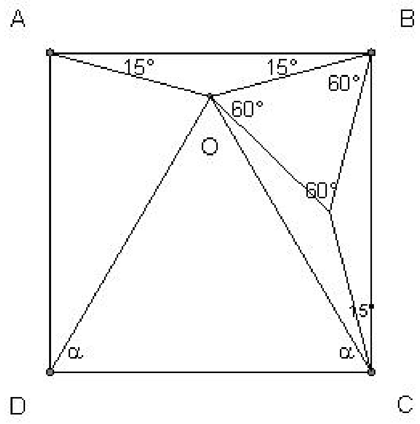


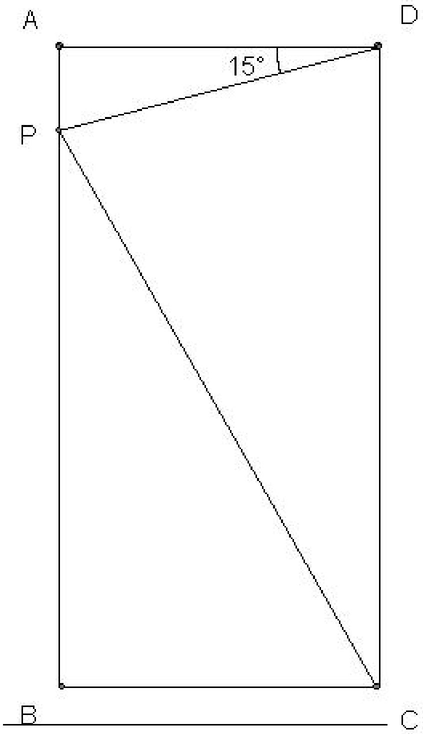
Figure 2

At the seminar, the speaker was unable to devote much time to the details of the students' responses to the task, although her paper (Zaslavsky, 2005) engages with them in depth. I became curious about how to prove that $\alpha=60$ during the seminar, and the feeling wouldn't go away. Orit did show us the student Bob's diagram (Figure 2) but I didn't find it very helpful. Moreover, I had the "I'd never have thought of that" reaction to it, which was quite alienating.

Tony P

Later the same week, I 'phoned a friend'. Well, e-mailed. Tony Pay read mathematics at Cambridge a long time ago, but has earned his living as a musician. I wrote[9]:

> OK, here's a little 'math' in case you find yourself without a crossword to do. ABCD is a rectangle with AB=2BC. P is a point on AB such that angle PDA = 15 degrees. The original problem says "prove that angle PCB = **degrees", but I'll say "what is angle PCB?". I then ask that you prove your assertion by Euclidean methods (i.e. methods known to Euclid, the stuff of Durell). By the way, I bought a copy of Durell from Abe[10] about 5 years ago



The reference here is to Clement V. Durell, whose textbooks had circumscribed, in a benign way, our East London Grammar School mathematical education between the ages of 11 and 15. Durell, the son of the Rector of Fulbourn, in Cambridgeshire, was a Cambridge wrangler[11] who spent most of his career teaching at Winchester school. He was a prolific and very successful writer. When he died in 1968, his estate was worth around five million pounds in today's values. His school geometry text (Durell, 1939), was actually rather good, and I think that Euclid himself would not have been disappointed with it. A certain Carl E. Linderholm rather unkindly dedicates his 1971 book *Mathematics Made Difficult* to "Clement V. Durell, M.A., without whom this book would not have been necessary".

[9] The email transcripts are nearly always verbatim; the few deviations from this rule are in the cause of brevity and clarity.

[10] An online "source for used, new, rare and out-of-print books".

[11] Those students who achieve a First Class pass in Part II of the Mathematical Tripos at the University of Cambridge are called 'wranglers'. The ranking of wranglers ceased in 1909.

I note here that I had reframed the original problem (Figure 3, see left). It was easier to describe this version in words, and my efforts to solve the original had convinced me that it would be sufficient to work with just one half of the original line-symmetrical figure. In a way, I now regret imposing that restriction on others. Tony replied:

> What was Durell called, exactly? I might try to do the same. What surprised me was that I didn't know this result. I suppose the 'right' way to do it is to prove directly somehow that if AB = 2, BC = 1, and PDA = 15 degrees, then PC = 2. What I did was: AP = tan 15 = t say, and then sin 30 = 1/2 = 2t/(1 + tsquared), solve quadratic for t giving AP = 2 - sqrt 3, from which PB = sqrt 3 and so triangle PBC is half an equilateral triangle, and the angle is 60 degrees. It's not very nice, but I can't think of a natural way to use the 15 degrees in a construction.

I replied with the name of Durell's book, adding:

> I have since found that connoisseurs of Eucl Geom drool over L. Roth - *Modern Elementary Geometry*, Nelson, 1948 … "such elegant riders ...").

The reference to Roth and to connoisseurs was prompted by an enjoyable experience just a few years earlier working on a government-funded secondary algebra and geometry project with Tony Barnard and a small group of talented teachers. Our contribution was to devise and trial materials for teaching Euclidean geometry. A guiding principle behind the 'scheme' – entirely novel to two younger members of the team – was that the 'starting points' were made explicit, and absolute rigour was demanded in the deductive arguments based on these axioms, although that makes it sound more scary than it was. The enthusiasm of this team was invigorating and highly infectious. Incidentally, Leonard Roth, another Cambridge wrangler, was born in Edmonton around 1905. He became a Reader at Imperial College and was only 64 when he died in a car accident.

Tony (Pay)'s reply:

> I get it. What clears it up is to look at it the other way around. Imagine a general rectangle ABCD with AB = 2, BC = y, say. Now *construct* CP = 2. It's not difficult to show angle PDA = 1/2 BPC. Then, what more natural for the examiner to choose y= 1, PDA = 15 degrees, and set the problem backwards. And in fact, all that can be done (backwards) just using Durell/Euclid :-)
>
> Thanks for telling me this -- it was fun to think about! And I've ordered my Durell...

As I read this, I thought that the observation that if CP=CD, then, *irrespective* of the dimensions of the rectangle, ∠BPC = 2∠PDA – easily proved by angle-chasing – was a significant and pleasing insight. In the case that PDA=15, ∠BCP would have to be 60, and BC=2 in turn. There were further exchanges, in which Tony was working hard to find a 'natural' context for the result - one of them very much in the spirit of dynamic geometry, with P sliding up and down AB, with corresponding consequences for the rest of the diagram.

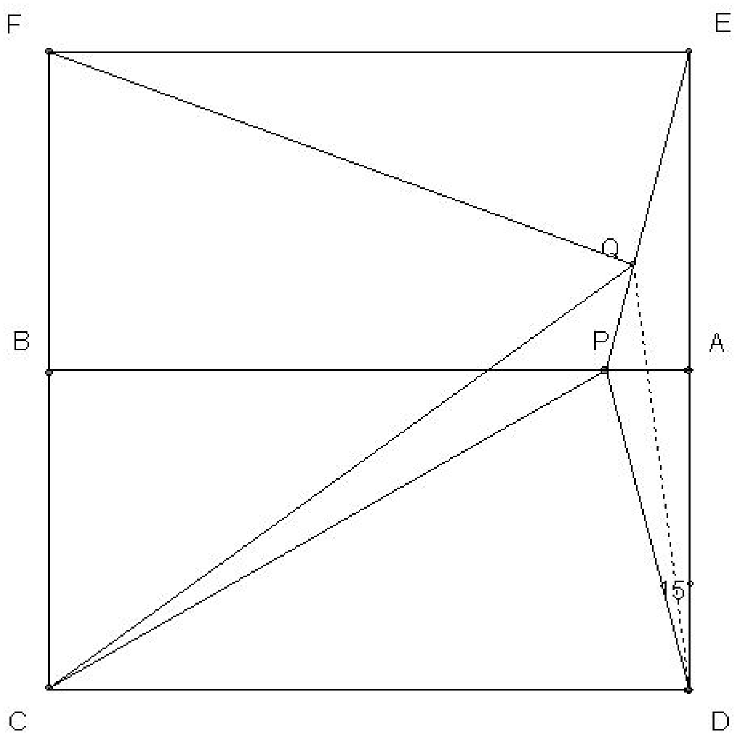
Tony B

By now, I was curious to know what the other Tony – Barnard – would make of the geometry problem. Tony is a mathematician, with an active interest in mathematics education – a kind of complement, or mirror image, of myself, in fact. I sent this Tony the problem, and the gist of my discussions with the other Tony. He replied:

> Dear Tim, This nice problem is similar to one we set in our annual problems drive at King's [London] many years ago. I have to admit that I'd forgotten about it until reminded by my friend John who has some nice solutions. The following is adapted from one of them and I've tried to make it as elementary as possible (in the sense that it uses the minimum amount of theory). In the description below, numbers will be degrees, but the word 'degrees' will be omitted for brevity.
>
> Consider the rectangle ABCD as the lower half of a square DEFC. The triangles PDA and PEA are congruent (SAS), so angle AEP = 15 and angle FEP = 75.

Given that I had posed my email version of the problem by reference to half of the original square, it was interesting to see that Tony had now re-instated the square

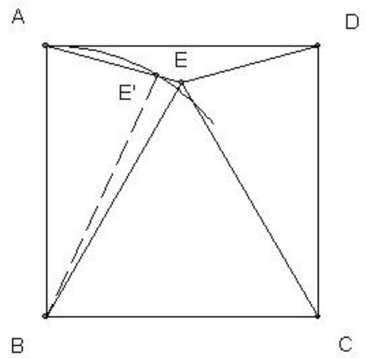
(See Figure 4 below)



> Now let Q be the point on EP (or EP produced) for which FQ = FE. Then angle FEQ = 75 (base angles of an isosceles triangle) and so angle EFQ = 30. Therefore angle QFC = 60. It now follows that triangle CFQ is equilateral (as FQ = side-length of the square DEFC). Therefore angle FCQ = 60 (hence angle DCQ = 30) and also CQ = CD (side-length of the square again). So angles CDQ and CQD are (also) the base angles of an isosceles triangle whose other angle is 30. Thus angle CDQ = 75. Hence angle PDQ = 0. This means that P and Q must be the same point. Therefore angle PCB = angle QCB, which is equal

to 60.

I recognised the care with which Tony justified each and every step of the deductive argument, even down to "Hence angle PDQ = 0", when it would be natural to say "But CDP also equals 75, so P and Q must be the same point". I was also intrigued by a strategy that proposes a point (Q here) that has certain desirable properties, and constructs an argument that inexorably identifies this point with one 'already' in the diagram.

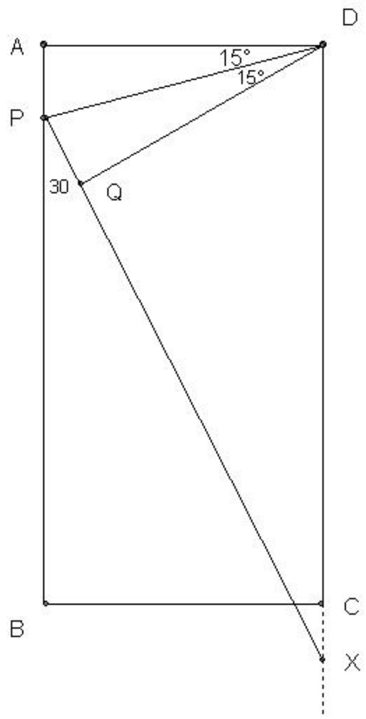


Tony explained that the original 'King's problem' had been formulated in terms of a square – essentially that in my Figure 1, with the point O re-labelled as E. He went on to give three more solutions, credited to his colleague, John. The second was this (See Figure 5, left)

Let the circle centre B through A meet AE in E': then by the alternate segment theorem, AE' subtends an angle of 15 degrees, so that ABE' is 30 and E'BC is 60, that is, E'BC is an equilateral triangle. By symmetry, E'=E.

That one needs some unpacking, and eventually I arrive at the following. AD is a tangent to the circle through A and E', centre B. By the alternate segment theorem (and the 'angle at the centre' theorem), $\angle$ABE'=2$\angle$ E'AD. But A, E and E' are collinear, so $\angle$ E'AD = $\angle$EAD=15, $\angle$ABE'=30 and $\angle$CBE'=60. Since BC(=BA)=BE', triangle CBE' is equilateral. The "by symmetry" seems to be saying that if E" is a point on ED defined in the same way as E', but with circle centre C, then both BE'C and BE"C would be equilateral triangles, "and this could only happen if E' and E" were the same point, at E". Specifically – $\angle$E'CE"=60-60=0. Given that E' and E" are defined to be points on AE and DE respectively, $\angle$E'CE"=0 when both points coincide at the intersection of the two lines i.e. at E. Again, I like the strategy of defining a point one way and then constructing an argument to identify it with another point. Well, not another, of course, in the end. John's[12] final proof was this:

[12] Strictly speaking, this is Tony B's strictly-Euclidean rendition of John's proof by transformations.



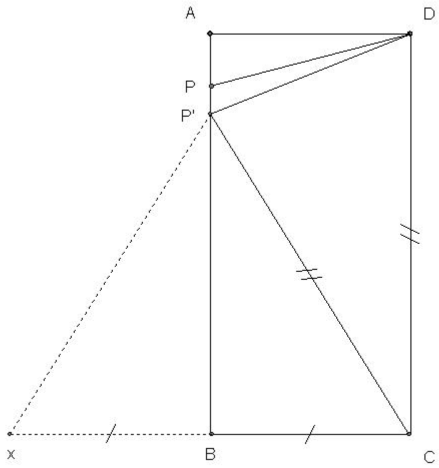
Let X be the point on DC (or DC produced) for which angle BPX = 30. Then angle DPX = 75 (angles on a straight line, because angle DPA = 75). Now let Q be the point on PX such that angle PDQ = 15. Then angle DQP = 90. Therefore triangles ADP and QDP are congruent (AAS). But DX = 2DQ. (This follows either from the fact that sin30 = ½ or by easily showing that, if R is the point on DQ produced for which QR = DQ, then triangle DRX is equilateral.) Therefore DX = 2AD = DC. Hence X = C and so angle CPB = angle XPB = 30. Therefore angle BCP = 60.

A little later, Tony helpfully remarked:

As you must have noticed, the third of John's solutions was directly for the one you sent, rather than for the 'King's' problem. (Figure 6, left).

Once again, I noted the construction of the point X, and its subsequent identification with C. It's time to give that strategy a name: perhaps the *'identical points'* strategy.

What would Tony the musician think of this proof? His reply:



How did we miss *that*?? Though I thought the proof would have been better with a variant P (P' say) to make the *length* right, as I suggested before [the dynamic geometry-like proof]:

Let X be the point on CB produced with CB = BX, and P' the point on AB such that CP' = CD = CX. Then CP'X is equilateral, DCP' = 30, P'DA = 15. PDA = 15. Hence P = P'.[13]

(Figure 7, left)

... but that again seems to require a knowledge of the answer, which the construction of X in 'solution 1' [the proof that I'd sent] doesn't really. (After all, you've got to get the 15 degrees involved somehow, and sticking a 60 degrees in to leave another 15 in the right angle is something you might think of.)

Off to OZ, NZ, Kuala Lumpur and Hong Kong ...

… and a little later – :

[13] No doubt the other Tony would insert a statement saying that angle PDP'=0 …

But before I go, I just wanted to say that the 20-20 hindsight, motivated proof number 0 is:

Rectangle ABCD, AB = 2BC, P on AB s.t. angle PDA = 15 degrees, what is angle PCB?

How do we use AB = 2BC? ... Construct perpendicular bisector of DC through midpoint X, say.

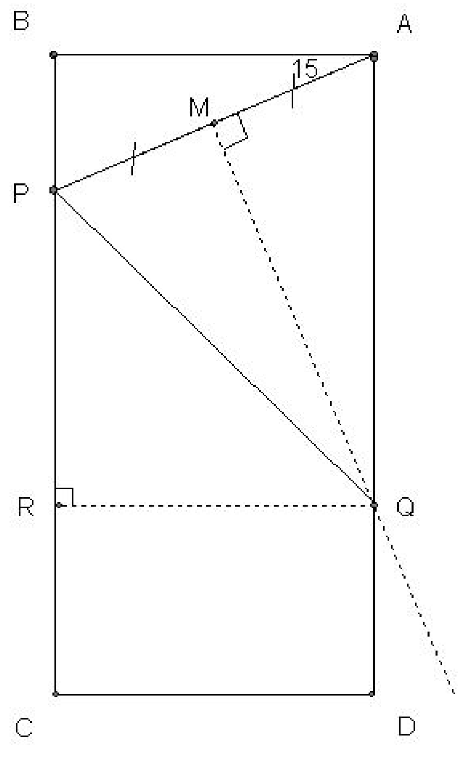
How do we use angle PDA = 15 degrees? ... Construct line through D at 15 degrees to DC meeting that perpendicular bisector at Y, say. Now join CY, PY.

This creates three congruent triangles DPA, DYX, CYX, so triangle DPY is isosceles with vertex 60 degrees, so equilateral. So DY = PY = CY, triangles YDC, YPC are congruent, PCD = 30 degrees, and PCB = 60 degrees.

That's what I was trying to say all along!:-) Tony

Sketching the diagram, as I hope you will, shows how Tony's efforts – with hindsight admittedly – reconstruct 'Bob's ingenious proof' (Figure 2). The proof remains ingenious, but my earlier sense of alienation has now gone.

Rex



A week later I emailed the problem to my (former) colleague, Rex Watson. Rex frequently sends me problems (inventions of his fertile and evidently restless mathematical intellect) and I regret having less and less time to dwell on them. I had slightly mis-remembered my earlier notation: now the rectangle ABCD has BC=2, AB=1, P on BC such that BAP=15. Rex's solution came quite quickly:

Let M be the midpt of AP and let the perp bisector of AP meet AD (produced if necc) at Q. Let R be on BC s.t. QR is perp to AQ.

Now AQM=15 (easy) and QMP, QMA are congruent (SAS), so AQP=30 and so PQR=60.

But QR=1 so PQ=2 (by the usual equilateral triangle argument). Therefore QA=2 i.e. Q is D, so PDC=60.

(See Figure 8, left)

Another 'identical points' proof, but a new one. I forwarded it to Tony B, with the comment:

> I think this is different from the others ... and there's something very direct about it.

Tony’s reply:

> Thanks for sending this. It's a lovely solution, but why bother with the congruent triangles? Why not just take Q on AD such that APQ = 75? The 30-60-90 triangle PQR forces PQ to be 2, and so Q to be D. This solution is equally direct in its alternative form (where you have a 15-15-150 triangle inside a square and have to show that the 'other' one is equilateral).

A characteristic of both Tonys is to appreciate a new solution, but to see how it might be refined and improved. Each solution is a kind of launch pad for yet another.

Vesna

Now fast forward nearly three years. The year is 2009. We have a good group of mathematics education masters students this year, among them Vesna Kadelburg – another Cambridge wrangler. Vesna completed her mathematics PhD not long ago, and her day-job is teaching in a local school. She is also the deputy leader of the UK team for the International Mathematics Olympiad. At a recent supervision, it occurs to me that she might enjoy The Problem, so I jot it at the end of my comments on one of her essays, and she hurries away to a Research Methods session. Now I should explain that one of the readings for our masters course is my account (Rowland, 2003) of solving a problem – one that I learned from Tony Pay, in fact – while walking along the river from Ely to Cambridge. In that paper, I argue that it mattered where I was and what I was doing: that the network of ideas and memories which we assemble in the course of solving a particular problem has temporal, spatial and emotional components. Vesna’s narrative account is a deliberate and playful imitation of the genre!

> Dear Tim, Here is my story about solving the geometry problem. I don't have your original diagram any more, so sorry if the labelling is not the same. [It’s similar but not quite the same]. ABCD is a rectangle with BC=2AB. P is a point on AD such that <ABP=15. Prove that <PCD=60.
>
> I started by thinking: if the claim is true, what else needs to be true? If <PCD=60, then PCD is a 30-60-90 triangle, which I spotted straight away, because it is my favourite triangle as it comes useful in many different contexts (from evaluating trig ratios of special angles to solving some pretty obscure geometry problems). I know that this is half of an equilateral triangle, hence PC=2CD, so PC=BC. Thus if I can show that PC=BC, then I'm done.
>
> I can see two ways of doing this: Pythagoras or trigonometry. First, label some sides: AB=a, BC=2a, AP=x so that PD=2a-x. I can play with Pythagoras on triangles ABP and PDC. As I know what I'm trying to get, I should be able to fudge the algebra. I try this for about two minutes, and then give up …

Would anyone like to try this idea for three minutes, just in case?

> … and decide to launch into some trigonometry. I started this towards the end of the break in a [research methods] lecture, then had a short interruption to complete a set task, and now have a few minutes spare before the next bit of the lecture starts.
>
> I decide to go with the trigonometry. I can find by using double-angle formula that cos15=(1+root(3))/2root(2) and cos75=(root(3)-1)/2root(2). I therefore write BP=acos15 [presumably a/cos15 intended] and then use cosine rule in triangle BCP to find that indeed PC^2=4a^2, so PC=2a, as required. The algebra is not too bad, especially as I know what I'm aiming for. This takes about 5 minutes.

Well, it looks good to me, though it took me more than 5 minutes to check the details. And yes, it is trigonometry. But there is more:

> I then get back to the lecture, but keep thinking that I'd now like to find a nice solution to the problem. As I want CP=CB, I start scribbling some circles, hoping to use circle theorems about angles. I try a circle with centre C passing through B (and so hopefully P), but don't see anything immediately. The lecture is over and I get on my bike. It's good that I have solved the problem, but I want a nicer solution.

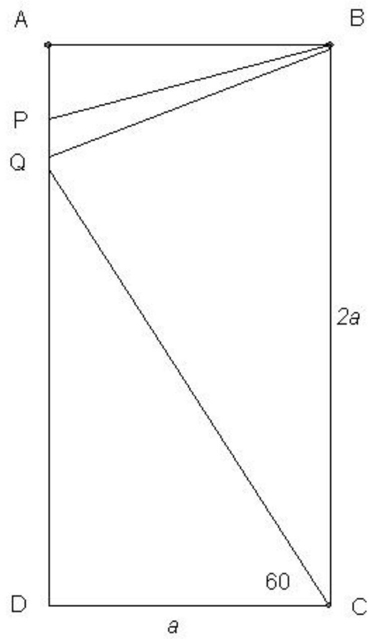
That elusive 'nice' quality … elegance. The aesthetics and the affective response to it are very much to the fore in this contribution, and that of the two Tonys. Read on.

> You wouldn't set me a problem which is plane trigonometry (although it is a nice example of how powerful trigonometry actually is, even if angles don't at first appear to be "nice"!).

I said something similar about Tony Pay's problem in Rowland (2003). Knowing the problem poser sets up certain expectations of the problem itself.

> I am cycling and still thinking about circles, when it hits me that I solved the problem a long time ago, I was just not using my logic in the right direction. In order to avoid accidents, I try thinking about something else until I get home, and then immediately get a piece of paper to get my thoughts down in writing (I like to see the flow of logic in writing, that's the way I check that it does actually work). I immediately fall in love with this problem - the geometry is trivial, but the logical construction is beautiful!

Any commentary from me must surely be superfluous.



My solution: Let Q be a point on AD such that <DCQ=60. There is clearly only one such point, as it is obtained by drawing a line through C making a 60-degree angle with CD, and intersecting it with AD. Then CDQ is a 30-60-90 triangle, hence CQ=2a. But then CQB is isosceles, and <QCB=30, so <QBC=(180-30)/2=75. Thus <ABQ=70-75=15. Hence point Q is the same as point P, and the assertion is proved.

(See Figure 9, left)

What Vesna proposes here meets an objection that I had voiced to Tony Pay's 'final' solution in our original correspondence, which I paste here for ease of reference:

> [Tim to Tony P.] This is very neat, but I'm still stuck! I see that if CP=CD and CD=2AD then it follows that ADP=15 - the crucial insight being your angle PDA = 1/2 BPC. But (and I hope this isn't being pedantic) I can't deduce CP=CD from the premises: ADP=15 and CD=2AD.

In many respects the two solutions – Tony P's and Vesna's – are very similar. Vesna defines Q by constructing an angle of 60 at C, whereas Tony, in effect, defines P as the intersection of AD and the circle through B with centre C (although the points were labelled differently then). In both cases, CDQ is a half an equilateral triangle, and CQB isosceles with base angles of 75. Vesna's logical trick is the 'identical points' strategy, a possibility that I had not recognised in my message to Tony P. (the one pasted immediately above).

Conclusion: creativity and elegance

These multiple solutions of The Problem illustrate well the scope for creativity offered by Euclidean geometry within the confines of relatively elementary mathematics, yet much of this has been lost from the curriculum in England. The same is true, incidentally, of conics in coordinate geometry, where, for example, problems about intersections of tangents and normals to ellipses with their axes offer possibilities of working with equations or parametric forms, for example. Creativity is an elusive and over-hyped notion in school mathematics (Huckstep and Rowland, 2001), but it can legitimately be associated with flexible and original approaches to problem solving. If these abilities can be learned, or at least acquired, this kind of geometry would seem to be a good training ground in which to develop them.

Various aesthetic judgements and expressions of affect are evident in the correspondence about The Problem, from Tony Pay's search for a 'nice' solution to Vesna's transparent expression of her feelings about the problem itself. Tony Barnard had also made comments (omitted here of necessity) expressing preference for 'light' proofs, making little or no reference to 'heavy' theorems. In contrast, 'heavy' proofs plunder the interconnected structure of the theorems of Euclidean geometry. This structure is displayed in what we had called The Map in our curriculum development project – a network of Euclidean theorems, displayed and partially ordered by logical dependence: a deductive family tree, as it were, originating in the axioms. Part of The Map is shown as a figure in Barnard (2002). This preference for 'light' proofs indicates, it seems to me, appreciation of the mathematician who brings a minimal toolkit to his or her work – who likes to travel light, in fact. Parallels come to mind in music and in fine art. This appreciation of the light touch might relate to the elusive notion of 'elegance' – the pleasing and seemingly-economical production of the argument; the rapier rather than the sabre.

In this paper, I have shared the enjoyment of doing the mathematics along with my email correspondents. My story illustrates once again the place of mathematics for mathematics' sake in the private lives of individual human beings, expressed in their willingness – eagerness even – to share their pleasing insights (or are they revelations?) with others. It shows, once again, that there are reasons for teaching and learning mathematics that lie beyond utility (Ernest, 2000), and perhaps beyond rational argument. I found the following quotation from Poincaré in the Preamble to David Wells' recent book (Wells, 2008, p. 9):

> Those skilled in mathematics find in it pleasures akin to those which painting and music give. They admire the delicate harmony of numbers and of forms; they marvel when a new discovery opens an unexpected perspective; and is this pleasure not esthetic, even though the senses have no part in it?

Acknowledgements

I acknowledge with thanks the inspiration of Orit Zaslavsky, and the contributions of Tony Pay, Tony Barnard, Rex Watson and Vesna Kadelburg: without which, as should be very apparent by now, this paper could not have been written.

References

Barnard, A. (2002). Starting points and end points. *Mathematics in School* 31(3), 23-26.

Durell, C. V. (1939). *A new geometry for schools.* London: G. Bell and Sons Ltd.

Ernest, P. (2000). Why teach mathematics? In S. Bramall and J. White (Eds.) *Why learn maths*? (pp. 1-14). London: Institute of Education.

Huckstep, P. & Rowland, T. (2001). Being creative with the truth? Self-expression and originality in pupils' mathematics. *Research in Mathematics Education* 2, 183-196

Rowland, T. (2003). Mathematics as human activity: a different handshakes problem. *The Mathematics Educator* 7(2), 55-70.

Wells, D. (2008). *What's the point*? London and Bristol: Rain Press.

Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics* 60, 297–321.