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Theme: *'Walking the Talk'* – Using Mathematics Education Research

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Acknowledgement

We thank the keynote speakers, paper presenters, and all participants in the second MEI conference on research in mathematics education in Ireland. In addressing the conference theme ‘Walking the Talk – Using Mathematics Education Research’ we sought to bring together, in St. Patrick’s College, those who have an interest in mathematics education, to consider future directions for mathematics education research in Ireland in the light of recent developments and international trends, and to consider ways of improving linkages with mathematics education communities within Ireland and in other countries. We hope we succeeded in this and that the conference provided participants with an interesting and challenging programme of presentations, and with opportunities for discussion and making contacts and renewing friendships. We are happy that Irish mathematics education is flourishing and we welcome links to the wider scientific research community.

We also express our sincere gratitude to all those who supported the conference including: Dr. Pauric Travers, President of St. Patrick’s College; our sponsors; St. Patrick’s College Research Committee, the Department of Education and Science, Science Foundation Ireland, and the Centre for the Advancement of Science, Technology and Learning in DCU.

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The Dutch Experience, Threat or Treat?

Truus Dekker, Freudenthal Institute, Utrecht, The Netherlands

It is impossible to use a foreign language after learning many words and grammatical rules. In the same way, engaging in mathematics should not be reduced to learning facts, definitions and procedures. Since its founding in 1971, the Freudenthal Institute has developed a theoretical approach towards the teaching and learning of mathematics known as 'Realistic Mathematics Education'. This does not mean students only solve real life problems but the problems given to them should be meaningful, challenging, and enable different levels of mathematical formalisation. Does this mean Dutch students "only" learn applied mathematics? Does it form a threat to studying mathematics at an abstract level? We will look beneath the tip of the iceberg.

> In realistic instruction the learner is given tasks that proceed from reality, that is, from within the learner's ever-expanding living world, which in the first instance require horizontal mathematising. One example: If, in the course of progressive algorithmatising, column multiplication is first carried out as successive addition, in order to gradually be shortened (by using the tables of multiplication and the positional system), individual learners – in the social context perhaps of the whole group – will eventually acquire the standard algorithm. (H. Freudenthal, Revisiting Mathematics Education, 1991)

Start with a real life situation

In mathematics education, contexts play many different roles. Meyer (2001) distinguishes the roles (a) to motivate, (b) for application, (c) as a source of mathematics, (d) as a source of solution strategies and (e) as an anchor for student understanding. However, in many mathematics textbooks, contexts are almost exclusively used for application and problem solving. As if pupils are studying a foreign language by first learning a large number of words and grammatical rules and in the end start speaking and writing fluent sentences and make themselves understood. We all know it does not work that way.

According to the PISA[1] 2003 Assessment Framework (2003):

> Literacy in the linguistic sense presupposes, but cannot be reduced to, a rich vocabulary and a substantial knowledge of grammatical rules, phonetics, orthography, etc. To communicate, humans combine these elements in creative ways in response to each real-world situation encountered. In the same way, mathematical literacy cannot be reduced to, but certainly presupposes, knowledge of mathematical terminology, facts and procedures, as well as skills in performing certain operations and carrying out certain procedures. Mathematical literacy involves the creative combination of these elements in respond to the demands imposed by the external situation. (p. 25)

De Lange (1999) discerns different orders for contexts. The zero-order context is a made up story as to make a bare problem look like a real-world problem and which is found so often in mathematics text books as well as in mathematics tests. Pupils should be taken seriously and this type of context should be avoided. The first order context is relevant, there is a real problem to solve and the answer should be judged within the situation. The distinction between the first- and second order context lies in the role of the mathematisation process, in order to solve the problem, in the second order context it is necessary for the pupil to first mathematise the situation. Third order contexts appear when the context serves the construction or reinvention of new mathematical

[1] PISA Programme for International Student Assessment (OECD)

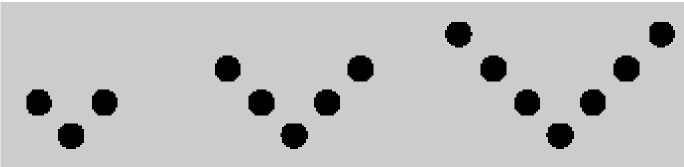
concepts. A simple example is the use of a bus ride as a model for addition and subtraction (van den Brink, 1989)

However, all of these views on contexts and their role in mathematics education do not show why teachers as a general rule should start within a real life situation. Therefore we need to go back to Freudenthal's (1991) citation at the start of this paper which reveals an important feature of the so-called RME, Realistic Mathematics Education. From this perspective, pupils are seen as reinventors of the mathematics involved, which initially requires horizontal mathematisation of this and similar situations. Pupils are encouraged to reflect on the process. The pupils' informal mathematical activities form a basis from which they construct more formal concepts and solution strategies. Vertical mathematisation takes place when pupils, guided by their teachers, realize that similar situations can be described by the same mathematical concepts and that in return mathematical rules and algorithms can be used to solve new and unfamiliar problems. The process –mathematising- is as important as the product – mathematics. That is why de Lange (1987) already emphasised the importance of starting a new topic within a real-life context problem as to develop 'conceptual mathematisation'. In doing so, pupils use all kinds of combinations of horizontal and vertical mathematising. The instructional guidance of teachers in this process is critical, as teachers gradually introduce and negotiate with the pupils the meaning and use of mathematical terms, signs, and symbols.
The learning theory based upon building knowledge as an individual construction as opposed to using the endpoint of the work of mathematicians as a starting point for instruction is known as constructivism. Note however, that constructivism as a learning theory differs from constructivism as a pedagogy where the use of textbooks and 'teaching' is discouraged. RME highly favors the role of the teacher (and the textbook) in 'guided reinvention'.

From informal to formal, conceptual understanding
According to a Dutch dictionary, *formal* means "keeping to formal rules". In the example below we see a set of *questions* that may lead to answers of increasing understanding of the concept from informal to formal. A pupil's answers show whether he or she still argues at an informal level or is able to use more formal, mathematical language and symbols. The example is adapted from the middle school curriculum *Mathematics in Context*, the unit Patterns and Symbols. The problem was used for pupils of about 11 years old. So far they have not yet developed formulas to represent a situation.
The situation to start with is birds flying in a V-pattern as shown below.

The drawing on the right shows how the situation is modelled into a “perfect” V. Making this assumption explicit is important for pupils as to start thinking about a mathematical model of the situation. This aspect of mathematical modelling is often neglected in textbooks.

Below are the three smallest V-patterns. To make drawing the pattern easier, dots are used instead of birds, the model becomes less concrete. Many pupils will refer to the dots as “birds” for a long time.



Here are some example of questions that may be discussed in class. To shorten the space needed for the example, many other questions were omitted.

Draw the fourth dot pattern.
Is it possible for a dot pattern as shown above to have 84 dots? Why or why not?
How many dots are in pattern number 6? In number 10?
The dot patterns can be extended as far as you wish. How many dots are in V-pattern number 100? How do you know?
Two groups of birds, each flying in a perfect V-pattern join together. Is it possible a new perfect V-pattern is formed after they joined?
I have chosen two whole numbers. After adding them, the result is odd. What do you know about the two numbers I chose?
The last question requires more formal reasoning about the concept of even and odd numbers, which was one of the goals of this section.

Examples of student answers
Question 2, Can a dot pattern consist of 84 dots?
No, because there is a bird in front (correct, informal reasoning, referring to the birds instead of the dots)
No, if you split 84 you have 42 on either side but there should be one extra in front. (correct, pre-formal. Thinking of number properties but also with the bird pattern in mind.)
No, each V-pattern is an odd number and 84 is even. (correct, formal reasoning, referring to number properties)

Question 4, How many dots are in V-pattern # 100?
One side 51 and one side 49 (incorrect answer based upon informal reasoning)
201, because you need the same number of birds on each side and a leader (correct, informal reasoning)
201, you double the number of the pattern and add 1 (correct, preformal reasoning)
2 × V-number + 1 (correct, formal)

Question 5 Can two V-patterns added together form a new V-pattern?
It is impossible to know, you do not know how these birds will fly. (incorrect, informal)
First I need to know how many birds in total. (incorrect, informal)

No, it is impossible, there is no leader. (correct, informal)
No, because odd plus odd is even (correct, formal)

Question 6 Adding two whole numbers, the result is odd.
Now the students need to generalize and reason in an abstract way; here abstract and formal get together.
I cannot know those numbers, they are in your head!
I know you chose 3 and 8 because 3 + 8 = 11
One must be odd, the other one is even or the other way around.

From informal to formal, solution strategies
There is another way to look at the dimension from informal to formal understanding of a concept. Consider the problem

$6 \div \frac{3}{5} = ...$

One could argue this is a formal problem, because a formal mathematical notation is used. However, the problem can be solved in different ways. Below some sample strategies are ordered from formal through pre-formal to informal showing pupils' understanding of the concept and procedure. The order of course may be discussed and more strategies are possible. The most formal solution is 'following the rules' or 'follow an algorithm' (which may not even be understood, the solution does not show understanding). Formal rules can be applied to any problem but they may not always be the most efficient strategy in a particular case. The most informal strategy may only apply to this particular situation and not in general.

$6 \div \frac{3}{5} = \frac{30}{5} \div \frac{3}{5} = 10$ Use a less known but still formal rule that may be better understood

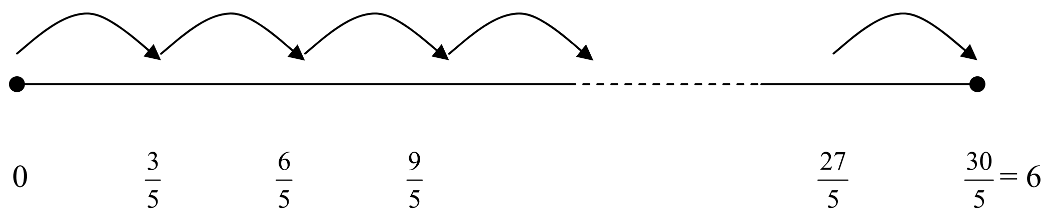
$6 \div \frac{3}{5} = ?$ The fraction $\frac{3}{5}$ can be written as a decimal, $\frac{3}{5} = 0.6$. The problem now becomes $6 \div 0.6 = 60 \div 6 = 10$ (Less formal)

$6 \div \frac{3}{5} = ?$ Use a ratio table as shown below. (Pre-formal)

6	30	10
$\frac{3}{5}$	3	1

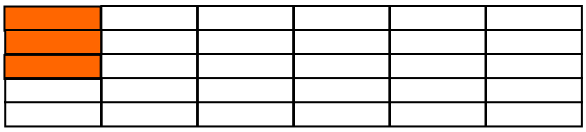
$6 \div \frac{3}{5} = (6 \text{ x } 5) \div (\frac{3}{5} \text{ x } 5) = 30 \div 3 = 10$

$6 \div \frac{3}{5} = ?$ Use jumps on a number line. Ten jumps are needed. (Pre-formal)

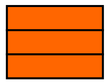


$6 \div \frac{3}{5} = ?$ Think of a context to represent the problem. Suppose I have to run 6 miles in total and each lap is $\frac{3}{5}$ miles or 0.6 miles. I will run 10 laps so the answer is 10. (more informal)

$6 \div \frac{3}{5} = ?$ Make a drawing as shown below. How many times does $\frac{3}{5}$ fit into 6? There are ten groups of ‘three out of five’.



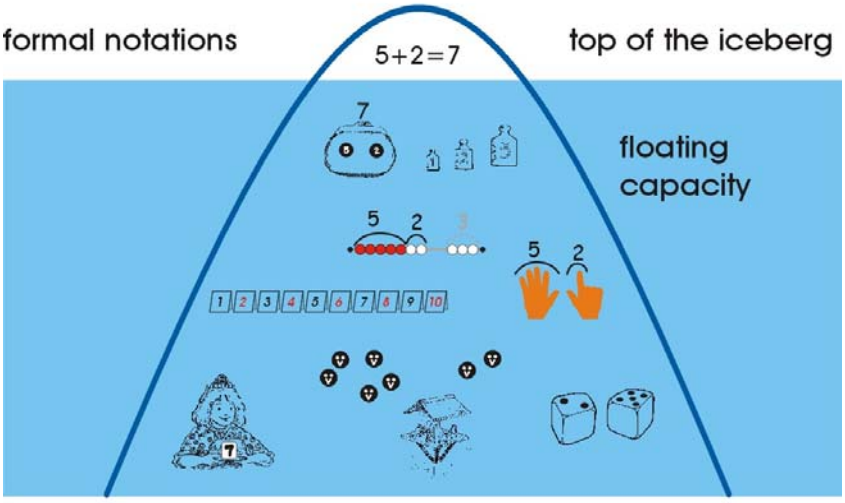
6



$\frac{3}{5}$

Note that informal strategies often involve contexts and drawings. This may be one reason why Dutch tests, for example the central exams, are very visual. Pre-formal solutions often make use of models. In both cases, the strategies used may not apply to any problem of the same type whereas the formal solution usually involves an algorithm and formal rules that will apply to any problem of the same type.
Some people think that ‘abstract’ and ‘formal’ are always the same. Here is an example of a formal question that can only be solved by using abstract reasoning, in contrast to the previous problem.
Why do fractions like $\frac{1}{2}$ and $\frac{3}{4}$ have terminating decimal forms?
In order to answer the question, the pupil needs to refer to formal definitions, such as the definition of a decimal (a fraction with denominators that are multiples of ten) and show that fractions like $\frac{1}{2}$ and $\frac{3}{4}$ can be rewritten as fractions with a denominator of 10 or 100 whereas other fractions like $\frac{1}{3}$ and $\frac{3}{7}$ cannot be rewritten with such denominators because 10 or 100 are not divisible by 3 or 7. (Or, there are no multiples of ten divisible by 3 or 7).

The tip of the iceberg
Does RME work for weaker pupils as well? Research with special education children in the Netherlands suggests that the process of learning mathematics takes more time (Kraemer et al, 2000). In many cases, special education pupils are promoted to higher grades with limited preparation for success at that grade level (Kraemer et al, 2000). All teachers know that many pupils will not reach a formal level of mathematisation. In many schools this results in providing a lot of practice for these pupils albeit at the same formal level they did not really understand to start with. For use with teacher professional development, researchers at the Freudenthal Institute developed the iceberg model (Boswinkel & Moerlands, 2001).This is a visual model to distinguish the role of informal, pre-formal, and formal representations used by pupils. The iceberg consists of the “tip of the iceberg” and a much larger area underneath, the “floating capacity.”



Before pupils reach the formal level of working with an algorithm, they need to build upon less formal representations as to develop conceptual understanding. For a pupil it is not obvious at all that two coins in a purse represent the same amount as five and two fingers or being seven years old. Once pupils have reached a formal level of understanding, they may go back to less formal ones when encountering a new situation, just as some pupils went back to the situation of birds flying in a V-pattern when talking about the 'leader' in a dot pattern.

Summer 2007, teachers in Boulder, Colorado (US) constructed their own iceberg models based on activities in the textbooks they used. Some of them soon found that their textbook provided hardly any 'floating capacity' which made some of them remark that they now understood why many of the weaker pupils had a lot of trouble to 'get it' and why just providing many practice problems of the same formal type and reteaching the algorithms again and again did not help. Practice may make perfect but practice without understanding is not really useful. Teachers and curriculum designers should look for playful ways to practice. Moreover, research showed that practice problems that allow for own productions are more effective than others. As Freudenthal stated: If children learn mathematics in an isolated fashion, separated from their experiences, it will be quickly forgotten and the children will not be able to apply it.

Summary

RME views mathematics as a human activity and mathematics teaching should not start at the endpoint of the work of mathematicians but rather as organising subject matter from the real world. Pupils should be active participants in the educational process, they should not be given "ready made" mathematics. The role of the teacher is crucial in guiding the pupils' reinvention of important mathematical concepts. Formal notations and formal algorithms are only the "tip of the iceberg". Informal and pre-formal understanding is needed to build formal understanding.

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Developing Knowledge for Mathematics Teaching: A Theoretical Loop

Tim Rowland, University of Cambridge, UK

What is the role of teachers' knowledge of mathematics 'content' in teaching, and how can this knowledge be developed? In the UK, official attention has focused on what Shulman has called 'subject matter knowledge', while recognition of the complexity and significance of the pedagogical dimension remains weak. In this paper, I propose and demonstrate an approach to the development of knowledge for mathematics teaching. This approach exemplifies the application to practice, of theory developed from research into practice - what Skott has called a 'theoretical loop'. In this case, we see an Irish novice teacher, Máire, having significant insights into her pedagogy even as she teaches: indeed, 'Maire's epiphanies' would have been a very apposite, alternative subtitle to the one I have given, and a fitting tribute to her.

Introduction

The title of the MEI-2 conference – "walking the talk" – is provocative, and deliberately so, I am sure. My response to the provocation is this paper, and I hope that I might be judged to have risen to the implied challenge by its conclusion. The broad theme of the research reported in this paper is *knowledge* for mathematics teaching. Shulman (1986) reasserted the importance of the subject-matter knowledge of teachers. He used the term 'missing paradigm' to refer to a blind spot with respect to subject-matter content that had characterised earlier research studies on teaching.

An uninformed perspective on subject-matter knowledge in relation to mathematics teaching might be characterised by the belief that secondary teachers already have it and elementary teachers need very little of it. But there is evidence from the UK and beyond to refute both parts of that statement (e.g. Ball, 1990; Alexander, Rose and Woodhead, 1992; Ofsted, 1994; Ma, 1999). Ma, in particular, presents compelling evidence that the adequacy of elementary teachers' knowledge of mathematics, for their own professional purposes, cannot by any means be taken for granted. Recent government initiatives to enhance the mathematics subject-matter knowledge (SMK) and pedagogical content knowledge (PCK) of prospective and serving elementary teachers have been taken in a number of countries. The rather direct approach to the 'problem' in England is captured by an edict in the first set of government 'standards' for Initial Teacher Training (ITT) issued in 1997:

> All providers of ITT must audit trainees' knowledge and understanding of the mathematics contained in the National Curriculum programmes of study for mathematics at KS1 and KS2 [2], and that specified in paragraph 13 of this document. Where gaps in trainees' subject knowledge are identified, providers of ITT must make arrangements to ensure that trainees gain that knowledge during the course … (DfEE, 1997, p. 27)

The process of audit and remediation of subject knowledge within primary ITT became a high profile issue following the introduction of these and subsequent government requirements (DfEE, 1998). Within the teacher education community, few could be found to support the imposition of the 'audit and remediation' culture. And yet the introduction of this 'testing' regime provoked a body of UK research on prospective primary teachers' mathematics subject knowledge (e.g. Rowland, Martyn, Barber and Heal, 2000; Goulding and Suggate, 2001; Jones and Mooney, 2002; Sanders and Morris, 2000, Morris, 2001; Goulding, Rowland and Barber, 2002). The proceedings of a symposium held in 2003 usefully draw together some of the threads of this research (BSRLM, 2003).

My work within this corpus of research, in association with colleagues in London, included an investigation of the relation between student teachers' SMK, as assessed by a 16-item written audit instrument, and their teaching competence. A simple chi-square test showed a significant association between audit score and teaching competence. This finding turned out to be robust when we replicated it with a different cohort of student teachers (Rowland *et al.,* 2001). Students obtaining high (or even middle) scores on the audit were more likely to be assessed as strong numeracy teachers than those with low scores; students with low audit scores were more likely than other students to be assessed as weak numeracy teachers.

We wanted to find out more about what was 'going on'. If superior SMK really does make a difference when teaching elementary mathematics, it ought somehow to be observable in the *practice* of the knowledgeable teacher. Conversely, the teacher with weak SMK might be expected to misinform their pupils, or somehow to miss out on opportunities to teach mathematics 'well'. In a nutshell, we wanted to identify, and to understand better, the ways in which trainees' knowledge of mathematics, or the lack of it, is evident in their teaching.

The Skima Video Study

By this time our project had been named *SKIMA* (subject knowledge in mathematics) and the study described here was undertaken in collaboration with colleagues in Cambridge. Our approach to investigating the relationship between teacher knowledge and classroom practice was to observe and videotape novice teachers teaching. This took place in the context of a one-year PGCE course, in which 149 trainees followed a route focusing either on the Early Years (pupil ages 3-8) or the Primary Years (ages 7-11). Six trainees from each of these groups were chosen for observation during their final school placement. Two mathematics lessons taught by each of these trainees were observed and videotaped, i.e. 24 lessons in total. The trainees were asked to provide a copy of their planning for the observed lesson. As soon as possible after the lesson (usually the same day) the observer/researcher wrote a brief (400-500 words) account of what happened in the lesson, so that a reader might immediately be able to contextualise subsequent discussion of any events within it. These 'descriptive synopses' were typically written from memory and field notes, with occasional reference to the videotape if necessary.

From that point, we took a grounded approach to the data for the purpose of generating theory (Glaser and Strauss, 1967). In particular, we identified aspects of trainees' actions in the videotaped lessons that seemed to be significant in the limited sense that it could be construed to be informed by their mathematics content knowledge or their mathematical pedagogical knowledge. We realised later that most of these related to choices made by the trainee, in their planning or more spontaneously. Each was provisionally assigned an 'invented' code (see below). These were grounded in particular moments or episodes in the tapes. This provisional set of codes was rationalised and reduced (e.g. eliminating duplicate codes and marginal events) by negotiation and agreement in the research team.

This inductive process generated 17 agreed codes. Next, we revisited the 24 lessons and, after further intensive study of the tapes, elaborated each descriptive synopsis into an analytical account of the lesson. In these accounts, significant moments and episodes were identified and coded, with appropriate justification and analysis concerning the role of the trainee's content knowledge in the identified passages, with links to relevant literature.

Walking the talk?

At this point we duly presented our findings to the research community, at the next BERA conference. Only after that did we realise the potential of what we had found for our work, and for that of our colleagues, in teacher education. In the UK, a large part[1] of the graduate initial training (PGCE) year is spent teaching in schools under the guidance of a school-based mentor. The proportion of time differs in other countries, but *practicum* placement is more-or-less universal. Placement lesson observation is normally followed by a review meeting between a school-based teacher-mentor and the student teacher. On occasion, a university-based tutor will participate in the observation and the review. Research shows that such meetings typically focus heavily on organisational features of the lesson, with very little attention to mathematical aspects of mathematics lessons (Brown, McNamara, Jones and Hanley, 1999; Strong and Baron, 2004). Our 'pure' research clearly offered a basis for us to develop an empirically-based conceptual framework for lesson reviews with a clearer focus on the mathematics content of the lesson and the role of the trainee's mathematics SMK and PCK. Such a framework would need to capture a number of important ideas and factors about content knowledge within a *small* number of conceptual categories, with a set of easily-remembered labels for those categories.

The identification of the 17 fine categories was a stepping stone with regard to our ambition to develop the framework for observing and reviewing mathematics teaching with student teachers. We didn't want a 17-point tick-list (like an annual car safety check), but a readily-understood scheme which would serve to frame an in-depth discussion between teacher and observer. The key to our solution was the recognition of an association between elements of subsets of the 17 codes, enabling us to group them (again by negotiation in the team) into four broad, superordinate categories, or 'units'. These four units are the dimensions or 'members' of what we call the 'knowledge quartet'. Each unit is composed of a small number of subcategories that we judged, after these extended discussions, to be of the same or a similar nature[2].

The Knowledge Quartet

We have named the four units of the knowledge quartet as follows: foundation; transformation; connection; contingency. Each unit is composed of a small number of

[1] Half in the case of university-based primary training, or two-thirds for secondary, and even more for the various school-based schemes now available.

[2] Specifically, the codes contributing to each of the fours units are as follows:

FOUNDATION: adheres to textbook; awareness of purpose; concentration on procedures; identifying errors; overt subject knowledge ; theoretical underpinning; use of terminology.

TRANSFORMATION: choice of examples; choice of representation; demonstration.

CONNECTION: anticipation of complexity; decisions about sequencing; making connections; recognition of conceptual appropriateness.

CONTINGENCY: deviation from agenda; responding to children's ideas; use of opportunities.

cognate subcategories. For example, the third of these, *connection*, is a synthesis of four of the original 17 codes, namely: *making connections; decisions about sequencing; anticipation of complexity,* and *recognition of conceptual appropriateness.* Drawing on the extensive range of data from the 24 lessons, we offer here a brief conceptualisation of each unit of the knowledge quartet.

The first category, *foundation*, consists of trainees' knowledge, beliefs and understanding acquired 'in the academy', in preparation (intentionally or otherwise) for their role in the classroom. The key components of this theoretical background are: knowledge and understanding of mathematics *per se* and knowledge of significant tracts of the literature on the teaching and learning of mathematics, together with beliefs concerning the nature of mathematical knowledge, the purposes of mathematics education, and the conditions under which pupils will best learn mathematics. The second category, *transformation*, concerns knowledge-in-action as demonstrated both in planning to teach and in the act of teaching itself. As Shulman indicates, the presentation of ideas to learners entails their re-presentation (our hyphen) in the form of analogies, illustrations, examples, explanations and demonstrations (Shulman, 1986, p. 9). Of particular importance is the trainees' choice and use of examples presented to pupils to assist their concept formation, language acquisition and to demonstrate procedures. The third category, *connection*, binds together certain choices and decisions that are made for the more or less discrete parts of mathematical content. In her discussion of 'profound understanding of fundamental mathematics', Liping Ma cites Duckworth's observation that intellectual 'depth' and 'breadth' "is a matter of making connections" (Ma, 1999, p. 121). Our conception of this coherence includes the sequencing of material for instruction, and an awareness of the relative cognitive demands of different topics and tasks. Our final category, *contingency*, is witnessed in classroom events that are almost impossible to plan for. In commonplace language it is the ability to 'think on one's feet'. In particular, the readiness to respond to children's ideas and a consequent preparedness, when appropriate, to deviate from an agenda set out when the lesson was prepared.

Our scrutiny of the data suggests that the knowledge quartet is comprehensive as a tool for thinking about the ways that subject knowledge comes into play in the classroom. However, it will become apparent that many moments or episodes within a lesson can be understood in terms of two or more of the four units; for example, a **contingent** response to a pupil's suggestion might helpfully **connect** with ideas considered earlier. Furthermore, it will become clear that the application of subject knowledge in the classroom *always* rests on **foundational** knowledge. This point is made very clear, e.g. in the discussion of Contingency in the lesson considered later in this paper: the factors involved in this teacher's decision to deviate from her planned course will already have been raised in the discussion of Foundation.

The notion of 'theoretical loop'

In a paper presented to a meeting of teacher education researchers, Skott (2006) highlighted the changing nature of the relationship between theory and practice in teacher education. He observed that the theories brought to bear on the task of *improving* teaching increasingly derive from studies *of* teaching, and coined the term 'theoretical loop' to capture this dialectical relationship between theory and practice in teacher education. In this instance, the knowledge quartet came into being as the outcome of systematic observation of mathematics teaching. Initially, we viewed it as a way of managing the complexity of describing the role of teachers' content knowledge in their teaching. In the spirit of Skott's theoretical loop, we subsequently developed ways of using the knowledge quartet as a framework to facilitate reflection on and discussion of mathematics teaching between prospective teachers, their mentors and teacher educators (see e.g. Rowland and Turner, 2006). The framework has also been successfully applied to supporting mathematics teaching development in early-career teachers (Turner, this volume) and structuring 'lesson study' in initial teacher education (Corcoran, this volume).

To make this point more explicit, I have chosen to apply the knowledge quartet to the analysis of a lesson which was not one of the 24 lessons from which the 'theory' – the knowledge quartet – derived. The lesson in this case was observed and videotaped by Dolores Corcoran. I am grateful for her permission, and that of her participant, 'Máire', to draw on this lesson for the purpose of exemplification and demonstration.

The way that the knowledge quartet is best used by observers – usually teacher-colleagues and by teacher educators – is to identify *for discussion* various matters that arise from the lesson observation, and to structure reflection on the lesson. Some possibilities for discussion with the student teacher, and for subsequent reflection, are identified by the observer. There has been a process of selection of potential matters for discussion in the commentary which follows, and even these would be too numerous for a typical, time-constrained post-lesson review meeting. Corcoran's (2007) earlier analysis of the lesson offers other issues for consideration, and embeds the student teacher's classroom practice in the Irish curricular context.

The Case of Máire

Máire, a student teacher in Ireland, was following an 18-month postgraduate diploma in education course (PGDE). She had a degree in modern languages and had taught English in Europe and the Far East. Máire's forty-minute mathematics lesson was observed during her final teaching placement in a girls' primary school. The focus of the lesson was on whole number division, and the analysis that follows is confined to a six-minute fragment near the beginning of the lesson. The class was a mix of 3rd and 4th Class girls (age 9-10 years). Máire had written separate worksheets on division for each age group, both set in a fantasy Harry Potter[1] scenario. The first problem for each of the two groups was as follows:

> 3rd Class: Ron has 18 Galleons[2] and a pack of cards costs 3 Galleons. How many packs can he buy?
>
> 4th Class: Fred and George want to buy magic worms to put in everyone's bed. They had 44 galleons, and each worm costs four galleons. How many worms could they buy? [This question continues: if there are 30 beds, how many more worms would be needed?]

Máire wanted the children to solve the problems with the aid of manipulatives which she brought to their attention at the beginning of the lesson. She provided Dienes' apparatus for the 4th class pupils, and reminded them at the outset about the relative values of the pieces, and how a ten could be exchanged for ten units. The manipulatives provided for the 3rd class pupils were simple tallies, in the form of butter beans. She implied that these would suit their purpose "because the numbers are much smaller".

[1] The *Harry Potter* novels by J K Rowling are well-known in Ireland.

[2] Galleons are the fictional currency in use at Harry Potter's school.

Conversely, when a 4th Class girl suggested that *they* might use butter beans, Máire commented that “the numbers you are going to be using are going to be too big.” She suggested instead that “You can draw them in your copy[1].”

Máire then set to work with the 3rd Class girls. One pupil, Rosin, read out the first problem, while Megan volunteered to count out 18 butter beans. Máire said a few words of explanation about the “wizard money”, and continued:

Máire:	Megan is counting out her 18 galleons, her 18 beans, and how many groups does she need to break it into and can you tell me why? Hannah what do you think?

Somewhat later, another pupil read out the 4th Class problem, and Máire discussed with her how to use the Dienes’ apparatus to solve it.

Máire:	So you had 44 there […] If we took that four away, that would be one worm. So you might need to turn those into units ...
Child:	That would be easy.

In the event, the processes of solution turned out to be problematic in both cases.

Foundation

I raise for consideration here two aspects of the lesson that fall within the scope of what the knowledge quartet delineates as foundation knowledge:

Máire’s beliefs about how mathematics is best learned;
The nature of division and the corresponding conceptual structures.

Máire’s beliefs

Concerning the former, Máire’s decision to locate the division problems in a Harry Potter context could be construed to point to certain beliefs about learning and motivation for learning – that mathematics should be set in relevant and meaningful contexts, and that it should be enjoyable. On the face of things, such beliefs seem to be entirely sensible and proper. They are shared by a great many novice elementary school teachers, no doubt in reaction to their own recollections of dull, meaningless mathematics as learners. One carefully-documented case is that of ‘Ms. Daniels’ in Borko *et al* (1992). Ms. Daniels believed that good mathematics teaching included making mathematics relevant and meaningful for students, so that teachers should incorporate applications from students’ everyday lives, as well as applications students might believe were useful to someone someday. She also thought it important to plan activities students might enjoy.

It is difficult to tell whether the Harry Potter context of the problems motivated the pupils as intended. Ironically, however, Máire was obliged to explain the ‘fun’ currency used in the problems in terms of the more prosaic euro. So with the 3rd Class pupils, she added:

[1] ‘Copy’ is a reference to the pupil’s notebook or exercise book.

Máire: That's 18 Galleons, you know that’s wizard money. That’s like 18 euros or 18 pounds. So…she has her…while Megan is counting out her 18 Galleons, her 18 beans …

With the 4th Class girls:

Máire: OK, so stop there, cause that’s the first part isn’t it? They have 44 galleons each, so they go into the shop and they want to buy the worms .They have 44 galleons … 44 euro. Now … each one costs four euro. So they’re standing there with their money, thinking about the worms, oh how many worms could we buy?

And later, with a group:

Máire: So the boys are in the shop and they have their money and ... 44 galleons, that’s the same as 44 euro. And how much are the worms?

We may also reasonably infer that Máire believed that the provision of manipulatives is important for children of this age. This is somewhat at odds with the trend across the water in England, where the emphasis would be on mental representations of the problem and the evolution of flexible mental strategies to solve it. Threlfall (1996) reviewed the arguments for the value of practical number apparatus and examined the practical activities for which it is used, to try to explain why the theoretical benefits do not seem to translate into practice. He observed in particular widespread, unhelpful use of the apparatus as an aid to calculation, and recommended a change of emphasis in the use of number apparatus. As we shall see later, there was a particular difficulty to do with the use of Dienes’ apparatus in connection with the problem Máire had set for the 4th Class pupils.

The nature of division

One of the ways in which children make sense of mathematical operations, procedures or concepts is by appreciating the range of different situations to which they apply. The various problems and scenarios relevant to a particular operation can be grouped into a small number of categories, each with the same fundamental structure. In the case of division, there are at least two[1] key problem structures, variously called partition/sharing and quotition/measurement/grouping. In the case of 20 divided by 4, the first structure, partition, is exemplified by the problem:

> Mary has 20 sweets. She shares them with Alba, Beryl and Connie. How many sweets will each receive?

For the second structure, quotition, the story is something like:

> Mary has 20 sweets, and gives 4 each to some friends at school. How many friends will get some sweets?

There is a number of differences between these two structures. Each involves some sweets and some children, but the ‘answer unit’ is sweets in one case, but children in the other. In the first case, we share 20 things among 4 people, and each receives the same quota of 5 things; in the second, we distribute a pre-set quota of 4 things to 5 people.

The distinction between these two structures is not generally known. Indeed, in her seminal doctoral study, Deborah Ball (1990) found that many beginning teachers - primary and secondary, some with mathematics degrees - lacked sufficient awareness of

[1] In a popular and well-respected teacher education text, Haylock (2006) refers to quotition as ‘inverse of multiplication), and adds a third ‘ratio’ structure, which compares two magnitudes with reference to the ratio of their sizes.

quotition to be able to explain *why* $1\frac{3}{4} \div \frac{1}{2}$ is $3\frac{1}{2}$, other than just saying "invert and multiply". This is no disgrace and should not come as a great surprise. Even a very good mathematician does not need *explicit* awareness of the existence of these two structures, or their names, in order to do their work. The awareness is likely to be implicit, for the purpose of solving particular problems. For the teacher of mathematics, however, the situation is different, as close examination of Máire's lesson reveals.

First, it is clear that in each of the two Harry Potter problems under discussion, the problem structure is *quotition*. They both begin with a certain supply of Galleons, and a fixed quota (3 Galleons, 4 Galleons), whereas the 'answer unit' is packs for one problem, worms for the other. However, in exploring how to resolve the problem, Máire draws on the language and concepts of *partition*.

Máire:	While Megan is counting out her 18 Galleons, her 18 beans, and *how many groups* does she need to break it into and can you tell me why? Hannah, what do you think?
Child:	Into three groups.
Máire:	Into three groups. Well done, and why? You can read the question again if you want.

Máire's reference here to the *number* of 'groups', and not their size, points to a partition structure, which is picked up by the child. This is hardly surprising since, as Brown (1981, cited in Dickson et al, 1984) found, primary school children have a propensity to opt for partition in preference to quotition when asked to supply a problem for a 'bare' division sentence. Indeed, as noted earlier, Ball (1990) found the same to be true of student teachers. Máire congratulates the child ("Well done") on her inappropriate suggestion. But, for some reason, Máire is inspired to ask the child to *explain* ("and why?"). The interaction then takes a different direction.

Child:	Because there's three packs of cards.
Máire:	It's *not* that there's three packs of cards. But what *is* it about the cards?
Child:	It costs three galleons.

It is interesting to note that the child's first justification would be entirely appropriate for the partitive division model that she proposed earlier. 18 Galleons, 3 packs of cards, 6 Galleons for each pack. But, for reasons that we can only speculate, Máire is pulled up short at this point. She knows that there are *not* three packs of cards. Máire has inadvertently directed the pupils to the wrong division structure, and she knows it. Homer Simpson would say, 'Doh!". She is in a hole, and she resolves to find a way out:

Máire:	It costs three galleons. So if you share out your 3 galleons, you see how many packs of cards you're able to buy […] Megan you're already ahead of us. You've got 18 and what are you doing?

Máire is attempting to alter the direction of the discussion. Her use of the word 'share' to point to quotition is perhaps unfortunate (and, remarkably, 'grouping' earlier when she had partition in mind!). The child who responds has not altered course:

Child:	Splitting them up into three groups ...
Máire:	Ahh …? Into groups of three [she nods]. And how many groups do you have?

Máire's response "Into groups of three" is a direct correction, and her language is now correctly aligned with quotition/grouping. The vessel (a Galleon, perhaps?) has altered course:

Child:	Six.
Máire:	So *how many packs of cards* could Ron buy?
Child:	Six.
Máire:	He could buy six packs of cards. Can everybody follow that? What sentence would you write to explain what we just did?

As we remark in the next section, this ability to change course as a result of reflection was quite rare in the lessons that we had observed in our original study.

Contingency

I have drawn attention to the moment when Máire asked the child to explain her interpretation (in terms of partition) of the packs-of-cards problem, and how Máire had the sudden insight that the very interpretation that she, herself, had suggested earlier, and praised when the child took it up, was not appropriate for the problem under consideration. As a consequence of her insight, Máire then steered the interaction in a different direction.

Child:	Because there's three packs of cards.
Máire:	It's *not* that there's three packs of cards. But what *is* it about the cards?
Child:	It costs three galleons.

This is an example of what we would call a 'contingent moment', and to re-visit that moment in a review of the lesson with Máire would surely be valuable. Many teachers of mathematics would attest that there is nothing unusual in the fact that she finds herself in a 'pickle'. Máire emerges from this one with credit, however, and that *is* unusual. Máire could not have prepared (in her planning) for what she did at that moment, but what she did say and do brought about a significant and pedagogically important shift in the discourse and the cognitive content of the lesson. This was possible because Máire seems to have experienced an *insight* of some kind, an 'aha' of a pedagogical kind.

As a matter of fact, this insight of Máire's has been significant in terms of our conceptualisation of the Contingency dimension of the knowledge quartet. This dimension was rooted, as it arose from the data in our original study, in the teacher's response to *children's* insights and misconceptions. In this instance we seems to have a moment where Máire herself suddenly realises that the problem, the child's suggestion, and her approval, simply do not 'stack up'. In our study of Máire's lesson, we apply a theory derived from practice back *to* practice – a case of the theoretical loop. This is also a case of what Glaser and Strauss (1967) call 'theoretical sampling', whereby the application of the theory lays it open to refinement, modification and possible improvement. Máire's moment of insight is an instance where theoretical sampling has found the current theory wanting, and caused it to be rethought and enhanced. Therefore, in terms of the theory itself, there is something rather special about Máire's response to this teaching situation.

Connection

This dimension of the knowledge quartet is essentially about the logical and psychological coherence of the lesson. One of the codes that contributes to our conceptualisation of this aspect of the lesson refers to *anticipation of complexity*. In our original study, we saw how well-informed and well-prepared teachers anticipated the 'tricky bits' of the lesson in their planning and in their presentation[1] of the lesson. An instance of this occurs in the opening moments of the lesson. Máire stands before the class with two Dienes' tens-pieces between her fingers, and says:

Máire :	Alright, fourth class you've got your Dienes' blocks. Remember what we said about those yesterday. If you have a ten or if you have two tens and you need to break it up among three people what do you do? What do you do?
Child:	You get the units ...
Máire :	You get the units. Yeah, and if you're having problems with the Dienes' blocks …what can you do if you don't like using these?
Child:	You can use the butter beans.

It would be fair to say that Máire's anticipation of the complexity to come is partial: she anticipates that "breaking up" Dienes' tens might necessitate exchanging one or more of them into units. On the other hand, her division language is that of partition from the outset "if you … need to break it up *among three people*". What she has not anticipated, at this stage at least, is that the tens will later need to broken up according to a pre-set *quota*.

Transformation

I want to consider here Máire's representation of the 4^{th} class problem (buying worms for the beds) using Dienes' apparatus. But first, I shall set the scene.
When Máire came to introduce the 4^{th} Class girls to their problem, she asked one pupil to read it, and the discussion then proceeded as follows:

Máire:	OK … they have 44 galleons each, so they go into the shop and they want to buy the worms. They have 44 galleons, €44 ... Now… Each one costs 4 euros. So they're standing there with their money, thinking about the worms – Oh, how many worms could we buy? How many worms … So what can we do to find out?
Child:	Ask the man.

This child reminds us that the 'realistic' word problem is a kind of educational charade, a genre (Gerofsky, 1996), that may connect with the child's reality in unexpected ways, and with unexpected consequences. I bring to mind the 'innumerate' adults interviewed by Bridget Sewell for her Cockcroft research (Sewell, 1981). What need had they of arithmetic, when the shopkeeper could be trusted to give them the right change? To sustain this particular game, however, Máire needs to plant the seeds of doubt in her pupil:

Máire:	You *could* do that but maybe the man would cheat them, so they want to know for themselves. So what could they do themselves instead of asking the man? Yeah ... So you had 44 there … so you don't need this for the moment. So we just have 44, and 4 so if we took this 4 away ...

If we took that four away, that would be one worm.

[1] What others might call 'delivery' …

I note at this point that it is difficult, if not impossible, to recover from the data the referent of 'this' in "you don't need this for the moment". Máire is leaning over the pupils' table and moving something on the table, but only Máire's face and shoulders are visible in the video[1]. It would seem that she is isolating 4 tens and 4 units from the pile of Dienes' material on the table, and that 'this' is the surplus material that she has moved to one side. One worm costs 4 Galleons, so Máire moves the 'quota' in the form of the four 'free' units to one side, in a classic demonstration of grouping-division. But now she wants 4 more units, but none can be seen since all the remaining 'currency' (40 Galleons) is now bundled into tens. Her intended approach is the one which Threlfall (1996) identified as questionable i.e. she is using the apparatus to perform the calculation. In this instance, this would be laborious in the extreme to repeatedly remove four units from four tens, with the necessity of four decompositions (of a ten into 10 units) along the way. Máire is aware of this.

Máire:	So you might need to turn those into units ...
Child:	That would be easy.

There then comes Máire's "epiphany" (Corcoran, 2007), as she struggles to deal with yet another key pedagogical insight:

Máire:	Actually ... couldn't we … no we couldn't do it that way ... that might be easier. Do you want to try that? Think about it and see how it goes and while you're thinking about it, think about any other way that you'd be able to do it.

Even reading the transcript, one can see the cogs turning in Máire's head! "… couldn't we … no we couldn't do it that way ... that might be easier.". Again, the 'that' is *deictic* – the referent cannot be recovered without more information about the context. My conjecture would be that Máire has the insight that the 4 tens and 4 units could very easily be *partitioned* into 4 equal parts, each with one ten and one unit. Since this is not shared with the children, and with good reason, the first sentence above is a kind of dialogue with herself, thinking aloud. "That way" is most likely partitioning, yet, especially since her very recent earlier epiphany into the structure of the 3rd Class problem, she realises that partitioning does not 'fit' the semantic structure of this worms problem either, and for this reason she says "… no we couldn't do it that way", even though "... that might be easier.". Her next utterance ("Do you want to try that? Think about it and see how it goes") is an admission that she can't see a way out of this situation (given the other demands on her time as class 'manager').

Corcoran (2007) writes:

> In discussion after the lesson, Máire disclosed that she realised at that point that it would be much easier to divide 44 into four groups of 11 but also feared that it wouldn't fit the problem context she had created and being unsure about how to proceed encouraged the children to "think about any other way that you'd be able to do it".

One pupil responds:

Child:	Can we just write the answers? ... that might be easier.

[1] This could be construed as disappointing, but I see it as a sharp reminder that the evidence for any inferences that we might make is always partial. In any case, Dolores Corcoran subsequently interviewed Máire about the lesson, and this substantially added to the data, and the possibility of validating (or refuting) any conjectures made from observation, and/or from the video.

Out of the mouths of babes and sucklings? Again we can't know exactly what this child had in mind, but it might convey the fact that they could do the calculation if they could set aside the Dienes' apparatus.

Conclusion

Research originally fuelled by curiosity about teacher knowledge and classroom practices led to the development of the knowledge quartet, a manageable framework within which to discuss actual, observed teaching sessions with trainees and their mentors. The framework is in use in teacher education programmes in Cambridge and elsewhere. Those who use it need to be acquainted with some details of its conceptualisation, as described in this paper. Initial indications are that this development has been well-received by teacher-mentors, who appreciate the specific focus on mathematics content and pedagogy. They observe that it compares favourably with government guidance on mathematics lesson observation, which focuses on more generic issues such as "a crisp start, a well-planned middle and a rounded end. Time is used well. The teacher keeps up a suitable pace and spends very little time on class organisation, administration and control." (DfEE, 2000, p.11).

It is all too easy for analysis of a lesson taught by a novice teacher to be (or to be perceived to be) gratuitously critical, and it is important to emphasise that the knowledge quartet is intended as a tool to support teacher development, with a sharp and structured focus on the impact of their SMK and PCK on teaching. A post-observation review meeting usefully focuses on a lesson fragment, and on only one or two dimensions of the knowledge quartet to avoid overloading the student teacher with action points.

In addition, we have recently been analysing lessons taught by secondary mathematics PGCE students, through the lens of the knowledge quartet. In a different development, colleagues working in English, science and modern foreign languages education have seen potential in the knowledge quartet for their own lesson observations and review meetings. It would be interesting to see what the conceptualisations of the dimensions of the quartet look like in these and other subject disciplines, and some potentially interesting times lie ahead.

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Reasons Behind the Finnish Success in PISA Mathematics Survey and the Role of Research and Developmental Activities in Improving Mathematics Education

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Extended abstract
One of the key questions in all school systems is what kind of curricula and instructional practices would yield the best learning experiences and outcomes. Because of the PISA-studies and their results in mathematics and in other literacy domains, the Finnish comprehensive school system and its outcomes received wide international attention at the beginning of the new millennium. Hundreds of foreign delegations have visited in Finland to learn the secrets of the high performing system. This great interest has been a somewhat puzzling experience to all those responsible for and making decisions about mathematics education in Finland. Traditionally, we have been used to thinking that the models for educational reforms have to be borrowed from abroad. The sudden change in our role from a country following the examples of others to one serving as a model for others has prompted us to think seriously about the special characteristics and strengths of our education and school system. This concerns also our mathematics education. What kind of reasons and explanations can be found behind our success in PISA mathematics? Are there some special "secrets"? What has been the role of research and developmental activities, implemented since 1995, in our PISA success?

The lecture will introduce and discuss reasons behind the Finnish success in PISA mathematics. At first, a brief account of the Finnish comprehensive school education and mathematics education is presented. The main part of the lecture takes a closer look at factors, activities, and processes which create a multidimensional "explaining network" for high level performances in PISA mathematics. Finally, some challenges for the development of Finnish mathematics education and teacher education in future are considered.

Finland has nine years of compulsory schooling and children generally start school at the age of seven. Usually, for the first six years of comprehensive school, the children are taught by a class teacher, who generally teaches all or at least most subjects. Then, during the last three years, the different subjects are taught by specialised subject teachers. Almost all of the age group (99.7%) completes compulsory schooling. The school network covers the whole country. Comprehensive schools are primarily run by local authorities, with the exception of a few private schools. The government contributes to the financing of all schools. For children, the teaching and educational equipment are free of charge. In addition, the pupils get a free warm meal at school. Presently, the smallest schools have fewer than ten pupils, and the largest ones about 900. There are some 4000 comprehensive schools in Finland.

The Finnish comprehensive basic education is, however, not only a system. It is also a matter of pedagogical philosophy and practice (Linnakylä & Välijärvi 2005). It accentuates the fact that school is for every child, and that the school must adjust to the needs of each child, not the other way round. The pedagogy has been developed to adapt to heterogeneous student groups; no student can be excluded or sent to another school. Students' own interests and choices are taken into consideration when selecting course contents, textbooks, learning tasks, strategies and methods, as well as assessment devices. Of course, for heterogeneous groups to be successful class size must be relatively small; PISA data shows that Finnish class sizes are among the smallest in the OECD countries. Heterogeneous groups also require a flexible, school-based and teacher-planned curriculum, student-centred instruction, and counselling and special support for students with learning difficulties.

According to the National Framework Curriculum (National Board of Education 1994), all students must have an opportunity to develop such basic mathematical knowledge and skills that create a necessary foundation in view of their further studies, employment and everyday life. The essential aim is to develop the student's ability to classify, organise, and model situations that come up in the surrounding world, with terms she/he has learned. Students should also have an understanding of the importance of mathematics in the past and present and its part in the development of our culture. In the new framework curriculum introduced in 2004, the weighting of mathematics contents has been slightly changed.

Finland's high achievement in PISA mathematics survey seems to be attributable to a whole network of interrelated factors. From my perspective, there are a couple of factors especially related to mathematics education (Kupari 2004, 2005). *The systematic development of a comprehensive school mathematics curriculum* could be seen as a significant feature behind the Finnish success in PISA 2003. Applications and problem solving have been important principles in the mathematics curriculum of our comprehensive school during the last 20 years, and they have step by step become well established in mathematics teaching practice. We know that the PISA approach focuses particularly on young people's capability to apply their mathematical skills and knowledge in situations that are as authentic and close to daily-life needs as possible. Thus, the Finnish mathematics curriculum has emphasized and also implemented similar goals and contents to which have been assessed in PISA mathematics surveys.

In 1996 the Ministry of Education launched *the LUMA programme* to promote mathematical and scientific competence in Finland (LUMA is an acronym for the Finnish 'luonnontieteet ja matematiikka', i.e. science and mathematics). LUMA was a six-year-long project and ended in 2002. The core operation environment of LUMA consisted of a development and information network involving 78 municipalities and 10 training schools. Great efforts were accordingly devoted in the following domains: increasing the number of university student places for mathematics, science, and technology; enhancing teacher training as concerns both subject and pedagogical studies; updating computer hardware and software as well as science laboratory equipment and material at schools; and increasing experimental activities. Even though it is not possible to establish numerically a causal link between the LUMA programme and Finland's mathematics performance in PISA, the programme has undeniably opened new educational opportunities and, above all, aroused new faith in and enthusiasm for the development of Finnish mathematics education.

In addition to the above mentioned factors, there are certain characteristics in the Finnish education system that may have contributed to the good mathematics performance of its schools and individuals. Features and dimensions such as equal opportunity, individualism and inclusion, culture of trust, well-trained teachers, curricular flexibility and pedagogical freedom and cultural homogeneity (Aho et al.

2006, Kupari 2005, Linnakylä & Välijärvi 2005, Simola 2005, Välijärvi et al. 2002) will also briefly be discussed in the lecture.

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Creating Mathematics Lessons

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It is said that there are only seven jokes, and the ones we hear are only variations on these jokes templates. The same is true for mathematics lessons. Drawing on perspectives on learning and classroom based research, templates for four distinct mathematics lessons are proposed and analysed. These templates can be used by beginning teachers and experienced teachers alike to create mathematics lessons with particular pedagogical purposes.

To create a lesson, first take a task …

Productive classroom learning is the result of an evolution from the genesis of the initial idea for the lesson, which in most cases is a task, through a range of steps until the student has an experience that results in awareness of new learning. Stein, Grover, and Henningsen (1996) provided a model of how an idea evolves into a learning experience. They analysed 144 tasks in terms of their features and cognitive demands, and they studied the implementation of the tasks in classrooms. Their model, converted from a diagram on p. 459 to text here, proposed that the sequence goes from

Mathematical task as presented in instructional materials

which, influenced by the teacher goals, their subject matter knowledge, and their knowledge of students, informs …

… mathematical task as set up by the teacher in the classroom

which, influenced by classroom norms, task conditions, teacher instructional habits and dispositions, and students learning habits and dispositions, influences …

… mathematical task as implemented by students

which creates the potential for …

… students' learning.

The focus of this argument is on the second of these steps, that of setting up a task to be used in a lesson. To get to this stage, first the teacher must choose a task, a part of which is imagining how the task can be used in the classroom. Ames (1992), for example, who studied motivation and achievement from a psychological perspective, argued that teachers can influence the students' approach to learning through careful task choice or design. Ames pointed out that students should see a meaningful reason for engaging in the activity, that there needs to be enough but not too much challenge, and that variety is important. This complements suggestions about tasks from Gee (2004), who formulated principles for task design from a literacy perspective, derived from the analysis of successful computer games. Included in his principles of task formulation were for: learners to take roles as "active agents" with control over goals and strategies; and ideally for tasks to allow learners to take on a particular identity (e.g., that of a scientist or author). Gee also suggested that tasks be able to be customised to match the readiness of the learner both for those who experience difficulty and those for whom the core task is not challenging.

There is also widespread recognition that it is through and around tasks that teachers and students communicate and learn mathematical ideas, so the tasks used by the teachers become the mediating tools. Christiansen and Walther (1986), drawing on the work of Leont'ev (1978), argued that the tasks set and the associated activity form the basis of the interaction between teaching and learning. Similarly, Brousseau (1997) proposed that "the teacher must imagine and present to the students situations within which they can live and within which the knowledge will appear as the optimal and discoverable solution to the problems posed" (p. 22). Hiebert and Wearne (1997) also proposed that "instructional tasks and classroom discourse moderate the relationship between teaching and learning" (p. 420). The next step for the teacher is to create a lesson.

From tasks to lessons

While there is much interest from the research community in task choice and design, there is perhaps less attention to how tasks can be converted into lessons. Van Den Heuvel-Panhausen and Teppo (2007) argued that it is necessary to go beyond tasks to consider teaching sequences and overall trajectories. Sierpinska (2003) reviewed a range of research reports from one volume of the Norwich PME conference and concluded that while most studies referred to tasks in some way, few problematised the task or its context, and few reported details of the lesson in which the tasks were embedded.

At least part of the impetus for a focus on lesson planning is the claim that variation in teaching patterns between countries can explain differences in mathematics achievement identified in the TIMSS results (e.g., Lokan, Ford, & Greenwood, 1997). The Learner's Perspective Study (Clarke, Emanuelsson, Jablonka, & Mok (2006) sought to inform practice, policy and theory about learning and lessons by the comparative study of teaching practices which includes comparisons and analyses of stereotypical lessons across a range of countries, including details of teachers' actions within classrooms. Stigler and Hiebert (1999) have earlier described patterns of teaching in the United States, Japan and Germany, and claimed that there are great differences between the cultures in approaches to teaching, but little variation within the cultures.

The lesson structure they suggested as representative of mathematics teaching in the U.S. is similar to that described by Good, Grouws, and Ebmeier (1983) which specified particular activities including: daily review of homework; development (including addressing prerequisite skills, lively presentations, assessing comprehension, controlled practice); seatwork; and homework assignment. Soeda (2001) argued that it is common in such lessons in the United States for the teacher to seem strict, to be the only user of the blackboard, to ask mainly easy questions, to pose many problems or exercises to be completed within the class, and to cope with many interruptions.

In contrast, Stigler and Hiebert (1999) characterised Japanese teaching as consisting of: review of previous work; the teacher posing a problem(s); students working individually and/or in groups; class discussion of selected student work; and the teacher highlighting and summarising key points. Soeda (2001) claimed that such lessons in Japan are often characterised by the teacher smiling, a pleasant classroom atmosphere, the class operating together, with few problems being posed, and few interruptions.

These descriptions, however, may be only part of the story. For example, Soeda (2001) cautioned against making inferences on the basis of these two stereotypical lessons. Arising from surveys of students, he argued that the Japanese lesson type, while common and part of the Japanese mathematics education literature for many years, is taught comparatively seldom and is representative of only a minority of lessons. Likewise, Soeda claimed that students in the U.S. reported that the standard mathematics lesson, even though characterised by Stigler and Hiebert as common across the country, is also enacted by teachers less than 50% of the time.

It seems that the American and Japanese lesson types may be useful in describing some lessons, but clearly more types are needed, and advice on how the lesson types can be used could be helpful for prospective and practising teachers.

Templates for mathematics teaching

As a way of providing advice on lesson structures to both prospective and experienced teachers, the notion of templates is proposed. The advantages of identifying particular templates for mathematics lessons are twofold: first, the templates can support teachers in creating their own lessons based on particular tasks; second, the templates provide pedagogical guidelines for particular types of lesson. The four lesson templates below seek to do this. In addition, the lesson templates are intended to be generic and adaptable.

In the following discussion, partly to clarify the pedagogical potential of the respective templates, and partly to illustrate how the templates differ, some key aspects of described.

First, the descriptors of the aspects of mathematics teaching summarised by Kilpatrick, Swafford, and Findell (2001) are used to describe the nature of the mathematical learning that is fostered. The aspects are: *conceptual understanding; procedural fluency; strategic competence; adaptive reasoning;* and *productive disposition.* These aspects are widely reported and their meaning is implicit in the terms.

Second, the factors that may influence students' responses to the lessons are considered. These include:
- the level of implied student choice, in that student choice in focus, approach, and level of demand contributes to motivation (see Middleton, 1995);
- potential for prompting communication, in that communication can contribute to effective learning (e.g., Wood, 2005), and that communication is more productive if there is more to discuss than the correctness of the answer;
- degree of risk, recognising that not all students will respond well when they are uncertain on how to proceed or the risk of failure is high (see Doyle, 1986, Dweck, 2000); and
- level of potential student engagement: Fredericks, Blumfield, and Paris (2004) described engagement in terms of behavioural, emotional, and cognitive responses. They argued that engagement is enhanced by tasks that are authentic, which is assumed to mean relevant, providing opportunities for students' sense of ownership and personal meaning, fostering collaboration, and drawing on diverse talent, and fun.

Third, the discussion also uses the verbs of mathematics proposed by Watson and Mason (1998) to describe the nature of the prompted mathematical activity in generic rather than content specific terms. These elaborate the descriptions of the following four possible templates for mathematics lessons.

Lesson Template 1: Active Teaching

The *active teaching* lesson template is similar to the stereotypical US lesson, and is suited to tasks and teaching where the teacher seeks to explain a procedure or technique.

To clarify the elements or phases of the template, suppose the teacher wants her class to learn that it is possible to multiply by 5 mentally by first multiplying by 10, then dividing by 2. The left hand column is intended to apply to all lessons of this type. The right hand column is a description of how this might work for this example:

Table 1: *Lesson elements for template 1*

Key lesson element	**In this example …**
The teacher revises pre-requisite content, assesses current understandings.	The teacher poses some examples such as 32 × 10, and 320 ÷ 2. These are the pre-requisite skills.
Teacher uses lively methods to illustrate or model aspects of mathematics or procedures, and gives one or two practice examples, which are reviewed.	Teacher writes some examples like 34 × 5, 48 ×5, and asks the students to work out the answer. Some students explain what they have done. Teacher demonstrates the method: $34 \times 10 = 340$ and $340 \div 2 = 170$ and poses some more practice examples.
Individually or in small groups, students complete further examples, activities or problems designed to give practice and consolidation of the content that is the focus of the lesson.	Further practice examples are posed, in sets of similar type tasks. The first set might be like 42 × 5, the next set like 68 × 5, and the next set like 57 × 5, and then extending to 346 × 5.
The teacher reviews the methods and answers of the students, and attends to particular problems or responses that assist in consolidating the purpose of the lesson .	The student responses to any set exercises are corrected and some further examples (e.g., 42 × 5) are posed to check student learning.

In terms of the Kilpatrick et al. (2001) dimensions, this lesson template is useful for building conceptual understanding and developing procedural fluency. In this lesson template, there would often be not much student choice, in that the focus and pacing is determined by the teacher, there would be limited potential for prompting communication, in that it is merely correctness that is of interest, the experience would usually be low in risk for the students in that the teacher guides the lesson, and the engagement would be through the energy and activity of the teacher, or a student's innate desire for mastery of their learning. The types of mathematical activity implied in this template, in terms of the Watson and Mason verbs, are: *exemplifying, specialising, completing, deleting, correcting, comparing, organising, generalising, explaining, verifying.* This lesson template applies to most tasks where the emphasis is on developing a specific mathematical skill or procedure.

Lesson Template 2: Imagined Representations

The *Imagined Representation* lesson template is an elaboration of the Japanese lesson described above, and is suited to investigations that use a "practical" context. The template is similar to, and indeed derived from, the activities suggested by Lovitt and Clarke (1988), titled *Estimation with fractions*, and, *How many people can stand in your classroom*? The name, *Imagined Representation,* relates to the key element of problem

solving associated with imagining possibilities. A task which could form the basis of a lesson based on this template is as follows:
The height of the Statue of Liberty is 46.05 metres. How long would you expect the arm to be?

Again, the left hand column in the following is intended to apply to all tasks and lessons of this type. The right hand column is a description of how this might work for this particular task:

Table 2: Lesson elements for template 2

Key lesson element	In this example …
After posing and clarifying the problem, the teacher asks the students to record an estimate.	The teacher asks the students to record an estimate of the length of the arm on the statue. The estimation fosters interest in the answer.
Students are invited to think about what strategies they might use to calculate an answer, first individually, then brainstorming in a group, and the groups report their strategies to the class.	After discussion, some groups might suggest comparing the length of the height and the arm on a photograph of the statue, others might suggest measuring a sample of people to get a common ratio of height to arm length, and others might suggest using common ratios used by artists in creating artworks based on people .
Groups choose (or are allocated) a strategy, they implement the strategy to find an answer, and prepare a report. The teacher monitors the work of the groups, ensuring that all students are involved in the strategy implementation.	The groups of students are allocated to a particular solution strategy, preferably one they have suggested. Using whatever resources are required, they implement the particular strategy and prepare a report.
The teacher leads a review of responses, including attending to issues such as efficiency of a strategy, and appropriateness of the degree of accuracy. Ideally the teacher will select few rather than all groups to report, particularly those that are likely to contribute to the purpose of the activity.	The students report on their strategy including indicating their estimate of the length of the arm. The teacher can ask questions such as ‘which do you think is the most accurate method?’, ‘how accurate is good enough?’, ‘which is the most efficient method?’

It may even be appropriate for students to complete some additional problems or exercises that consolidate the principles identified in the investigation.	Some similar ratio tasks can be posed that allow students to practice the skills, or prompt for transfer to alternate situations.
The teacher summarises the main mathematical ideas addressed in the activity. One key aspect of the teacher's role is to emphasise the "dimensions of variation" (Mason, Drury, & Bills, 2007) inherent in the range of strategies and modes of communication of solutions that arise.	The teacher would emphasise the process for calculating ratios, which is presumably the purpose of posing the task in the first place, as well as the steps necessary to ensure that data collected are accurate.

While it is difficult for the teacher to predict what will happen, this particular activity provides a context for introduction of proportional reasoning and probably for application and reinforcement of estimation and measurement of length. It includes a strong metacognitive element, in that it is contributing to students' awareness not only of the possibility of multiple appropriate strategies, but also the usefulness of planning.
It is noted that this activity could be varied to suit a range of classes, anywhere from year 4 to year 9, depending on responses sought and additional questions asked. As an aside, the length of the right arm of the statue is 12.80m, which is shorter than might be expected. Students can try to suggest why this might be. A lesson in this format focuses explicitly on strategic competence and adaptive reasoning. There is student choice in the strategy to be used, in that they suggest which strategy to use, it is high in potential for prompting communication, in that students would be keen to explain what they found and how they found it, there would be medium risk in that, while the students have some degree of choice, what they are required to do is ambiguous, and the engagement would be through both the potential for choice, and the inherently interesting nature of the task.

Examples of mathematical actions implied in this template are:

> Comparing, sorting, organising, varying, reversing, altering, conjecturing, explaining, justifying, verifying, refuting

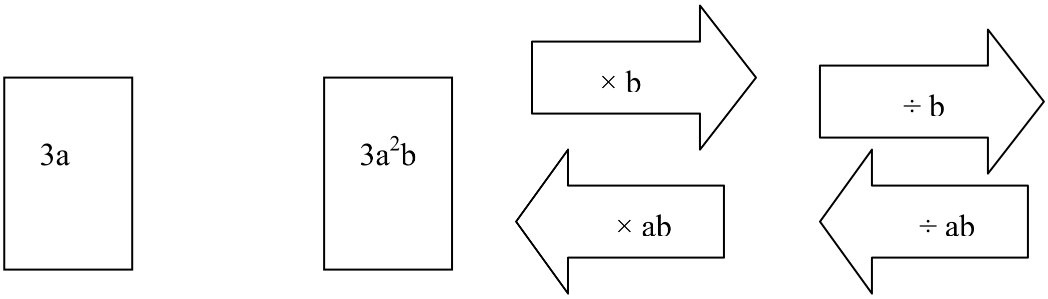
This lesson template is applicable to any practical or realistic task that requires investigation or consideration of strategy by the students, especially where there is a need for students to imagine a representation. Note that this template is also suitable for extended practical investigations, such as the following:

Find out about the cost for the types of services on your mobile phone, or a plan that has been advertised. Explain the costing of the services on the mobile phone plan to others. Describe the way you use your phone (voice, text, time of calls, etc). Listen to the descriptions of others, and evaluate your plan. Prepare a report comparing the cost of your plan with the plans of others.

Lesson Template 3: Purposeful Games and Puzzles

Purposeful games and puzzles (PGP) have potential to form the basis of meaningful mathematical experiences. It is suspected, though, that mathematical learning does not occur optimally merely as an incidental part of engaging in the PGP. In this case, the intent of the template is not only to emphasise the mathematical purpose of the PGP, but also to facilitate making mathematical connections in ways that have potential for future use.

The following is an example of a mathematical puzzle, adapted from a suggestion by Swan (no date). The puzzle involves a set of term (or number) cards and operation cards, a subset of which could be



In this case, the puzzle is to choose the two operation cards that can be placed between the two term cards to represent the connection. The point is that students have to look for the appropriate operation card to connect the terms, and by doing that have to evaluate a range of possible operations simultaneously. It is also self correcting, in that there are unique operations connecting the terms.

The generic lesson template, and possible actions for this puzzle, are as follows:

Table 3: Lesson elements for template 3

Key lesson element	In this example …
After explaining the rules and purpose of the PGP, the teacher demonstrates the PGP to the class.	The teacher explains that there are cards on which are mathematical terms, and arrows on which there are operations. The intention is to connect the terms using the operation cards. The teacher might model the process using different but related cards.
Students engage in the PGP for a short while, after which there is a teacher led class discussion of the strategies and or mathematical point of the PGP.	After the students have worked for a while on the task, there can be a class discussion of the processes for deciding which operation card is placed where, after which the students can continue with the puzzle. Students can be invited to describe how they made decisions on operations.

The students are then offered further opportunity to engage with the PGP. There can be additional discussion and activity as needed. The teacher or the students can suggest variations, such as making the PGP more challenging for some, or less complex for others. It is possible to group students based on their success at the PGP, so that, for example, students who complete the activity quickly might be grouped together for the next implementation of the PGP.	The teacher monitors the students' work as they arrange the cards of the puzzle. It is possible to have both harder and easier sets of cards, for those who finish quickly, and those for whom the puzzle as it is, is too difficult. Some cards sets might just involve numbers, or easier operations such as addition and subtraction. If necessary, it may be helpful to pose questions such as $3a \times ? = 3ab^2$
The teacher leads a discussion of the strategies and mathematics of the PGP. Specific problems can be posed that allow students' practice to focus on the mathematical point, or even extend their thinking.	The students could be asked to complete a set of practice exercises. For development, students can be asked to create their own sets of term and operation cards.
Finally, the teacher summarises the main mathematical ideas. The teacher has an active role to find commonalities, patterns, and principles that can form the basis of the formalisation of the intuitive insights developed during the engagement with the PGP .	The teacher can ask for the students to suggest rules that can guide operations using algebraic terms, and to illustrate how the principles in choosing the operation to connect 3a and $3a^2b$ can be extended to other operations.

To illustrate how this lesson template applies to mathematical games, the following is a description of the game *Race to 10* (Brousseau, 1997). In this game, players take turns to add either 1 or 2 to the previous total. Assuming that the players start at zero, a possible sequence could be as follows:

Player One	1		4		6		8		
Player Two		3		5		7		10	

One of the mathematical points in this game is that counting on is an efficient way to add. Another aspect of the game, in this case, relates to the existence of a winning strategy. Rather than detracting from its effectiveness, the existence of the strategy enhances the search for mathematical connections.

After playing the game for a short while, it is preferable to have a class discussion of the strategies used. This can assist by clarifying the rules and method of playing, to the process of counting on, and to the existence of a winning strategy. After more playing, possibly with winners matched with winners, there can be a review with some specific problems posed. Note that this game can be played in many ways, including for algebra, for example, by playing Race to $5x + 5y$, adding on x, y, or $x + y$.

A lesson in this format focuses explicitly on all of the Kilpatrick et al (2001) elements, especially conceptual understanding, strategic competence and productive disposition. There is student choice in the strategy to be used, in that students choose not only the game or puzzle strategy but also the ways they solve the mathematical aspect of the game or puzzle, it is medium for prompting communication, so the teacher must take an active role in encouraging students to talk to each other about the choices they make, it is low in risk in that students have some degree of choice and the game format is self correcting, and the engagement is through the competition or challenge associated with the activity or game.,

Examples of mathematics verbs implied in this template are: completing, comparing, sorting organising, changing, varying, conjecturing, justifying, verifying.

Lesson Template 4: What if?

The *What if*? template is useful for open-ended and investigative tasks (see Sullivan, 2007, for discussion of the ways that open-ended questions contribute to mathematics teaching and learning; see Sullivan & Lilburn, 2002, for a range of examples).

The following is an example of a task that is suitable for this template:

When 3 dice were rolled, the sum was 10. What might the dice have shown?

The generic lesson template, and possible actions for this example task, are as follows:

Table 4: *Lesson elements for template 4*

Key lesson element	In this example …
Teacher poses and clarifies the purpose and goals of the task. If necessary, the possibility of multiple responses can be discussed. .	The teacher might simulate the task by rolling dice and covering them up. The teacher could explain that the total of 10 could be reached in various ways. The students might be invited to record their answers systematically .
Students work individually, initially, with the possibility of some group work. Based on students' responses to the task, the teacher poses variations. The variations may have been anticipated and planned, or they might be created during the lesson in response to a particular identified need. The variations might be a further challenge for some, with some additional scaffolding for students finding the initial task difficult.	The teacher monitors the work of the students. For students who have difficulty answering the initial question, the teacher might ask: What if there were only 2 dice? What if you had some counters to help? What if you actually did it with dice? For students who produce one or more correct responses, the teacher might ask: What if you were asked to find all the answers? What if the sum was 11? What if there were 4 dice?

The teacher leads a discussion of the responses to the initial task. Students, chosen because of their potential to elaborate key mathematical issues, can be invited to report the outcomes of their own additional explorations.	Some students with simple strategies might be invited to demonstrate those to the class. Next, the teacher might choose a student who had produced an organised response to summarise their answers to the whole group. Students who have different responses can be invited to contribute their answers.
The teacher finally summarises the main mathematical ideas.	Finally, the teacher can summarise the successful strategies and the collective responses. Again this is the key part of the lesson for drawing out the patterns, commonalities, and generalisations.

A lesson in this template focuses on conceptual understanding, strategic competence, and adaptive reasoning.

There is student choice in the strategy to be used, in that they can choose the degree of difficulty, and the mode of representation, it is high for prompting communication, in that students have the products of their own explorations to contribute, it is low in risk in that students have choice in strategies and the level at which they work, and the engagement would be through their personal choice and the challenge of the task.

Examples of mathematics verbs implied in this template are:

completing, comparing, organising. changing, varying, reversing, generalising, conjecturing, explaining, justifying, verifying.

Summary

The basic argument is that, by seeking the commonalities or key features of mathematics lessons, it may be possible to identify the key elements of lessons, and so facilitate the creation of further lessons by both prospective and practicing teachers, as well as making it clearer how teachers can support student learning within such lessons. The four templates proposed above can be used flexibly, and they provide a useful guide.

The first step in the process is choosing a task, and then converting the task to a lesson. It is possible that any particular task is suitable and could be used as the basis of more than one template, but the templates are sufficiently idiosyncratic that usually the task is better suited to a particular template.

One issue is whether the templates proposed above are actually different. For this purpose, and treating the common phases of each lesson to be an introductory phase, a student working phase, and a review phase. The following is a tabulation of key aspects of these templates.

Table 5*: Comparison of the lesson elements across templates*

	Introduction	Students working	Review
Active teaching	Energetic teaching of particular concepts	Supervised practice, with teaching of small groups, possibly with like needs students	Correction and summary
Imagined representation	Posing problem and establishing context	Collaborative work, in mixed groups	Synthesis of student products and summary teaching
Purposeful games and puzzles	Explaining rules and demonstrating game	Student playing games in mixed groups, with like needs pairs evolving	Synthesis of student products and summary teaching
What if?	Posing clarifying questions	Individual students work on task with subsequent variations in reaction to student responses. Some selective grouping.	Synthesis of student products and summary teaching

It is clear that there are differences in emphasis in each of the lesson parts, and to further illustrate the differences between the templates, Table 6 summarises the purposes of the templates using the categories described above.

Table 6: *Summary of the learning features of the respective templates*

	Learning focus	Student Choice	Prompting communication	Degree of risk	Engagement	Key maths verbs
Active teaching	Conceptual understanding Procedural fluency	Low	Limited	Low	Through the energy of the teacher	completing, correcting, explaining
Imagined Representation	Strategic competence Adaptive reasoning	Medium	High	Medium	Through interest in the task	comparing, conjecturing, justifying,
Purposeful games and puzzles	Conceptual understanding Strategic competence Productive disposition	Medium	Medium	Low	Through the challenge	completing, comparing, sorting

What if?	Conceptual understanding Strategic competence Adaptive reasoning.	High	High	Low	Through student choice	changing, varying, generalising

These templates are proposed tentatively as a prompt for teacher educators and teachers in their planning of lessons.

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How the 'Stumbling of Ideas Across Each Other' Facilitates Insight in Primary Mathematics

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In this paper there is a description of a mathematics lesson which extended over two class periods and in which 5th class pupils attending a school designated as disadvantaged experienced mathematical insight. The moments of insight were not experienced by all pupils in the group but, at the same time, were facilitated by the whole class collaboration that took place at the end of the lesson. Construction of mathematical understanding in this way has been likened to jazz and theatrical improvisation in which the performers build on each other's input and where the outcome is the property of the group rather than of any one individual. The data from this lesson will be analyzed through an improvisational lens and implications for primary mathematics teaching and learning will be discussed.

Introduction

In her keynote address to the participants of MEI 1[1], Leone Burton (2005) posed the following challenge:

> [W]hy should we think that lack of mathematical sophistication precludes an appreciation of aesthetic pleasure – anyone who has worked with young children knows this not to be the case. So the argument in favour of teaching the 'basics' before permitting learners to touch complexity, seems to me extremely shallow and a misunderstanding of the nature of learning *and* of mathematics. (p. 10)

Although mathematics is recognized as a creative activity in the Irish primary mathematics curriculum (DES/NCCA, 1999a), there continues to be an emphasis on lower order skills and the 'basics' in primary mathematics classes (DES, 2005; NCCA, 2005; Shiel, Surgenor, Close, & Millar, 2006). For example, in the 2004 National Assessment of Mathematics Achievement (NAMA 2004) (Shiel *et al*., 2006), it was found that 95.5% of the sample of 4th class pupils (n=4171) are in classes where the text book is used on a daily basis. Over 80 per cent of these pupils are taught by teachers who use a class text book as the main source for the selection of problems and for applications in mathematics classwork and homework. Starko (2005) argues that pupils will experience the 'Aha' of mathematical creativity by thinking as mathematicians do. He points out that mathematicians seldom contemplate solutions to algorithms but rather that they look for patterns and try to understand them. Furthermore an emphasis on basic number operations inhibits mathematical insight as it induces a mental 'set' or 'algorithmic fixation' (Dooley, 2005; Haylock, 1987).

One of the recommendations of the NAMA 2004 report is that

> Schools and teachers should place a stronger emphasis on teaching higher-order mathematics skills, including Applying and Problem Solving to all pupils by implementing in a systematic way the constructivist, discussion-based approaches outlined in the *Guidelines* accompanying the 1999 [Primary School Mathematics Curriculum] (Shiel *et al*., 2006: p.155)

Implicit in this recommendation is the notion of equity, that is, that all pupils, regardless of race, gender or social class have the right to 'good mathematics taught well' (See Dooley & Corcoran, 2007 for elucidation of this). Implicit in it also is a focus on collective rather than individual understanding. Indeed the emergence of collective mathematical understanding is a subject that is receiving much attention by the mathematics education community (See, for example, Davis & Simmt, 2003;

[1] The First National Conference on Research in Mathematics Education held in St. Patrick's College, Dublin in September 2005.

Hershkowitz, Hadas, & Dreyfus, 2006; Martin, Towers, & Pirie, 2006). How it plays out in the normal classroom situation, is however, less clear.

Theoretical Framework

Consistent with a view of mathematics as a socially and culturally situated activity, the point of reference in mathematics education is not the discipline per se but rather the classroom mathematical practice, a perspective that has been described by Cobb (2000) as *emergent*. Sawyer (2004) maintains that this perspective implies that teaching must be improvisational and 'that the most effective learning results when the classroom proceeds in an open, improvisational fashion, as children are allowed to experiment, interact, and participate in the collaborative construction of their own knowledge' (p.14). If a teacher strictly controls a class and limits students' input, co-construction by members of the classroom community cannot occur. He maintains that like the improvisation that occurs in theatre or in a jazz ensemble, creative teaching is both emergent and collaborative. It is emergent because the outcome cannot be predicted in advance and it is collaborative because the outcome is determined not by any one individual but by the participants of the group.

In theatrical improvisation, a group of actors creates a performance without using a script. Because it is characterized by a high level of unpredictability, the performance has associated with it what Sawyer describes as a 'moment-to-moment contingency' (Sawyer, 2006: p.153). As the actors play their parts, several potential possibilities are brought into the frame. What emerges is not decided by any one person but rather is a phenomenon that is produced by the group. In jazz improvisation, each soloist is assigned a number of measures to play before the next soloist takes over. Due to the rapidity of the transition, a player will rarely develop a completely new idea but rather will respond to and build on the previous player's input: 'In their musical conversations, musicians constantly balance coherence and innovation, borrowing material from the previous phrase and then transforming it' (Sawyer, 2006: p.150).

Martin, Towers and Pirie (2006) used the improvisational lens to analyse collective mathematical understanding . They describe collective mathematical understanding as the kind of learning and understandings that occur when a group of any size work together on a mathematical activity. Central to their analysis is the idea of *co-acting* which they define as

> …a process through which mathematical ideas and actions, initially stemming from an individual learner, become taken up, built on, developed, reworked, and elaborated by others, and thus emerge as shared understandings for and across the group, rather than remaining located within any one individual. (p.156)

They make a distinction between co-actions and interactions. While in interactions there is an emphasis on reciprocity and mutuality, co-action concerns actions that are dependant and contingent upon the actions of other members of the group (Towers & Martin, 2006). Through this co-acting, an understanding emerges that is the property of the group rather than any individual. It is not that all individuals bring the same understandings to the scene but rather that individual contributions will result in something greater than the sum of the parts. Neither is it that an individual will not make his or her own personal advancements. Davis and Simmt (2003: p.146) compare this to the discovery of some major mathematical ideas:

> At various times and in varied circumstances, intellectual movements can arise spontaneously and may quickly exceed the possibilities of any of their members – at the

> same time as they provide the conditions for these members to advance their personal understandings and insights, The knowledge is a property of the collective (p.146)

They contend that the implications of this for a mathematics class are that 'concepts and understandings must be made to stumble across each other' (ibid., p.156)

There is a danger that, in using the metaphor of improvisation to describe creative teaching, an 'anything goes' approach will be assumed. What is required is a balance between structure and flexibility. Too little structure and the students become anxious, too much and they are constrained. Sawyer (2006) suggests that the teacher should lead the classroom in group improvisations rather than act as solo performer in front of the class audience. He or she creates a dialogue with the students giving them freedom to construct their own knowledge. According to Neyland (2004), a teacher has to amplify the contributions that are considered mathematically fruitful and bracket others that may not be so productive. However the outcome emerges through the group collaboration and is not predetermined by the teacher. It is the shared insight rather than the teacher or student that becomes the focus of attention (Davis & Simmt, 2003).
Martin *et al.* (2006) used the improvisational lens to analyse the construction of mathematical understandings by small groups. They suggest that this framework might be relevant to a whole class situation. While they recognize that it would be quite difficult for larger numbers of students to establish a 'group mind', they posit that not all participants have to be equally involved at any one time and, in this context, point to the entrances and exits of actors as a necessary part of improvisational theatre sketches. In this paper there is a description of a lesson that took place over two class periods and in which primary pupils constructed insight during a plenary session. The focus is on the way that the structure imposed on the task and the co-actions between individuals facilitated these insights.

Methodology

The aim of my research is to investigate the factors that contribute to the development of mathematical insight by primary school pupils. The methodology is that of 'teaching experiment' which was developed by Cobb in the context of the emergent perspective and in which students' mathematical development is analysed in the social context of the classroom (Cobb, 2000). Between April 2007 and June 2007, I worked with a class of thirty pupils, aged 10 - 11 years. There are eighteen girls and twelve boys in this class. The school is designated as disadvantaged. In my pilot study I had found that pupils were most likely to experience 'Aha' moments when they worked on investigations that had potential for generalization. I entitled the investigation outlined in this paper 'Count the Ways'. It is an extension of the 'Make Ten' problem described by Rowland (2000) and concerns the number of ways an even number might be formed as a sum of two positive integers[1]. The lesson took place over two class periods, each lasting approximately forty minutes. All phases of the lesson were audiotaped. When children were working in pairs, audio tape recorders were distributed around the room. Pairs of pupils were encouraged to write about their thinking in prepared work sheets. Each pupil maintained a reflective diary. Follow-up interviews were held with pairs of students, at least one of whom had shown evidence of reaching new understandings over the course of a lesson.

[1] If x is a positive even integer and f(x) is the number of pairs (a,b) where a,b ε N and a + b = x then, if a + b is considered to be equivalent to b + a and if zero in included as one of the addends, f (x) = ½ x + 1 or ½(x + 2) (Rowland, 2000)

Although the focus of this paper is on the plenary session that took place towards the end of the second period, I will describe other phases of the lesson as they played a significant part in creating the structure that facilitated the ‘group improvisational performance’. In the transcripts below, the following codes will be used, T: The researcher/teacher (myself); Chn: simultaneous contributions from two or more children - these different contributions are separated by the symbol >; Ch: a single child whose name could not be identified, otherwise pseudonyms are used. Three dots (…) represent a pause and comments are inserted in brackets.

Day One: Setting the Scene

I introduced the investigation by asking the pupils to give examples of ways in which numbers such as 2, 4 and 6 might be ‘made’[1]. Children’s suggestions were listed on a whiteboard. The following is an example of a discussion that took place after pupils had suggested 3 + 1 and 2 + 2 for the number ‘4’:

T:	We’ve got 3 + 1; 2 + 2, Sean?
Sean:	Ah, 1 + 3?
T:	Now, you could say…but because we’ve got 3+ 1…
Chn:	Oh, I know (*voice raised*)
T:	We have 1 + 3 because we don’t count the ones in which the order changes but good.
Chn:	Ah>teacher
T:	Ehh....
Ch:	4 + 0 (I write this on whiteboard)
Ch:	Ah
T:	Yes, anything else?
Chn:	No>no
T:	Is there any other way? Yes?
Ch:	Could you do 2 + 1 + 1?
T:	That’s a great example….I actually hadn’t thought about that one but in this particular example we are just looking at [times] when you add two whole numbers but, good, that’s really thinking…

Thus the pupils were familiarized with the rules of the investigation, i.e., the inclusion of zero as an addend, the principle of commutativity and the focus on pairs of numbers. Although the child’s suggestion of 2 + 1 + 1 might be accepted in a different kind of investigation[2], the above structure was given to stimulate the generalisation concerning addition of pairs of positive integers. It was designed to maximise the potential for creative output. This is akin to the provision of props for theatrical improvisation or the chordal structure that is decided in advance by a jazz ensemble. Worksheets in which pupils were asked to consider the number of ways that even numbers between 2 and 20 and also 100 might be made were then distributed (see Appendix A). Pupils worked on these in pairs and the class teacher and I circulated the room. When Cian and Lee had looked at the number of ways up to 20, I asked them what it might be for a hundred. Cian suggested that it could be 60 on the basis that there were 6 ways for the number 10. In other words, he was (incorrectly) applying a multiplicative pattern to establish the

[1] This was explained to the pupils as the number of ways an even number might be found through addition. The word ‘made’ will be used hereon for the purpose of succinctness.

[2] For example, investigation of the number of ways a positive integer might be written as a sum of positive integers. This corresponds with the classic notion of ‘partition’ in number theory.

solution. I asked him to think about a means of checking this conjecture. In the plenary session at the end of this class period, the numbers of ways to make even numbers up to twenty were listed on the whiteboard. I asked the pupils to predict how many ways there might be to make 100. Lucy suggested 55 on the basis that there were 11 ways for 20. Other numbers mentioned were 65, 99 and 12. As it was nearing break-time, I asked the pupils to think about that particular question overnight.

Day 2: Continuing the Performance

At the start of the lesson on the following day we revisited the findings of the previous day's lesson, i.e. a listing of the number of ways from 2 – 20, as follows:

Number	Number of ways
2	2
4	3
6	4
8	5
10	6
12	7
14	8
16	9
18	10
20	11

The pupils experienced little difficulty when I asked them what the number of ways for 22 and 24 might be. When we went on to consider 100, Ella suggested continuing with the list in their diaries, a suggestion that found favour with the other members of the class. They proceeded with this and although, there was a hum of discussion around the room at this time, any new understanding I observed seemed to emerge at an individual level. For example, Ella noticed that the numbers on the left-hand column were going up faster than those on the right-hand column. She had thought that the number of ways to make 100 might be 100 but now began to change her mind:

Ella: the way it goes up in even numbers, there are two like, there's 22 is 12, 24 is like another 1, so you are only on like tens like when you are on 20s, so it won't be a hundred ways, it would be like 200 would be a hundred…

As pupils completed the list, I encouraged them to think about a rule that could be applied to establish the solution for any even number. Cian correctly made a list of the number of ways but continued to apply the multiplicative pattern as he had done on the previous day:

T: So do you see any rule there?

Cian: 50 is 26.

T: So is there anything that will help you to see something there?

Cian: Yeah I saw that em that 40 is also eh 22 is like …20 is 11, eh 40 is...

T: 20 is 11, 40 is?

Cian: is 22

T: Is it? (21 has been listed in his diary)

Cian: Ah no, it's 21.

T: 21, right

Cian: Ah, no

T: Is there any rule you see there, just think about that, 20 is 11, 40 is 21…

Cian: Now, I'm stuck

During the plenary session at the close of this lesson, I continued the list up to 40 and then we again looked at 100. Most children who made the list from 2 to 100 found that there were 51 ways to arrive at 100 (see an example of this in Appendix B). However some children erroneously omitted one of the even numbers and thus arrived at 50 rather than 51 ways. There follows a transcript of the discussion after this error was noticed:

Ella: The first time I done my, em, my eh ways for doing it, I got 50 for a hundred cos I left out a number but I was just thinking that half the number there is 50…

Cian: You left out a number (*stressing 'a'*)

Ella: …so 2 like say in fractions is a half and then a half of a hundred is fifty

T: Ok, I am just going to ask you to do this, there's something very, very important here, you are on to something really important, right …Ok, if you look at the numbers here, 2,4,6,8, is it always half the number?

Lee: Yeah

Chn: No>no>no

Ella: No because it's the number that's next to them, like 4 and 3…1, 2, 3, 4

Ch: That's half a, ah no

Ella: Ah no..

T: Is it always…is it half of eight, half of ten?

Chn (Chorus): Oh>oh>Therese (*excitedly*)

Lucy: See the way you have eight take away five is three and then half of eight is four

Ella: Yeah, see the way six take away two cos 1,2,3 , Therese see there, it goes four take away one is three, so that's one…

Cian: Ohh (*voice raised*)

Ella: …and then six take away four is two, eight take way five is three, ten take away six is 4 and then 12 take way

T: Ok

Cian: Oh,oh,oh

T: You are onto something there

Ella: So that's…

Cian: Oh Therese, I know the quick way of doing it.

Voices in background

Cian: I know the quick way.

Ella: You see 1,2,3,4

Ch: Ok

Cian: Therese, I know the quick way.

T: Yes

Cian: See the way when you have half of ten, you get five, just plus one you have six, when you half twelve you've got eh six you plus one you have seven, so if you half a hundred you have fifty you plus one you have fifty one, I think that's how you…

T: What do you think, Lucy? (Lucy had hand raised but shakes head)

Cian: Cos if you half the number you just plus on one and…

Lee: Well done, Cian.

Cian: …like twenty, half of eh twenty is ten plus one is eleven so you have eleven ways of doing it

Lee: Cian got that right

Ch: Yeah, he did, didn't he?

T: Do you think it always works, do you think Cian's idea always works? He says..

Ch: I think..

T:	He's looking at a pattern and he says the answer is always half the number plus one
Chn:	yeah>yeah>

Ella began to see some connection with the half but when I drew attention to the list on the whiteboard, she questions her thinking. However, Lucy picked up Ella's idea of half but noticed this by looking at the difference between the pairs of numbers in the two columns. Ella then latched onto Lucy's observation of this difference while Cian built on the idea of a half further and discovered 'the quick way', i.e., the number of ways to make an even number is half the number plus one. This was immediately affirmed by Lee and another pupil. Ella then began to elaborate on the difference pattern she had found:

Ella:	I just notice another pattern as well, you know the patterns when we were doing twenty two take ten it goes two, four three one and that's , cos six take away four is two and it goes three, four, five..
(Cian in background singing: 'I got the pattern, I have a pattern')	
T:	You are looking at the differences?
Ella:	…the whole way down
T:	That's right, if you look at the differences you can, yeah, that would be…
Ella:	Cos two take zero and then four plus plus, four take away three is one,
T:	Yeah
Ella:	six take away four is two,
T:	Two
Ella:	eight take away five is three, ten take away six is four,
T:	Four
Ella:	twelve take away seven is five and then all the rest of the way down
T:	so you could just keep going that way, that would be another way that's another way of seeing the pattern
Ella:	When you are getting to 20 you start all over again

As I was drawing the lesson to a close, Roisin interjected:

Roisin:	I'm after noticing there. You know the way it has, you know the way the second number on the even side is four, well if you look on the ways the first one it's always half it, cos half of four is two…
T:	Hm, hm
Roisin:	…then half of six is three, half of eight is four (*ch joins in*)
T:	So you are looking at, can you come up and show me what numbers…you are looking at diagonal is it…ok so you are looking at half of six, half of eight, oh I haven't seen that before.
Leah:	That's good
T:	That's another one, so we have seen another pattern
Ch:	And half of ten is five
T:	So you are looking at the diagonal that way…great…

Roisin has now reworked Cian's idea and has established that the number of ways to 'make' an even number is half of the next even number (see Appendix B). This is affirmed by Leah and picked up by another pupil. The lesson now finished but during break time some pupils continued to search for patterns. Other pupils gathered around Cian to discuss his finding. Later I interviewed pairs of pupils (Cian and Kevin and also Ella and Amy) about the lesson.

Discussion

The transcripts from this lesson show that the improvisational lens is a useful means of analyzing collective understanding at whole class level. Although the plenary session that took place at the close of the second period was the main site for the emergence of insight, the preparatory work that preceded it was a significant factor in the generalization. It allowed pupils to become familiar and comfortable with the problem conditions. It also facilitated ‘possibility thinking’ (Craft, 2001) and led to a quest for a solution. Cian, in a follow-up interview, said he was excited at finding an answer ‘after thinking about it for a few days’. This has implications for the way in which a ‘lesson’ is perceived. The weekly time allocated to mathematics in 1st to 6th classes is three hours (DES/NCCA, 1999b) which means that the duration of class periods for the subject is approximately 35 minutes per day. Very often lessons are designed to fit these class periods and homework assignments are based on the topic covered. This leaves little space for the thinking time that can facilitate ‘Aha’ moments. In this regard, teachers might consider leaving a lesson ‘hang’ at the end of one period and continuing with it the following day. The maintenance of reflective diaries by pupils may encourage them to give consideration to the topic in the interim.

One of the difficulties of analysing group understanding is to determine the extent to which all members of the group have advanced their mathematical thinking. This difficulty is exacerbated in the context of a large group such as the one discussed in this paper. There was evidence that co-acting took place in this lesson. In particular, Lucy picked up the thread of Ella’s input. This was elaborated by Cian and reworked by Roisin. Cian noted this in his diary entry:

> When I got it I was very excited. The way I got it was I heard Lucy saying half the number and then it came to me.

Although Cian is aware that he has advanced his own mathematical understanding he does not take full responsibility for finding the solution. He recognised that understanding has emerged at a collective level. There were certainly some ‘main’ players who made several contributions over the course of the lesson. Cian and Ella were among these. Roisin, on the other hand, seemed to be observing from the sidelines and made her entrance when she felt it was appropriate. The fact that there were other observers of the action is evident from the affirmation that was given to both Cian and Claire when they gave their solutions. It is interesting to note that their solutions are those accepted by the mathematics community and the pupils seemed to recognise their elegance. There may have been other pupils who decided to exit completely during the final plenary session. This may have been due to apathy or to the challenge imposed by ‘finding the rule’. What is interesting is that, at break-time, some pupils who did not take part in the final discussion talked to Cian about his finding. While these individuals may not have grasped the generalisation themselves, Cian’s obvious excitement at making a mathematical discovery seemed to have an impact on them. What is evolving is a group understanding of what it is to come to know mathematics: ‘an exhilarating ride – at times risky, infused with uncertainty, at other times replete with ‘approaching completeness’’ (Rowland, 2000: pp. 117 - 118). In subsequent lessons other students took central stage.

These mathematical insights occurred in a school designated as disadvantaged. The children who constructed them are not regarded as having high mathematical attainment on the basis of standardized test results. This suggests that these results say little about the capacity of these pupils to engage in higher order mathematical thinking and to experience ‘Aha’ moments in the subject. Other broader forms of assessment would paint a fuller picture. It might also be the case that positive experiences of mathematics would impact on the standardized test results. It is true that the facilitation of such lessons is demanding of teachers. It has implications for their mathematical knowledge and for their understanding of what is worthwhile in mathematics teaching and learning (see, for example, Corcoran, Rowland, Turner, this volume). It means that mathematics lessons cannot be guaranteed to be 'quiet, busy' times. However, in today's society where there are increasing demands for creative and flexible thinking, there is a need to give consideration to approaches that allow mathematical ideas to 'stumble across each other'.

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Appendix A: Pupil's worksheet

Count the Ways 1

Names: ______________________________

A. How many ways are there to write 2 as an addition sum?

It can be written as 2 + 0 and as 1 + 1. There are 2 ways.

B. How many ways are there to write 4 as an addition sum?
Write the different ways here:

There are ___ ways.

C. How many ways are there to write 6 as an addition sum?
Write the different ways here:

There are ___ ways.

D. How many ways are there to write 8 as an addition sum?
Write the different ways here:

There are ___ ways

E. How many ways are there to write 10 as an addition sum?
Write the different ways here:

There are ___ ways

F. How many ways are there to write 12 as an addition sum?
Write the different ways here:

There are ___ ways

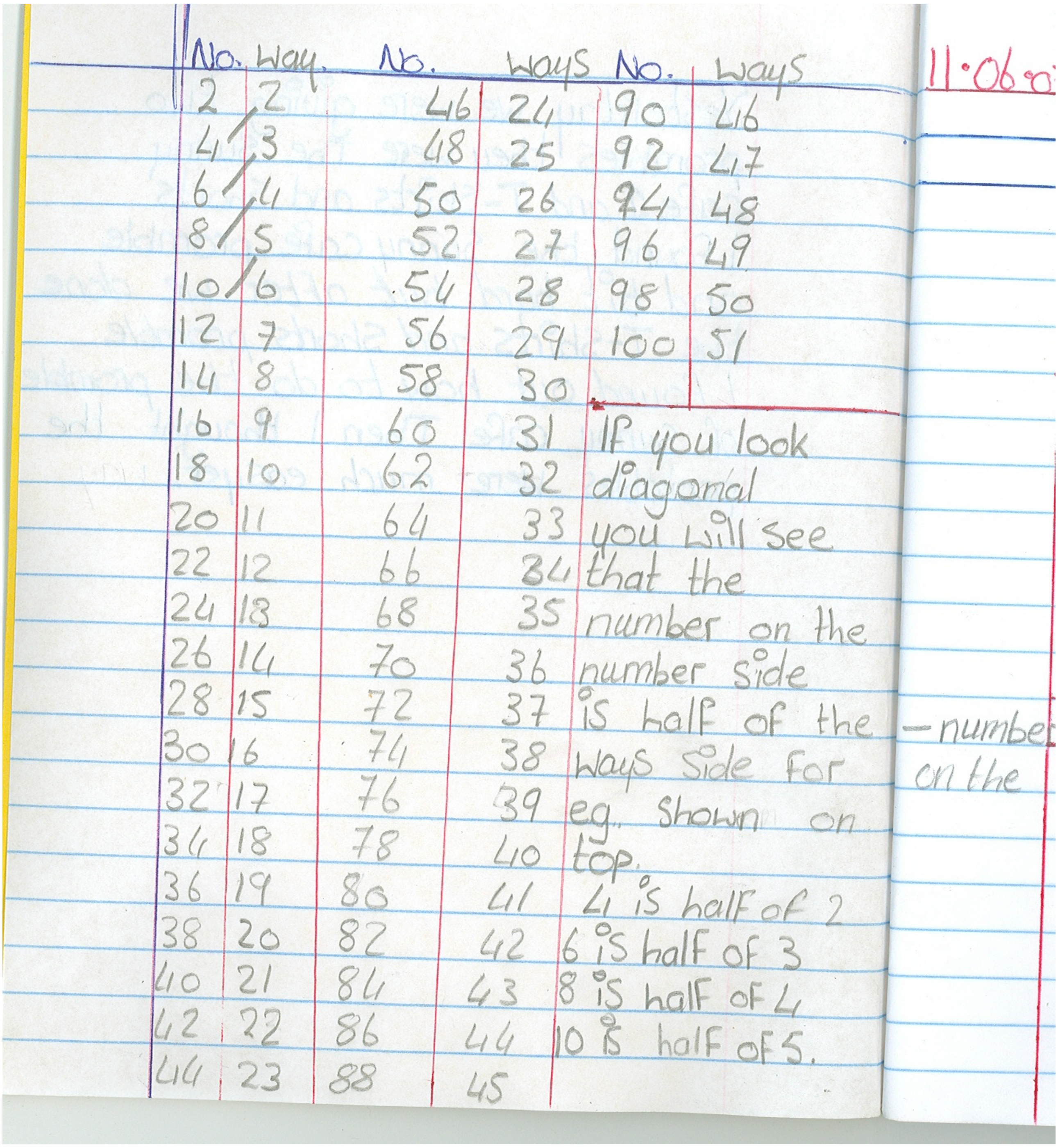
How many ways are there to write 14? 16? 18? 20? You can write it in a table like this:

Number	Number of ways to write as addition sum
2	
4	
6	
8	
10	
12	
14	
16	
18	
20	

Do you notice a pattern?

How many ways are there to write 100 as an addition sum?

Appendix B: Roisín's diary entry



11·06·0

No.	Way.	No.	Ways	No.	Ways
2	2	46	24	90	46
4	3	48	25	92	47
6	4	50	26	94	48
8	5	52	27	96	49
10	6	54	28	98	50
12	7	56	29	100	51
14	8	58	30		
16	9	60	31		
18	10	62	32		
20	11	64	33		
22	12	66	34		
24	13	68	35		
26	14	70	36		
28	15	72	37		
30	16	74	38		
32	17	76	39		
34	18	78	40		
36	19	80	41		
38	20	82	42		
40	21	84	43		
42	22	86	44		
44	23	88	45		

If you look diagonal you will see that the number on the number side is half of the ways side for eg. shown on top.
4 is half of 2
6 is half of 3
8 is half of 4
10 is half of 5.

— number on the

The Problem-Solving Strategies of Fourth Class Children in a Designated Disadvantaged School

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This research study focused on the strategies children used to solve non-routine mathematical problems. The problems were based on a thematic approach influenced by the principles of Realistic Mathematics Education. The children were not given prior formal instruction on problem-solving strategies or models. It emerged that children of all abilities were capable of solving non-routine mathematical problems, but that the author and class teacher had a vital role to play in maximising success, particularly in the early stages. As the study progressed children worked more independently, relied less on concrete materials and produced a wider variety of more sophisticated and refined strategies. Perhaps the most significant finding was that the normally lower-achieving children performed beyond expectations, often outperforming their higher achieving classmates.

Introduction

The purpose of this study was to investigate children's problem-solving strategies in mathematics. The research was initially influenced by the author's personal concerns about pupils' problem-solving abilities; in particular their dependence on teachers during problem-solving and their general lack of interest in word problems. Problems in standard textbooks very often only provide computational practice. This research allowed pupils the opportunity to experience an alternative type of mathematical problem-solving, which would be more realistic, motivating and relevant to their lives. This paper provides the rationale for the research and subsequently explains how the study was carried out. The paper presents the findings of the research and discusses their possible implications for school planning and policy.

Background

Problem-solving is central to the aims of the 1999 revised Primary School Curriculum for Mathematics, which advocates (DES, 1999a, p.13) that mathematics should "develop problem-solving abilities and a facility for the application of mathematics to everyday life" and enable the child to "plan and implement solutions to problems, in a variety of contexts". However the issue of whether children in Irish schools are successful mathematical problem-solvers is debatable. There is increasing concern that a significant number of children may not be equipped with the skills necessary to tackle realistic mathematical problems. The National Assessments of Mathematics Achievement (NAMA) held in 1999 and 2004 found that pupils performed poorly on items assessing higher-order thinking skills, including problem-solving (Shiel and Kelly, 2001; Surgenor, Shiel, Close and Millar, 2006). In 2004, the mean score for problem-solving was slightly lower than the previous assessment in 1999, whilst there was some improvement in the five remaining mathematical skills. The 2004 assessment also involved examining the responses of principals, teachers and inspectors in relation to the teaching of fourth-class mathematics. Inspectors reported being least satisfied with the teaching of higher-order skills, recommending "more systemic and explicit approaches to teaching problem solving, and less emphasis on textbooks in favour of activities designed to develop the language of mathematics and to apply mathematical concepts to real-life context". (Surgenor et al, 2006, p.35).

Mathematical achievement and disadvantage

The 1999 and 2004 National Assessments also found that children in designated disadvantaged schools achieved a significantly lower mean score than pupils in non-disadvantaged schools. These findings are supported by the study of Literacy and Numeracy in Disadvantaged Schools (DES, 2005b), which indicated that almost two-thirds of pupils in the most disadvantaged schools achieved at or below the twentieth percentile on standardised tests, compared to the one-fifth national average. This finding is supported by the standardised test results of the participating class. Seventeen of the twenty-four pupils scored below the twentieth percentile on the SIGMA-T standardised test administered in June 2006, with four of the remaining seven children scoring between the twentieth and thirtieth percentiles.

Problem-solving in textbooks

The 1999 revised Primary School Curriculum emphasises a shift from textbook-dominated to constructivist-based lessons, which favour practical and mental mathematics activities over traditional pen-and-paper activities. However NAMA 2004 found that ninety-five percent of teachers continue to use the mathematics textbook *every day* during lessons. If this is the case it raises concerns over the type of problems with which children are presented. Mathematics textbooks, particularly at middle and senior level are dominated by blocks of word problems which typically appear at the end of a chapter. Such problems generally require the child to select and apply the appropriate computational operation. New concepts and skills are contextualised, in an attempt to make them more relevant and realistic for the child. However such pages often fail to motivate. The amount of language may be off-putting. Problems are often poorly phrased, unchallenging and irrelevant to the child. For example it is unlikely a child would be motivated to solve the following problem which appears in the textbook used by the project class:

> There are 235 pages in an English book.
> (a) How many pages are in 39 English books?
> (b) How many pages less than 9500 is that? (Barry et al, 2002, p.110)

With such continued dependency on textbooks, it is questionable whether pupils in Irish schools are experiencing a wide variety of problem types which give them the opportunities to develop higher-order thinking skills as required by the curriculum. Schoenfeld (1992, p.343) warns that continuously providing typical textbook problems to students can have harmful long-term implications for the student's mathematical competence. He warns that children can develop and accept a passive role to the learning of mathematics, believing that mathematics is a body of skills and procedures to be passed on from teacher to pupil and that mathematical problems should be quickly solvable by applying a readily available solution.

Non-routine problems

This research engaged children in a two-month intervention, during which the children participated in weekly hour-long problem-solving sessions. During each session the children solved non-routine problems. Non-routine problems differ from routine problems in that the method of solution is not immediately obvious. Schoenfeld (1992, p. 357) describes them as "problems for which there is no standard algorithm for extracting or representing the given information". In some circumstances, there can be several correct solutions. The underlying principle of such problems is that they lead the child towards developing a more advanced level of mathematical thinking. Such problems are supportive of Hiebert's (1998, p.142) understanding of rich mathematical problems as those which "engage student's curiosities and provide opportunities to wrestle with important mathematical ideas…[and] are often non-trivial, multifaceted and solvable using a variety of strategies".

Non-routine problems not only allow children to recall and apply mathematical skills, but they can also enable children to construct new mathematical knowledge and make important links between mathematical concepts and processes. They can encourage stimulating and productive discussion amongst children, which in turn can benefit the development of communication skills and mathematical language and allow children to come to appreciate the different ways in which mathematical problems can be solved.
To increase the level of motivation the problems used for this research were based on a theme, reflective of the principles of Realistic Mathematics Education (RME). The theories of RME represent a radical shift in thinking about how mathematics should be taught and how much involvement children should have in their own learning. DeCorte (1995, p.42) explains that RME "conceives mathematics as a human activity focused on problem solving and construction of meaning". The use of RME methodologies allow children to be actively involved in constructing mathematical knowledge and principles, rather than passively learning rules and procedures as was characteristic of traditional didactic approaches to the learning of mathematics.

Problem-solving strategies
This research placed major emphasis on what children could do without prior formal instruction on problem-solving strategies. While the mathematics curriculum strongly endorses a constructivist learning approach, it also recommends children be *taught* a variety of problem-solving strategies. (DES, 1999b, p.36) Several authors on problem-solving (Wheatley: 1983, Duncan: 1993, Jones, C: 2003, Foster and Ankers: 2004) suggest the teaching and modelling of specific problem-solving strategies (e.g. draw a diagram, make a list) rather than allowing children to construct their own. However Schoenfeld (1992, p.352) warns that if children do not have a deep understanding of the principles underlying their mathematical procedures, they are more likely to misuse or forget them. This theory is supported by Roh (2003, p.2), who adds that students' creative thinking skills can be stifled by instruction. Lester (1994, p. 666) reports that one of five major findings from problem-solving research up to 1994 was that "teaching students about problem solving strategies and heuristics and phases of problem solving…does little to improve student's ability to solve mathematical problems in general". Other researchers and authors such as English (1993), Trafton and Midgett (2001) and Windsor (2003) argue against the formal teaching of strategies, agreeing that problem-solving skills, such as strategy development, are learned naturally through solving problems. Such authors promote a problem-based approach to learning. The principles of constructivist learning theories are evident in their suggested teaching approaches; their methodologies are based on the central belief that children will use and build on existing knowledge and skills in order to solve a problem.

Aims of the study
This research particularly sought to investigate and provide information on the following research questions:

- What strategies do fourth-class children use when solving non-routine mathematical problems?

- Do children's strategies change over the course of a problem-solving intervention?
- Does the problem-solving performance of higher-achieving children differ from that of lower-achieving children?

Methodology

It was decided that qualitative research techniques were best suited to the needs of the research project. Such techniques included analysing interviews, observational field-notes and video-recordings. During a two-month period from February to March 2007, non-routine problems were used weekly with fourth-class pupils in a school designated as disadvantaged in a suburb of Dublin, in which the author worked as resource teacher for mathematics. The children were observed as they worked in pairs on the non-routine problems. The problems were based on the theme of the popular 2006 Disney production "High School Musical"; the use of the thematic approach influenced by the principles of Realistic Mathematics Education, whereby children are actively involved in solving mathematical problems in a context which is either real to them or can be easily imagined. The theme of High School Musical was suggested by the class teacher, as the children showed great interest in this film. Non-routine problems were then sourced and adapted to suit the musical theme. The problems were primarily sourced from Foster and Anker's (2004) *Can Do Problem Solving* resource book for teachers. Problems were also sourced from various problem-solving journal articles. They were carefully chosen on the basis of their cognitive requirements and learning potential. Consideration was given to the required strategy and efforts were made to choose a wide variety of non-routine problems.

The Participants

Twenty-four fourth-class pupils, all female, participated in the intervention. Four children were selected by the teacher to become the focus group for the study. The teacher regarded two of these children as high mathematical achievers and two as low-achievers. It should be considered that the teacher was gauging their abilities in relation to the performance of the rest of the class. In fact the two higher-achieving pupils, Natasha and Joanne, only scored at the 32nd and 25th percentiles respectively on their most recent standardised test, the SIGMA-T, conducted in June 2006. In the same test the two lower-achieving pupils, Tara and Katie, scored at the 5th and 14th percentiles respectively. In other words the two higher-achieving pupils may not necessarily be considered high-achievers on a national scale, particularly in non-disadvantaged areas[1]. Preliminary interviews held with the class teacher and the focus group revealed that the class were generally very dependent on their teacher when solving mathematical problems and their typical reaction to word problems was to ask someone for help. The teacher reported that generally when the children see word problems, they are instantly put off, often saying, "I can't do them. What do I have to do? Do I add, subtract, divide or multiply?" The class had not previously worked with non-routine problems.

Procedure

Each week the author video-recorded the focus group working on a new theme-based non-routine problem in the maths resource room, separate to the rest of the class. The

[1] For the remainder of this paper, the two pairs of pupils will be referred to as the 'higher-achievers' and 'lower-achievers' respectively.

higher-achievers were filmed at a different time to the lower-achievers. By video-recording the focus group working on the problems, the author was able to observe differences and similarities in the problem-solving approaches of lower and higher-achieving children. On those same days, the author facilitated a problem-solving session with the remainder of the class. The author was concerned that language and reading difficulties might prevent children from progressing and create feelings of negativity towards the problems. Therefore before the children started working, they had the opportunity to ask about new or difficult words.

The author and the class teacher circulated the room offering support and assistance. Children had access to a variety of resources and manipulatives. Each session focused on the children working independently, developing their own strategies and justifying their reasoning. The class teacher and author acted as facilitators, encouraging discussion and equal participation. They did not suggest strategies to the children. When possible the author documented her observations, which were later typed as field-notes. The class teacher's comments were also recorded. Each session concluded with a whole class discussion, where the sharing of ideas and strategies was central. A home problem was then distributed. To build confidence the home problem generally required the application of the strategy used earlier that day.

Towards the end of the intervention it was recognised that the video-recordings combined with field-notes had provided a wealth of data. It was therefore decided that analysis would focus on the children's progress on three particular problems; "Musical Instruments", "Dance Routines" and "Practice Days". The three problems, included in Table 1, were selected as representative of different types of non-routine problems. They each required the recollection and application of different aspects of mathematical knowledge, concepts and skills. The focus group and class were subsequently asked to solve three non-routine problems, unrelated to the musical theme, but each of a similar style to the three problems named above. Table 1 shows the problems which were used. For example "Book Giveaway" was used to help assess the retention of knowledge and skills gained in "Musical Instruments", which had been solved weeks earlier.

Table 1: Non-routine problems chosen for analysis

Theme-based problem	Follow-up problem
A: Musical Instruments Ms. Piper asks Siobhan to carry the tambourines from the storage room to the stage for music practice. On the way she has to drop off some of the tambourines to different classes. She gives half of them to 3rd class. She drops off 15 to 4th class. When she reaches the stage she has thirty tambourines left and she gives them to Ms. Piper. How many tambourines did she have when she left the storage room? **(90)**	**A1: Book Giveaway** Bethany has won a €100 voucher for the local book shop, so she decides to give away all her old books to children who need them. She gives half of them to the Children's Hospital. She sends a box of 12 to the junior infant class in her school. She gives 8 to her next door neighbour's son. Finally she gives 2 books to her little baby brother. How many old books did Bethany give away altogether? **(44)**
B: Dance Routines There are six girls taking part in one of the dance routines. During the dance each girl must pair-dance with every other girl once only. How many pair-dances will take place altogether? **(15)**	**B1: Six Nations** The Six Nations Rugby Championship has started. Six teams are taking part: Ireland, Scotland, Wales, England, France and Italy. Each team must play each other team once. How many games will be played during the Championship? **(15)**
C: Practice Days The girls practise for the musical between March and April. Every ten days dancers are called for practice. Every fifteen days singers are called for practice. Every six days actors are called for practice. If they all started together for a big practice on the 1st of March, on which date would they next be all together for a practice? **(March 31st)**	**C1: Dublin Airport** Dublin airport is a very busy place. Every 10 minutes the airport bus leaves for the city. A helicopter takes off every 12 minutes. An aeroplane takes off every 6 minutes. If they all started working at 8 a.m., when would be the next time a bus, a helicopter and an aeroplane would set off together? **(9am)**

Children's Approaches to Problems C and C1

Within the limitations of this paper it is not possible to describe how the children approached all of the above problems. Therefore a description of how the children approached Problem C is provided. This problem has been selected as it showed evidence of positive progression in the children's level of independence, persistence and strategy development. During this session, the teacher commented that the children were more focused and less reliant on her help. All pairs managed to solve the problem within the time available, which was a first since the intervention began. The class also generated a variety of more refined strategies, which included various diagrams and the use of a table for the first time. Significantly, the children who attended Learning Support for mathematics contributed more useful suggestions and actually reached solutions before others in the group

Strategies used by the class to solve the problem "Practice Days" (see Table 1)

The most common approach taken initially by the class was to add the fifteen, ten and six, thus totalling thirty-one and suggesting that the thirty-first of March was the answer. Coincidentally this was in fact the answer, but the real concept behind the problem had been overlooked. Through whole class discussion, the children were encouraged to revisit the problem and imagine the situation, the author emphasising the words "*every* ten days…*every* six days…*every* fifteen days". Gradually the idea that the problem needed further working began to dawn on the children and they realised a simple sum wouldn't suffice. Notably a child (Roisin) who attends Learning Support for maths really helped progress the problem; she was the first to openly realise that if the dancers started on the first of March than they would be back again on the seventh, then the thirteenth and so on. This helped to 'kick-start' other pairs. Natalie admitted to the class teacher; "I just added the three numbers until I heard what Roisin said and then I knew there must be more to it". After that the children chose different ways to figure out the answer, but all used a written approach.

Michelle and Jane (who both attend Learning Support) were first to finish. They drew a diagram using numbers and arrows. Three horizontal lines under each other displayed when the dancers, singers and actors needed to come to practice:
DANCERS: 1st ➔ 7th ➔ 13th ➔ 19th ➔ 25th ➔ 31st
SINGERS: 1st ➔ 16th ➔ 31st
ACTORS: 1st ➔ 11th ➔ 21st ➔ 31st

The girls saw that the 31st was the first date to appear in all three lines and reasoned that this was the answer. Other pairs represented the information differently. One drew a concise table, using numbers rather than dates. This was significant as it was the first use of a table since the intervention began.

Dancers	Singers	Actors
1	1	1
7	16	11
13	31	21
19		31
25		
31		

One pair solved the problem by representing the pattern in sum format using plus and equals symbols: 1+6= 7+6 = 13+6 = 19+6 =25+6 = 31
Another pair solved the problem in a rather laborious way, by writing it all as a story using full sentences for all the dates on which practices were held. However this method possibly helped to clarify the problem for them:
The actors will come back on the 7th of March, then on 13th of March, then on the 19th of March and then on the 25th of March and then on the 31st of March.
A small number needed more specific help on getting started and devising a strategy and were more dependent on the teacher. Children who finished quickly were further challenged to calculate the next dates on which the musical groups would once again meet for practice. Three pairs subsequently managed to reach this stage, solving for practices in April and May.

Strategies used by the focus group
The focus group experienced the same initial difficulties as their classmates, in that they began by adding the fifteen, six and ten. They failed to understand that the practices were happening simultaneously, reasoning that when one group finished practising, the

next group would start. The author needed to emphasise what the problem was asking, particularly with Katie and Tara (lower-achievers) who struggled to understand what was being asked:

Researcher:	So what we're trying to find out is ….. If they follow this routine every 10 days, every 10 days, every 10 days, and the singers are every 15 days, every 15 days, every15 days, and the actors every 6 days, every 6 days, every 6 days…..
Katie:	(interrupts) so you're counting in your 10's, 15's and 6's.
R:	Excellent. So we're trying to find out when they will arrive back together.

(Pauses- Children still appear confused about what the problem is asking)

R:	The problem tells us there will be a date when they will all arrive back in the hall together. When will that date be?
Tara:	You have to add up 10 and then 10, 15 and then 15, and then 6 and 6.

It took much discussion and questioning as above to bring the children to an understanding of what was being asked. The girls devised a strategy unlike any other used in the class. They represented the month of March on a page with dates (similar to a calendar) and circled the practice dates for the singers, actors and dancers. Even during this process, misconceptions were evident:

K:	On the 1st, actors are acting, singers are singing, dancers are dancing. Then 6 days later the actors are back, 10 days later dancers are back, 15 days later singers are back, then 6 days after that the actors are back.
R:	It's every 6 days the actors come back. See, the actors don't really have to worry about when the singers and dancers are back. They just know they have to be there every 6 days. So let's have a look at when the actors go.

By circling the dates on which the actors came back (7th, 13th and so on), the girls gradually began to understand what was being asked. They used the same list of dates to circle the dancers' and singers' practices, writing A, D and S over the dates to differentiate between the groups and finally identified the thirty-first as the date common to all three groups.

After some initial misconceptions, the higher-achievers made sense of what was being asked. While they understood the concept that there were simultaneous practices, Natasha ignored the fact that dates were involved and proceeded to list multiples of 10 up to 150, 6 up to 72 and 15 up to 150. At this point Joanne intervened:

J:	Wait. I know I should have told you this earlier but it's dates and there's no 36! There's only 31 days in a month.
N:	Oh yeah. I never thought of that. What will we do now?
J:	Start again.

Natasha begins to start the list of 10's again.

Researcher:	What date did they all have their first practice on?
N:	The 1st of March
R:	So 10 days later they have to come back again…so when would that be?
J:	Oh yeah! The 11th, so it's 1 and then 11 and then……21.

N writes 1 then draws a hopping arrow, and then writes 11, then an arrow, then 21 as below

1 ➔ 11 ➔ 21 ➔ 31, then stops.

R:	Why have you stopped, would you not go to 41?
J:	No because there's only 31 days in March.

Progress from this stage was straightforward, with some simple questioning from the author; the girls solved the problem efficiently using three horizontal lists.

Retention of strategies

Four weeks later, Problem C1, At the Airport, was used to help assess children's retention of the strategies used for Problem C. This problem had similar requirements to Problem C, though perhaps slightly more difficult due to the fact that knowledge of time was necessary.

Comments were made comparing this problem to the "one with the practice days". One pair's reaction was to first add the six, ten and twelve, but the rest of the class was able to recall the method used to successfully solve the Practice Days problem, which was to generate a horizontal or vertical list of the times. The problem required the children to recall the multiples of 6, 10 and 12. Of those who chose to use horizontal listing, most used hopping arrows between the times.

8.00 ➔ 8.06 ➔ 8.12 ➔ 8. 18 ➔ …

The children understood the concept of the problem, which was to identify a common time in all three lists and they worked efficiently until they reached the solution. The success rate was high and the time spent on the problem short in comparison to Problem C. Some minor errors were made as digital time was incorporated into the problem.
Similarly the focus group realised that the problem was similar to Problem C, Natasha commenting "Ah! This is like the one where they all go back for different dates of rehearsal". However both pairs still displayed the same initial misconception, which was to add the 10, 12 and 8. When the author encouraged the girls to think of how they had solved Problem C, Natasha exclaimed "Oh, now I know how to do it", while Katie suggested "We could do it the same way as we did the other one, like start at 8 and go on 10, go on 10, go on 10".

Natasha and Joanne (higher-achievers) made several unsuccessful attempts to solve the problem mentally. It was only with encouragement from the author that they decided to record the different times on paper. Using three vertical lists to represent the planes, buses and helicopters they solved the problem. Katie and Tara (lower-achievers) needed no encouragement to document their work. They independently chose to make three horizontal lists and solved the problem relatively easily, with only some minor confusion about multiples of six.

Findings and Discussion

This section of the paper discusses the primary findings from the research and simultaneously suggests how they may have implications for school planning and policy.

The study found that children of all abilities could successfully solve complex mathematical problems without prior formal problem-solving instruction. Children were capable of generating their own problem-solving strategies and most were able to retain these strategies and use them to solve similar problems weeks later. As the intervention progressed manipulatives were used less often. Diagrams and other written approaches such as lists dominated children's approaches. It was evident that children were beginning to refine their strategies and were becoming less concerned with the surface features of problems and more concerned with the structural features. For example, one of the early problems entitled *Costume Design* involved combining different items of clothing to find the total number of unique costumes. Many children chose a time-consuming approach, drawing detailed leggings, t-shirts and eye-masks. However as the intervention progressed they began to represent elements of problems in more efficient ways, using symbols, colours and initials. It was evident that children were spending more time planning their approaches to a problem and recording their progress in more logical ways.

Such findings imply that whole-school planning for mathematics should be centred on constructivist teaching methodologies, which emphasise active rather than passive participation by children of all abilities. Mathematics lessons need to be structured so as to allow discovery learning to occur. Children need to be provided with opportunities to discover mathematical concepts by themselves, while supported and challenged by their teacher. The author and class teacher had a vital role to play, particularly during the first month of the intervention. In the initial stages they focused on encouraging the children to persist at the problems and to evaluate their answers. Later in the intervention, as the children showed more persistence and independence, the author and class teacher had more time to listen to the children explaining their thinking and to challenge them further. Their use of questioning and discussion was important in maximising the full learning potential of each problem. Both Jones (L., 2003, p.89) and Schoenfeld (1992, p.354) emphasise that knowing how to use this type of problem-solving can be challenging for teachers. It is reasonable to suggest that teachers may be unsure of how best to support children during the problem-solving process - how much or how little to say and what type of questioning to use. Professional development courses may assist teachers in implementing this approach to problem-solving in the classroom. This suggestion is supported by Surgenor et al (2006, p.41), who recommend that as part of in-career development, teachers should be supported "in implementing approaches to higher-order mathematics skills including Applying and Problem Solving".

At first, children showed little evidence of self-evaluation skills. In the early sessions they were totally reliant on the teacher and author to tell them if they were right or wrong. By consistently asking them to reason and support their solutions, they gradually learned that they could use the problem to check their solutions. Their level of persistence and independence improved. They realised that there was a process involved in each problem and were less likely to make random guesses. The majority of the class were consistently able to spot similarities between problems and thus they drew on previous knowledge and skills to help solve new problems. This finding implies that more time could be dedicated during mathematics lessons to allowing children opportunities to explain and justify their reasoning. Whilst the role of the teacher has been recognised as crucial to the success of the non-routine problem-solving approach, so too is the role of discussion amongst the children themselves, the value of which cannot be underestimated. The role of discussion is a crucial element of the constructivist approach advocated by the curriculum (DES, 1999a, pp.3-4). The participating children benefited from listening to each other's suggestions and explanations. This finding implies that children be given frequent opportunities during mathematics lessons to participate in peer discussion, so that they can come to value and respect each other's mathematical approaches, opinions and ideas.

The use of the High School Musical theme was highly motivational and helped to sustain a high level of interest throughout the eight-week intervention. The children, their parents and the class teacher expressed positive comments on the connection that was made between mathematics and real-life. This finding suggests that teachers could benefit from adopting more integrated approaches to the teaching of mathematics over the school year, by basing learning around areas which interest the children such as sporting or music events and by integrating mathematics with other subjects.

The study found that ability differences between higher and lower-achievers were less apparent during non-routine problem-solving. The non-routine problems promoted a high level of inclusion. Children of all abilities could fully participate and were highly motivated by the problems. In fact lower-achieving children often worked at a more practical level, showed more persistence and outperformed their higher-achieving peers. Video transcripts of the focus group at work allowed the author to observe certain characteristics in the problem-solving behaviour of lower and higher-achievers. Generally, higher and lower-achievers shared the same misconceptions in a problem. Higher-achievers seemed more concerned with right answers and how quickly they solved a problem. Natasha (higher-ach.) frequently questioned how Katie and Tara (lower-ach.) had succeeded on a problem and how long the problem had taken them, whereas the reverse never occurred. Natasha and Joanne often spent a long time trying to solve the problems mentally, whereas Katie and Tara were more likely to keep a written record of their progress and this worked in their favour. Katie and Tara were also more likely to try out a new approach if the first was unsuccessful. Such findings suggest that the ability gap between higher and lower-achievers may not be as wide as indicated by standardised tests. Schools therefore, need to be aware of the limitations of standardised tests and use multiple methods of assessment in order to measure a child's actual mathematical ability and potential. The Primary School Curriculum for Mathematics (DES, 1999a) recommends using alternative methods of assessment as well as the standardised test, such as teacher observation, teacher-designed tasks and tests, work samples, portfolios and projects, curriculum profiles and diagnostic tests.

Conclusion

Although this study was of a small-scale nature, the author believes that long-term regular use of realistic non-routine problems could help to boost children's mathematical achievement and help to foster positive attitudes to mathematics. This is particularly important in disadvantaged areas, where mathematical achievement levels continue to be disappointing. The author and class teacher involved in this study were genuinely surprised by what the participating children achieved over the course of this intervention. In particular, children who had previously scored poorly on standardised tests performed beyond expectations and were highly motivated by the problems. Children frequently asked for extra problems to bring home and some began to write their own problems. Parents too became involved and reported that they enjoyed working on the problems in the evenings with their children.

The theme of "High School Musical" was very important in arousing and sustaining the children's interest in the problems. School planning in the future should begin to embrace alternative approaches to the teaching of mathematics such as Realistic Mathematics Education. However, if such approaches are to be successfully incorporated into Irish schools, teachers must firstly examine and determine the role of textbooks in their lessons and be willing to make the transition from textbook-orientated to constructivist-based lessons.

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An Optimum Maths Program for Junior Infants in Designated Disadvantaged Settings

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The mathematical attainment of pupils from disadvantaged backgrounds, both internationally and in Ireland, gives cause for concern. Typically, the mathematical performance of these pupils is less than their performance in reading, both of which compare unfavourably with the average achievement of their peers. There are grounds for optimism, however. Pupils from "at-risk" backgrounds show enhanced performance when they have the opportunity to attend high quality preschool programs and when their formal schooling takes place in small classes. In Ireland the first year of formal schooling presents a unique opportunity. In many disadvantaged areas children of preschool age, namely five or under, attend school, usually in small classes. The current project represents a prototype of an optimum way to deliver the Mathematics component of the Revised Curriculum to these Junior Infants. It consists of two activity based, constructivist mathematics programs, Number Worlds and Big Maths for Little Kids. The project was implemented in six Junior Infant classes during the school year 2005-2006. Data obtained enabled comparisons to be made between pupils' performance at the beginning and at the end of the school year. It was also possible to compare some aspects of performance with that of comparable pupils receiving regular mathematics instruction.

Introduction

My interest was originally in the assessment of young children's learning. It quickly became apparent, however, that assessment, learning and teaching are inseparable processes. Moreover, optimum teaching should enhance pupil motivation, as well as pupil learning (Tschannen-Moran et al., 1998). Consequently, my research evolved into an investigation of how best to jointly facilitate the development of young children's understanding of, and enthusiasm for mathematics.

I chose to focus on young children attending schools designated as disadvantaged, for a number of reasons. Firstly, there is considerable evidence, both internationally and in Ireland that the scholastic attainment of such pupils is low, especially in the area of mathematics. Young children from lower SES backgrounds perform less well than their peers from other SES backgrounds on a range of mathematical tasks, including addition, subtraction and complex counting (Ginsburg and Russell, 1981), concrete addition (Hughes, 1986) and mental number line tasks (Case and Okamoto, 1986). In Ireland, inspectors from the Department of Education (2005) noted that two-thirds of pupils in designated disadvantaged schools were extremely weak at mathematics. Such low attainment in the area of mathematics is somewhat intriguing given the mathematical competence observed in very young infants. For example, babies determine how many objects are in a small set (Starkey and Cooper 1980) and can engage in a basic form of addition and subtraction (Wynn 1992). More recent research indicates that infants can do much more than simply count: they can match the number of human voices with the number of people seen in a video display. (Jordan and Brannon, 2006). Given this apparent hard wiring for mathematics observed in infancy, the performance gap is intriguing, This conundrum of early mathematical competence followed by later low attainment is not helped by teachers' beliefs about their own capacity to teach the subject. This belief, called teacher efficacy, has been defined as "teachers' belief or conviction that they can influence how well students learn, even those who may be difficult or unmotivated (Guskey and Passaro, 1994). Teachers judge themselves less capable of influencing pupil learning in mathematics than in literacy (Bandura 1997).

Finally, the assessment of learning in young children has not been well researched. This is partly a reflection of the challenges associated with assessment generally, one of which relates to the relationship between children's underlying knowledge and their performance on tasks designed to assess that knowledge. Whereas this is a difficulty with all forms of testing, it is particularly so in the case of young children, whose cognitive development is fragile and uneven. Furthermore, young children can be difficult to motivate, especially when the reason for assessment is not apparent to them. Factors such as these may result in a child performing "poorly, adequately or extraordinarily" (Bandura, 1993), despite having the same knowledge and skills.

Project Origins

The introduction to the revised Primary School Curriculum (1999) notes that appropriate, special interventions should be introduced in Junior Infants, where there is reason to believe that the level of development reached by some children, at school entry, may not reflect their potential.

Obviously, any such intervention must be compatible with the constructivist approach to teaching and learning underpinning the revised curriculum. The corner-stone of this philosophy is that children construct new knowledge by actively engaging in tasks that are meaningful and motivating. The curriculum notes the importance of children manipulating real-life materials during learning experiences that are informal, play-based and characterised by an explicit emphasis on oral language.

While it is easy to embrace constructivism in principle, it is not always easy to translate the philosophy into classroom practice (Griffin and Case 1997). There are indications that some teachers, at least, continue to use transmission type approaches to teaching. The recent evaluation of designated disadvantaged schools carried out by the Inspectorate of the Department of Education (2005) indicates that dependence on textbooks and workbooks is unhelpful to pupils. With regard to younger children, it has been shown that workbook competency does not necessarily indicate mathematical competency (Hughes, 1986). Combining these findings would seem to indicate that traditional, workbook-based approaches are suboptimal, especially for children from so-called disadvantaged backgrounds.

The same evaluation by the Department makes another interesting observation. It notes that high levels of numeracy are not observed, even where optimal teaching occurs. This further underlines the urgent need to introduce innovative ways to enhance pupil interest in, and knowledge of mathematics.

Project Design

The design is quasi-experimental, being implemented in six Junior Infant classes in three urban schools. All three schools are designated disadvantaged and are part of the Giving Children an Even Break scheme. The schools represent an opportunity sample consisting of the researcher's school and schools local to it. Three classes from another urban school with disadvantaged status constituted the comparison group. In this school, also part of the Giving Children an Even Break scheme, teachers delivered the maths curriculum in the regular manner and no special intervention took place.

All teachers were qualified and experienced. Although the maximum class size in schools in the Giving Children an Even Break scheme is twenty, the maximum class size in each project class was fifteen. In the three control classes, the maximum class size was 20. Five of the intervention classes were single sex: one boys' class and four classes of girls. The remaining class was a mixed class of boys and girls. All children in the three control classes were in mixed classes of boys and girls.

The intervention consisted of implementing the Junior Infant maths curriculum in six classes using activity-based programs from the United States. In the three control classrooms, the teachers implemented the maths curriculum in the regular manner.
The two activity-based programs were Number Worlds and Big Maths for Little Kids. These programs share a number of important features. Both programs are specifically designed for four-year olds from so-called disadvantaged backgrounds. Both programs place an explicit emphasis on the development of children's language, and are designed to be motivating, as well as challenging. They are designed to foster the development of a robust understanding of mathematics and a positive attitude towards the subject.

Number Worlds, as the name suggests, deals only with the development of number knowledge. Big Maths for Little Kids covers number along with the remaining strands of the maths curriculum. Number Worlds was implemented during the first two terms of Junior Infants, with Big Maths being implemented in the final term.

Number Worlds
Griffin and Case, the authors of this program, argue that young children display two different types of mathematical competencies. Four-year olds can reliably count a small set of objects (Gelman 1978), but they also display considerable non-numerical knowledge about quantity (Starkey 1992). Initially these two systems are separate. Integrating these two distinct systems, a process fundamental to the development of formal mathematical understandings, poses a considerable challenge for young children. The Number Worlds program was developed to facilitate this process.

A basic aim of this program is to facilitate the development of essential mathematical concepts and skills. In order to do this, the program explicitly addresses the different ways in which number is represented in our culture: as three-dimensional objects, as two-dimensional sets, as positions on a line (horizontal or vertical) and as points on a dial. The authors note, for example, how the same mathematical relationship can take a variety of forms: having three sweets and receiving another four; driving three kilometres and then driving a further four; eating lunch at three o'clock and then eating dinner four hours later. The Number World activities are designed to facilitate children making connections between these representations, representations that superficially appear unrelated.

Many of the Number World activities are dice based and provide numerous opportunities for children to subitise (recognise the number of dots on a die face by the dots' spatial configuration rather than by counting). Moving counters along a variety of different pathways, depending on the results of the die throw, provide multiple opportunities for counting. Thus, the children experience the set representation (n dots on dice), and the accompanying movement of n steps around a circular skating rink, or along a linear path. Experiences such as these provide considerable opportunity for the development of related language and for metacognition, the process of thinking about thinking. For example, if a child is three steps away from a finishing line, the teacher can ask a variety of questions concerning the optimum roll of the die, (3), dot patterns that would be useful (1 or 2) or dots patterns that would not be useful (4,5,6).

Big Maths for Little Kids

The Big Maths for Little Kids (Greenes, Ginsburg and Balfanz 2004) program is designed to foster the development of the informal mathematical knowledge children bring to school, by capitalising on children's interests. The program immerses children in a rich mathematical environment to facilitate the development of robust mathematical knowledge and skills. The program, described as a joyful mathematics program, consists of activities and stories covering the domains of number, shape, pattern, logical reasoning, measurement and space. There is an explicit emphasis on language and on the expression of mathematical thought.

Like Number Worlds, this program emphasises multiple forms of representation. Children are exposed simultaneously to numerals, word names and tallies representing a given number. Similarly, children experience a variety of ways of representing the same pattern. For example, an abc pattern can be represented by colour cubes as a red-green-blue, red-green-blue …, as actions hop-clap-sit, hop-clap-sit…., as musical pitches, low-medium-high, low-medium-high…., or as numerals
1-2-3, 1-2-3…. The program emphasises mathematical thinking, based on research evidence indicating that young children are more mathematically competent than is generally believed.

Assessment

Assessment of the intervention consisted of three components. Firstly, a Number Knowledge Test was administered to intervention and comparison pupils, once at the beginning of the academic year and again towards the end of the year. This test is part of the Number Worlds program and consists of fourteen items designed to assess the children's intuitive, informal knowledge of number. This test can also facilitate the assessment of children's underlying knowledge and strategies.

The second assessment instrument used was a pupil profile, designed specially for this project. The twenty-four items in this profile were drawn directly from the Revised Maths Curriculum. These items represent the six strands of the curriculum: Early Maths Activities, Number, Algebra, Shape and Space, Measurement and Data. Teachers were required to rate pupil performance on each of twenty-four items, once at the beginning of the year and again at the end of the year. The administration of these profiles took approximately thirty minutes per pupil, per administration. Consequently, there was insufficient time to administer these profiles to the comparison pupils. The profiles assessed the children's number knowledge, the target of the Number World program, and also algebra, shape, space and measurement the domains targeted by the Big Maths for Little Kids program. Recent analyses by the author indicate that both the Number Knowledge test and the assessment profiles are reliable instruments.

Results

The cohort

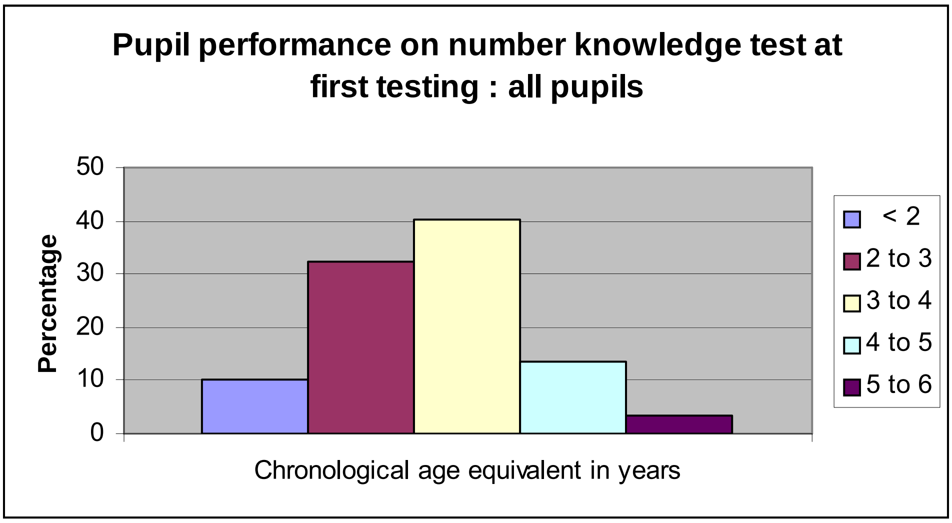
Data was collected on a total of 117 Junior Infant pupils attending four designated disadvantaged schools in Dublin. The mean age of these pupils was 4.5 years. However, eleven of these pupils, approximately 9.4 % of the total, had not reached their fourth birthday on 1st of September of the year they started school. All of these pupils attended intervention classes. In one of these classes, for example, a third of the pupils were under four on 1st September 2005. These pupils began school once they reached their fourth birthday. In contrast, all pupils in the control classes were at least four years old at the beginning of the school year.

Boys comprised 26.5% of the cohort, with girls comprising the remaining 73.5%. Of the intervention pupils, 86.6% were native Irish with the remaining 13.3% being New Irish. In contrast, 61.6% of the control pupils were native Irish with the remaining 38.2% being New Irish. Most of the New Irish national children had little fluency in English.

Mathematical performance of pupils on first testing

As they began school, there was no statistical difference between the performance of the comparison and the intervention pupils on the test of number knowledge and so data on these pupils can be combined for analysis, as follows. A total of 10.3% of pupils scored below the level of the average 2 year old. A further 32.4 % of pupils performed at the level of the average 2 – 3 year old, while 40.1% of pupils performed at the 3 – 4 year old level, in a cohort whose average age was 4.5 years at school entry. Thus, 82.8% of pupils were underperforming mathematically at the start of formal schooling. In contrast, only 17.2% of Junior Infants were performing at the level of their chronological age. A caveat to these findings is that the scale used to convert raw scores to chronological age equivalents was computed in the U.S and may not reflect exactly the level of development of Irish children.

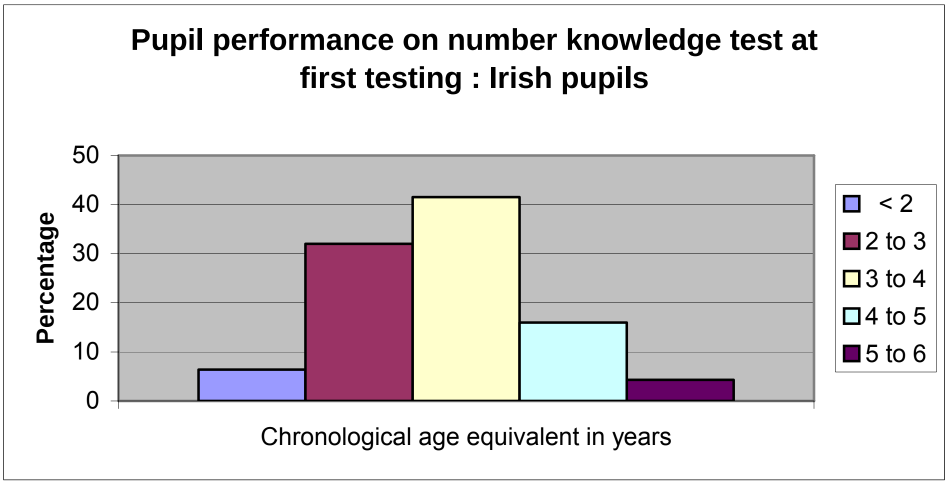
Figure I



The disappointing performance at school entry could be due, in part, to language difficulties experienced by New Irish pupils. Such pupils composed 19.66% of the total sample, or almost one in five pupils. For this reason, it was decided to remove the New

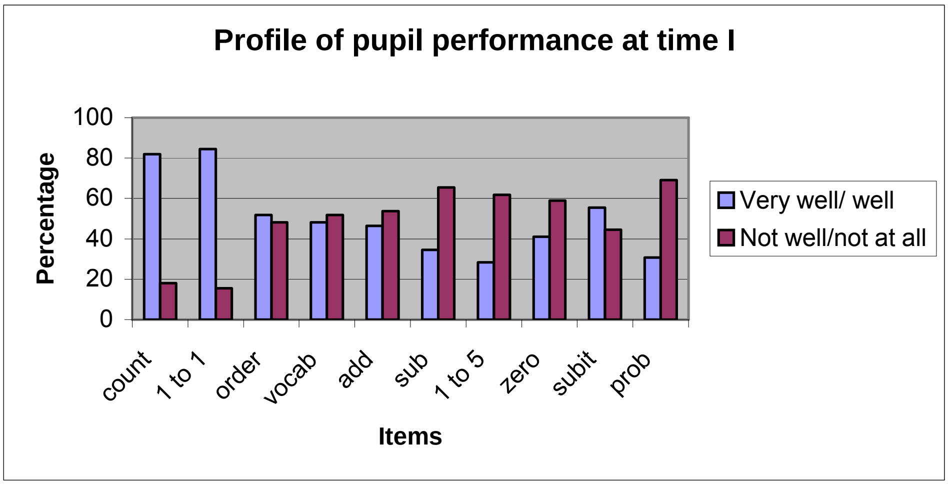
Irish and plot the performance of Irish pupils. However, the resulting distribution (Figure II) is very similar to that obtained from data on the combined sample of Irish and New Irish. For example, 32% of Irish pupils performed at the 2-3 level compared with 32.4% of all pupils. Similarly, 41% of Irish pupils performed at the 3-4 year old level, compared with 40.1% of Irish and New Irish combined. It seems reasonable to conclude that the inclusion of New Irish pupils is not responsible for any large-scale distortion of the distribution of scores.

Figure II



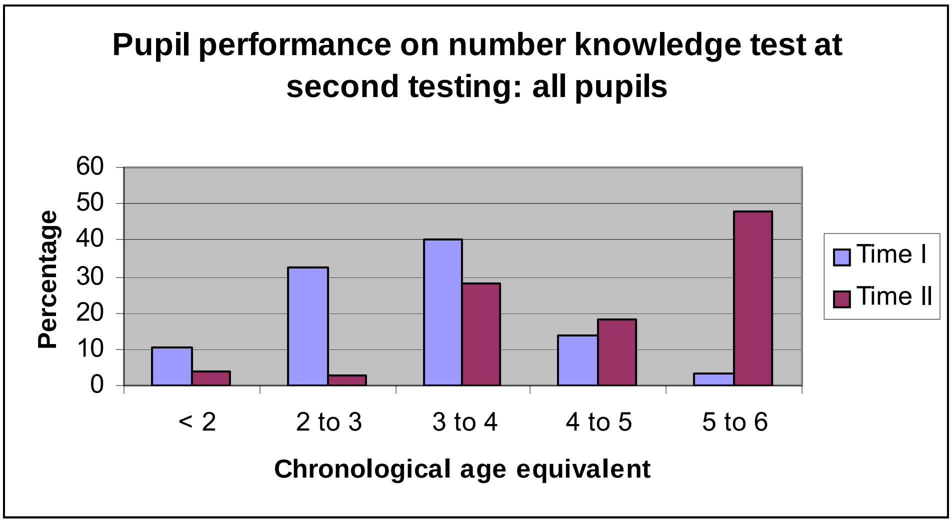
Subsequent profiling of pupils as displayed in Figure III indicates the knowledge and skills underlying pupils' performance on the number knowledge test. (See appendix A for key to Figure III). Pupils performed well on two tasks, namely counting and set comparison: 81.9% of pupils could count sets of objects and 84.5% could compare equivalent and non-equivalent sets using one-to-one correspondence. However sizeable proportions of pupils demonstrated difficulties in every other aspect of number profiled. For example, when asked to order sets of objects by number, 48.2% of pupils performed poorly on the task. Similarly, on simple subtraction tasks, on familiarity with the zero concept, on the ability to read, write and order numerals 1 to 5, on subitising, that is identifying the number of objects in a set without counting, and on simple problem solving, pupils' performance indicates the need for considerable development.

Figure III



Mathematical performance of pupils on second testing
When tested for a second time, the difference between the project pupils and the comparison pupils was not significant. For this reason it is possible to combine the data from both groups, as per Figure IV. While pupil performance had undoubtedly improved, it should be remembered that the children's average age had also increased from a mean age of 4.54 years at first testing to 5.2 years at second testing. While the percentage of pupils performing at a level equivalent to a chronological age of 3 years or less has decreased greatly, nevertheless as they approached the end of their first year at school 27.9% of pupils were still performing at the 3 to 4 year old level.

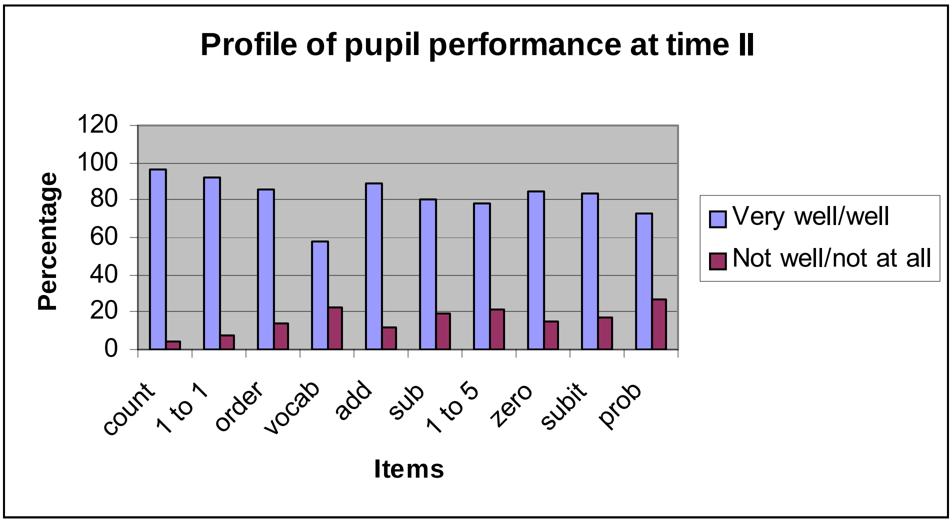
Figure IV



The second set of profiles, in Figure V shows pupils' performance at second testing compared with performance assessed at time I. The improvement from first to second profiling can be seen across the entire range of mathematical knowledge assessed. Yet approximately one fifth of pupils continued to experience difficulties using ordinal language, performing simple subtraction, reading, writing and ordering numerals 1 to 5 and calculating the answers to simple problems. It should be noted that due to the time needed to profile individual pupils, only those in the intervention group were assessed, both at the beginning and at the end of the school year.

Conclusion
It was somewhat disappointing that the gains made in the mathematical knowledge of pupils involved in the project were not significantly greater than those made by the comparison group. However, without random allocation of pupils, even if such differences had emerged they could not have been reliably attributed to the project itself. Furthermore, the intervention and project pupils differed on a number of variables, including the age of pupils, with comparison pupils being significantly older than project pupils. Comparison classes also contained greater proportions of New Irish pupils and there is some evidence that these pupils demonstrate high levels of scholastic attainment once they develop some competence in English.

Figure V



Apart from the above differences that may have favoured the comparison pupils, there were some limitations in the way in which the project was implemented. Ideally, intervention teachers should have been provided with a programme of professional development to support delivery of the programme, but for logistical reasons this was not possible. This was unfortunate since both the Number Worlds program and Big Maths for Little Kids require considerable redefinition of the teacher's role. The programs are designed so that learning occurs as children engage with each other in range of small-group activities. The teacher's role is predominantly that of supporting children as they co-construct their own knowledge. Teachers also indicated that the absence of an additional adult in the classroom as children participated in the maths activities and games may have reduced the opportunities for learning, particularly in relation to language development. This is interesting given that the maximum class size in intervention classes was 15, small by Irish standards.

However if one of the goals of teaching is to enhance motivation, the program was a resounding success. Pupils not only participated enthusiastically in the group activities but they looked forward to these and regularly asked teachers about when they could next play what they termed maths games. Teachers were also extremely positive about the program's activities, in part because of the pupils' positive response. Teachers were also very positive about the profiling process, which though time consuming, often revealed unexpected competence. Such revelations are of particular importance in relation to raising teacher expectations.

The revised Primary School Curriculum (1999) calls for appropriate, special interventions in infant classes when there is reason to believe the level of development reached by pupils may not reflect their true potential. This project is an example of a prototype of such an intervention.

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Private and Public: Doing Mathematics in Different Places

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With the rise in both Internet access and content, mathematics resources are no longer limited primarily to being school based - the home could also have the resources openly available. When taken up, dual locations of private/home and public/school are formed. Whilst there is the potential for the two to be linked, evidence suggests that this is not yet necessarily the case. This paper reports on findings from a recently posted web-based questionnaire exploring pupils' current practices in using the Internet for mathematics problems and perceptions of classroom teaching. On the private side, 49% of respondents who do mathematics problems at home reported that they thought their teachers were unaware of this. For the public face, findings show that pupils feel that undertaking Puzzles and Games in mathematics gives the feeling of doing mathematics in a different way, making the work more interesting and enjoyable. From listening to pupils talking in this way there is a greater chance for all involved in education to be working together. Analysing the collection of responses to the questionnaire is the first step along the walk.

Introduction

The work described in this paper builds on a particular finding from an evaluative study of a mathematics (NRICH[1]) web-site[2] [Jared 1998] that provided evidence that some (half of the respondents from 40 replies) young people[3], who were accessing the mathematics problems at home, were only working on them at home - a 'curriculum without walls' [Furlong et al 2000:108]. Adding the inference that it was likely that some respondents were not necessarily 'telling their teacher', this finding raised many questions. If, as it appears, that there were young people making an individual choice to do work in their free time at home, is there, with easy access to the Internet, a possibility of a shift in teaching and learning environments – where is the *pupil-learning place*? If, in simplistic terms, at school a pupil is generally reliant on the teacher to 'dictate' the work to be studied but at home the individual has greater freedom to decide what work they would like to do, should school mathematics be changing to include more 'popular' mathematics? This paper does not set out to provide definitive answers to these two questions but rather, through asking young people for their views, offers supporting data for consideration.

To set the work in context a review of current literature, a brief synopsis of the NRICH web-site and a description of the research methodology are first included. Findings (from pupil only responses) relating to (i) home use, (ii) understanding and learning mathematics and (iii) puzzles and problems within the school curriculum are then discussed before concluding with implications for further consideration.

Review of literature

For decades, concerns have been expressed about the general unpopularity of, and poor achievement in, school mathematics [see e.g. Cockcroft 1982, Brown 1999, Hatch 2002] although simultaneously interest has remained in finding the 'best' way to teach the subject. The seemingly constant reviews of the National Curriculum since its

[1] NRICH is not strictly an acronym. The web-site **enriches** mathematics and was set up by four people from **N**orfolk LEA, the **R**oyal **I**nstitution, **C**ambridge University and **H**omerton College Cambridge.

[2] www.nrich.maths.org.uk

[3] The terms young people, pupils and students- are used variously throughout this report, depending upon the emphasis of a home or school environment. All are referring to that part of the population aged up to 18.

inception [DES 1989, DfES/QCA 1999] along with the National Numeracy Strategy [DfEE 1999] and the National Numeracy Framework [DfES 2002], all have in some form or another embraced the now infamous *'Mathematics Counts'* paragraph 243 which listed six different areas of teaching that pupils should experience [Cockcroft 1982]. Alongside the most common forms of teacher exposition and practice of routine calculation, a place should be found for more open-ended problems and investigations. Now, in 2007, *'the consensus is that learners need to be encouraged and supported in actively taking initiative in their learning'* [Mason & Johnston-Wilder 2004:5]. However, in pursuit of good results within an examination system in an overcrowded curriculum, lessons are all too often reduced to the exposition and practice with teachers feeling they cannot afford to give time to 'play' with the subject. [Boaler 1997, Watson 1994]. Such short-term gains are unhelpful in engaging the interest of pupils who would like to pursue higher studies. The relatively recently published *Making Mathematics Count*[1] cites the *'perceived quality of the teaching and learning experiences'* and the *'failure of the curriculum to excite interest and provide motivation'* as two of four factors contributing to the decline in pupils opting to study mathematics beyond age 16 [Smith 2004:pp12-13].

This perceived 'unhappiness' with the subject is reflected in a survey undertaken by Miller et al [1999] involving over 6000 pupils in nine secondary schools responding to one of three, sixty item questionnaires (the English National Curriculum core subjects of English, mathematics and science). Out of the three subjects mathematics had the lowest enjoyment factor - only a third looked forward to their mathematics whilst a quarter wished they did not have to go to mathematics lessons at all. Only just over a quarter thought that mathematics was fun. Of equal concern is the one-third of pupils who never found mathematics interesting (in the other two subjects there were some who felt the same, though numbers for these had dropped to one in four). With such findings it would appear that there is substantial work to be done to increase the popularity of the subject.

Such concerns are not the exclusive domain of the English education system. In the United States, Schoenfeld is renowned for his belief in having problem-solving activities as the core of any mathematics course [Schoenfeld 1994]. Although some ten years earlier than the study reported above, Schoenfeld's research is nevertheless uncannily similar. Using a questionnaire of seventy closed and eleven open questions administered to 230 mathematics students enrolled in Grades 10 to 12, (amongst other things) comparative perceptions were made between mathematics, English and social studies, again with mathematics scoring poorly. Schoenfeld bemoans the fact that students reiterated the teachers' rhetoric that the most important thing to pass exams is to memorise. He takes this further in suggesting that people differentiate between memory-school mathematics and creative-outside mathematics where memory gives way to pursuit of the problem [Schoenfeld 1989] – a difference, if it exists, similar to the dual location theory proposed in this paper.

In summary then the picture painted is rather a depressing one. The studies show that school mathematics is not overly highly regarded by its pupils and the packed curriculum time-pressurises teachers to forgo more open-ended activities. Although

[1] This is commonly known as The Smith Report and took its title from Mathematics Counts (The Cockroft Report) published some 22 years earlier.

such activities might take longer to execute [Romberg & Kaput 1999] they are likely to increase interest and 'fun' and ultimately better situate pupils to perform equally well, if not better, within the constraints of the examination system. This view was propounded some eighty years earlier [Hogben 1938 cited in Mason & Johnston-Wilder 2004]. It appears that currently little use is being made of the opportunities that Internet access could afford [Somekh et al 2002]. *'The New Landscape of Learning: ... enrolling in a school would no longer mean entering an institution that would educate within its walls'* [Bentley 1998:185] has yet to arrive.

The suggestion being made here is that a web-site accessed from home (as well as the potential for school access) providing more open-ended, investigative problems might be able to redress some of the bleakness reported upon in the literature. A brief description of the NRICH web-site as an exemplar site (though in reality it is unique in its scope), illustrating the opportunities that the Internet can afford follows.

The NRICH web-site

The web-site was established in 1996 with the intention that it would have a multitude of uses, appropriate to learners and teachers (aged 5 to 95+). Now under the umbrella of Cambridge University's Mathematics Millennium Project[1], the web-site received 70 million hits and three million visits last year. Additionally, since 2004, the archive had become sufficiently large (and still growing) that any pupil entering school aged 5 and leaving aged 18 could find a new problem (and more) for every day they attended school. Such statistics are indicative of NRICH's potential. Stated within its mission statement, the project aims to "*enrich mathematical experiences by giving students the opportunity to explore and engage with mathematical ideas, offer challenging activities ... to develop their mathematics and foster a community where students can be involved and supported in their own maths learning"* [NRICH 2007 web reference].

The site has a 'magazine content style' - alongside the mathematics problems to try there are 'games' (mathematical) to play, articles to read, news of mathematical events and at various times specific topic activities such as LOGO and Excel activities had delegated slots. A new edition of the magazine is published on the 1st of each month. Everyone world-wide, (individuals or groups), is invited to submit a solution by the 21st with the best of these solutions published within the following month's solutions section and subsequently archived. The interpretation of 'best' can be judged at several levels – e.g. age, explanation and justification of method.

To enable a virtual meeting place for young minds to share ideas and collaborate there is web-board conferencing between school pupils and university students - personal mathematical questions can be answered through 'Ask NRICH', the 'Ask a Mathematician' Answering service. There is also 'NRICHtalk' a self support 'chat' group where views and experiences can be exchanged.

This then is the essence of NRICH. It is driven to provide a place where young people can 'play' at being a mathematician in the way that professional mathematicians act. By considering the project's policy to *'focus on mathematical reasoning rather than to meet the needs of the 'standard curriculum', not to teach mathematical skills and basic*

[1] www.mmp.maths.org.uk

concepts in isolation but to use them, and see a purpose for learning them, through problem-solving contexts' [Piggott 2006:15], it is clear that the way of working and the type of problem presented is different from that found in the majority of classrooms.

Having described the fundamental ideas of the NRICH site in order to illustrate the type of mathematics available, attention is now turned to the research conducted using an on-line questionnaire.

Methodology

The overall research involved a combination of both qualitative and quantitative data collected from an on-line questionnaire consisting of 32 questions. 21 questions were multi-choice (with 37 statements needing a degree of agreement), 6 questions required open-ended responses, and 5 related to personal data (e.g. age, gender etc.). There were six sections to the questionnaire: 'using the Internet at home', 'our future with the Internet', 'using the NRICH website', 'general information', 'doing puzzles and games in mathematics lessons' and 'understanding and learning mathematics'. This questionnaire had been trialled in three previous developing states in a paper-based format in five schools.

NRICH established a link to the questionnaire from its home page for three months in early 2007 and information about it was included in the site's Newsletter. Other than this, there was no further advertisement of its existence. Although any replies would not be able to be viewed publicly, the responses were anonymous. There was no requirement to answer any specific question and the number of responses dropped from a starting point of 250 respondents to around 150 'die-hards' who painstakingly went on to complete all questions. There was little movement in results between the period when around two-thirds of the total had been collected and the full set, which suggests consistency and replicability.

From the 160 responses indicating gender, slightly more were from females (56%) than males (44%).

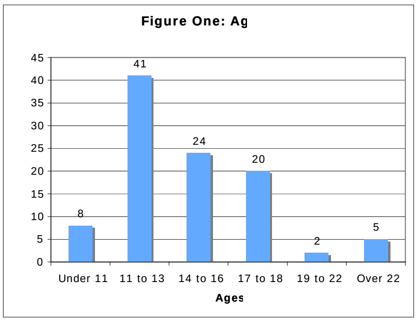


Figure One shows a percentage breakdown of the age of these 160 respondents across key stages. Responses from sixth formers (ages 17 to 18), accounted for one-fifth of the total. The maturity of these replies was notable when compared with responses from younger age groups, aiding the quality and depth of written responses for the open-ended questions. The spread of ages provided opportunities to investigate different views according to experience.

Findings presented in this paper have been taken from data from the questionnaire sections 'using the NRICH website', 'understanding and learning mathematics' and 'doing puzzles and games in mathematics lessons'. Where appropriate, analysis of data from previous trials signalled above is discussed alongside the findings of the on-line questionnaire. Each of the three sections is now presented and discussed in turn.

Findings – using the NRICH website

Unlike the other two sections to follow, given that in this section it is the web-site users that are under review, data here can only be confined to web respondents. The relevant questions from the questionnaire appear in Appendix 1.

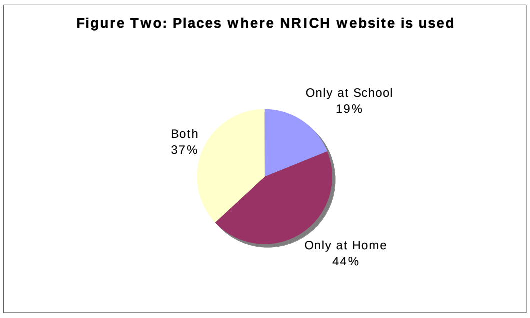


Figure Two: Places where NRICH website is used

163 responded to the question asking where the participants used the website: only at school, only at home, or both at home and at school. The results (as percentages) are given in Figure Two. With over two-fifths (44%) of the school-aged respondents doing NRICH problems only at home and nearly a further two-fifths (37%) indicating that they did the work both at home and at school, this leaves (from this sample) just under one fifth of the respondents doing the problems only at school. Although this study is small scale, at the very least it can be seen that some mathematics work is being undertaken in the home, not all of which is teacher set. However whether the teacher knows or not, is not yet determined.

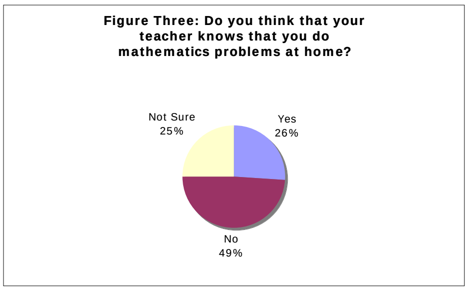


Figure Three: Do you think that your teacher knows that you do mathematics problems at home?

Figure Three charts the 129 (out of a possible 132) replies to a subsidiary question targeting those who had responded either ‘only at home’ or ‘both at home and at school’ on the point of the teaching knowing (or not). Even if all the ‘not sure’ options were actually ‘yes the teacher did know’, in keeping with the original 1988 evaluation referenced in the introduction, these latest results show that just under a half (49%) definitely do not think that their teacher knows. The argument is not to say that the teacher should know - indeed it will be shown later that analysis of qualitative data indicates that for some respondents they have no wish for their teacher to know - but rather if the teacher is unaware then less can be done to personalize the school experience.

For those students who undertake the mathematics problems at home, a further question asked for information about who was involved when working on a problem. A range of options: alone, a group of friends, a friend, siblings, adults was provided, with multiple selection allowed. Of the 132 ‘home only’ and ‘both home & school’ replies noted above, 120 answered this question.

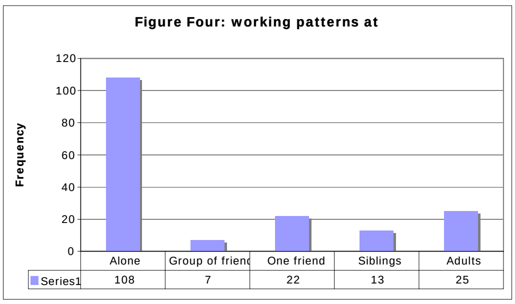


Figure Four: working patterns at

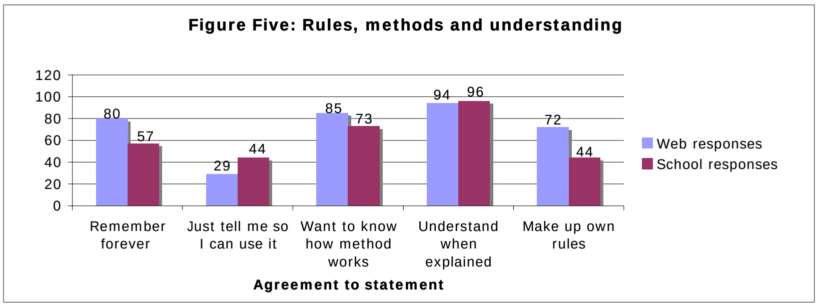
Figure Four illustrates the frequency of each option, although as more than one option could be selected the choices are not mutually exclusive. However delving deeper, of the 108 occurrences of ‘alone’, 73 chose no other option and of these, 43 had indicated in an earlier question that they only did the problems at home – that is, 59% of home only users only did the problems on their own. Further investigating these 43 ‘alone-and-only-at-home’ respondents, 30 (70%) had suggested that their teacher did not know that they did the problems and a further 6 (14%) were not sure whether their teacher knew or not. This leaves a mere 16% (approximately one pupil in six) who thought that their teacher certainly did know.

One interpretation of these results would be to indite the teaching profession as neither caring nor interested in what their pupils are doing out of school. This would be far too simplistic and has the potential to mislead. There must be speculation as to who really knows whether the teacher does know and indeed teachers, respectful of privacy, may not feel it is their business to know. In fact, analysis of qualitative data shows the importance that some respondents place on privacy. Across the age ranges, some respondent comments seemed quite adamant that the teacher did not need to know e.g.: *"Why should teacher have to know?"* [Male KS5], *"I need to learn but my teacher does not need to know"*, [Female KS3], and *"It makes no difference, whether my teacher knows or not"* [Male KS4]. Another respondent was quite clear about individual responsibility: *'It is me who is leading my life, the teacher is, for sure, a guideline but she need not say everything that is perfectly right as she is also a human. I know to choose between good and bad"* [Female KS4]. One comment acts as a bridge between the two 'sides': *"[I do puzzles] because I'm curious! and my teachers don't need to know, they know I love mathematics but are not so interested in discussing anything not on the A-level syllabus"* [Female KS5]. However note that even here, there is a portrayal of a student acceptance of an examination-dominated curriculum.

Do these results indicate a shift in the *pupil-learning place*? For the moment, who can tell? But is this private work at home, different in some way to that done in public mathematics classrooms? In school, teachers' pedagogy influences, or even - it might be contested - limits, opportunities for pupils to develop personal strategies for learning. A section of the questionnaire therefore sought views on the role of rules within this process and it is to this area that the discussion now moves.

Findings – understanding and learning mathematics

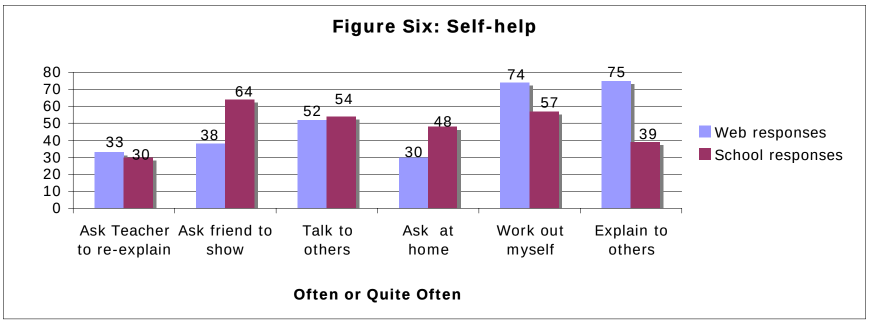
A series of five statements (Appendix 2) were based on memory/understanding of rules and methods [Skemp 1976, Hewitt 2002] and asked respondents to select their level of agreement on a four-point frequency scale that they felt most comfortable with. These four options have been compressed into a two-point scale of agreement ('definitely' and 'I mainly think so') and disagreement ('no, not really' and 'certainly not'). As the same statements were given to both web-based (web group) and the paper-based (school group) the 144 web responses can either be compared with (or added to) a previous 51 responses from two classes (aged 12 and 15) who had trialled the final paper version of the questionnaire. Figure Five shows the percentage of each group opting for one of the two agreement options for each of the five statements.



These results show that there is a strong belief (an agreement of at least 94%) from both groups that, if the rule or method is explained, then they will understand how it works. In addition, over four-fifths as a combined sample were interested in knowing why it works. In keeping with this wish, far fewer (a combined one-third) wanted just to be told the rule in order to use it, though as the percentage sum for these two supposedly mutually exclusive statements is greater that one hundred, a few respondents obviously were interested but nevertheless wanted to be told how to use it. The web respondents appear to be more self-sufficient in that they are more likely to make up their own rules and to some extent (a difference of 15%) are less content just to be told the rule. It is debatable as to whether any of these findings show 'brightness' in the same way as the self-help findings later illustrate. It could be foolish to suggest that the increase in percentage of the web group who feel that they can remember 'most of the time' is akin to anything but a good memory, though this resonates with Schoenfeld's memory-school mathematics alluded to earlier. However, if recall is an asset to solving problems, it may in part explain 'success'– but only in the sense that they seem to be able to remember better.

A further question forced a decision to be made in selecting from two statements the one that they thought was the more important of the two: (1) Understanding the mathematics so you have a way of doing it for yourself or (2) Remembering the rule or method. Both school and web responses resulted in a (overwhelming) ratio of almost 4 to 1 in favour of understanding.

Having focused on aspects of understanding based on classroom experience, a further question (Appendix 3) asked respondents to indicate a frequency of use ('often', 'quite often', 'a bit, or 'never') of six suggested ways that could aid 'self-helping'. Figure Six below shows percentages recorded by each group for the more frequent pairing of 'often and quite often' for each of the six ways suggested.



There are only small differences in the combined frequencies for both asking the teacher to explain again - even though only about one third do so - and talking to others in class. However, web respondents are less likely to ask a friend for help and (in keeping with the previous statement) will actually be more likely to do the explaining to peers. As web respondents also seek out asking at home less, there is consistency with the finding that they should be more likely to try and work things out for themselves. Unlike the previous findings on frequency, the inference being made here is thus the web respondents generally appear to be more self-sufficient and maybe 'brighter'. Believing it is likely that web respondents would consist of members of the population who enjoyed and would be successful at the subject, this might be considered an obvious statement, though in a previous study, NRICH users stressed self-sufficiency 'to do better in my exams' rather than any level of 'brightness' [Stanford et al 2001].

If, as it appears, pupils would prefer to gain understanding rather than just relying on memory, would a greater emphasis on problem-solving (and puzzles) provide better opportunities? The questionnaire does not explicitly answer this question, though responses were clear on some aspects of this type of work as the following discussion indicates.

Findings - doing puzzles and games in mathematics lessons

Findings here are based on a series of nine statements (contained within Table One of Appendix 4) relating to aspects of undertaking puzzles and games in mathematics lessons. As before, each pair of positive/negative agreements was combined to suggest general agreement/disagreement and data was again available for the two groups from above (web and school). From the raw data (tabulated in Appendix 4) the results from each group appear similar suggesting that perceptions differ little from respondents in the two different locations – a point which will be returned to in the concluding remarks.

Six of the nine statements had over two-thirds of the percentage score in one of the agree/disagree bandings. On this basis the following trends predominate (percentages stated relate to web responses). Undertaking Puzzles and Games in mathematics gives the feeling of doing mathematics in a different way (85%), making the work more interesting (78%) and more fun than otherwise (82%). Work with Puzzles and Games in mathematics is undertaken more collaboratively (70%), involves working in a more problem solving way (77%) than in other mathematics lessons and generally does not make it harder to learn mathematics (69% do not think it is harder).

The remaining three statements are more ambivalent as results show that Puzzles and Games do not substantially make people think more for themselves (only 59% believe it so), such work is not more challenging (48% think that it is) and there is not a greater tendency to work harder at it (51%). On the surface, such results bring into question some of the reasons given for including such work in the curriculum [see e.g. Fennema & Romberg 1999, Schoenfeld 1994] but, at best, the numerical data presented here is illustrative of simply placing a tick. Quantitatively it fails in any consideration of reasons behind the choice of position. However even when looked at qualitatively by way of an example here, the notion that doing this type of work is 'fun' needs treating with a little caution.

The Miller et al study referred to in the literature review had students suggesting that there was little fun within Mathematics, with the implication that it would be advantageous to have more. However, such an argument might not be so clear-cut. Comments from an earlier pilot study with 31 pupils aged 11 to 14 had included comments such as e.g. *"it is easier to learn about what you are doing and it is more fun to get on with"*[middle set] and *"people are more interested as it is fun"* [lower set]. Two further comments address the layman's question 'if you are having fun, can you be learning?' One pupil clearly thought so as they wrote *"we are learning AND having fun"* [respondent's own capitals, Upper set]. However a note of skepticism was already surfacing when a second pupil had the completely reverse notion: *"they are just fun games not learning games"* [upper set].

Similar comments appear in the latest survey involving web respondents. 12 of the 46 written responses included the word fun and nine of these again made a link to learning e.g: *"I find them a lot of fun and find that they really excite me and help me to learn and to apply what I learn in normal lessons"* [Female KS4], and *"they are fun and offer a more relaxed type of mathematical learning"* [Male KS3]. Some (generally, younger respondents) mentioned that the whole class were 'better behaved' e.g. *"every one does the work more willingly"* [Female KS3], and *"hold interest, so they get better done"* [Male KS4]. Having found such enthusiasm and positive beliefs, there is a degree of opposition to this outlook in some older (sixth form) replies where 'fun' was further cautioned as presenting some masking effect: "*I'm afraid whenever we've done puzzles and games in maths it's always been a poor attempt to make maths "fun""* [Male KS5] and *"I learn the subject less thoroughly, with less retention, than if doing exercises from a text book. Despite being fun to solve, they lack a unifying theory"* [Male KS5]. At this point then, any rhetoric that 'fun' mathematics is the solution to all the concerns that abound for mathematics in school needs to be tempered.

As Mason and Johnston-Wilder have stated *'it is much easier to see what is wrong with education than to do something about it'* [2004:35]. So even where the findings discussed here are not overwhelmingly consistent or 'clear-cut', they have nevertheless attempted to address some of the current issues whilst, at the same time, opening up further avenues to explore. Such a small-scale study cannot on its own make any definitive claims on the situation of present-day mathematics teaching but it can be viewed as playing a part in gathering together a range of evidence that when it is all taken together can present some powerful arguments for change.

Conclusion

Disquiet and dissatisfaction with school mathematics continues. This paper reports on a series of findings, principally from a web-based questionnaire, that explored pupils' current practices in using the Internet for mathematics problems and perceptions in classroom teaching. The underlying premise to the work was to investigate the potential for school 'to tap into' the work that some pupils choose voluntarily at home (and which they may keep a secret) and to suggest that the Internet has provided new and further opportunities than previously for the *pupil-learning place* to be extended beyond the classroom walls. Problems that are not standard textbook problems and generally more open-ended do allow the 'solver' to work in a way similar to that of a professional mathematician. Results from the study show that many young people are keen to understand the mathematics behind the question and are not overly keen just to be told the method to obtain a solution. They also respond generally with greater enthusiasm to problems that they do not associate with more traditional methods of teaching. If pupils have these positive feelings then maybe it really is the time for teachers to have greater confidence and to feel less pressurised to deliver a curriculum solely focused on examination questions. It is suggested that the private way of working that young people enjoy can be embraced in the public place of the classroom. If so, then related strategies might well enhance both pupils' attitudes to mathematics and place them in a better position to attempt 'real' mathematics even within the constraints of a prescribed system.

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Appendix 1

7. Please select the option for where you use the website:

only at school ☐ only at home ☐ both at home and at school ☐

8. If you use the website at home do you think that your maths teacher knows that you do?

yes ☐ no ☐ not sure ☐

11. If you are doing maths from the NRICH website at home, do you try the problems:
(you may tick more than one)

alone ☐ with a group of friends ☐ with a friend ☐
with brothers/sisters ☐ with adults in your family ☐

Appendix 2

29. For each statement in the table below choose the option that you feel you are most comfortable with.

	Statement	Definitel y	I mainly think so	No, not really	Certainl y not
	If I am given a rule or method I can generally remember it forever				
	I like to be told the rule and method to be used, without any explanation, so I can simply use it				
	I am interested in knowing why the rule or method works				
	If the rule or method is explained I generally understand how the rule and method works				
	Making up my own rules and methods helps me to understand the work				

Appendix 3

31. For each statement in the table below choose the option that you feel you are most comfortable with.

	Statement	Often	Quite Often	A bit	Never
	I ask my teacher to explain it again to me				
	I ask a friend to show me what they are doing				
	I talk about it with other people in my class				
	I ask someone at home to help me				
	I try and quietly work it out for myself				
	I explain the work to others in my class to help them				

Appendix 4

Table 1: Percentages per first two and last two options accumulated for statements relating to perceptions when undertaking puzzles and games in mathematics

	Statement Doing puzzles and games in maths...	Agree: Definitely I mainly think so		Disagree: No, not really Certainly not	
		Web	School	Web	School
	… makes me feel I am doing mathematics in a different way to my other mathematics	85	96	15	4
	… is more interesting than my other mathematics work	78	96	22	4
	… is less fun than other mathematics	18	2	82	98
	[When] … I think for myself more than when I am doing other mathematics work	59	57	41	43
	… makes it harder to learn maths	31	23	69	77
	[When] … more of us work together than with other mathematics work	70	82	30	18
	[When] … I feel I am working in a more problem solving way than at other times	77	84	23	16
	… is generally more challenging than other mathematics questions	48	35	52	65
	[When] … I try harder to solve the problems than with other mathematics questions	51	65	49	35

Coordinating Student Learning and Teacher Activity – The Case of Savannah: Motivating an Understanding of Representativeness through Examination of Distributions of Data

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This research study investigated the measures chosen by five 4th to 8th grade students when selecting statistical measures to describe distributions of data. Over the course of eight weeks of instruction, individual teaching experiments were conducted to investigate the development of understanding of distribution. The results indicate that consideration of representativeness was a major factor that motivated modification of approaches to constructing indices of distributions. This paper outlines the case of a 4th grade student who participated in the study and narrates her journey as she grappled with increasingly complex statistical ideas relating to representativeness.

Theoretical Perspective

Recent approaches to curriculum design and research in statistics education engage students in examining *distributions* of data rather than working with isolated data values devoid of context. A distribution refers to the arrangement of data values along a scale of measurement (Hardyck & Petrinovich, 1969) and distributions may be depicted using a variety of graphical representations. Distributions are commonly described using *parameters*: measures of centre, variability, skew (symmetry) and kurtosis ("peakedness"). The parameters that are the focus of study at primary school level are center (mean, median, and mode) and variability (range).

An important skill when examining a distribution of data is the ability to find trends and patterns in the data and thus infer properties. This is in essence what the parameters of a distribution describe – underlying signals (Konold & Pollatsek, 2002) or messages communicated by the data. In primary school mathematics one way to focus students on the signals emitted by a distribution is to ask them to elect a value that best represents the distribution – often referred to as a *typical value*. This notion of typicality is referenced in many curricula and by national bodies such as in the Principles and Standards for School Mathematics (NCTM, 2000).

Recent research in statistics education examines primary students (Cobb, 1999; Konold, Robinson, Khalil, Pollatsek, Well, Wing & Mayr, 2002; Leavy & Middleton, 2001; Lehrer & Schauble, 2002; Mokros & Russell, 1995) and preservice teachers (Leavy, 2004; Leavy & O'Loughlin, 2006) understandings of and efforts to represent (or typify) distribution. Research examining the measures children use to represent and summarize distributions of data indicates that a variety of measures are used. The most frequently identified approaches used to determine representative values are the measures of center and spread such as modal clumps, ranges, and intervals. Student understandings of appropriate measures to represent data vary and include conceptions of average as modal (values occurring with the greatest frequency) and middle (akin to the median).

Purpose of the Study

The research from which this paper was motivated examines student's decision-making processes when selecting statistical measures to represent a distribution of data. In this line of inquiry, effort was made to ascertain what (formal or informal) measures students utilize in representing distributions, and how these measures develop and contribute to an understanding of distribution shape and structure. For the remainder of the paper, the measures students select to represent distributions are referred to as *representative* measures.

Design and Methodology

Participants

Five participants from a variety of urban schools in the Southwestern United States participated in the study. Students had not received any specialized instruction or experiences with data prior to participation in the study. The 4th grade student, who is the focus of this paper, had few experiences with data, limited mainly to pictogram and pie chart interpretation. All other students had received curricular instruction relating to graphical representations in addition to calculating measures of central tendency.

Procedure

Teaching experiment methodology was used to uncover student's conceptual understanding of distribution. Teaching experiment methodology consisted of a clinical interview phase, a teaching phase and an analysis phase. Clinical interviews lasted 60 minutes and were video taped. Two clinical interviews were conducted with each student, one at the outset of the study to determine initial understanding of distribution and another at the end of the study to assess changes in understanding of distribution.

Teaching episodes took place following the initial clinical interview and consisted of each student working individually on a variety of model-eliciting activities. Each teaching episode was characterized by the presentation of specific tasks designed to elicit models of students understanding of certain statistical phenomena. The tasks used were derived from a variety of sources, the majority being developed by the instructor and informed by a previous pilot study (Leavy, 2003), in addition to original tasks used by permission of the Cobb & McClain group at Vanderbilt (McGatha, Cobb, McClain, 2002). Problems were presented in several forms (e.g. graphical, tabular etc.) and students were asked to describe distributions and determine representative values for the distributions. In other cases students collected their own data and constructed typical values for the data. All data sets were presented predominantly on line plots or on stem and leaf plots chosen due to their structure facilitating the display of individual data values, the visibility of which were considered important for the construction of representative values. Eight teaching episodes, lasting approximately 50 minutes each, were carried out with Savannah; each episode was audio taped and several sessions were videotaped. These episodes were used to construct a model of her mathematical thinking and guide her to develop more sophisticated ways of reasoning about data.

Data Analysis

The analysis phase involved examining the data resulting from the clinical interviews and the teaching episodes. Data from the clinical interview were used to construct a hypothetical learning trajectory (Simon, 1995) for Savannah. Ongoing analysis occurred between teaching episodes, and a retrospective analysis focused on the cumulative episodes. Results from the ongoing analysis were used to inform the direction of the study and the construction of tasks for subsequent teaching episodes. The main sources of data were the researcher's field notes, audio and video records of the interactions, and samples of the student's written work. Savannah's verbalizations, inscriptions, and models were analyzed as to how they revealed her schemes for structuring and depicting distributions. Hence, the teaching episodes provided the means to generate in-depth knowledge of Savannah's conceptual understanding of data and how it was fostered in a coherent instructional sequence.

Results: A Case Study of Savannah

In the reporting of the results emphasis is placed on understandings of the statistical concepts examined in the initial clinical interview and the subsequent development of student's conceptual schemes over the course of the teaching experiment. Each of the five participants in the teaching experiment constitutes a separate developmental case. For the purpose of this paper, I will focus on the case of Savannah the 4th grade student. This case casts light on particular statistical understandings and misconceptions the roots of which have important implications for the design of statistical instruction particularly within the Irish primary education system.

Background on Savannah

Savannah was 9 years old and stated that mathematics was her favourite subject. She was in a 4th grade non-differentiated mathematics class in public school. Her in-class performance in mathematics placed her in the upper 10% of the students. As a result, Savannah would be placed in a group for more mathematically able students in 5th grade.

Initial approaches to constructing representative measures

During the initial clinical interview and first teaching episodes Savannah constructed representative measures based on her *own experience of the phenomenon* under investigation. In these situations Savannah seemed not to comprehend that data referred to an actual event, this was indicated by her inability to attend to the data presented on the graphs. The following task (Task 1, Appendix A) is an example of one such episode in which Savannah demonstrated the ability to read values from the graph but the inability to use the data to answer the question.

> Teacher: How many gummi bears would you say are in a packet of candy?
> Savannah: Ah.. like … maybe 25 or no .. probably 26.
> Teacher: Why do you say 26 gummi bears? …… (long pause)
> Teacher: And did those students (pointing to the graph) get as many as 26?
> Savannah: No, but … I think you'd get more. It really depends on … just … I always get more.

As can be seen from the example above, Savannah used her own experience of the size of a packet of Gummi Bears to answer the question. The value Savannah elected actually fell outside the range of values presented on the line plot and resulted in the presentation of an *idiosyncratic* and non-representative data value.

Initial analysis of Savannah's response suggested that the situation presented in the task had particular relevance for Savannah and influenced her response. To test this hypothesis, Savannah was presented with a number of tasks and asked to interpret the data presented on them. As with the 'Gummi Bear' task, she continued to use her own experiences of the problem situations to construct representative values. It seemed that Savannah (erroneously) perceived that graphs were merely display tools used to present a 'picture' of data. In her view, graphs were objects the purpose of which was to read off discrete values rather than presenting a means to examine distributional structure – hence explaining her difficulties with questions that demanded an examination of structure. The following transcript (Task 2, Appendix A) arising from the first teaching episode provides another example of her tendency to revert to her own experiences when determining representative values.

> Teacher: Based on this graph, generally, how many medals do you think a country won in the Olympics? …
> Savannah: Yeah .. em .. (sighs) okay 8.
> Teacher: Why 8?
> Savannah: Because .em .well I watched the Olympics and they said that it had been on for a while and they said she has already won 3 medals. And actually I would probably say about 6.
> Teacher: 6 okay. Why 6?
> Savannah: Because I was watching the Olympics and a girl had .. this swimmer had only had .. em .. only 3 medals .. and it was almost over so she couldn't get that much more.
> Teacher: Okay.

Teacher Activities to Promote the Development of Understanding

Due to the difficulties in getting Savannah to attend to the data presented in tasks, several teaching episodes were devoted to data collection activities. The intent was that by providing experiences in collecting data and constructing graphs, Savannah would begin to understand that the data presented in graphs represented real situations. These data could then be used as information to determine representativeness. Examples of such activities were counting the numbers of raisins in boxes of raisins, the number of M&Ms in packets, recording resting and active heart rates etc. The data were collected and graphed on line plots from which Savannah was required to describe and interpret her data. As a result of engaging in such data modeling (Lehrer & Romberg, 1996) activities Savannah began to attend closely to the structure of the data, and midway through the teaching experiment paid attention to the structure of data sets that she had not constructed. It cannot be overemphasized that, in Savannah's case, the activity of collecting and graphing data was a necessary precondition to understanding data. However, the transition was not effortless and as can be seen from the following transcript (Task 3, Appendix A) she did at times refer back to her own data collection experiences and neglected the data on display.

> Teacher: Based on this data, if you were the general manager how many raisins would you say are generally in a box of raisins?
> Savannah: I'd probably say 17.
> Teacher: Tell me why.
> Savannah: Because last week I counted up and measured and most boxes I had got were 17.
> Teacher: Now imagine that you didn't count them up last week .. that you didn't know there were 17. So imagine you were Anne Browne and you only had those 30 boxes …
> Savannah: Then I'd probably say 26.
> Teacher: Why?
> Savannah: Most people got the .. 7 that's what it was. There is more of that [26] than that [17]. And it would probably be closest to that [26].

As can be seen from the example above, the results of the data collection activities in which Savannah had been involved were more salient than the data presented in the task. Hence her experiences took on greater importance and influenced her responses. Savannah made the transition from collecting her own data and making inferences from these data to being presented with data sets collected by others. Great care was taken in explaining how these data were collected and Savannah was often presented with descriptive scenarios related to the problems. The provision of context to problems facilitated Savannah in understanding and attending to the data. Her representative values eventually became less related to her own experiences and she began to construct representative values using one particular strategy.

Emerging Understandings of Representativeness

The nature of the representative values provided by Savannah remained steadfast throughout the teaching experiment. All her typical values shared the same characteristic: they were modal values. Regardless of the shape of the distribution presented to Savannah, she always chose the mode as the value that represented the data set. Savannah never adjusted the representative value from the mode to allow for outliers, skew or gaps in the data. Even when her attention was drawn to the shape of the distribution and to the inability of a modal value to represent all the data, she remained unwavering in her decision that the mode represented the majority of the data.

When asked to construct a representative value from a multimodal data set, Savannah's initial strategy was to name all the modes that appeared in the data set. This pattern occurred across several tasks (Task 4, Appendix A), and Savannah was reluctant to examine the data so as to choose one specific typical value from the several possibilities.

> Teacher: If somebody came into Ms. Murphy's class and said "Generally how tall are the students in your class. What would Ms. Murphy say?"
> Savannah: em .. 48
> Teacher: Okay. Why would you say 48?
> Savannah: 48 actually 48 or 43 because they have the most amount of people .. heights.

In subsequent teaching episodes Savannah was presented with scenario's that required her to choose one of the several possible modal values as the representative value. Savannah examined the data sets and chose modal values that had the largest number of data values clustered around it. This strategy indicated that she had some notion of the most frequent value being a representative measure (Task 5, Appendix A).

> Teacher: So it says that Jodie told the principal that the typical height of a 6th grade student was 43 or 48 inches. Why do you think she said that?
> Savannah: Because 43 and 48 em have the highest .. em .. number inches .. eh .. of a person in 6th grade.
> Teacher: However, the principal said: "... You must choose either 43 or 48 inches. Look at the data so that you can make your decision ..." Which value do you think Jodie should choose .. – 43 or 48 inches? Why?
> Savannah: 43 because it comes up first.
> Savannah laughs
> Savannah: Oh I have a better idea. 48 because there are more numbers that are closer to 48 than 43.
> Teacher: Great. For example?
> Savannah: Em next to 48 there's 49, 50, 47.
> Savannah: For 43 there's only 44 and 42.

Teacher Activities Highlighting Attention to the Limitations of the Mode as a Representative Measure

While the mode is an appropriate representative measure for many distributions of symmetric shape, it becomes less representative once the distribution becomes skewed. Savannah used the mode as a blanket measure regardless of distributional shape resulting in non-appropriate representative measures on several occasions. As a result, it was considered important that Savannah's attention be drawn to the limitations of the mode and that she be encouraged to consider other measures that represent a data set.

Many attempts were made to draw Savannah's attention to the importance of a typical value being representative. Savannah understood that the mode was not accurate in describing all the values in a distribution and expressed this understanding on a number of occasions. On one occasion she was presented with a scenario of another student deciding not to choose a modal value and adjusting the typical value from the mode to reflect the data more accurately. Savannah admitted the value of adjusting scores to increase representativeness but voiced concerned regarding the *truth* of such approaches (Task 6, Appendix A).

> Teacher: What do you think is the typical value? How high do you think a 4th grader is?
> Savannah: 30 .. 31
> Teacher: Why?
> Savannah: Because most kids are.
> Teacher: Okay now if you look at the next part of the question.
> Teacher reads the explanation
> Teacher: Do you understand that even though 31 was the most it was also the smallest height?
> Savannah: Yeah.
> Teacher: She says there were lots of values higher than 31 so she moved the typical value up higher to reflect more of the students heights…. Do you think that is a good strategy?
> Savannah: No. Because it just wouldn't be .. it wouldn't be true.
> Teacher: Okay?
> Savannah: The .. the .. if most students are 31 … but well it's more fair.

It was thought that following introduction to the <u>median</u> that Savannah may have found it appropriate to use the median as a representative value. She chose, however, to select modal values even in the cases of distributions in which the median would have been the appropriate measure and the mode inappropriate. Savannah was presented with the situation of a student her age choosing a median value as the typical value and asked what she thought of the strategy. While she admitted that there were merits associated with using the median she stated that her first choice would remain the mode.

Data from the clinical interview indicated that Savannah had no conception of the <u>mean</u>. In the teaching episodes, the mean presented difficulties for Savannah and it took several episodes for her to construct a meaningful understanding of the mean. She was initially introduced to the mean through a fair share model. She was encouraged to carry out fair share problems by physically distributing candy fairly amongst a number of agents. Savannah succeeded in solving each of the problems by distributing the candy. However, she often found it difficult to relate the answer back to the question and understand why the answer made sense. Savannah was also introduced to a balance approach to the mean. While she was not encouraged to balance values at equal distances on either side of the mean, she was exposed to the notion that the mean is the point on which the distribution balances. Initially it seemed as if Savannah had grasped the notion, as she was quite accurate at eyeballing the balance point of graphs, however, subsequent probing identified that she was in fact gauging the median. She was identifying a balance point of the distribution by finding the point on the graph that divided all the data values in half. She was not conceptualizing balance as a weighted quantity. Savannah never grasped the concept of the mean as a representative measure and did not suggest that the mean could be used to summarize a distribution of data.

Final Understandings of Representativeness

In the final clinical interview Savannah was presented with the *Gummi Bear* task. Her response to this task indicated that she was attending to the presented data and not making references to her own experiences. The actual representative value provided by Savannah represented the mode of the distribution. Her choice of the mode reflects her strong belief that the mode is representative of the data because it is the most frequently occurring data value. Savannah's justification for the mode being the typical value is more sophisticated than her previous arguments in that she incorporates the notion of the importance of the typical value being surrounded by other values. This indicates a transition to the notion of variability in that she has incorporated a 'cluster' idea, and may signal an attempt at coordinating the notions of location and variability.

Savannah did, however, realize that the mode was not representative of all the values in the distribution. The terms she used to express this notion were the concepts of "fair" and "true." She considered that the modal values told the truth about the data, the truth being that most values occurred at the mode. Hence in some ways the term typical meant the most frequently occurring value. The mode, however, according to Savannah was not always 'fair' because it did not represent everyone in the data set. Truth, however in Savannah's mind, was primary to fairness. As the mode told the truth, even to the detriment of being fair, she felt compelled to use the mode in determining representative values (Task 7, Appendix A).

> Teacher: How long can a 3rd grade student hold their breath?
> Savannah: 14 or 7 seconds.
> Teacher: Okay. Why?
> Savannah: Because they both .. have the most amount of people .. 3 people have 14 and 3 people have 7 and one person has 5 .. 3 people held their breath for 14 which is the most. But still 3 people held their breath for 7 seconds so I think that 14 and 7 are the only ones that matter and 24 doesn't matter 'cause it is all the way down there and only one person got it.
> Teacher: Great thank you.
> Savannah: They already got they [person who got 24 seconds] got a lot of stuff and then .. it's not really fair.
> Teacher: Oh okay. On who?
> Savannah: Okay the people who held the most [the value at 24]. They beat that persons scores [14].. but still most people got it and ..
> Savannah: It is not really fair but it's true.

Summary of Savannah's Schemes

When asked to construct a representative value, Savannah displayed the tendency to ignore the data presented to her and rely on her own experiences of the presented situations. This resulted in the construction of *idiosyncratic* and thus non-representative measures. This predisposition to focus on her own experiences was overcome by engaging Savannah in data modeling activities. As a result of modeling data Savannah began to understand the meaning of data values presented on graphs, and with this comprehension came the keenness to examine graphically represented data.

Subsequently, Savannah's construction of representative values was exclusively focused on identifying *modal values* in a data set. Savannah considered the modal value as constituting an exact and accurate typical value, whilst not always being a fair value, she did consider the mode as embodying an authentic and truthful representation of the data. It seemed that the mode was truthful because when used as a typical value it represented the ‘most’ occurrences of data in contrast to the median (or mean) which did not necessarily represent the most frequently occurring data values. In a sense, Savannah considered the mode as accurately portraying or reflecting the majority of the data values. This frequentist view of data is not atypical and has been found in other studies. When presented with multimodal data Savannah used the distribution of data around each mode as a determining factor indicating an understanding of a typical value being representative.

Savannah also exhibited understanding of the *median* as constituting the middle of a data set and could identify the median when presented with a variety of forms of data. She did not, however, utilize the median as a representative measure. Thus she demonstrated procedural understanding of the median without the corresponding conceptual understandings (Hiebert and Lefevre, 1986). In other studies, this development of skills rather than concepts (Resnick & Ford, 1981) has been shown to result in students being able to calculate medians but not necessarily recognizing medians as measures of center or as group descriptors of data (Bakker, 2004; Konold & Higgins, 2003). In fact, other studies indicate that students see the median as a feature associated with a particular data value in the middle of the group rather than as a characterization of the entire group (Bakker, Biehler & Konold, 2005).

Attempts were made to introduce Savannah to the concept of mean. However, she demonstrated difficulties in constructing a rich understanding of the mean and did not conceive of the mean as a representative measure. Following completion of the teaching experiment, Savannah’s understanding of the mean was limited to a fair share approach to the mean, and calculation of the mean could be carried out only through the use of concrete materials. Similarly, other studies have indicated that an overriding feature of the mean that presents difficulty for children is representativeness. A study of fifth through eighth graders found that students did not recognize instances in which the mean could be used to typify a data set, as indicated by the lack of instances where the mean was used to compare two groups of unequal size (Hancock, Kaput & Goldsmith, 1992). The ability, of students to compute representative values when specifically instructed (Mokros & Russell, 1995) compared to their inability to construct representative values in other situations (Hancock et al., 1992) suggests that students do not typically understand the role that representative values play in data analysis.

In summary, many experiences in collecting and graphing data were required before Savannah understood the meaning of data. The use of activities that required data collection, organization, and depiction resulted in Savannah demonstrating the ability to construct and interpret data using a variety of graphical representations. Her limited experiences in school mathematics coupled with her age and grade level resulted in Savannah never moving beyond her use of modal values in representing a data set. She understood the limitations of such measures but was reluctant to relinquish use of the mode. The concept of variability was not explored in any detail with Savannah due to her reluctance to consider the importance of any features of a data set apart from the mode. She did examine the distribution of data around the mode indicating her awareness of variability, however this was only in the context of the mode. However, it does indicate the incorporation of a ‘cluster’ idea, which might be thought of as a precursor or first look at variability.

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Appendix

Task 1 *Number of 'Gummi Bears' in a packet*

A class of students were interested in examining the number of Gummi Bears in a packet. They each counted the number of bears in a packet and put their results on a line plot.

What does the graph tell you about the number of Gummi bears students counted?

From examining the graph, generally how many Gummi Bears would you expect to find in a packet of Gummi Bears? Why? Here are their results:

			X							
	X		X							
	X		X							
X	X	X	X	X	X		X			
X	X	X	X	X	X	X	X	X		
7	8	9	10	11	12	13	14	15	16	17

Task 2 *The Winter Olympics Problem*

The 1984 Winter Olympics were held in Sarajevo, Yugoslavia. The data were reorganized into a line plot. Describe the data. What is the typical number of medals won by a country?

X																								
X																								
X	X		X				X																	
X	X	X	X	X	X		X	X				X											X	X
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

Task 3 *The Raisin Data Controversy*

The general manager of sun maid raisins, Ann Brown, has received an inquiry regarding the average number of raisins that are in a box of sun maid raisins. She went down to the factory floor and picked up 30 boxes of raisins, counted the number of raisins in each box and the put the results on a line plot. Here is the number of raisins that he found in the 30 boxes:

									X	
									X	
							X		X	
							X		X	
		X	X				X	X	X	
	X	X	X	X	X	X	X	X	X	
X	X	X	X	X	X	X	X	X	X	
17	18	19	20	21	22	23	24	25	26	27

Based on this data, if you were the general manager how many raisins would you say are generally in a box of raisins?

Task 4 *Class Height Data*

On school sports day, Miss Murphy's 5th grade basketball team played a game of basketball against Mr. Cody's basketball team. Miss Murphy's team won the game.
Jamie, a student in Mr. Cody's class, believes that the game was not fair. She believes that students in Miss Murphy's class are taller than students in Mr. Cody's class. So, when choosing a basketball team, Miss Murphy has a lot of tall students from which to choose. Jamie decided to measure the height of each student in both classes and construct a line plot for each class. Examine the graphs.
What is the typical height of a student in Miss Murphy's class?

Height of Miss Murphy's students

			X					X										
			X				X	X	X	X								
		X	X				X	X	X	X								
X		X	X	X			X	X	X	X			X	X			X	X
40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58

Task 5

A student in the 6th grade class, Jodie, was also asked to calculate the height of the students in the 6th grade class. So, Jodie measured each member of the class using a tape measure and recorded it on a line plot. This is the line plot of the student's heights.

Height of 6th grade students

			X					X										
			X				X	X	X	X								
		X	X				X	X	X	X								
X		X	X	X			X	X	X	X			X	X			X	X
40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58

Jodie told the principal that the typical height of a 6th grade student was 43 or 48 inches. However, the principal said:
"Tell me one number. You must choose either 43 or 48 inches. Look at the data so that you can make your decision between the two."

Which value do you think Jodie should choose as the typical height – 43 or 48 inches? Why?

Task 6 *What do you think?*

X																								
X																								
X	X		X				X																	
X	X	X	X	X	X		X	X				X											X	X
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	4	4	4	4	4
1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5

Another student who did this task during the summer chose 34 inches as the typical height even though 31 inches was the height that occurred most. Maria said that she chose 34 because:

"I originally chose 31 as a typical value because it occurred the most. But then I looked at the data and saw that there were lots of people who were taller than 31 inches. In fact, even though 31 was the one that occurred most it was also the smallest value. So, I didn't think that I should give the smallest height as the height of most students. There were lots of values higher than 31 so I moved the typical value up higher to reflect more of the students heights. However, I made sure that I kept my new value close to the 31 inches"

What do you think? Is this a good strategy? Why?

Task 7 *How long can you hold your breath?*

The 3rd grade classroom in the same school decided to carry out a similar study. They wanted to see how long they could hold their breaths. They also made a line graph of their results. What is the typical length of time a 3rd grader can hold their breath?

		X							X										
	X	X	X					X	X	X									
X	X	X	X	X			X	X	X	X	X								X
5	6	7	8	9	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2
					0	1	2	3	4	5	6	7	8	9	0	1	2	3	4

Task 6 *What do you think?*

X	X		X				X																	
X	X	X	X	X	X		X	X				X											X	X
X																								
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	4	4	4	4	4
1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5

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What do you think? Is this a good strategy? Why?

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		X							X										
	X	X	X					X	X	X									
X	X	X	X	X			X	X	X	X	X								X
5	6	7	8	9	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2
					0	1	2	3	4	5	6	7	8	9	0	1	2	3	4

Approaches to Reporting Performance on the National Assessment of Mathematics Achievement

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National Assessments of Mathematics Achievement have been implemented in primary schools in Ireland on five occasions since 1978. This report describes three approaches to reporting achievement on the most recent assessment, which was implemented at Fourth class level in 2004. First, performance is described with reference to mean scale scores that are linked to an earlier national assessment at the same grade level in 1999. Second, percent correct scores are used to describe performance on mathematics content areas and processes, and on calculator-appropriate test items. Third, performance is reported with reference to mathematics proficiency levels. The paper discusses ways in which these different scores can be used to interpret achievement and draws attention, where relevant, to differences in outcomes between the 1999 and 2004 studies. It is suggested that proficiency levels in particular may be useful in understanding the strengths and weaknesses of pupils in Fourth class.

Introduction

The 2004 National Assessment of Mathematics Achievement (NAMA 2004) was administered to a representative sample of Fourth class pupils in 130 primary schools in May 2004. The survey – the first since implementation of the 1999 Primary School Mathematics Curriculum began in schools in 2002 – involved the administration of a test of mathematics to pupils, and the completion of questionnaires by the pupils, their parents, class teachers, learning support teachers and school principals.

The target population for NAMA 2004 comprised all pupils in Fourth class in primary schools in Ireland with the exception of those attending private schools, special schools, or special classes in ordinary schools. Pupils with a learning/physical disability that would prevent them from attempting the test, and 'newcomer' pupils whose proficiency in English was so low they could not attempt the test, were also exempted. In addition to six selection strata,[1] schools were sorted according to disadvantaged status, Gaeltacht status, and the proportion of female pupils in the school. Using a random-start, fixed interval selection procedure schools within each stratum were selected with probability proportional to size. Of the 136 schools selected to participate, 130 participated, with completed test information available for 4171 pupils.

The test used in NAMA 2004 consisted of 150 items. One hundred and sixteen of the items were carried forward from a previous national assessment, conducted at Fourth class level in 1999 (NAMA 1999), when the item set had been prepared in anticipation of implementation of the revised curriculum. Thirty four new items were added – nine items to replace those dropped from NAMA 1999 (because they were poorly functioning), and an additional 25 items for which pupils would have a calculator available to them. The items were grouped into sections, each with 25 items. Each of five test booklets included three sections, with all sections, except Section B appearing

[1] Schools were selected in conjunction with the 2004 National Assessment of Reading in First and Fifth classes. Three school sizes (small, medium, and large) by two class options (having either 4th and 5th classes, or 1st, 4th, and 5th classes) created six strata.

once in the initial and final sections of a test booklet. Section B, which was common to all five booklets, always appeared in the middle position in a booklet. (Table 1). Booklets were assigned at random to pupils in participating classes (up to two classes per school). Pupils who were assigned Booklets 4 and 5 had access to a calculator for Section F only. Thirty minutes was allocated to each section, with short breaks between sections. Test administration – which was conducted by pupils' class teachers – was overseen in each school by an Inspector of the Department of Education and Science.

Table 1 *Structure of Test Booklets – NAMA 2004*

Booklet	*First Section*	*Second (Common) Section*	*Third Section*
1	*A*	*B*	*C*
2	*C*	*B*	*D*
3	*D*	*B*	*E*
4	*E*	*B*	*F**
5	*F**	*B*	*A*

***Calculator Section (pupils had access to calculator for this section only).*

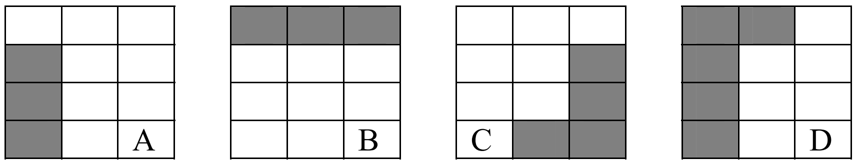
The test items themselves were based on the 1999 Primary School Mathematics Curriculum (DES/NCCA, 1999). Two item types were used – multiple-choice (53.3%) and short-answer (46.7%) items. Table 2 shows the distribution of items across the mathematics content areas and processes in the revised Primary School Mathematics Curriculum.

The following are examples of types of items used in the test. For each item, the content area and principal processes are identified, as well as the principal process skill (see below). The percentage of pupils who answered the item correctly is also given. Additional sample items may be found in Shiel et al., 2006).

Figure 1: Sample Item: Connecting modes of representing fractional numbers

Content strand: Number
Skill category: Integrating & Connecting
Principal process skill: Connect modes of representing fractional numbers
Item type: Multiple-choice

Which of these figures has one third shaded? *(Circle one letter, A, B, C, or D)*



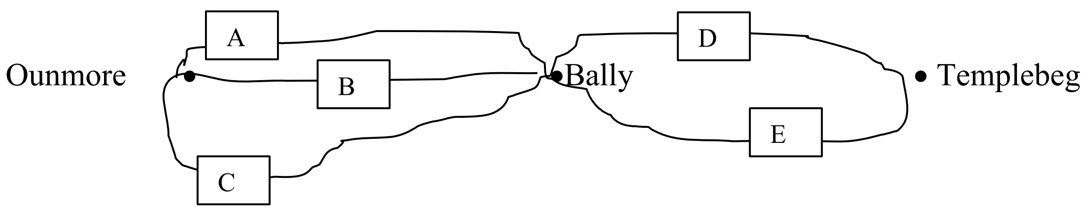
Answer: C

Similar Item	N	% Correct
A20	1669	39

Figure 2: Sample Item: Listing systematically all possible routes on a map

Content strand: Data
Skill category: Reasoning
Principal process skill: List systematically all possible routes on a map
Item type: Multiple-choice

This part of a map shows five roads, A, B, C, D, E. What are all the different ways you can drive from Ounmore to Templebeg?



A CD, BD and AE
B CD, AD, BD and AE
C AE, CD, BD, AD, and BE
D AE, AD, BD, BE, CD, and CE

Answer: D

Performance on the Overall Scale

NAMA 2004 was scaled using Item Response Theory (IRT) methodology. An IRT model was implemented using the BILOG programme (Mislevy & Brock, 1990). For multiple-choice items, three parameters – item difficulty, item discrimination, and guessing – were estimated. For short-answer items, strong prior distributions were set on the guessing parameter. One item was removed from the analysis as it had a negative item/total correlation. The likelihood-ratio chi-square statistics generated by BILOG flagged 13 of the remaining 149 items as potentially fitting poorly to the underlying IRT model. However, an examination of the response curves for the items indicated no substantial deviations from the theoretical curves. Since classical item statistics for these items were satisfactory, and other reasons for eliminating them such as a large percentage of missing responses were discounted, it was decided not to omit any of them. A regression of the derived IRT scale scores on the number of items answered correctly (raw scores) resulted in an R^2 of 96.5%, indicating that the IRT scale provided a satisfactory representation of pupils' achievement on the test.

To facilitate comparisons between results from the 1999 and 2004 NAMA surveys, the 2004 scores were set on the scale used in 1999 using the following procedures: Item parameters were calculated and pupil scores derived for all 2004 items (i.e. both new items and items common to NAMA 1999 and 2004). The new item parameters for the common 1999/2004 items were used to re-score the cases in the 1999 survey and the mean m_n and standard deviation s_n of the new scores were computed. The cases in the 1999 survey were then scored using the 1999 item parameters for the common items, and the corresponding mean m_o and standard deviation s_o of these scores were computed. Pupil scores for the 2004 survey (calculated at step 1) were rescaled to obtain scores for the new test on the old scale using $y_i^* = (s_o / s_n)(y_i - m_n) + m_o$, where y_i is the ith pupil's score on the 2004 scale and y_i^* is the 1999 equivalent score.

Since performance in 2004 is reported on the scale used for NAMA 1999, it is possible to compare performance across the two assessments. In 1999, the mean and standard deviation were set at 250 and 50 respectively (Table 3). In 2004, the obtained mean score was 250.8, and the standard deviation 49.03. The mean score difference (0.8 points) is not statistically significant (SED = 3.23; 95%CI = -7.24 to 5.64), indicating that overall performance did not change between 1999 and 2004.

Performance at Key Benchmarks

Performance was examined at key benchmarks – the 10th, 25th, 50th, 75th and 90th percentiles (Table 4). No statistically significant differences were observed between 1999 and 2004 scores at any of the benchmarks examined. Hence, neither high achievers (those scoring at the 90th percentile) nor low achievers (those scoring at the 10th percentile) registered a significant change in performance between the two years.

Table 3 *Mean Scores on Overall Mathematics Scale, 1999 and 2004*

Year	N	Mean Scale Score	SE	SD
1999	4747	250.0	2.20	50.0
2004	4171	250.8	2.36	49.03

Table 4 *Performance at Key Benchmarks on the Overall Mathematics Scale, 1999 and 2004*

	1999		2004	
Percentile	Score	SE	Score	SE
10th	181.0	3.01	182.9	3.04
25th	217.3	2.73	221.2	2.96
50th	256.3	2.39	253.6	2.78
75th	284.4	1.82	285.0	2.89
90th	308.7	1.75	311.2	2.81

Performance on Mathematics Content Areas and Processes

Each item in the assessment was categorised according to the mathematics strand it assessed. The distribution of items across strands reflects their representation in the 1999 Primary School Mathematics Curriculum. This section describes performance on the mathematics content areas (strands), mathematics processes, and calculator appropriate items.

Mathematics Content Areas

The weighted mean percent correct scores for each of the five mathematics strands, aggregated across the five booklets, are presented in Table 5. Pupils achieved the highest mean percentage correct score on items relating to Data (68.8% correct), and lowest on items dealing with Measures (49.2% correct). Mean scores for Number, Algebra, and Shape & Space items were close to the overall mean of the test (57.6%).

Table 5 *Percentages of Items and Mean Percent Correct Scores, by Mathematics Content Strand, 2004*

Strand	% of Items	Mean % Correct	SE
Number	38.7	55.6	0.92
Algebra	4.7	55.9	1.19
Shape& Space	14.0	55.9	1.08
Measures	32.0	49.2	1.09
Data	10.7	68.8	0.86
Total	100	57.6	0.94

Comparisons between performance in 1999 and 2004 are limited to the 116 items that were common to both years. Significant differences were obtained for two strands only (Table 6). The mean scores for items assessing Shape & Space, and Data are significantly higher in 2004 than in 1999. For both, the increase was in the order of 5%.

Table 6 *Mean Percent Correct Scores on Common Items, and Mean Score Differences, by Mathematics Content Strand, 1999 and 2004*

Strand	1999 Study		2004 Study		Diff with 1999	SED	95%CI	
	%	SE	%	SE				
Number	57.4	.93	56.7	.901	–0.74	1.30	-2.7	4.2
Algebra	58.3	.94	61.0	1.16	2.68	1.49	-6.5	1.3
Shape & Space	50.8	.87	55.7	1.10	4.96	1.40	-8.7	-1.2
Measures	54.1	.89	54.0	1.07	–0.03	1.40	-3.4	3.7
Data	66.0	.81	71.3	0.92	5.30	1.22	-8.5	-2.1
Total	57.3	.78	57.6	0.94	0.23	1.22	-2.1	2.7

Significant differences in **bold**; SE = Standard Error; SED = Standard Error of the Difference

Performance on Mathematics Processes

The percentage of items assessed in each mathematics process tested in NAMA 2004, and the mean percent correct score for each are presented in Table 7. Pupils achieved the highest mean scores on items assessing basic mathematics skills – Understanding & Recalling (61.7%) and Implementing (57.9%). They performed least well on items assessing higher-order skills, including Applying & Problem Solving (48.2%).

Table 7 *Distribution of Common Items by Mathematics Skill, and Mean Percent Correct Scores, 2004*

Skills	% of Items	Mean % Correct	SE
Understanding & Recalling	12.3	61.7	1.03
Implementing	28.0	57.9	0.92
Reasoning	20.3	57.0	0.94
Integrating & Connecting	7.3	55.5	1.27
Applying & Problem Solving	32.0	48.2	0.98
Total	100	57.6	0.94

SE = Standard Error

A comparison of performance on the 116 items common to the 1999 and 2004 assessments revealed a significant increase in 2004 for one skill, Reasoning, for which the mean score increased by almost 4% (Table 8). In both years, pupils achieved the highest scores on Understanding & Recalling, and the lowest on Applying & Problem-Solving.

Table 8 *Mean Percent Correct Scores on Common Items, and Mean Score Differences, by Mathematics Skill, 1999 and 2004*

Skill	1999 Study		2004 Study		Diff with 1999	SED	95%CI	
	%	SE	%	SE				
Applying & Prob. Solving	50.5	0.90	49.0	0.99	–1.5	1.32	-1.96	5.02
Integrating & Connecting	53.2	1.00	55.0	1.37	1.5	1.72	-6.57	2.57
Reasoning	56.8	0.87	60.7	1.00	3.9	1.31	-7.52	-0.58
Implementing	58.8	0.89	59.4	0.92	0.5	1.29	-3.88	2.96
Understanding & Recalling	61.9	0.88	63.8	1.05	1.8	1.36	-5.42	1.78
Total	57.3	0.78	57.6	0.94	0.2	1.22	-2.14	2.74

Significant differences in **bold**; SE = Standard Error; SED = Standard Error of the Difference

Performance on the Calculator Appropriate Items

One of the main differences between the 1999 and the 2004 national assessments was the inclusion of 25 calculator-appropriate items (included as the last section in one booklet, and the first in another) in 2004. Such items were designed to be done using a calculator. Pupils performed less well on the calculator section than on other sections, with an overall percent correct score of just under 40%, compared to an average of 55% across all sections. Here, we look at the characteristics of selected calculator-appropriate items. Table 9 indicates, for each of several selected items, the percent correct score, and the level of difficulty of the item within the calculator section.[1]

Table 9 *Descriptions of Selected Calculator Items, Percent Correct Scores, and Item Difficulties, 2004*

Item	Descriptor	% Correct	Difficulty
F2	Identify the number in a number sentence that should be left out to make it correct (e.g. 175 +236 + 318 + 240 = 733)	66.1	Easy
F8	Supply a missing digit to make a number sentence correct (e.g. 4_5 ÷ 9 = 45)	42.9	Difficult
F9	Indicate the missing operation to make a number sentence correct (e.g. 27 _ (31 _ 11) = 540)	49.1	Moderate
F12	Compare the performance of athletes over two rounds of a competition, where distances arepresented as decimal numbers.	40.1	Difficult
F16	Identify the next number in a sequence (e.g. 4.2, 8.4, 16.8, ?)	13.8	Difficult
F18	Find the perimeter of a field where the length and width are decimal numbers	37.1	Difficult
F24	Solve a routine problem involving operations with fractions (e.g., The normal price of atoy is €13. Its price is reduced by a quarter in a sale. What is the sale price?)	25.8	Difficult

Among the items that students found difficult were supplying a missing digit in a number sentence to make it correct, identifying the next decimal number in a sequence of decimal numbers, and solving a routine problem involving operations with fractions.

[1] Difficulty levels were arrived at by identifying the 33rd (46.8%) percentile and 67th percentile (65.4%) on the distribution of percent correct scores for items on the calculator section.

It may be that pupils taking the test had insufficient practice with calculator usage to enable them to apply their mathematical knowledge in attempting to answer these items.

Performance on the Proficiency Scale

A broad indicator of overall performance, such as a mean score, or a score at a key benchmark, tells us nothing about the mathematics skills that pupils can use. For this reason, mathematics proficiency levels were developed for NAMA 2004.

An item descriptor was generated for each test item. The item descriptor details both the content of the item and the underlying process skills. The item descriptors for the two example items were: Connect modes of representing fractional numbers (Figure 1) and List systematically all possible routes on a map (Figure 2). Item descriptors were repeated as required where they were linked to two or more items. The steps in developing the proficiency levels were as follows:

Logit values for each item/descriptor (derived from IRT scaling) were examined with a view to identifying appropriate cut-off points on the standard score scale between clusters of items. Using an iterative process, cut-points were identified.

Duplicate item descriptors *within* each band were eliminated. Where two items had different scale values, the item at the highest point on the scale within a band was the one retained on the basis that if pupils are likely to be able to do the upper item of the pair then they are even more likely to be able to do the lower one(s) on the scale.

Next, duplicate items/item descriptors *across* bands were eliminated. Again, the item at the highest point on the scale was the one retained.

Where two items within the same proficiency band were judged to belong to substantially overlapping item domains the item descriptor was modified to include both items and the item lower on the scale was then eliminated.

The item descriptors within each proficiency level were then grouped and ordered, firstly by content area/strand (Number and Algebra; Shape & Space; Measure; and Data), and then by cognitive process skill (Recall and Procedural Skills; Reasoning with and Connecting Conceptual Knowledge; Problem-Solving). This was done to ensure that each proficiency level had the same underlying organisation.

Since IRT places pupils and items on the same scale, it was possible to assign a level to each pupil, based on his/her overall performance on the test (also expressed as a logit value). Pupils scoring at a particular point on the scale have a 50% chance of getting items at that point on the scale correct. They have a greater than 50% chance of getting items at lower points on the scale correct, and a less than 50% chance of getting items at higher (more difficult) points. As the range of item values is narrower than the range of pupil values, pupils with logit values that were lower than the item with the lowest logit value could not be assigned to a level. These pupils were considered to have scored below Level 1, and hence it was concluded that their achievement was not assessed by NAMA. Table 10 shows the distribution of pupils by proficiency level.

Table 10 *Definition of Proficiency Levels*

Level	Descriptor	Interval
Level 5	Advanced	>1.05 (logits)
Level 4	High	>0.35 and <=1.05
Level 3	Moderate	>-0.25 and <=0.35
Level 2	Basic	>-0.95 and <=-0.25
Level 1	Minimum	>-2.05 and <=-0.95
Below Level 1	Achievements not assessed	<=-2.05

Table 11 provides a summary of the process skills assessed at each proficiency level. This is an abbreviated version of the full scale, which details the principal (or dominant) process skills assessed by all 149 valid items.

Table 11 *NAMA 2004 Proficiency Levels – Summary Descriptors*

Level 5 (>= 1.05) Advanced Level of Mathematics Achievement
Implement procedures for estimating sums and quotients
Connect decimal and fraction notation in measure contexts
Extend more complex patterns in number
Hypothesise and test answers for correctness (mixed operations number sentences)
Apply concepts of ratio and proportion in practical contexts
Solve non-routine multi-step problems involving fractions and measures
Level 4 (< 1.05 ≥ 0.35) High Level of Mathematics Achievement
Recall and use definitions of parallel and perpendicular lines
Identify angle types in 2-D shapes
Partition 2-D shapes using fractions
Add measures of length
Identify missing information in problems
Identify a fraction between two fractions
Make informal deductions about properties of 2-D shapes
Apply concept of scale to reading maps
Hypothesise and test answers for correctness in multiplication or division sentences
Convert fractions to decimals
Solve routine problems involving calculation of perimeter
Level 3 (< 0.35 ≥ – 0.25) Moderate Level of Mathematics Achievement
Calculate a fraction of a number
Divide a decimal by a whole number
Round four-digit numbers
Estimate products of whole numbers
Implement procedure for division of whole numbers
Order fractions in terms of magnitude
Identify fractional areas of regular 2-D shapes
Visualise properties of 3-D shapes from 2-D nets
Complete number sentences involving associative and distributive properties
Connect verbal, diagrammatic and symbolic representations of problems
Hypothesise and test answers for correctness (single operation number sentence)
Solve non-routine one-step problems involving operations with fractions and measures

Level 2 (< – 0.25 ≥ – 0.95) Basic Level of Mathematics Achievement Calculate area of regular shapes using a grid Identify decimal between two decimals Implement procedures for multi-digit subtraction and long multiplication Select appropriate units of measure Connect diagrammatic and verbal representations of problems Visualise and identify properties of 2-D and 3-D shapes Extend decimal number patterns Reason with place value and notation of 4-digit numbers and decimals Hypothesise answers and test them for correctness (addition number sentences) Apply Unitary Method in everyday contexts Make informal deductions about simple graphical data Analyse tables of data to solve routine and non-routine problems Solve routine problems involving operations with whole numbers, fractions, and measures Solve non-routine problems involving operations with whole numbers
Level 1 (< – 0.95 ≥ – 2.05) Minimum Level of Mathematics Achievement Recall basic multiplication and division facts Identify place value in four-digit numbers and in two-place decimals Identify properties of 2-D shapes Implement procedures for multi-digit addition and short multiplication Order simple events in terms of likelihood of occurrence Read and interpret bar charts, line graphs, tables, decimal scales, and area diagrams Identify and extend simple number patterns Combine and partition 2-D shapes into sets of specified shapes Solve simple, routine word problems involving multiplication/division facts; calendar; subtraction; chance
Below Level 1 (< - 2.05) Level of mathematics knowledge not assessed by this test

Table 12 shows the distribution of pupils by proficiency level.

Table 12 *Numbers and Percentages of Pupils at Each Proficiency Level on the Overall Mathematics Scale, 2004*

Level	Label	Number	% of Pupils	SE
Level 5	Advanced	488	11.7	1.11
Level 4	High	1072	25.7	1.50
Level 3	Moderate	1087	26.1	1.33
Level 2	Basic	906	21.7	1.47
Level 1	Minimum	512	12.3	1.06
Below Level 1	Below Minimum Level	106	2.6	0.55

SE = Standard Error

Just under 12% achieved at an ‘Advanced’ level (Level 5). Pupils performing at this level are likely to succeed on the most complex tasks in NAMA 2004, such as implementing procedures for estimating sums and quotients, and solving non-routine problems involving fractions and measures. Pupils achieving at this level would also be expected to do well on items at lower levels. Pupils are evenly distributed at the centre of the distribution of proficiency levels, with 26% achieving a 'High' level (Level 4), 26% a 'Moderate' level (Level 3), and 22% a 'Basic' level (Level 2). Pupils achieving a High level can be expected to succeed on items such as recalling and using definitions of parallel and perpendicular lines, partitioning 2-D shapes using fractions, and solving routine problems involving the calculation of perimeter. Pupils achieving a Moderate level can be expected to succeed on items that involve rounding whole numbers, estimating products of whole numbers, and solving non-routine one-step problems involving operations with fractions and measures. Pupils at the Basic level can be expected to visualise and identify properties of 2-D and 3-D shapes, solve routine problems involving operations with whole numbers, fractions and measures, and solve non-routine problems involving operations with whole numbers. Twelve percent of pupils achieved a 'Minimum' level (Level 1). These pupils can be expected to recall basic number facts, identify place value in whole numbers and decimals, and solve simple, routine word problems involving multiplication/division facts, subtraction and chance. The mathematics achievements of pupils scoring below Level 1 (3%) are not assessed by NAMA 2004. These pupils have a less than 50% chance of answering Level 1 items correctly.

It is also possible to select a specific aspect of mathematics (e.g., 2-D shapes) and follow its progress through the proficiency levels. For example, pupils scoring at Level 1 can be expected to identify the properties of 2-D shapes. Those scoring at Level 3 can be expected to identify fractional areas of regular 2-D shapes, while those at Level 4 can be expected to identify angle types in 2-D shapes, and make informal deductions about the properties of such shapes. Similarly, problem solving advances from solving simple, routine word problems involving operations with whole numbers (Level 1), to solving non-routine problems involving operations with whole numbers (Level 2), to solving non-routine one-step problems involving operations with fractions and measures (Level 3), to solving non-routine multi-step problems involving fractions and measures (Level 5).

Conclusions

This paper reviewed approaches to reporting on the outcomes of the National Assessment of Mathematics Achievement, and looked in trends in mathematics achievement between 1999 and 2004. .

First, consideration was given to the use of scale scores to report on performance. Scale scores and scores at key benchmarks such as the 10th or 90th percentiles are useful for describing achievement in an overall sense. In 2004, pupils at Fourth class level achieved a mean score that was not significantly different from the mean score of pupils at the same class level in 1999. This suggests that changes in curriculum between 1999 and 2004 had not yet impacted on performance in mathematics. Further, since no changes were observed at key benchmarks, such as the 10th and 90th percentiles, between the two years, it can be concluded that changes in curriculum did not impact differentially on pupils at different levels of performance.

Percent correct scores were used to describe the performance of students on the mathematics content areas and processes in NAMA 1999 and 2004. The improvement in performance on Shape & Space between the two years is encouraging, particularly as students in Ireland have done poorly in this area in recent international assessments at post-primary level (e.g., Cosgrove et al., 2005). The significant improvement on Data is also welcome, and may reflect improvement in handling probability, which is included in the Data strand on the revised curriculum. However, improvement in performance in Shape & Space and Data did not have a large impact on overall scores. Some of the gains in these areas may have been offset by a drop in achievement (albeit not statistically significant) in Number and Measures – two areas that are more strongly represented in NAMA than Shape & Space or Data.

Although improved performance between 1999 and 2004 is welcome on items categorised as assessing Reasoning, it has to be acknowledged that performance on Applying & Problem Solving is still low (48%) correct, compared with, for example, Understanding & Recall (62%). Other studies have also pointed to difficulties with problem solving. The DES Inspectorate (DES, 2005) reported weaknesses in teaching problem solving in one-third of classrooms across all levels, while the NCCA (2005) noted that teachers over-relied on traditional textbook problems when teaching problem solving. Concerns about problem solving are not confined to primary level. In PISA 2003 mathematics, just 11% of Irish 15-year olds were able to solve complex problems (those at Levels 5 and 6 on the PISA proficiency scale) compared to an OECD average of 15%, and 24% in Finland. This seems to suggest that pupils would benefit from additional attention to this important aspect of mathematics.

The performance of pupils in NAMA 2004 on calculator-appropriate items was disappointing given that calculators now form part of the primary school curriculum. However, according to data from the Teacher Questionnaire administered as part of NAMA, one-third of pupils in the study were taught by teachers who reported that they hardly ever or never used calculators in mathematics classes, while a further 45% were taught by teachers who did so just once or twice a month. Clearly, calculator usage is not widespread at Fourth class level, and this may have impacted on the ability of pupils to use this tool to solve problems on the Calculator section. It may be that, as curriculum implementation in mathematics accelerates in the future, greater attention will be given to the frequency of calculator usage and the nature of mathematics problems that are presented to pupils as they work with calculators

Proficiency levels were used to describe the types of items that pupils were likely to be successful on at different points along the mathematics achievement scale. Five levels of proficiency were described, as well as ‘below Level 1’, which is assigned to pupils who are unlikely to succeed on Level 1 items. Proficiency levels are useful to the extent that they describe what pupils can do (and, by inference, what they cannot do). They also provide a useful benchmark against which to judge performance in future national assessments of mathematics achievement. As teachers become familiar with proficiency levels, they may begin to interpret test performance in terms of the specific mathematical content and processes.

Preparations are already underway for the next National Assessment of Mathematics Achievement in 2009. This will be administered in the Second and Sixth classes, as well as in Fourth class. Subsequent assessments will be administered in Second and Sixth classes only.

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A Team Approach to Number Concept Development

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Number Worlds (Griffin and Case, 1997) is based on the Rightstart intervention (Griffin, Case and Siegler, 1994) which was used successfully to close the number knowledge gap between children in schools in low-income, high-risk communities and their more affluent peers in the United States. It is an in-class early intervention programme beginning in Junior Infants. Findings from a research study carried out in 2004/05 to assess the impact of Number Worlds in an Irish context merited a replication of the study on a larger sample of pupils. Therefore, during the school year 2005/06, a similar study was carried out on a larger sample of pupils in disadvantaged schools in Dublin 8, 10 and 22. The research involved teams of parents and teachers (class and learning support/resource) working together to help children develop number concepts. The purpose of the research study was to assess the effectiveness of the programme in relation to the number element of the revised mathematics curriculum. Drawing on both quantitative and qualitative data including video analysis the study sought to ascertain the key elements across different contexts that defined the effectiveness or otherwise of the programme. The potential of the programme to facilitate a team approach to in-class support in mathematics and involve parents was also a key focus of the study.,The research report highlights the main differences between traditional approaches to teaching number and the Number Worlds approach. The results point to positive learning outcomes in number concept development and in social skills for children who were taught number concepts using the Number Worlds programme. Its potential as a vehicle for in-class support and parental involvement also emerged as a key finding.

Introduction

There is a general acceptance of the importance of mathematics education. It is perceived as a core area of learning in most educational systems throughout the world. Irish students appear to perform relatively well in terms of mathematical achievement when compared to their international peers (TIMSS, 1995, PISA, 2003). However, within Ireland, the discrepancy between the mathematics achievement of pupils in schools designated as disadvantaged and in non-disadvantaged schools has been a concern for many years (Greany and Close, 1989; Sheil and Kelly, 2001, Weir 2003, DES, 2005a; Shiel, Close Surgenor and Millar, 2006). This study is about a mathematics intervention called Number Worlds, which aims to address this discrepancy. The purpose of the research was to assess the effectiveness of the intervention in relation to the number element of the Revised Mathematics Curriculum (DES, 1999) and it builds on a pilot project (Mullan and Travers, 2007). The intervention uses a combination of group work and play as an alternative to traditional teaching approaches and is based on Griffin, Case and Siegler's Central Conceptual Structure theory (1994).

Central Conceptual Structure

What does a child need to know about a number in order to understand the concept of that number? Many prominent researchers have addressed this question, the most influential being Piaget (1952). According to Piaget, the development of number was closely linked to the development of logic and conservation and this led to an emphasis on developing the skills of ordering, classifying and matching. Griffin, Case and Siegler (1994) proposed the Central Conceptual Structure (Fig. 1) to encapsulate all that a child needs to know in order to understand number concepts.

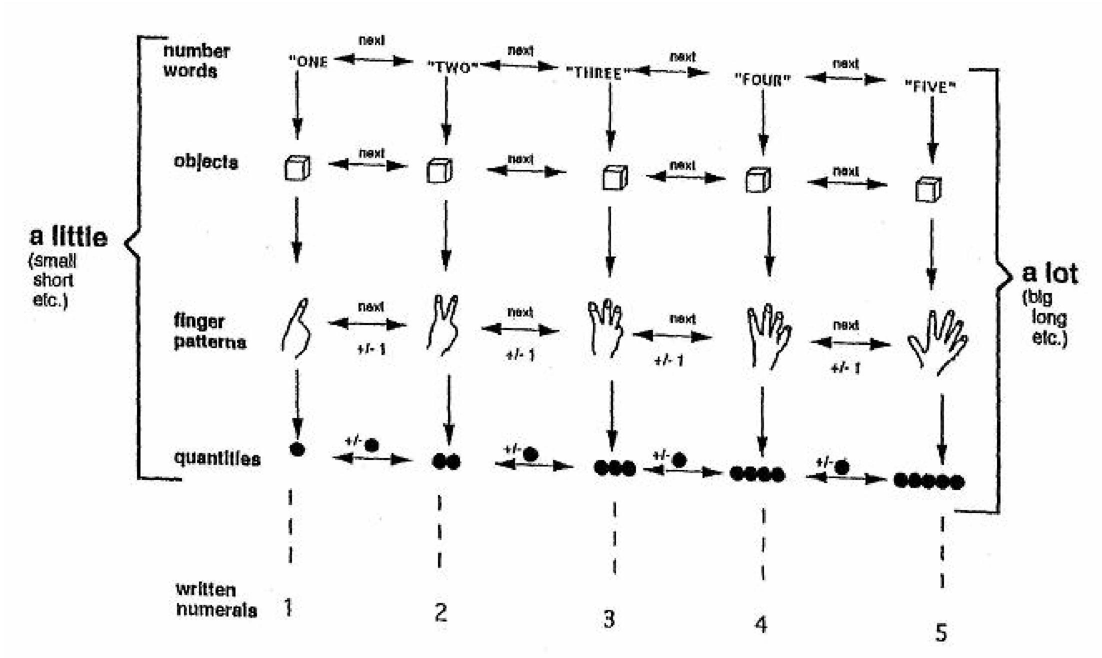

Figure 1: "Mental Counting Line" Central Conceptual Structure

Griffin, Case and Siegler (1994) proposed that children need to have a representation of number that is akin to a mental counting line and that children must: 1) be able to generate the verbal label for each number; 2) understand 1-1 correspondence; 3) understand that each verbal label has a set size associated with it which has a certain canonical perceptual form; 4) understand that movement from one of these set sizes to the next involves the addition or subtraction of one unit and 5) recognise the written numerals 1- 10. In the pilot study, this knowledge which is implied in the Central Conceptual Structure was found to be easily turned into teaching objectives and to be more useful than obscure theories which are not always of immediate relevance to teachers (Mullan and Travers, 2007).

Number Worlds games were designed to teach the knowledge implied in the Central Conceptual Structure. Griffin, Case and Siegler (1994) argue that this knowledge is essential to performance on a broad array of mathematical tasks and that the absence of this knowledge constitutes the main barrier to learning arithmetic. If there has not been a heavy emphasis on counting or quantity in early home environments then, according to Griffin, Case and Siegler (1994), counting and quantity *should be* the core focus of the school mathematics curriculum. Counting is one of the strands of the Revised Mathematics Curriculum (DES, 1999) but it is not included in the Early Mathematical Activities of the Curriculum.

Play and Group Work

The report of the most recent assessment of performance in mathematics at fourth class level in primary schools *NAMA 2004* (Shiel *et al,* 2006) advised that the recommendations in the LANDS (DES, 2005a) report be implemented. One of the recommendations is that teachers should incorporate differentiated approaches to a much greater degree in order to cater for all pupils, including those with special educational needs. Differentiated approaches however, are easier said than done and require detailed planning, a high-level of organisational skills and a thorough knowledge of children's levels of understanding (Westwood, 2007). The Number

Worlds programme provides teachers with detailed structured lesson plans with reachable teaching objectives for different levels of ability and understanding. Objectives are reached through play in a whole class setting, in group settings and through a language round-up at the end of each session.

There should be provision for ample time, materials and teacher support for children to engage in play (NAEYC and NCTM, 2002). The NAEYC suggest that over time, children can be guided from an intuitive to a more explicit conceptual understanding of ideas through play. When children become intensely engaged in play, pursuing their own purposes, they tend to tackle problems that are challenging enough to be engrossing but not totally beyond their capacities. In addition, when several children grapple with the same problems they often come up with different approaches and learn from one another (NAEYC, 2002). Kirk (2003) argues that in recent years the television and the computer screen occupy children's attention for more time than the amount of time they spend interacting with family and friends. Thus children lack basic social skills such as listening, sharing, taking turns and resolving conflicts constructively (Kirk, 2003). When children play together in groups they learn necessary social and communication skills from one another and different backgrounds, values and abilities enhance learning opportunities (Sharan and Sharan, 1976). Additionally, the opportunity to explain material/ideas to their own classmates in a simple manner helps children to internalise new ideas, formulate new concepts and monitor their own progress (Reynolds and Muijs, 1997, Gettinger and Stoiber, 1999).

Method

The purpose of this evaluation of the Number Worlds programme was to assess its effectiveness in the Irish context in terms of academic and social outcomes in designated disadvantaged contexts and its potential to facilitate more in-class support work from learning support/resource teachers and parents. This necessitated a multi-method study. To assess the academic outcomes a quasi-experimental design was used. Seventeen classes were assessed on the Number Knowledge test, a validated test of early number ability (Clarke and Shinn, 2004). Fourteen of these classes were in designated disadvantaged contexts and the remainder in non-designated settings. Nine of the classes in the designated contexts implemented the Number Worlds programme while five classes in these schools continued using their current programmes. All of the three control classes in the non-designated schools did likewise (Tables 1 and 2).

To assess the social, other academic and in-class effects, qualitative data in the form of group and individual interviews were held with all class teachers and learning support/resources teachers implementing the programme along with a sample of parents. In addition, a video analysis of one complete lesson was also conducted in the class which produced the highest mean increase in scores on the Number Knowledge test. All teachers received professional development input on the aims of the programme, the psychology of early number development and a video demonstration of a lesson taken during the pilot phase. This amounted to one day in total. Each class involved in the study was visited at least once during the period of the intervention. There were variations in the level of implementation fidelity to the format of the programme and such variations were recorded.

Fidelity is a major issue in the evaluation of any intervention. In the complex environment of the classroom there is a balance between following a programme exactly as intended by the authors and exercising professional judgment in terms of adapting the programme to the differing classroom contexts. Ideally Number Worlds lessons should have three parts. The first ten minutes should be spent on whole-class games and the next fifteen to twenty minutes should be spent on small group games. The final part of each lesson involves the whole class listening while one child from each small group is prompted by the teacher to recount what happened during small-group games. Table 3 summarises the number of times per week that groups spent on aspects of Number Worlds lessons.

In assessing an intervention in a real world context it was recognised that not all variables can be controlled for. It also very difficult to pinpoint what exactly makes the difference in an intervention. Every effort was made to build up an accurate picture of the organisation of the teaching of number in all of the classes involved. In this way we hoped to describe the different circumstances in which the implementation of the programme was played out and the effects it had. Table 5 is a summary of teachers' experience.

Pawson and Tilly (1997) recommend focusing on contexts, mechanisms and outcomes in evaluation research. Contexts in this study refer to variables outside the control of the evaluation: the setting of the school, the experience and expertise of the teachers, the motivation of the teachers and class size. Mechanisms refer to the actions taken by teachers and others in using the programme to produce outcomes. In this study this encompasses the level of additional in-class support and how it was utilised across classrooms, the degree of implementation fidelity and the emphasis placed on various aspects of the programme by the teachers. Outcomes in the study refer to the level of engagement of the pupils with the programme and their scores on the Number Knowledge test at the end of the intervention. It also includes the level of in-class support facilitated by use of the intervention. The evaluation seeks to describe the contexts that produce desirable outcomes**.**

The sample of schools involved in the study was purposive. Schools designated as disadvantaged were invited to participate and invest in the programme based on a positive pilot study in one school (Mullan and Travers, 2007). Others were invited to be control groups with the option of using the programme later. Three of the schools (Schools 2, 4 and 6) were matched with *Number Worlds* groups in terms of age, class, and community. Three further control groups (Groups 7, 7a and 8) were matched with the *Number Worlds* groups in terms of age and class but not in terms of community as they were in two non-disadvantaged schools. Quantitative data were analysed using ANOVA tests on the pre and post test scores from the Number Knowledge test. The percentage increase in the mean score of each class was also tabulated. Interview data and observations from video analysis were coded and categorised into themes arising from the purposes of the evaluation.

Table 1: *Type, status, enrolment and number of groups per school.*

	Type	Status	Enrolment 05/06	Groups
1	Mixed School Junior Infants to 6^{th} class	DEIS Urban Band 1	214	1 NW Junior Infants (1)
2	Mixed Junior School Junior Infants to 2^{nd} class	Disadvantaged	522	1 NW Junior Infants (2) 1 Control Junior Infants (2a) 1 NW Senior Infants (S2) 1 Control Senior Infants (S2a)
3	Mixed School Junior Infants to 6^{th} class	DEIS Urban Band 1	211	2 NW Junior Infants (3) and (3a)
4	Mixed School to 1^{st} class Girls only from 2^{nd} to 6^{th} class	Urban Band 1	256	1 NW Junior Infant (4) 1 Control Junior Infants (4a)
5	Mixed School Junior Infants to 6^{th} class	DEIS Urban Band 2	Need to confirm	2 NW Junior Infants (5) and (5a)
6	Junior Mixed School Junior Infants to 2nd class	DEIS Urban Band 1	260	1 NW Junior Infants (6) 2 Control Junior Infants (6a) and (6b)
7	Mixed School Junior Infants to 6^{th} class	Non-Disadvantaged	457	2 Junior Infants (7) and (7a)
8	Mixed School Junior Infants to 6^{th} class	Non-Disadvantaged	705	1 Junior Infants (8)

Table2: *Summary of groups by school status*.

Junior infants designated disadvantaged implementing Number Worlds	Junior Infants designated disadvantaged control classes	Junior infants non-designated control classes	Senior Infants designated disadvantaged implementing Number Worlds	Senior infants designated disadvantaged control classes
8 classes	4 classes	3 classes	1 class	1 class
1, 2, 3,3a,4,5,5a,6	2a, 4,6a,6b	7,7a,8	S2	S2a

Table 3: *Number of children in each group.*

Group	1	2	2a	S2	S2a	3	3a	4	4a	5	5a	6	6a	6b	7	7a	8
Class Size	23	25	22	20	22	17	15	20	18	18	17	20	19	19	31	31	27

Table 4: *Mean age of children in each group at pre-test*

Group	1	2	2a	S2	S2a	3	3a	4	4a	5	5a	6	6a	6b	7	7a	8
Mean Age	4.5	4.6	4.7	5.9	5.9	4.6	4.9	4.8	4.7	4.5	4.4	4.9	4.4	4.6	4.9	4.9	4.9

Table 5: *Teaching experience of teachers of each group***.**

Groups	**1**	**2**	**S2**	**3**	**3a**	**4**	**5**	**5a**	**6**	**2a**	**S2a**	**4a**	**6a**	**6b**	**7**	**7a**	**8**
Teaching Experience	3	6	3	3	7	12	3	1	2	3	4	4*	2	7	3	2	17
Current class	0	1	2	0	1	3	0	0	0	3	2	4	0	3	1	*0*	2

*Teaching experience in Montessori School

Table 6: *Number of aspects of NW lessons played per week in each group.*

NW groups	1	2	S2	3	3a	4	5	5a	6
Whole Class Games	2	2	0	4	4	3	5	5	5
Small Group Games	3	4	3	4	4	3	4	4	5
Language Round Up	0	2	0	0	0	3	2	2	5

Table 7: *Number of adults in each group per week*

	Number Worlds Groups									**Control Groups**							
Group	1	2	S2	3	3a	4	5	5a	6	2a	S2a	4a	6a	6b	7	7a	7b
No. of adults	2	2	2	3*		4	4	4	2	2	2	1	2	2	1	1	1
Days per week	3	2	2	4		3	4	4	2		2		2	2			

**Groups 3 and 3a worked together and had support from a Resource Teacher 4 days per week.*

Findings and discussion

Table 8: Mean pre-test, post test and percentage increase in mean score of each group

Group	Pre-test	Post -test	% increase
1	6.35	7.9	24%
2	4.92	8.21	67%
2a	3.86	5.57	44%
S2	8.9	14.8	66%
S2a	9.27	11.35	22%
3	5.06	9.94	96%
3a	6.67	10.31	55%
4	4.15	9.78	136%
4a	4.56	7.76	70%
5	5.06	9.39	86%
5a	4.82	8.89	84%
6	6.15	9.6	56%
6a	6.47	9.06	40%
6b	4.58	6.89	50%
7	8.71	11.37	31%
7a	8.71	10.79	24%
8	9.11	11.18	23%

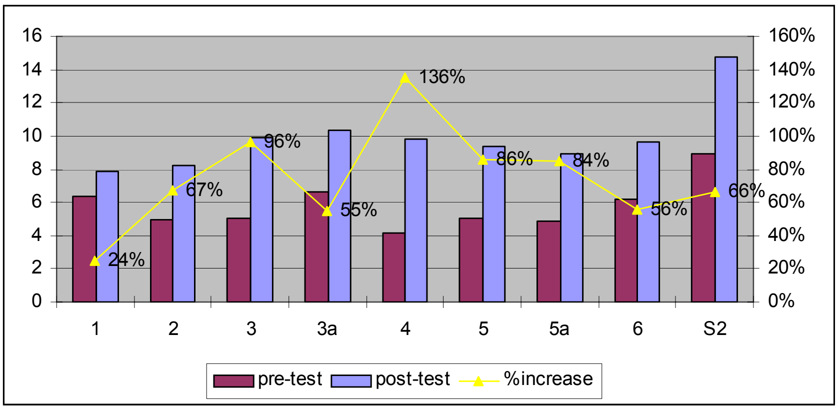

Figure 2: Percentage increase of mean NKT scores of Number Worlds Groups.

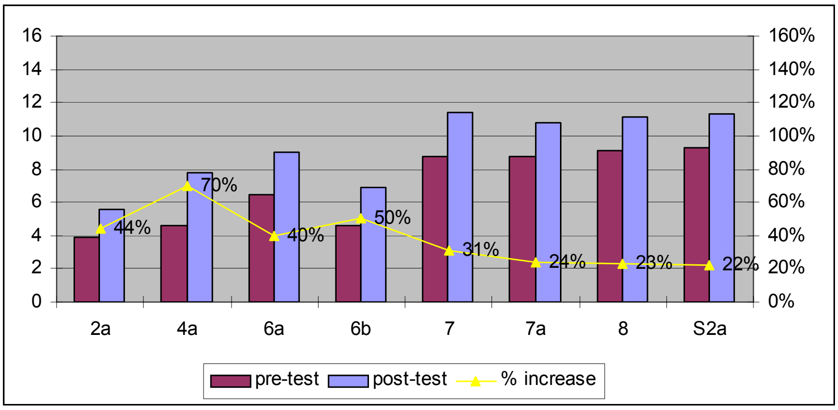

Figure 3: Percentage increases of mean NKT scores of Control Groups.

The mean pre-test and post-test scores of each group can be seen in Table 8, along with the percentage increase in score of each group. In Figures 2 and 3 the same data is presented separately for Number Worlds and Control Groups. Mean scores of children in Number Worlds Groups ranged from 24% (1) to 136% (4) and the mean percentage increase was 74%. In Control Groups the percentage increase ranged from 22% (S2a) to 70% (4a) and the mean percentage increase was 38%. Significant differences in mean scores that had existed at pre-test between the Junior Infant Number World disadvantaged groups (2,3, 4,5 and 5a) and the Junior Infant non-disadvantaged groups (7, 7a and 8) were reduced and were no longer significant. However, this was not the case for three of the four Junior Infant disadvantaged control groups (2a, 4a and 6b) whose mean post-test scores were significantly lower than the non-disadvantaged groups (7,7a and 8). The mean post test score of Group 2a was also significantly lower than those of Number Worlds groups (3, 4, 5, and 6).

It is difficult to disentangle the reasons for differences in scores. There are many possible influences on children's learning including teacher influences, children's language ability; cognitive ability; illness; parental support and cultural differences. However, it is possible that the higher increase in mean scores of most Number Worlds groups was due to differences in the ways in which children experienced number. These differences are outlined in the following sections under the headings, Exceptions, Numbers in Context, Group Work, Adult Support, Written Work and Language.

Exceptions

The percentage increases of Groups 1 (24%) and 4a (70%) stand out as exceptional within their groups as Group 1 was the only Number Worlds Group to have an increase of mean score lower than 55% and Group 4a was the only Control Group to have a mean increase in score higher than 50%. The following points are noteworthy, as they may have influenced scores in groups 1 and 4a.

Group 1: Group 1 spent less time per week on aspects of Number Worlds than any of the other Junior Infant Number Worlds Groups (Table 6). The language round-up section of a Number Worlds lesson was omitted. However, Groups 3 and 3a also omitted the language round up and their increases in mean scores were 96% and 55% respectively. The teacher of Group 1 was supported by a Special Needs Assistant for small group work three days per week but reported that she could have done with more adult help:

> "It was great to have her (Special Needs Assistant) … but I probably could have done with more help. I found it very hard to supervise the other groups – if you were teaching a new game to one group …the others…they'd be fighting about whose turn it was or losing the dice or whatever, you know."(Group 1)

However, the teacher of Group 1 was not alone in this view. Several teachers expressed the same opinion:

> "Overall it was easier when the ratio was smaller" (Groups5 and 5a)

> "It would have been brilliant to have a person with each group" (Group 2)

Group 4a: Group 4a were taught number by teacher with a Montessori degree whose teaching methods included lots of play and sensorial materials. One of the principles governing Montessori methods is that concepts should be conveyed to children "not so much through the eyes and ears, but through the child's hands….cognition is born from manual movement" (Lillard, 2005). Children in Group 4a experienced number not as individual units but as wholes, the length of different sized rods, which children arranged in order of size. When children had mastered the ordering of rods, they were introduced to the abstract symbol (numeral) and were shown how to place a sandpaper number card beside each rod. Individual numbers were introduced after the rods using a

Spindle Box. The teacher allowed children to play games on a number line on the floor with a large dice –a game similar to the Number Worlds Line Land game.

Numbers in Context

The most striking difference for the observer between both groups was the way in which numbers were introduced to children. The emphasis in Number Worlds classes was on counting, matching and discussion. Numbers were not taught individually. Children met several or sometimes all of the numbers in the 1-10 range when playing games in small-groups and in whole-class settings. Children were required to count and sometimes to match sounds, objects, counters and pictures. They learned about numbers in the context of other numbers on number lines and in games.

In Control Groups, numbers were introduced individually. At least one week was spent on teaching the concept of each number. Children experienced each number by being physically active e.g. jumping or clapping the number, by making sets from concrete objects and by recording sets in copies and workbooks. Teachers involved the whole class or groups of children in clapping, jumping or performing a physical activity a set number of times. Children sang songs and number rhymes using counters/cubes or a physical activity to reinforce number concepts. After at least one week of making and recording sets, children learned to decompose individual numbers into number stories e.g. the number story of number 2: 2+0=2, 1+1=2, 0+2=2. Teachers used workbooks, worksheets, copies, songs and rhymes to reinforce the stories of numbers. Thus the emphasis in Control Groups was on individual numbers and composition/decomposition of numbers, while the emphasis in Number Worlds groups was on numbers and their context in relation to other numbers.

Group Work

A second striking difference between Number Worlds and control groups was group work. Number Worlds groups spent approximately 50% of each class playing games within a group of four or five children. During small group games children played together and learned social, communication and game skills from one another. Control Groups did not use group work. There were opportunities for children to interact with each other and with teachers during whole class teaching. However, most teacher-child interaction and questioning in Control Groups tended to be in 1-1 settings. When children were working on activities, copies or workbooks, teachers went around the room giving individual support to children, asking questions and supporting children to do an activity or workbook page. However, time constraints and pupil teacher ratio meant that only a minority of children could benefit form this type of interaction.

Adult Support

There was more adult support in most Number Worlds groups than in Control Groups (Table 7). Each small group in Classes 4, 5 and 5a was supervised by an adult and the increases in mean scores of these groups were amongst the highest. In other classes (1, 2, 3, 3a and 6) adult support was divided between groups. Number worlds provided a structured way for parents to be involved:

> "I think it was brilliant too to involve the parents 'cause sometimes you have some parents who want to be involved but they don't know what to do… It was a way of involving them and it gave those (parents) a structure (Group 4).

Adult support facilitated learning about number *and* about social skills:

> "Now they obviously, some of the children would get distracted or stop playing or fight… And I found it worked quite well in my class anyway with just two adults."(Group S2)

> Definitely it was easier when we had more adults for the groups of children" (Groups 3and 3a)

> If my SNA or myself had to look after 2 groups it went a bit haywire (Group 1)

Written Work
A further difference between Number Worlds and Control Groups was the amount of individual written work that took place:

> I think the practical was so important for early maths because so much of commercial schemes, there is so much emphasis on the written and it's the practical that is needed for Junior Infants." (Group 6)

Number Worlds lessons did not include written work. Children in some Number Worlds Groups did written work on days when they did not do Number Worlds (Table 6). All control groups did written work almost everyday during mathematics lessons. In one class (6a) children practised writing numerals with fingers in the air, on the backs of the person beside them, in copies and in workbooks. Children drew sets to match numerals and illustrated stories of numbers in copies and workbooks. One teacher (7) reported that she followed the workbook page by page because Junior Infant children have difficulty finding pages. Two teachers (2a, 8) expressed dissatisfaction with workbooks in use and said that it was necessary to supplement the class workbook with photocopied pages from alternative workbooks or to make up their own worksheets. Teachers of Number Worlds groups commented that when they began to work on number pages in workbook after (and some during) the project, children did not seem to need to be taught how to do workbook tasks

> "It made any kind of written work kind of a lot easier because they covered all the topics beforehand and …they recognised them (numbers) in a different format."(S2)

The ease with which Number Worlds children undertook symbolic work in workbooks may be explained by the fact that children had first learned about numbers enactively as Bruner (1966) suggests should be the case. Our observations in classrooms suggest that there continues to be a heavy reliance on workbook activity in Junior infant classrooms, despite the fact that children's use of workbooks has been found to dissipate rather than to intensify the quality of teaching and to reduce opportunities for children to learn (Reynolds and Muijs, 1997).

Language
Language development was an integral part of mathematics lessons in both Number Worlds Groups and Control Groups. The questions asked by teachers about numbers were similar e.g. "How many more do you need?" Do you have enough?" However there were differences. Firstly, Number Worlds Groups 2, 4, 5,5a and 6 included a language round-up session at the end of lessons in which one child from each group summarised a small group game. Secondly, small group work and extra adult support in Number Worlds Groups meant that there were more opportunities for communication.

Thirdly, teachers reported that counting and talking about numbers spilled over into other periods of the day:

> The children enjoyed the games and their oral knowledge of numbers improved. They appeared to count more and notice numbers more in the environment (Group 1)

> There was enthusiasm towards counting …..Yeah, they'd count things, they'd count their lunch boxes together, and they talked about counting. They had the vocabulary for it and used it in their own private conversations. (Group 2)

> It led to children engaging with numbers and you know you could hear it in their private conversations (Group 5a)

The additional opportunities for talk afforded to children in Number Worlds Groups may have helped children to internalise new ideas and formulate new concepts (Gettinger and Stoiber1999).

> "You could see especially in the workbook sessions (after the project) the improvement in their mathematics language. You know terms like minus and plus and more and less… and they got a real sense of numbers you know…internalised and their visualisation of numbers… was really apparent" (Group 5)

Conclusion

This research focused on an intervention that teaches number in a different way to conventional approaches. The main emphasis was on counting - a skill that is thought increasingly to be the precursor to the development of the four operations - addition, subtraction, multiplication and division (Zevenbergen, Dole and Wright, 2004). A second distinctive feature of the intervention was that children learned through play. Data from teachers, classroom observations and results of the Number Knowledge Test indicate that the children in the Number Worlds intervention had a better understanding of number than their peers following conventional approaches.

It is difficult to disentangle the reasons behind the success of the intervention. However, we believe there is sufficient evidence to suggest that the emphasis on counting, small group structured play and the deployment of in-class support in a purposeful manner has the potential to raise achievement levels in early mathematics for many children. It is certainly worth implementing and evaluating over a longer time frame as the methods we are currently using to teach mathematics are failing many students in designated disadvantaged schools.

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The Importance of Identifying Children's Counting Types: An analysis of Steffe's counting types in first class children in disadvantaged schools in Ireland

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The paper examines the utility of identifying Steffe's counting types as a central element in identifying the best intervention strategies that might be used to enhance and advance first class children's number faculties in disadvantaged schools in Ireland. Using these types, this paper identifies the significance of establishing Steffe's counting types as baseline material, as a core dimension of meeting the project's central aim. of enabling children to acquire early number knowledge more rapidly by focusing on individual children's levels of knowledge and strategies and using those already acquired levels as building blocks for future development. This paper will examine counting strategies employed by pupils in these settings in the context of perceptual, motor, abstract, figural and verbal counting strategies, categorized by Steffe's types.

Introduction:

Levels of mathematics and numeracy attainment in Irish students have been gaining increased attention in education, media and industry circles in recent years. Recent reports pitch Irish students on national and international mathematics scales; reviews of teacher education approaches at primary level have been circulated; discussion papers on post-primary mathematical needs are published, while studies of numeracy and literacy levels in disadvantaged settings have generated huge interest. In responding to the low numeracy levels in disadvantaged schools, the Department of Education and Science, through its DEIS (Delivering Equality of Opportunity in Schools) initiatives, has supported the introduction of the Mathematics Recovery Programme. In 2005, pupils from a number of disadvantaged schools in Limerick were interviewed using the Mathematics Recovery Assessment schedules. This paper reviews the theoretical background and framework of Mathematics Recovery and its relationship with Steffe's research. It also examines the utility and suitability of using the Mathematics Recovery Assessment A Schedule with first class children in disadvantaged settings in Ireland. It looks at the type and amount of information gleaned about each individual pupil's numerical knowledge and counting strategies by examining their responses to the additive task in the Mathematics Recovery Assessment A Interview. It also seeks evidence of Steffe's counting types in these first class pupils.

Steffe's Counting Types and Mathematics Recovery:

Mathematics Recovery is an early intervention programme developed by R.J. Wright at New South Wales focusing solely on the area of number. It draws on research into children's early number knowledge development (Aubrey 1993, Wright, 1991, Young-Loveridge 1988, 1991) and uses as its basis the early mathematical investigations and theories of Steffe and colleagues. (e.g. Steffe, von Glaserfeld, Richards, & Cobb, 1983; Steffe & Cobb, 1988) and related work by Wright (e.g. 1991a; 1991b; 1992; 1994).

Mathematics Recovery is an approach to early number learning that integrates interview-based assessment, documenting students' current knowledge and administering instruction (Wright, 2003). It is a research-based, evolving programme. The key features of the Mathematics Recovery Programme are: (a) intensive, individualised or small group teaching of low-attaining first class pupils by specialist teachers for teaching cycles of length 10 to 15 weeks; (b) an extensive professional development course to prepare the specialist teachers, and on-going collegial and leader support for these teachers; (c) use of a strong underpinning theory of young children's mathematical learning; and (d) use of an especially developed instructional approach, and distinctive instructional activities and assessment procedures.

Critical to an understanding of Mathematics Recovery is the ability to observe, analyse and build on the child's number knowledge and strategies when solving tasks. Children progress through a process of construction of numerical knowledge and reconstruction of strategies. The relative sophistication of the strategies employed by the pupils and their progression through them is outlined in the Stages of Early Arithmetical Learning (SEAL). Simply put, a child who can say that nine counters and three counters is twelve because nine and one is ten and two more makes twelve is using a far more sophisticated strategy than a child who must count out the nine counters from one, count the three counters from one and then count all from one to twelve.

Mathematics Recovery and Steffe's research all use Von Glasersfeld's theory of cognitive constructivism as their primary point of reference. In this sense, they are more concerned with how humans come to know than an approach to teaching. Wright (2003) states that a fundamental question in Steffe's research and for Mathematics Recovery is 'What kind of instruction supports students' construction of arithmetical knowledge?'

Steffe's research was based on providing intensive, twice-weekly, individualized teaching of early number over extended teaching cycles (18-20 weeks) to students in their second or third year of school (6- to 8-year-olds) who had been assessed as being less advanced in number learning than many of their classmates. His research closely observed the strategies which a student uses in a problematic situation. These strategies, or schemes in Piagetian terms, are observed as they evolve, develop and become re-organised over the extended teaching cycle in the teaching sessions as well as in the pre- and post-assessment interviews.

Steffe's stages describe children developing a number sequence through counting. This counting progresses through counting perceptual items to counting abstract unit items. Pre-numerical children may perhaps be able to produce a correct number-word sequence but may not be able to count a number of items correctly or to arrive at the same conclusion in repeating the act of 'counting'. When they can count a collection by coordinating their number-word sequence with pointing at visible objects, they are described as perceptual counters. Figurative counters have internalised the items. They can count the imagined items – they may use finger patterns, draw pictures of the concealed objects or point to the cover. Neither perceptual nor figurative counters have interiorised their counting. This is evidenced in their need to 'count all'. This is what distinguishes a pre-numerical child from a numerical child. Being able to 'count on' exhibits a progression to the Initial Number Sequence. This implies that the child has counted the first sets of items and the outcome is now a numerical composite. It stands for a collection already counted. Now, the child is ready to count on from that point. The facility to manage to keep track of what the child is counting on by some form of double-counting suggests the interiorisation of the Initial Number Sequence into the Tacitly Nested Sequence. As Olive (2001) puts it, 'The elements of the INS (Initial Number Sequence) were interiorised counting acts. The elements of the Tacitly Nested Sequence are now *countable abstract unit items*.' When the child becomes aware of part-whole numerical relations, the child has established an Explicitly-Nested Number Sequence.

Wright et al. (1993), designed specific assessment tasks using Steffe's research to ascertain the level of the child's number knowledge and the relative sophistication of the child's strategies for counting, addition and subtraction.

The Learning Framework in Number (LFIN) in the Mathematics Recovery programme provides practitioners with a theory-based, integrated view of the key aspects of children's early numerical learning. The LFIN provides guidance for assessment and directionality in teaching early number. Those of particular interest to this paper are the Stages of Early Arithmetical Learning (SEAL), Forward Number Word Sequences (FNWS), Numeral Identification and Backward Number Word Sequences (BNWS). The clear distinction is made between counting which involves the coordination of each number word with an item, real or imagined, and the reciting or saying of a sequence of numbers.

The most important aspect of the LFIN is Stages of Early Arithmetical Learning (SEAL). The Stages are as follows:

Table 1: Stages of Early Arithmetical Learning (SEAL)

Stage 0	Emergent Counting
Stage 1	Perceptual Counting
Stage 2	Figurative Counting
Stage 3	Initial Number Sequence
Stage 4	Intermediate Number Sequence
Stage 5	Facile Number Sequence

The SEAL model has been adapted from research by Wright (1989; 1991a) and earlier work by Steffe and others (Steffe, 1992; Steffe and Cobb, 1988; Steffe et al., 1983). The similarity is clear from the table below.

Table 2: Comparison of terms to describe counting stages

Steffe's Counting Types	**Mathematics Recovery Stages of Early Arithmetical Learning**
Pre-numerical	Emergent
Perceptual	Perceptual
Figurative	Figurative
Initial Number Sequence	Initial Number Sequence
Intermediate Number Sequence	Tacitly-nested Number Sequence
Facile Number Sequence	Explicitly-nested Number Sequence

Ascertaining the pupil's numerical knowledge and understanding requires close observation and analysis of their verbal and non-verbal responses to counting, additive and subtractive tasks involving counters which may be screened using covers or visible to the pupil. Table 3 below provides a summary description of each of the stages in the SEAL model.

Table 3: SEAL Descriptors

Stage	**Behavioural Indicator**
Emergent	The child cannot count visible items. The child either does not know the Number words or cannot coordinate the number words with items .
Perceptual	The child can count perceived items but not those in screened collections. This may involve seeing, hearing or feeling items.
Figurative	The child can count items involving screened collections but counts from one when doing so. This will involve figural, motor or verbal counting.
Initial Number Sequence	The child counts-on to solve additive and missing addend tasks involving screened collections .
Intermediate Number Sequence	The child counts-down-to to solve subtractive tasks and can choose the more appropriate of counting-down-to and counting-down-from .
Facile Number Sequence	The child uses a range of non-count-by-ones strategies which include procedures such as compensation, using addition to solve subtraction, and using known facts such as doubles and sums which equal ten, an awareness of the 'ten' in a teen number.

Adapted from Wright et al. 2006

Other aspects of the LFIN which may be less important than the SEAL are Forward Number Word Sequences (FNWS), Numeral Identification and Backward Number Word Sequences (BNWS). They do, however, add significantly to the pupil's profile in early number knowledge. Models for the construction of the forward and backward number sequences and the development of numeral identification are provided. They are self-explanatory from the tables below.

Table 4: Model for the construction of Forward Number Word Sequences

Level 0	Emergent FNWS
Level 1	Initial FNWS up to 'ten'
Level 2	Intermediate FNWS up to 'ten'
Level 3	Facile with FNWS up to 'ten'
Level 4	Facile with FNWS up to 'thirty'
Level 5	Facile with FNWS up to 'one hundred'

Table 5: Model for the construction of Backward Number Word Sequences

Level 0	Emergent BNWS
Level 1	Initial BNWS up to ‘ten’
Level 2	Intermediate BNWS up to ‘ten’
Level 3	Facile with BNWS up to ‘ten’
Level 4	Facile with BNWS up to ‘thirty’
Level 5	Facile with BNWS up to ‘one hundred’

Table 6: Model for the development of Numeral Identification

Level 0	Emergent Numeral Identification
Level 1	Numerals to ‘10’
Level 2	Numerals to ‘20’
Level 3	Numerals to ‘100’
Level 4	Numerals to ‘1000’

Context and Purpose of the Study:

This paper reports on a study which took place as part of a piloting of Mathematics Recovery Assessment Schedule A in a number of disadvantaged schools in Limerick city in 2005. One of the objectives of the piloting at that point was to ascertain the type and amount of information regarding the pupil’s early number knowledge and strategies which would be yielded by such an instrument. This paper attempts to give a qualitative flavour of typical responses from first class children. In particular, it reports on the stage of early arithmetical learning as well as levels of knowledge of the number sequences and numeral identification from pupils from the same class in one school. Examples of pupils who are emergent counters, perceptual counters and figurative counters are discussed. It analyses the pupils’ responses to additive tasks and identifies the features of the particular counting type evident. Therefore, the objectives of the paper are

To ascertain the suitability of using the Mathematics Recovery Assessment A Interview with first class children in disadvantaged settings in Ireland.

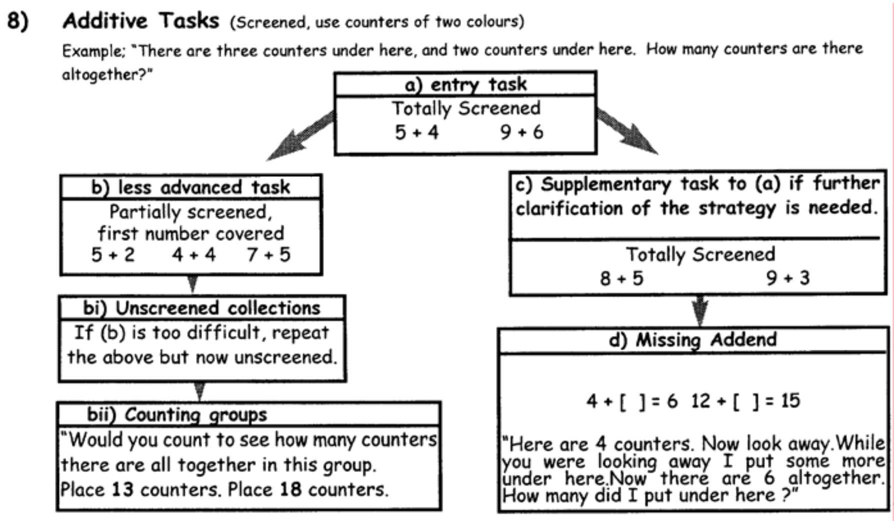
To evaluate the type and amount of information gleaned about each individual pupil’s numerical knowledge and counting strategies by examining their responses to the additive task in the Mathematics Recovery Assessment A Interview.

To look for evidence of Steffe’s counting types in these first class pupils.

It is important to note one further dimension of the pilot study which is beyond the scope of this paper: the identification of existing counting strategies used by individual pupil’s with a specific emphasis on how *some* existing strategies can be counter-productive in terms of their limitations. This is a key element in the overall research project which will have significant impact on curriculum implementation.

Methodology:

A number of tasks from Assessment A of the Mathematics Recovery Schedule were administered. These included tasks relating to the FNWS, BNWS, Numeral Identification and the Additive Tasks. The objective was to ascertain the pupil’s stage levels in terms of the LFIN by eliciting the most advanced numerical strategy. Each child was interviewed individually. As is the norm in the Mathematics Recovery assessments, it was an oral interview, with no reading or writing for the pupil. The assessment was videotaped. Videotaping of assessment and teaching sessions is a routine occurrence in Mathematics Recovery. It frees up the assessor to concentrate on the interview where probing and judicious questioning are vital in determining the strategy the pupil is using or endeavouring to use. There is a flexibility of approach in posing the tasks, with additional tasks being posed on the basis of the pupil's responses. The Additive Task which is the focus of attention here involves addition up to 20 using counters and screens. The task is presented below:



Interview Samples:

Jack – Evidence of Emergent Counting. Children at the emergent stage are not able to count perceptual items. They may know the number sequence up to ten or even beyond but they cannot coordinate the saying of the number sequence with the items. They may omit one of the counters or count one twice. A child at this stage may not be able to conceptualise the combination of the two collections into one.

This excerpt begins with Jack experiencing difficulties solving the sample task which involved adding 3 + 2 with both collections screened. This is meant to serve as a task to settle the child and to familiarise them with the task, the counters and the screens.

> Teacher: Ok, if we had three counters under this side (putting three counters under first screen) … right? And I put two more under that side (putting two counters under second screen)…
> Jack: Four (immediate response).
> Teacher: How many under this one (pointing to the first collection)?
> Jack: Three (immediately).
> Teacher: Right, and how many's under here (uncovering the second collection)?
> Jack: Two (immediately).
> Teacher: Right, so can you start now and how many are there altogether?
> Jack: Four (immediate response,).
> Teacher: (uncovered the first collection) now…
> Jack: (counted both collections using lip movements and head nods) Four.
> Teacher: We'll put them all together and count them up and see (combining both collections into one).

Jack: (using loose finger points) One, two, three, four. Four.
Teacher: Is there? Check again.
Jack: One, two, three (using finger to point to counters loosely) … One, two, three, four (recounting counters with finger points). Four.
Teacher: What about this one?
Jack: Five.
Teacher: So how many have we altogether?
Jack: Five.

Jack's responses were very fast. He did not show any concern about the difficulty of the task and seemed confident that each of his answers were correct. He stood firm that four was the correct answer. Even when the second screen was removed, he still said four. He used finger pointing in an incoherent and inaccurate manner. Combining the groups physically and re-counting made no difference. His one-to-one counting of the combined groups showed him omitting the last counter. It was only as his teacher drew his attention to the fifth counter that he included it.

Martin: Evidence of Perceptual Counting. A typical perceptual counter will be able to solve additive tasks where the material is perceptually available, usually visible, where one or both of the collections are openly visible rather than screened.

This extract shows Martin undertake 5 + 4 with both collections covered.

Teacher: I'll put five under there and I think I'll put four I think under here. (covering them)
Martin: (attempts to handle counters)
Teacher: Hold on. Hold on. (covering counters). How many under here?
Martin: Five.
Teacher: How many under here?
Martin: Four.
Teacher: How many altogether?
Martin: Nine (immediately).
Teacher: How did you do that, Martin?
Martin: Because…, we… we count… we count them…
Teacher: How did you count them?
Martin: because (takes a breath)… because I have it in my head.
Teacher: (Put nine counters under first screen, six under the second). Can you tell me how many there are altogether?
Martin: Umm… nine and six. Umm…(thinking pause, looking a bit anxious)
Teacher: Count it out loud if you like.
Martin: Umm…umm…is it? ...is it 16?
Teacher: (pauses) Try again.
Martin: 13, 14
Teacher: No, take your time. Take your time. How many are under here? (pointing to first collection).
Martin: Nine
Teacher: And how many here?
Martin: Six.
Teacher: Right.
Martin: (pauses) I'll count it. (Uses fingers). One, two, three, four, five, six, seven, eight, nine, (slight pause) ten, eleven, twelve, thirteen.., fourteen, fifteen, sixteen, seventeen…is it? (smiling) too fast?
Teacher: (removes second cover) How many are there? (pointing to first screen).
Martin: Nine.
Teacher: Ok, so how may altogether then?
Martin: Ten, eleven, twelve, thirteen, fourteen, fifteen. (finger pointing to each counter) Fifteen.

Martin quickly responds correctly to the first task above. His lack of ability to say how he came to his answer is typical of many pupils. The second task above (9 + 6) which required going beyond ten, where dependence on ten fingers becomes an issue caused problems for Martin. He proposes that the answer is 16. There was no evidence of what strategy he might be using and when he realised that that 16 was not the correct answer, he did what many perceptual children will do, he guessed 13 and then 14. Eventually, he decides to 'count'. He counted using his fingers from one to nine, takes a slight pause and counts onwards but has no strategy for keeping track of the second collection and counts up to 17. Taking away the second screen helped him solve the problem. With any hesitation, he counted on from nine, pointing to each of the second collection of counters to arrive at the correct solution.

Conor: Evidence of Figurative Counting A figurative counter can count items in screened collections. They will typically count from one. The term 'figurative' is used because the counting is based on some form of re-presentation of the material rather than the direct perception of sensory-motor material which occurs at the perceptual stage. Three types of counting have been observed during this stage of counting: figural, motor or verbal. Figural involves attempts at visualising the collection; motor involves uttering each number word after movement e.g., raising fingers sequentially; and verbal involves double counting to keep track of the second collection. Figural counting is accepted as the least advanced of the three, with verbal as the most advanced.

In this excerpt, Conor was presented with 8 + 5, both screened.

> Teacher: (Puts eight counters under first screen and five under second screen)
> Conor: Eight… and five… eh… (raises fingers sequentially saying) One, two, three, four, five, six, seven, eight… (pause)… nine, was it nine?
> Teacher: There was eight over here and five here (pointing to first and second screens respectively).
> Conor: (pauses to think … raises fingers sequentially saying) One, two, three, four, five, six, seven, eight … (pauses, looks briefly at first screen)… nine, ten, eleven, twelve, thirteen.

Conor exhibits a solid, robust approach to solving this task. He can be described as a figurative counter because he can count screened collections, counting from one each time. He presents as counting motor unit items - as he raises each finger to re-present the counters which remain screened all during the task, he says each number word in sequence after each movement.

Discussion and Conclusion:

Piloting the assessment schedule confirmed for the author and for the teachers involved the value of knowing and understanding the counting types outlined by Steffe and later incorporated into the Mathematics Recovery programme. Even from the limited but representative excerpts outlined above, it is clear that the Mathematics Recovery assessment schedule is a vital instrument in ascertaining the stage of early arithmetical learning at which the child is operating. The simplicity of the tasks used does not in any diminish the information found. In fact, their simplicity allows for a clear focus on the strategies being attempted or employed by the pupils.

Denvir & Brown (1986, 1986a) warned against thinking that every child's early number development proceeds along one path. Indeed, when looking at individual children's profiles, it is clear that each child has a unique profile and while the term 'typical' is used often in the course of this and other related papers, close inspection of each videoed interview demonstrates the strengths and weaknesses of each pupil assessed. The detail of information gleaned is invaluable in profiling the pupil. More importantly, it is vital in designing a teaching plan to meet the child's individual early number needs.

Steffe (1992) stated that 'the learning stages can be characterized by a progressive decrease in children's dependence on their immediate experiential world when creating countable items.' The counting stages developed from a solid research base can now provide for teachers insight into a learning trajectory in the area of early number. A knowledge and understanding of the counting types by teachers gives a thorough and perhaps alternative insight into how pupils learn and develop number and should influence how teaching number is conceptualised. 'Mathematics teaching in the early years of schooling essentially ignores children's number sequences and the further constrictions they make possible, and proceeds as if there were no rational systems of numerical operations available to the child.' (Steffe 1992) Have we moved on?

Mathematics Recovery embodies the philosophy of the Steffe approach. It is essentially a professional development programme with the expressed intention of equipping teachers in the recognition and analysis the pupils' number knowledge and counting strategies. Knowing where the child is at in terms of number knowledge and stages, while interesting in its own right, is important only in so far as it informs the teaching that will support and improve the pupils' learning.

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"We Don't Need Those Cubes": Place Value Understanding of Fourth Class Children

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This paper reports on a pilot study, which aimed to investigate the question 'What is the level of place value understanding of children in 4th class?' Fourth-class children were selected for this study, as they would have explored the concept of place value after the introduction in schools of the Primary School Curriculum (DES/NCCA, 1999) and the related in-service. Many researchers have identified that it is not until fourth grade (fourth class) that most children begin to demonstrate understanding of the concept of place value (C.Kamii, 1986; Kamii & Joseph, 1988; Ross, 1986, 1990, 2002). Using clinical interviews, children were asked to complete a series of tasks based on research findings to ascertain what level of understanding of place value concepts they possess. The results were analysed with reference to Sharon H. Ross's five-stage model of place value development (Ross, 1986). The paper concludes with a discussion of relevant findings of the pilot study and future research ideas are presented.

Background

Children's understanding and development of place value has been researched extensively worldwide. Both large-scale international comparative studies and smaller scale qualitative studies have been undertaken. Analysis of the influence of language, place value development, teaching methodology, use of materials, curricular placement, and textbook presentations have been studied in the attempt to better appreciate children's understanding of place value. With the current concerns on mathematical achievement in Ireland it is important to look at what is happening in Irish primary schools mathematics education. As number work permeates all strands of the primary mathematics curriculum it makes sense to begin with place value, which is the basis of our number system. The area of interest of this paper is the stages of development children go through in their understanding of the concept of place value.

Stages of Development

Many researchers have attempted to analyse the stages or levels of understanding that children possess about place value. While there is no general agreement on the stages and levels of development, there are some common themes identified. It is accepted that place value understanding is developed gradually over time. Starting with unitary conceptual structures in which children make the connection between the numbers one to nine, the numerals and a group of objects (Fuson, 1990a). Children then begin to partition numbers into tens and units, but do not yet have the multiplicative concept of place value, that is one ten is made up of ten units (Cobb & Wheatley, 1988; Thornton et al., 1994). The next step is understanding that one ten is made up of ten units and being able to count the tens as one group while knowing that it is made up of ten units (Cobb & Wheatley, 1988; Fuson et al, 1997) Children show no flexibility of representation at this stage and only use canonical representation (Miura & Okamoto, 1989; Ross, 1990) i.e. can only represent thirty-three as three tens and three units. As children's understanding develops they begin to use non-canonical representation that shows thirty-three as two tens and thirteen units or one ten and twenty-three units (Ross, 1990; Miura et al., 1993).

Sharon H. Ross (1986, 2) attempted to develop a "more comprehensive model of children's conceptual development in the place value domain". She used eighteen tasks to interview sixty children in grades two to five about their understanding of 'place

value and part-whole relations'. Six of the tasks were prerequisite task (concepts required for the child to develop understanding of place value concepts) and six were digit-correspondence tasks. All tasks related to two-digit numbers and materials were used for all tasks. Her findings were that children's understanding of place value was poor even at fifth grade level.
Based on the results of her study and previous research carried out by others, Ross developed the following five-stage model to explain how place value understanding of two digit numbers develops:

- In Stage I, '*whole numeral',* the child sees twenty-five as twenty-five units and places no meaning on the individual digits.

- In Stage II, '*positional property',* the child sees the five in twenty-five as in the units place and the two in twenty-five as in the tens place.

- In Stage III, '*face value',* the child uses "face value' as in seeing the two in twenty-five as two of the ten blocks and the five as five of the unit blocks, or seeing the two as two ten cent coins and the five as five one cent coins.

- In Stage IV, '*construction zone'*, children represent numbers using the number of tens blocks that correspond to the tens digit and the number of unit blocks to correspond to the units digit. No more than nine units are placed in the unit's position for e.g. twenty-five is represented as two tens and five units only (i.e. the child uses canonical representation). Ross classified this stage as a transitional stage.

- In Stage V, '*understanding*', the child understands that each digit (tens and units) represents part of the whole quantity. Children represent numbers using less than the number of tens blocks that correspond to the tens digit, and more than nine units are placed in the unit's position, for e.g. twenty-five can be represented as one ten and fifteen units (i.e. the child uses non-canonical representation).

Research by Ross and others showed that many children do not show stage V understanding until fourth grade. Ross contends that it is not until age 8 or 10 years that most children have the cognitive ability to understand place value.

Irish Context
While much research has been carried out on children's understanding of place value worldwide (Ross 1986; Kamii & Joseph 1988; Van De Walle 1990; Fuson 1990; Fuson et al. 1997; Thompson & Bramald 2002), little research has been done on Irish children's understanding of any mathematical concepts. Large-scale national and international research projects have focused on achievement across the whole primary mathematics curriculum as well as teaching and assessment of mathematics (Beaton et al, 1996; Sheil & Kelly, 2001; Sheil et al., 2006; DES, 2005b; NCCA, 2005).

The NCTM Learning Principle of *Principles and Standards for School Mathematics* states "Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (NCTM 2000, 20). Understanding is essential to allow students to deal with the new situations they will have to address in our changing world. The need for developing an understanding of mathematical structure and content was first explicitly recognised in an Irish curriculum in the 1971 Curaclam na Bunscoile.

The Primary School Mathematics Curriculum-Mathematics (DES/NCCA, 1999) continues this emphasis and has as one of its five aims “to enable the child to acquire an understanding of mathematical concepts and processes to his/her appropriate level of development and ability” (DES/NCCA 1999, 12) and one of the broad objectives is that children “understand, develop and apply place value in the denary system (including decimals)” (1999, 13). Place Value is first introduced in first class with the objective that “The child should be enabled to explore, identify and record place value 0-99” (1999, 41) It is suggested that children ‘group and count in ten and units’ using a variety of materials. In second class it is extended to 199. It suggests that children should be enabled to rename numbers and represent numbers in different ways. In third class it is extended to 999 and one place of decimals, and in fourth class to 9999 and two places of decimals. The significance of 0 as a placeholder is not emphasised until third class. Much emphasis is placed on the use of groups of objects, base ten materials, notation boards, money, number lines, hundred squares and the abacus.

In the recent Primary Curriculum Review-Phase One the majority of teachers at third to sixth class cited ‘operations’ and ‘place value’ as the strand units they found most “useful in their planning for, and teaching of Mathematics” (NCCA 2005, 136). While it is not clear what exactly was meant by ‘useful’ in this context, it perhaps suggests that operations and place value are the strand units that teachers see as most important. In light of this it is interesting to note that many teachers also identified number as the area they would like to prioritise in furthering their own implementation of the Mathematics Curriculum. When asked how often children in their class were enabled to use estimation strategies in number, approximately 60% of teachers responded less than once a week for front-end strategy, approximately 37% responded less than once a week for rounding strategy and approximately 57% responded less than once a week for clustering. These are all strategies that build on and help develop understanding of place value and number (NCCA, 2005, 139).

Irish primary mathematics textbooks put considerable emphasis on place value with pictorial representations of grouping into tens of different materials. This is extended into pictorial representation of Base ten materials (Dienes blocks) and money. There are many examples of the use of an abacus and hundred squares in representing place value. However, number lines are rarely used other than to demonstrate rounding up and down and filling in the blanks (O’Loughlin, 2002, Whelan & Collins, 2002). Both the Primary School Curriculum and the textbooks examined mostly emphasised what Thompson (2000) calls ‘column value’ as opposed to ‘quantity value’ in place value. He relates column value to seeing sixty-four as six tens and four units and quantity value as sixty and four. Column value is emphasised in the use of notation boards, abacus and standard algorithms while as quantity value is more identified with mental arithmetic, informal written strategies and empty number lines. Thompson (2003, 188) argues that both are necessary and that teachers need to “ensure that connections are made between quantity value and column value”. The NAMA 2004 study found that textbooks were used by over 95% of fourth class children in mathematics class on a daily basis and that over 50% of fourth class pupils are taught by teachers who report that text books and accompanying teacher’s manuals are the main resource used for deciding on what topic to teach and how to present a topic for planning mathematics lessons (Shiel et al., 2006). These findings go against the recommendations from the LANDS report (DES, 2005a, 38) that “textbooks should be used as resource materials to support the implementation of the curriculum, rather than the basis of the planning for curriculum delivery”. These findings also indicate the strong influence that textbook companies have on the curriculum that is actually implemented in Irish classrooms. Valverde et al. (2002) reported on this influence in their cross-national study of textbooks across TIMSS nations, which included Irish textbooks from third and fourth classes.

Pilot Study

The purpose of the pilot study was to test the protocol for the task-based interview, including aspects of video-recording the children. Qualitative research methods were employed in this study, i.e. semi-structured clinical interviews. Children completed a series of tasks based on research findings to ascertain what level of understanding of place value concepts they possess. This research looked at place value understanding of whole numbers only. The results were analysed with reference to Sharon H. Ross’s five-stage model of place value development.

Pilot Study Sample

The subjects of the pilot study were thirty-two children in fourth class, from two schools in Dublin City. Both were co-educational schools, which were designated disadvantaged. The sample was a non-probability sample based on convenience.

Pilot Study Interviews

The children were interviewed individually in a quiet area (the computer rooms) in their own school. This researcher conducted all interviews. Interviews were videotaped for further analysis. The child sat at the end of a small table and the interviewer sat to the side of the table to create an informal situation in which the child would feel at ease. The video camera was placed on a tripod and focused on the child. After the first set of interviews the focus of the camera was changed from the child’s face to the child’s hands as it was deemed more important to see what the child was doing with the materials and hear his/her responses. This was also decided to guarantee anonymity of the child.

Prior to conducting interviews, consent was sought for interviewing and video taping from the parents/guardians of all children and the Board of Managements of each school. Both parties were given information on the nature of the research and how the results will be used. At the beginning of the interviews children were told about the nature of the interview and that it would be videotaped. Any child who did not wish to participate was excused even if parental permission had been obtained. (One child chose not to participate). The results of the study do not identify any individual child and will be used for educational purposes only.

Interviews took place in the second half of the school year. All interviews took place in the mornings as it was felt that the children would be more alert at this time. Interviews took between 12 and 20 minutes depending on the child. The interview was moved on or terminated if the child became uneasy about a question. A variety of materials including Base-ten blocks (Dienes blocks), unifix cubes, place value board (HTU) and number cards were used. All numbers on number cards were presented in large font (72). All classes had previous experience of using the materials. Each child was asked if they were familiar with the materials used. All but three children said that they had used them before.

Pilot Study Tasks

Children were asked to perform a number of tasks to show their understanding of place value. These tasks were classified according to objectives of the Primary Curriculum-Mathematics for place value at the end of fourth class: Read and order whole numbers (questions 1-3), explore and identify place value in whole numbers 0-9999 (Questions 4-16), understand the significance of zero (Q10 and 14), round whole numbers to the nearest thousand (Q17). The tasks were based on questions from previous studies of children's understanding of place value, i.e. 'Practice Rooted Design' as identified by Zazkis & Hazzan (1999, 436).

Preliminary Findings

This paper will focus on the findings from tasks 7-10, all, which are digit-correspondence tasks.

Task 7

This was based on a task used by Constance Kamii in her 1986 study of place value understanding. Similar tasks have been used in other international research projects (Ross, 1986, Hiebert & Wearne, 1992, Price, 1998, Thompson, 2002). The purpose of this task was to ascertain whether the child knew that the one in seventeen represents ten objects.

(Give child seventeen unifix cubes)
Please count these for me. (Italics indicate words spoken by interviewer.)
(Show 17 on a card after the child counts cubes)
That's right, it's seventeen.
(Circle the 7)
Does this have anything to do with the number of cubes you counted?
Show me the cubes, which refer to this.
(Circle the 1)
Does this have anything to do with the number of cubes you counted? Show me the cubes, which refer to this. (Circle the 7)

Most children (twenty out of thirty-two children) showed one block to represent the one (tens) in seventeen and seven blocks to represent the seven (units) in seventeen (which was consistent with international results). When asked "what about these cubes?" (Nine cubes left over), most children said "We don't need those cubes" or that they had nothing to do with the seventeen. Others knew that they had something to do with the number but could not say exactly what. When probed further only two children self-corrected as indicated in the transcript below. These results demonstrated that for this task most children were in Stage III of Ross' five-stage model of place value understanding, i.e. they used 'face value' strategy. These results were similar to the results of Ross' study (1989).

The following transcript is from an interview with a fourth class boy:
Interviewer: *Could you count all these unifix cubes for me?*
Child: (Counts out loud) *1, 2, 3, 4…17.* (Counts again silently moving each cube from one pile to another) *Yeah, seventeen.*

Interviewer: *See this digit here* (circles the seven*), does this have anything to do with the number of cubes you have there? Can you show me the cubes that refer to this?*
Child: *Seven of these* (Puts seven cubes in a straight line at top of mat.) *There's seven.*
Interviewer: *Ok! Now this here*. (Pointing to one in seventeen). *Does that have anything to do with the number you counted here? Can you show me?*
Child: *One of them, will I put it beside them?* (Picks up one cube and puts it to left of the seven cubes)
Interviewer: *Ok, yeah.*
Child: *Is that how you do it?*
Interviewer: *Well what do you think?*
Child: *No 'cos that's eight then. That's seventeen isn't it* (pointing at digit card)? *Is it?*
Interviewer: *What do you think?*
Child: *There's seven isn't it* (pointing at seven cubes at top of mat). *Where else will you put your one, you'll have to put it there. Pointing to left of seven cubes.*
Interviewer: *What about all these here?* (Pointing to nine cubes left over).
Child: *Don't know.*
Interviewer: *Ok, that's fine.*
Child*: But, but like if you're trying to make. ...Seven, 1, 2, 3, 4, 5, 6, 7, one, that's seventeen.*
Child: *Ok, this would be…do you mean that this would be the… one (pointing at single cube),*
Interviewer: S*o how much is this worth? (Pointing at single cube)*
Child: *Yeah.*
Interviewer: *If you were showing me with these cubes (Dienes' blocks) how would you show it?*
Child: *Points to long (Ten).*
Interviewer: *That's right.*
Task 8This was a similar task to Task 7 and was based on a task used by Sharon Ross in 1986 and Thompson and Bramald in their 2002 study. The purpose of the task was to ascertain if grouping the cubes in a non-standard way would give different results. Would the children still use face value strategy?

(Give child six lots of four cubes joined together and two loose cubes)
Count the number of cubes here.
(If child answers twenty-six, show card with 26 on it)
Can you show me with the cubes what this part of the number means?
(Circle the two)
Can you show me with the cubes what this part of the number means?
(Circle the six)

The results of this were not consistent with Ross' findings that children found tasks with non-standard groupings more difficult than ungrouped or standard groupings. In this study sixteen out of thirty-two children showed two cubes to represent the two (tens) in twenty-six, when probed further eight out of sixteen of these children self corrected. This suggests that these children showed Stage IV (construction zone) development. Further investigation and analysis are warranted to explain the significant differences in results between Tasks 7 and 8.

Task 9

This was based on a task from Ross (1990), which asked children to represent a two-digit number using Base ten blocks (Dienes' blocks). The purpose was to see if children could represent the number in different ways, i.e. use canonical and non-canonical representation.

Show Base ten blocks (Dienes' blocks). *Have you used these before?*
Can you show me thirty-three with these blocks? (Show 33 on a card)
Can you show me another way, and another way?

Children who showed three tens and three units and were then asked to show another way usually showed two tens and thirteen units. When probed further children could then show one ten and twenty-three units, and then thirty-three units. Few children had any difficulty with this task. Most children could represent thirty-three in four different ways, i.e. using canonical and non-canonical representation. Only four children could not. This result would indicate that the children showed Stage IV ('Construction Zone') or Stage V ('Understanding') understanding for this task. This favourable finding could be due to Irish children's familiarity with the use of Base ten blocks (Dienes' blocks) in Irish textbooks for place value and regrouping in subtraction.

Task 10

This was based on a task from Resnick & Omanson (1987). The purpose was to ascertain if children understood the difference between a digit in the tens place and a digit in the units place.

(Show 35 and 83 on cards).
Can you read these numbers to me?
Which "three" is worth more? Which "three" is bigger? (Circle each 3.)

Most children (twenty out of thirty-two) said that the three in eighty-three was bigger than the three in thirty-five. Most argued that it was bigger because it was in eighty-three and eighty-three was bigger than thirty-five. Only one child self corrected. This is consistent with Ross' Stage III ('Face Value') understanding. "Students do not recognize that the number represented by the tens digit is a multiple of ten" Ross (1989, 3).

The following transcript is from an interview with a fourth class girl:
Interviewer shows two cards with thirty-five on one and eighty-three on the other.
Interviewer: *Can you read these two numbers for me please?*
Child: *35 and 83*
Interviewer: *Do you see this three here and this three here? Which three is bigger? Which three is worth more?*
Child: *This three is bigger.* (Points at eighty-three)
Interviewer: *Why is this three worth more?*
Child: *Because it's in eighty and that's in thirty.*
Interviewer: *Ok. So this three* (points at three in eighty-three*) is bigger than this three* (points at 3 in 35).
Child: *Yeah.*

Conclusion

Based on the preliminary findings of these four tasks most children in these two fourth classes are in Ross' Stage III or IV, though some children did exhibit Stage V characteristics in Task 9. This is broadly consistent with the finding of international research for fourth-class children. Some children displayed uncertainty about their own thinking which may indicate a development in their understanding, which is not quite solid yet. These results may be influenced by the children's familiarity with some tasks and not with others. Tasks 7, 8, and 10 are not tasks that children would have experienced before from textbooks, while Task 9 would be familiar. The preliminary findings are mostly consistent with international results. The pilot study was carried out in schools designated as disadvantaged; the results may be quite different if administered in non-disadvantaged schools. As the main aim of the pilot study was to test the interview protocol, it will remain to be seen if the results of the main study will be similar.

Main Study

The main study will contain three phases; Phase one will entail using clinical interviews to ascertain 'What is the level of place value understanding of children in fourth class?' As it is a qualitative research project a small sample of children will be taken from twelve schools selected nationwide to reflect urban, rural and suburban populations as well as different socio-economic backgrounds. Students will be selected at random. Phase two will entail a single-case intervention experiment involving a teaching programme in a fourth class using alternative methodologies and materials based on the Realistic Mathematics Education model. Phase three will entail a second interview of the children in the instruction class to ascertain if using different methodology had an impact on children's understanding of place value.

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Are Calculators Detrimental to Students' Mathematical Development? : The Effects of Introducing Calculators into the Irish Junior Cycle Mathematics Curriculum and Certificate Examinations

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In September 2000, a revised Junior Certificate mathematics syllabus was implemented in Irish post-primary schools. For the first time it explicitly included the use of calculators in mathematics classes. In line with this, students were permitted to use calculators in the Junior Certificate mathematics examination from June 2003 onwards. In the context of the introduction of calculators to the curriculum, the Department of Education and Science commissioned a research study on the effects of calculator use on mathematics in the Junior Cycle curriculum and in the Junior Certificate examination. The study was conducted over two phases. Phase I (2000-2003) involved a cohort of Third-year students and their teachers who worked under the pre-2000 syllabus (where calculators were not mentioned in the syllabus, and were not permitted in the Junior Certificate mathematics examination). Phase II (2003-05) involved a cohort who had studied the revised syllabus, and who had access to calculators in the Junior Certificate mathematics examination. Phase I of the study involved the administration, in 2001, of three calculator tests and an attitude questionnaire to a representative sample of students in Third year in post-primary schools; the administration of a questionnaire about calculator usage and attitude to their teachers; and the linking of student performance on the calculator tests to their performance on the Junior Certificate mathematics examination, which they sat in June 2002. Phase II involved the administration, in 2004, of the same tests and questionnaires as in the Phase I study, with minor revisions,. This presentation briefly outlines the background to and design of the complete study and discusses key findings.

Introduction

Although calculators have been widely and cheaply available for the past quarter of a century there is still considerable debate about the appropriateness of their use in school mathematics, particularly in primary school and the early years of post-primary school. The issue is further complicated by the different kinds of palmtop calculators and computational devices available, ranging from simple arithmetic calculators, through scientific calculators and graphics calculators to symbolic calculators and palm top computers with a variety of calculating modes and functions including algebra systems and spreadsheets. The focus of the study described in this paper is on the use of scientific calculators in mathematics in the first 3 years of post-primary school, i.e. in the Junior Certificate Mathematics curriculum and accompanying examinations with a particular emphasis on the Third year.[1]

In general, most of the research data relating to the use of calculators in the context of school mathematics comes from two main sources. One source is large-scale survey research (usually done at a national or international level and often addressing other issues as well as calculator use) in which representative samples of students and/or teachers are asked, by means of questionnaire or interview instruments, about their attitude to and use of calculators in mathematics. The resulting data may be correlated with the students' achievement in

[1] More detailed accounts of the two phases of the study can be found in Close et al. (2004) and Close et al. (2007), on which this paper is based.

mathematics as well as with other characteristics. The other source is the experimental type of study, in which groups of students participate in new or controlled experimental mathematics programmes in which calculators form a special part of the teaching/learning/assessment involved. In this case both process and outcomes may be monitored, and often compared with those of similar students following programmes that do not involve calculators or involve different uses of them.

Large scale research surveys

A number of recent international and national surveys of mathematics education in schools provide us with information on calculator use within and across countries and its relation to achievement. The IEA's TIMSS 95 (Beaton et al., 1996) involved over 40 countries at Seventh and Eighth Grades. The study found that mathematics teachers in many countries reported a high frequency of calculator use in their classes, often for checking answers, routine computation, and solving complex problems, but that there were other countries such as Korea, which scored highly on the mathematics tests, and notably, Ireland, where little use was made of calculators in mathematics classes. Again, in TIMSS 2003 (Ireland did not participate this time) there was a wide variation in the degree of usage of calculators across countries and no clear-cut relationship between achievement in mathematics and degree of calculator usage, although there was a decline in usage in the years preceding 2003. (Mullis et al, 2004). It should be noted that the items included in the TIMSS surveys did not necessarily require usage of a calculator, i.e. they could be expected to be done by most students using pen and/or paper and mental methods.

The IEA's TIMSS tests were designed to test mathematical skills common to the curricula of countries participating in the study and so had many items that tested these skills individually and in isolation from realistic contexts. The OECD PISA 2000 and 2003 international surveys of the mathematical knowledge of 15 year-olds were designed to test mathematical skills as the need for them arose, often in conjunction with other skills, in realistic problem contexts or situations, and were not constrained by the need to reflect the mathematics curricula of the participating countries. However a decision was taken not to include items that required use of a calculator**.**

The studies reported similar results with regard to calculator usage. In PISA 2000 and 2003, as with TIMSS 1995, 1999 and 2003, it was found that the percentages of students with access to a calculator differed across countries, and differed between survey dates (OECD, 2001; OECD, 2004; Cosgrove et al. 2005). An example of the latter finding was Ireland where 24% of the students in the PISA 2000 survey said they used calculators frequently in mathematics classes whereas 78% of the students in the 2003 survey said that they did so (Cosgrave et al., 2005). This reflects the revision, in the interim, of the Junior Cycle mathematics curriculum to include calculators. It is worth noting that this dramatic increase in calculator use between 2000 and 2003 was not accompanied by any significant improvement or deterioration in the mathematics performance of Irish 15 year-olds on PISA, either in absolute terms or relative to other countries.

Controlled Studies

With the advent of calculators in schools across most countries in the eighties and nineties many controlled studies were carried out to examine the effects of calculators on mathematics achievement and attitude. These studies varied considerably in terms of size of samples, grade levels included, and type and range of tasks or test items involved; however, all involved some form of control of the use of calculators in the teaching/learning or testing aspects of the studies. Most of these were supportive of (non-graphic) calculator use in mathematics classrooms in both primary and second level schools (Dunham, 2001). Hembree and Dessart (1986 and 1992) and Smith (1997) carried out meta-analyses of the effect sizes of many of these studies that met certain statistical criteria. They concluded that students who used calculators possessed better attitudes and had better self-concepts in mathematics than non-calculator users, and that testing with calculators produced higher achievement scores at all grades and ability levels. In general, this body of research says that students who use calculators in conjunction with traditional mathematics instruction perform better on pen-and-paper tests of basic skills and problem solving.

Irish Situation
Calculators were permitted in the Irish Leaving Certificate mathematics examination from 1986 onwards. There was no corresponding change in syllabus content or in the style of the examination questions at the time. Thus, numbers in examination questions were still chosen to facilitate pen-and-paper computations, so that the advantage to students who chose to use calculators was minimised. In practice, however, use of a scientific calculator became the norm.

In the 1990s, a major revision of the Irish Primary Curriculum was undertaken, and the revised curriculum was phased in from 2000. Its mathematics element incorporated calculator use from Fourth Class upwards; typical objectives state that the children should be able to perform [various operations and computations] 'without and with a calculator'[1] (DES/NCCA 1999). Students arriving in second-level schools after the revised curriculum was implemented should, therefore, be expected to be accustomed to, and competent at, calculator use. In the light of this development, and with the body of research broadly in favour of calculator use, a revised Junior Certificate mathematics syllabus was introduced in 2000 for first examination in 2003 (DES/NCCA 2001, 2002). The types of calculators sanctioned for use in the state examinations are four-function and scientific (non-programmable) machines, though there is no embargo on the use of graphics calculators as teaching and learning tools.

The introduction of calculators into the Junior Certificate mathematics curriculum provided opportunities for developments in teaching and learning, and for improvements in mathematics performance. However, it also raised concerns about the maintenance of computational skills for which calculator use is not appropriate. It was as a result of these concerns that the DES commissioned a research study to examine the effects of introducing calculators into the Junior Certificate curriculum and examinations on students' basic skills and understandings in arithmetic and data handling. This would be done by assessing students' levels of performance in calculator related areas of the mathematics curriculum both before and after the introduction of the revised Junior Certificate syllabus and examinations.

Goals of the Study
Greenes and Rigol (1992) classified the items used in U.S. College Board standardised tests into three main types of based on calculator relevance: *calculator inactive* items in which there is no advantage to using a calculator; *calculator neutral* items which can be solved without a calculator but for which a calculator may be used; and *calculator active* items which require the use of a calculator. For the purposes of the present study, the first category of item was not considered relevant and the second category, calculator neutral items, was seen as composed

[1] Recent national assessments of primary mathematics (Shiel and Kelly, 2001; and Shiel, Surgenor, Close and Millar, 2006) suggest that these objectives are not being met.

of two types of item: items that Junior Certificate students should be able to do without a calculator, and therefore for which a calculator is *inappropriate* (e.g. 7 x 5); and items for which use of a calculator might be advantageous though not necessary, and so could be considered *optional* (e.g. 1.25 x 8). The calculator optional items were thus seen as the items on which the effects of calculator access could be directly measured by assigning, in a random manner, one half of the students to doing them with a calculator and the other half to doing them without a calculator. Three tests would then be developed to test different aspects of students' mathematical skills in the presence or absence of a calculator. These were described as:

- A *Calculator Inappropriate* test (taken without access to a calculator by all students in the sample, and containing items that students should be able to do without a calculator, e.g. (3 x 4)/2);
- A *Calculator Optional* test (taken by half the sample with access to a calculator and by the other half without such access, and containing items that students should be able to do with or without a calculator, e.g. (3.1 x 25)/2); and,
- A *Calculator Appropriate* test (taken by all students with access to a calculator, and containing items that would normally require a calculator, e.g. (3.12 x 24.75)/0.2052).

Questionnaires would also be prepared to assess students' and teachers' attitudes to, and use of, calculators before and after their introduction into the curriculum.

Arising from the above considerations, the following research objectives were formulated:

1. To examine, both before the introduction of calculators in the Junior Cycle mathematics curriculum (2001) and after their introduction (2004), the performances of Third-year cohorts on three types of mathematical tasks: (i) tasks for which a calculator is inappropriate, (ii) tasks for which a calculator is optional, and (iii) tasks for which a calculator is appropriate;
2. To examine, on both occasions (2001 and 2004), the effects of calculator access versus non-access during the testing of student performance on the calculator optional tasks, and to compare the results for 2001 and 2004;
3. To examine the nature and extent of calculator usage by the 2001 and 2004 students and their teachers in Third-year mathematics classes;
4. To examine the attitudes of teachers and students in 2001 and 2004 towards calculator usage and their relationships to performance of the students on the three types of calculator task;
5. To examine the relationship between the performance of the 2001 and 2004 students on the three types of calculator task and their performance on the Junior Certificate mathematics examination that they took in 2002 and 2005 respectively.

Method

The study was carried out in two phases. The first phase (2000 – 2003) involved the administration, in November 2001, of three types of calculator-related mathematics tests (Calculator Inappropriate, Calculator Optional, and Calculator Appropriate) and questionnaires to a nationally representative sample of the final cohort of Third-year post-primary students to experience a mathematics curriculum in which calculators were not included. The second phase (2003 – 2006) involved the administration, in November 2004, of the same three tests and revised questionnaires to a nationally representative sample of the second cohort of Third-year post-primary students to experience a mathematics curriculum in which calculators were included. In this way the effects of calculator availability on mathematics performance and attitudes could be studied in a quasi-controlled experiment in which the performance and attitudes of a sample of the students who experienced a non-calculator mathematics curriculum (the ‘control’ group) could be compared with an equivalent group who subsequently experienced a calculator friendly mathematics curriculum (the ‘experimental’ group), all other relevant aspects being assumed to equal. Concerns about the validity of these comparisons between the 2001 and 2004 students include: (i) possible confounding effects of minor content changes in the revised curriculum introduced in 2000; (ii) changes in the numbers taking the three course levels (Higher, Ordinary, and Foundation) between 2001 and 2004, (iii) modest response rates of 73% in both phases, and (iv) any differential effects across the two phases of demographic or cultural trends. Steps were taken to allow for the second and third of these factors, including statistical weighting of data, but results should be viewed in the light of these limitations.

Within this larger design a second experimental study was implemented in which the students taking the Calculator Optional test were randomly assigned to two test conditions, one without calculators, and the other with calculators, to examine further the effects of calculators on items for which calculators may or may not be used. Table 1 summarises the design of the study.

Table 1: *Design of the Overall Study*

<table>
<tr><th>Phase/Treatment</th><th>Calculator Test</th><th colspan="2">Test Condition</th></tr>
<tr><td rowspan="3">Phase I 2001
‘Control’ group – experienced a mathematics curriculum which did not include calculators</td><td>Calculator Inappropriate</td><td colspan="2">No calculators</td></tr>
<tr><td>Calculator Optional</td><td>Half of sample: No calculators (control group)</td><td>Half of sample: Calculators Available (experimental group)</td></tr>
<tr><td>Calculator Appropriate</td><td colspan="2">Calculators available</td></tr>
<tr><td colspan="4">Revised calculator ‘friendly’ curriculum introduced in 2000 and examined in 2003 for first time</td></tr>
<tr><td rowspan="3">Phase II 2004
‘Experimental’ group – experienced a mathematics curriculum that included calculators</td><td>Calculator Inappropriate</td><td colspan="2">No calculators</td></tr>
<tr><td>Calculator Optional</td><td>Half of sample: No calculators (control group)</td><td>Half of sample: Calculators Available (experimental group)</td></tr>
<tr><td>Calculator Appropriate</td><td colspan="2">Calculators available</td></tr>
</table>

Development of the Calculator Tests

As described above, the design for the test specified three measures: a Calculator Inappropriate test (in which calculators were not available to any students); a Calculator Optional test (in which calculators were available to half of the cohort and not available to the other half); and a Calculator Appropriate test.

Test items fell into one of two cognitive process categories: those that assessed ‘knowledge of mathematical facts, procedures and concepts’, and those that assessed ‘knowledge of applications to real-life contexts’. They included both multiple-choice and short constructed-response items, and displayed an overall ‘gradient of difficulty’: that is, overall, the Calculator Inappropriate test was intended to be easier than the

Calculator Optional test, which, in turn, was intended to be easier than the Calculator Appropriate test. Test items focused chiefly on assessing the Junior Certificate syllabus content area Applied Arithmetic & Measure (because of its relevance for the use of real-life data), followed by Number Systems and Statistics (as these are the most calculator sensitive topics accessible to all Third-year students), and to a limited extent on Algebra (mainly on the solution of simple equations) (Table 2).

Table 2 *Number of Items on Each Calculator Test, by Mathematics Content Area*

Test	**Number of Items**				
	Number Systems	**App. Arithmetic & Measure**	**Algebra**	**Statistics**	**Total**
Calculator Inappropriate	13	10	1	1	25
Calculator Optional	12	15	4	2	33
Calculator Appropriate	7	15	0	8	30

Items were located in textbooks, tests, and examination papers, or written when necessary, and were then assembled into tests. Two pilot studies were conducted: one in a convenience sample of 7 schools in March 2000, and a second in a more representative sample of 15 schools in October 2000. Further details of the pilot tests are available in the main Phase I report (Close et al., 2004).

A high proportion of the items in the Calculator Inappropriate test were ‘pure’, emphasising basic numerical skills, while the Calculator Appropriate test contained items predominantly of the ‘applied’ type (emphasising the use of the calculator with real-life data). Consideration was also given to the placement of items in the test by content area. Where appropriate, items testing a given content area were grouped together, to avoid arbitrary shifts of focus from one topic to another within the tests. Examples of items similar to those on the tests (‘parallel items’), along with estimates of their difficulty levels, are given in Table 3.

Table 3 Sample Parallel Items for Each Calculator Test Booklet

Calculator Inappropriate Items	
Jane bought a CD for €5 and sold it for €7. What was her percentage profit? (A) 2% (B) 4% (C) 20% (D) 40%	*Content Area: Applied Arithmetic & Measure* *Difficulty Level: Moderately Difficult (45%)*
A class has 25 students. The ratio of boys to girls is 3:2 How many girls are in the class? Answer_______________	*Content Area: Number Systems* *Difficulty Level: Average (51%)*

Calculator Optional Items	
A pack of 120 identical cards is 3 cm thick. How thick is one card? (A) 0.0025 cm (B) 0.025 cm (C) 0.25 cm (D) 0.4 cm	*Content Area: Applied Arithmetic & Measure* *Difficulty Level:* *With Calculator – Average (65%)* *Without Calculator – Moderate (40%)* *Overall – Average (53%)*
Multiply: 6.4×2.5 Answer______________	*Content Area: Number Systems* *Difficulty Level:* *With Calculator - Easy (94%)* *Without Calculator – Average (64%)* *Overall – Moderately Easy (79%)*

Calculator Appropriate Items	
Evaluate: $\frac{(9.8)^3 - (29.2)^2}{0.0025}$ Answer______________	*Content Area: Number Systems* *Difficulty Level: Difficult (26%)*
How many 700 millilitre bottles of port can be filled from a 350 litre barrel? Answer______________	*Content Area: Applied Arithmetic & Measure* *Difficulty Level – Difficult (28%)*

After the completion of Phase I, minor changes were made to the tests in order to amend or remove items on which students had performed badly. In particular, two very difficult items were removed from the Calculator Appropriate test, and a simpler item testing the same skill was added to the Calculator Optional test. Two faulty items on Applied Arithmetic & Measure were removed from the Calculator Appropriate test and replaced by new items in Statistics, a content area that had, perhaps, been under-emphasised in the original tests. A Statistics item with potentially confusing wording in one of the Calculator Appropriate forms was re-written. It should be noted that these minor changes did not obviate comparisons between scores in Phase I and Phase II because IRT scaling was used in calculating the scores (as described below).

Development of Questionnaires

Teacher and Student Questionnaires were written for Phase I and developed further for Phase II. These were designed to investigate variables that might be associated with student performance on the tests, and to provide background data on participating students and their teachers and information on school policies on calculators and arithmetic skills.

The Teacher Questionnaire sought to ascertain teachers' attitudes towards calculator usage by students in a variety of contexts, including the home, the classroom and the Certificate examinations. For Phase I, the Teacher Questionnaire also sought information about the relative emphasis that teachers placed on various aspects of school mathematics. For Phase II – carried out when calculator use was allowed in the Junior Certificate examinations – some of the questions on this topic were replaced by questions seeking information in two areas: school or teacher policy with regard to numeracy issues, including calculator use; and teachers' experience of aspects, benefits and problems of calculator usage. The results of the Teacher Questionnaire are not addressed in this paper; details of these can be found in the main report (Close et al., 2007).

As well as background information, the Student Questionnaire sought information on students' calculator usage at home and at school in other subjects as well as mathematics, and asked about students' attitudes to mathematics in general and towards calculator usage in particular. In Phase II, additional questions were asked in order to investigate students' experience of calculator use in the revised curriculum.

The Sample of Schools and Students

Both phases of the calculator study were implemented at the same time of year (November) so that valid comparisons could be drawn between the 2001 and 2004 samples. The target population consisted of students in Third year in schools on the Department of Education and Science's post-primary schools database. Students in special schools, or in full-time special (resource) classes in ordinary schools, were excluded. The Department's database provided a listing of schools and of the numbers of male and female students in Third year in each school. Schools were stratified by type (Secondary, Vocational, Community/ Comprehensive) and size (large, small). Within each stratum, schools were sorted by the percentage of female students in Third year and by school size. Schools were then selected using a probability proportional to size (PPS) systematic sampling. Replacement schools were also selected during this procedure. One class in each school was then selected at random to participate in the study.

66 out of the 90 schools selected for Phase I (2001 test), and 73 out of the 100 schools selected for Phase II (2004 test), agreed to participate in the study (including replacements). Given the tight time-frame within which the study was conducted, it was not possible to recruit additional replacement schools. This meant that 1469 students in 2001 and 1459 students in 2004 completed the calculator tests, while 64 teachers in 2001 and 71 teachers in 2004 completed the teacher questionnaires. Weights were computed and applied to compensate for the somewhat unequal distribution of students in different strata in the sample, using procedures applied in the 1995 Third International Mathematics and Science study (TIMSS), which also involved sampling of intact classes in schools (see Foy, 1997).

Table 4 compares the achieved main study samples for Phases I and II. The table shows that, across the two phases, the achieved samples are broadly similar, though there were proportionately fewer large Secondary schools (and students) and proportionately more large Vocational schools (and students) in the Phase II sample, relative to the Phase I sample.

Table 4 *Numbers of Schools and Students in Achieved Sample, by Stratum – Phase I (2001) and Phase II (2004) Main Studies*

Stratum	Phase I (2001)				Phase II (2004)			
	Schools		Students		Schools		Students	
	n	%	n	%	n	%	n	%
Secondary (large)	34	51.5	792	53.9	32	43.8	671	46.0
Secondary (small)	7	10.6	132	9.0	7	9.6	115	7.9
Vocational (large)	8	12.1	175	11.9	17	23.3	329	22.5
Vocational (small)	3	4.5	60	4.1	4	5.5	75	5.1
Comm/Comp	14	21.2	310	21.1	13	17.8	269	18.4
Total	66	100	1469	100	73	100	1459	100

Implementation in Schools

Participating schools appointed co-ordinators to liaise with the Educational Research Centre and oversee the administration of the testing. In all cases, the 30-minute Calculator Inappropriate test was administered first. Students did not have access to a calculator for this test. After a short break, the 40-minute Calculator Optional test was administered to students. The calculator and non-calculator versions of the test (only the directions with respect to calculator usage were different) were distributed to alternate students to ensure random assignment to the two calculator conditions. After another short break, the 25-minute Calculator Appropriate test was administered. Following another short break, the Student Questionnaire was administered. Teachers were encouraged to complete the Teacher Questionnaire at the same time as students were working on their questionnaire.

Scaling the Calculator Tests

In 2001, Item Response Theory (IRT) methodology was used to derive scores for each examinee on the Calculator Inappropriate, Calculator Optional and Calculator Appropriate tests (Close et al., 2004). Analyses carried out on the 2001 data indicated that a small number of items were problematic and it was agreed that these needed to be deleted, replaced or improved for the 2004 administration, as described above. One of the key reasons IRT methodology was used was that it allowed changes to be made to the 2004 tests without affecting the validity of the comparisons that needed to be made between student performance in 2001 and 2004 (Baker, 2001). This important advantage of using IRT derives from the fact that IRT models result in student ability estimates that are not dependent on a particular set of items, as is the case when percent correct scores are used. The difficulty values for all items used in 2004 were then used to estimate student ability 'logit' scores (the units used in IRT scaling) in 2004. The 2001 student logit scores for each of the calculator tests had been transformed to a scale with a mean of 250 and a standard deviation of 50 to facilitate reporting. The scaling parameters used in 2001 were applied to the logit scores for test takers in 2004. In this way scores derived from both administrations of the tests can be considered comparable.

Analysis of the Data

Mean raw scores, scale scores, percent correct scores, and percentages of students, which are used to report results in the next section, are weighted population estimates that take into account the unequal representation of students from different schools and school types in the sample. They were obtained by applying weights to students' scores during analysis. Mean and percentage scores in this report are often accompanied by a standard error. A standard error is a measure of the extent to which an estimate derived from a sample (for example, a mean score) is likely to differ from the true (unknown) score in the population. Using these standard errors, it is possible to estimate whether differences (for example, between scores in Phase I and Phase II) occur due to chance, or whether the differences are 'real' (or statistically significant). The 5% level is used in determining statistical significance. If differences are identified as being statistically significant, therefore, we can say that we are 95% confident that the differences are real, and not simply due to chance alone.

Results and Discussion

There follows a brief description of the results of the study over its two phases along with comments where appropriate. As mentioned above, the results from the teachers' questionnaire are not included but can be accessed in the main report. The three calculator tests were scaled separately in 2001 using Item Response Theory methods, and the mean score and standard deviation for each test was set at 250 and 50 respectively. As explained above, scores on the 2004 tests were placed on the same scale as in 2001, using IRT methods. Results are presented first by overall performance on each test and then by content areas and key items. Results are also reported for the students' questionnaire and correlations between performance on the calculator tests and on the Junior Certificate mathematics examination are given.

Performance on the Three Calculator Tests for 2001 and 2004

The mean scale score and mean percent correct scores for the three tests for both 2001 and 2004, along with the results of tests of the significance of the differences, are given in Table 5.

Table 5 *Mean Scale Scores and Standard Errors on the Calculator Inappropriate, Calculator Optional, and Calculator Appropriate Tests (2001 and 2004)*

Test	2001		2004		04-(	SED	95%CI	
	Score	SE	Score	SE	Di			
Calculator Inappropriate*	250.0	4.54	243.2	4.48	-6.8	6.32	-5.79	19.45
Calculator Optional (Access)**	266.2	3.36	271.5	4.59	5.27	4.87	-15.00	4.46
Calculator Optional (No Access)**	235.8	4.68	227.1	5.35	-8.70	7.11	-5.50	22.90
Calculator Appropriate***	250.0	4.45	263.4	2.39	13.4	5.05	**-23.50**	**-3.34**

*Statistically significant differences are in **bold**. SE = Standard Error; SED = SE of the Difference*
** 25 items; ** 32 items; *** 27 items*

The effects of calculators can be seen by comparing the data for 2001 and 2004 in Table 5. Performance on both the Calculator Inappropriate test, and on the Calculator Optional test for the no calculator condition, declined slightly between 2001 and 2004, but not significantly so. On the other hand, performance improved significantly on the Calculator Appropriate test and slightly (but not significantly) on the calculator Optional test for the calculator condition. Altogether, therefore, performance without calculators declined slightly but not significantly, and performance with calculators increased, significantly in the case of the Calculator Appropriate test. The negative difference of only 7 points between 2001 and 2004 on the Calculator Inappropriate test suggests that 2004 students did not lose out significantly on basic mathematical skills while following the revised Junior Cycle mathematics curriculum, compared to 2001 students who followed the previous curriculum. The improvement in performance on the Calculator Appropriate test, which includes the items most likely to bring calculators into play, suggests that students' ability to make use of the calculator in solving problems improved over the three years. While items in this test were intended to be somewhat challenging, as described earlier, the overall low performance in both 2001 and 2004, raises concerns about how well students can use their mathematical knowledge to solve realistic problems.

Effects of Access to Calculators on Performance on the Calculator Optional Test, for 2001 and 2004

In both 2001 and 2004, prior to examining the effects of calculators on performance of the calculator group relative to the non-calculator group on the Calculator Optional test, student scores on the Calculator Inappropriate and Calculator Appropriate tests were used as measures of mathematics ability to investigate whether the students in the two groups were of equal overall ability in mathematics. For both 2001 and 2004, the small, insignificant differences between the mean scale scores on the Calculator Inappropriate and Calculator Appropriate tests, of those who had access to calculators for the Calculator Optional test, vis-à-vis those who did not, indicates that the two groups were well matched in terms of overall performance in mathematics (Table 6).

Table 6 *Mean Scale Scores of Calculator Group vs Non-Calculator Group on the Calculator Inappropriate and Calculator Appropriate Tests (2001 and 2004)*

Calculator Optional	Calculator Inappropriate		Calculator Appropriate	
	2001	2004	2001	2004
Calculator Access	249.5	241.6	250.5	262.4
No Calculator Access	250.4	244.8	249.5	264.5

In 2001, the difference in overall achievement between students with and without access to a calculator on the Calculator Optional test was 30.4 scale points, and in 2004 it was 44.4. On both occasions the difference was in favour of the calculator access group. Both of these differences are statistically significant, showing clearly the advantage conferred on students with access to calculators compared to those with no access (Table 7).

Table 7 *Mean Scale Scores, Standard Errors and Difference Scores for 2001 and 2004 on the Calculator Optional Test, by Access to Calculator*

Calculator Optional	2001				2004			
		Mean	**SE**	**Diff**		**Mean**	**SE**	**Diff**
Calculator Access	731	266.2	1.63	30.4	742	271.5	4.59	44.4
No Calculator Access	732	235.8	2.57		718	227.1	5.35	

These findings across time are in line with results from other studies that showed that students' test scores in mathematics improve to a significant extent when calculators are made available to them.

Performance on Mathematics Content Areas in 2001 and 2004

The lack of statistically significant changes in percent correct scores on each content area (on the three calculator tests combined) suggests that the overall mathematical performance of students has not significantly changed on any of the major content areas over the three-year period of the study (Table 8). When the results for each calculator test are analysed separately, however, several significant differences are discernible.

Table 8 *Mean Percent Correct Scores, by Mathematics Content Areas*

Content Area	2001		2004	
	%	**SE**	**%**	**SE**
Number Systems*	60.6	1.79	60.8	1.20
Applied Arithmetic & Measure**	46.4	1.87	45.7	2.00
Algebra***	46.2	2.92	48.3	1.67
Statistics****	50.1	2.11	49.8	2.04

** 32 items; ** 41 items; *** 5 items; ****9 items*

On the Calculator Optional test, the 2004 group who had access to calculators did significantly better than the 2001 group with access to calculators, on both Number Systems and Algebra, and slightly better (though not significantly so) on Applied Arithmetic & Measure and Statistics. This finding indicates that students were able to use calculators with somewhat greater effect on calculator optional tasks in Number Systems and Algebra in 2004 than in 2001. On the other hand, students without access to a calculator for this test did significantly less well on Number Systems in 2004 than in 2001.

In both 2001 and 2004, items on the Calculator Appropriate test were more difficult for students than items on the other tests, giving rise to a steeper gradient of difficulty than had been intended in the design of the study. Items that caused most difficulty were mainly in the area of Applied Arithmetic & Measure, though there was a significant increase in percent correct scores on this content area between 2001 and 2004.

Performance on Individual Items

An item-by-item comparison within each calculator test identified items on which overall student performance significantly increased or decreased between 2001 and 2004. On the Calculator Inappropriate test, students in 2004 did significantly better on five of the seven items on which there was a significant difference between the two years. Items parallel to items on the test for which the largest increase and decrease in scores were recorded are presented in Figure 1.

Figure 1: Sample Parallel Items on the Calculator Inappropriate Test on which there was a Significant Increase or Decrease in Percent Correct Scores between 2001 and 2004

Increase 2001 to 2004		
0.4956 corrected to 2 places of decimals is: _______		
Difference = 10.0%	SED = 2.5	95% CI = -14.9 to -5.1
The length of a rectangle is 6 cm, and its perimeter is 16 cm. What is the area of the rectangle in square centimetres?		
Difference = 11.0%	SED = 5.1	95% CI = -21.2 to -0.8
Decrease 2001 to 2004		
The Smith family uses about 6000 litres of water per week. Approximately how Many litres of water do they use per year?		
Difference = -8.0%	SED = 1.1	95% CI = 5.7 to 10.3
If $x = 0.3$, find the value of $5x$.		
Difference = -8.0%	SED = 3.1	95% CI = 1.8 to 14.2

SED = Standard Error of the Difference

On the Calculator Optional test when students had access to calculators, there was a significant increase in performance on 13 of the items. Descriptions of two parallel items are presented in Figure 2.

Figure 2: Sample Parallel Items on the Calculator Optional Test (Calculator Available) on which there was a Significant Increase in Percent Correct Scores Between 2001 and 2004

Increase 2001 to 2004		
Find the value of x if $3(2x – 8) = 39$		
Diff = 28.0%	SED = 4.7	95% CI = -37.4 to -18.7
Evaluate: $\sqrt{}(0.25 + (0.3)^2)$		
Diff = 37.0%	SED = 3.0	95% CI = -43.1 to -30.9

SED = Standard Error of the Difference

On the *Calculator Optional test* when students had no access to calculators, students in 2004 recorded significant increases in percent correct scores on 9 of the 14 items and a significant drop on 5. Descriptors for parallel items to those on which the largest increases and decreases in performance were noted are presented in Figure 3.

Figure 3: *Sample Parallel Items on the Calculator Optional Test (Calculator Not Available) on which There Was a Significant Increase or Decrease in Percent Correct Scores between 2001 and 2004*

Increase 2001 to 2004		
Find the value of 3.8 + (3.2 × 6)		
Difference = 15.0%	SED = 3.6	95% CI = -22.3 to -7.7
In a sale the price of a piece of furniture was reduced by 10%. The sale price was €12 What was the price before the sale?		
Difference = 16.0%	SED = 5.7	95% CI = -27.5 to -4.5
Decrease 2001 to 2004		
Multiply: 9.2 × 2.5		
Difference = -20.0%	SED = 1.4	95% CI = 17.3 to 22.7
Divide: .04 / 2.456		
Difference = -11.0%	SED = 2.8	95% CI = 5.4 to 16.6

SED = Standard Error of the Difference

On the Calculator Appropriate test, students in 2004 performed significantly better than students in 2001 on 18 items, and significantly less well on one. Descriptions of selected parallel items, including the item for which there was a significant decrease, are presented in Figure 4.

Figure 4: *Sample Parallel Items on the Calculator Appropriate Test on which there was a Significant Increase or Decrease in Percent Correct Scores between 2001 and 2004*

Increase 2001 to 2004		
Fill in the missing operations (+,–, ×, ÷): 27 □ (36 □ 11) = 675		
Difference = 30.0%	SED = 1.2	95% CI = -32.5 to -27.5
Find √524 and give your answer correct to two decimal places.		
Difference = 30.0%	SED = 3.2	95% CI = -36.4 to -23.6
Decrease 2001 to 2004		
Two successive electricity meter readings were 84 015 and 85 228. All units are charged at 7.14 cent per unit. Calculate the bill for the period, including VAT at 10%.		
Difference = -6.0%	SED = 1.2	95% CI = 3.6 to 8.4

SED = Standard Error of the Difference

Students' attitudes to calculators

The Student Questionnaire was administered to all the students participating in both the 2001 and 2004 tests. A brief account of the results is presented here. Readers are referred to the main reports for further details (Close et al., 2004, and Close et al., 2007).

Gender and Socioeconomic Status. There were no statistically significant gender differences in performance on any of the calculator tests. Based on parents' occupations, each student was identified as belonging to the upper, middle, or lower socioeconomic group (SEG). In both 2001 and 2004, students in the upper SEG achieved mean scores that were significantly higher than those of students in the lower SEG on all three calculator tests. In 2004, but not in 2001, students in the middle SEG significantly outperformed their counterparts in the lower SEG on all three tests. These findings are similar to findings on the SEG variable from other studies of school achievement in Ireland (e.g. Cosgrove et al., 2005; Surgenor et al., 2006)

Students' Use of Calculators. In 2004, almost all students reported that they owned, or had access to, a calculator at school and at home. Scientific calculators were most frequently owned, and less than 1% of students used, or had access to, a graphics calculator.

Students were asked about the frequency of calculator use in four subjects: Mathematics, Business Studies, Science, and Technology. While 81% of students used a calculator 'often' in Mathematics and 62% did so with the same frequency in Business Studies, less than 3% reported using a calculator 'often' in Science, and three-quarters of the respondents with respect to Technology 'never' used one in that subject.

When asked about calculator usage at primary school, 72% of students reported that they never used a calculator in their primary Mathematics classes, while 3% reported using one 'often'. With regard to the frequency of calculator usage in different areas of mathematics at post-primary level, just over 10% of students reported that they never used a calculator in their First year Mathematics classes, though for Second year this figure dropped to 5%. The frequency of calculator use in some mathematics content areas is displayed in Table 10. On average, students who reported using a calculator 'a lot' in a particular area tended to achieve higher scores on the Calculator tests than students who did not.

Comparisons with 2001, not surprisingly, show increased calculator usage in Mathematics classes, from fewer than 1% using a calculator 'often' in 2001, to over 80% in 2004. Calculator use in Business Studies classes remained about the same, while there was a slight increase in use in both Science (0.6%) and Technology (5.2%) classes.

Table 9 *Percentages of Students Indicating Various Levels of Calculator Usage in Different Mathematics Content Areas (2004)*

Content Area*	n	A lot	To some extent	Never
Fractions, Decimals & Percentage	1409	52.0	42.6	5.4
Length, Area, Volume & Time	1402	54.2	40.9	4.9
Algebra	1407	25.0	49.6	25.5
Statistics	1394	30.6	56.8	12.7
Geometry	1361	23.1	53.0	24.0
Trigonometry	1332	42.3	42.9	14.8
Graphs	1398	13.9	50.2	36.0

* *These areas, rather than the content areas referred to in the syllabus, were used in the student questionnaire*

Students' Attitudes to Calculators. The proportions of students who 'agreed' or 'strongly agreed' that a calculator can help with performance and that its use should be allowed for class and homework increased from two-thirds in 2001, to over 90% in 2004. A similar proportion in 2004 believed that calculators were not only for students with difficulties in mathematics, and that a calculator should not replace the need for pen-and-paper calculations.

Student attitudes to calculator use in mathematics and other subjects were mixed. Students who supported the availability of calculators across subjects, including mathematics, tended to perform better on the Calculator Appropriate test than those who did not. On the other hand, students who were more negatively disposed to calculator usage, and felt it could make them lazy at school mathematics, tended to do less well on each of the calculator tests than students who did not hold such views.

Aspects of calculator use that students enjoyed most were the ease and speed of computation and the convenience for basic operations, algebra, and fractions. The main perceived disadvantages of calculator use were the greater potential for making mistakes, difficulty in using the calculator, and a fear that calculators 'do not engage the brain'. The proportion of students who believed a calculator could make them lazy at school mathematics decreased from 55% in 2001 to 41% in 2004, perhaps owing to their greater familiarity with calculators and their benefits in doing mathematics.

Students' performance on the calculator tests compared to their performance on the Junior Certificate Mathematics examination

As part of the study, the scores of students in the 2001 and 2004 samples on the three calculator tests were related to their performance on the Junior Certificate examination in mathematics both by examination level and by a performance score, where a student's grade is converted to a points score. In both 2001 and 2004, the significant differences in performance among the three calculator tests were maintained when students were classified by the Junior Certificate mathematics examination level (Higher, Ordinary, or Foundation) that they said they intended to take and by the level actually taken, further affirming the validity of the tests and the study design. The grades the students received in the Junior Certificate examination in mathematics were used to place students on a Junior Certificate mathematics performance scale and these scale scores were related to their performance on the calculator tests. As in 2001, strong positive correlations were found between 2004 students' Junior Certificate Mathematics performance scores and their scores on the Calculator Inappropriate (0.7), Calculator Optional (0.66), and Calculator Appropriate (0.69), tests.

For the Calculator Optional test, the mean scale score of Ordinary level students with calculator access (243) approaches (and is not statistically different from) that of Higher level students without access (251). Similarly, the performance of Foundation level students with calculator access (200) is not significantly different from that of Ordinary level students without access (203). This suggests that calculator access enables students to perform at a level higher than they would otherwise attain on the types of task assessed by the test.

Conclusions

The study set out to examine the effects of introducing calculators into the Irish Junior Certificate Mathematics curriculum and Certificate examinations. This was achieved by carrying out the study in two phases. Phase I of the study, implemented in 2001, was designed to assess Junior Cycle students' performance on key areas of numeracy in the mathematics curriculum that was in place at the time, in which calculators played no part in the syllabus or the Junior Certificate Mathematics examination. Phase II of the study was carried out in 2004, following implementation of a curriculum in which calculator usage was actively promoted, to obtain data in the same key areas using the same instruments, for comparison with the data from Phase I. Major outcomes with regard to student performance are summarised here. Further results, and recommendations based on the findings, are presented in the main report (Close et al., 2007).

In 2004, the performance of students who did not have access to a calculator (hence, in the Calculator Inappropriate test and in the Calculator Optional test for the no access condition) was lower than that in 2001, though not to a significant extent. However, the performance of students who did have calculator access (hence, in the Calculator Optional test for the access condition and in the Calculator Appropriate test) was higher for the 2004 cohort – significantly higher in the case of the Calculator Appropriate test. Students' overall performance on the major mathematical content areas assessed did not change significantly between the Phase I and the Phase II stages of the study, although similar patterns emerged with regard to somewhat higher performance with calculators and somewhat lower performance without them.

Where Junior Certificate mathematics examination level is concerned, the benefit of calculator access is evident. On the Calculator Optional test in 2004, the mean scale score of Ordinary-level students who had access to a calculator approached, and was not significantly different from, the mean score Higher level students without access to a calculator, and a similar result was obtained with respect to Foundation and Ordinary level.

The non-significant decrease in performance when students do not have calculator access should be monitored over time, but should not be cause for concern, unless this trend is shown to continue. There are many factors besides calculator availability that should be taken into consideration in evaluating this. Firstly, changes in the syllabus were not limited to that involving calculator usage. Moreover, perceptions that the revised syllabus is 'shorter' than its predecessor may have led to reduced time allocations for mathematics in the Junior Cycle. Secondly, professional development for teachers with regard to the revised syllabus is still ongoing, and much work remains to be done with regard to promoting good use of calculators as learning tools as well as for computational purposes. Thirdly, cultural and demographic changes in the three years between each phase and marginal differences in the achieved samples may also be associated with the outcomes. Further evidence with regard to trends in numeracy may be obtained from the forthcoming results of the OECD PISA 2006 study. In any case, as indicated above, it is important that the situation is monitored over the coming years, so that good practice in the teaching and learning of mathematics can be appropriately developed.

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Language, Mathematics Learning and Gaeilgeoirí: A Study of Educational Transitions in Irish Education.

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Language is an essential element of learning, of thinking, of understanding and of communicating. The content of mathematics is not taught without language and educational objectives advocate the development of fluency in the mathematics register. This research is primarily focused on students (Gaeilgeoirí) in the transition from Gaeilge-medium maths education to English-medium maths education. The theoretical framework underpinning the research design is Cummins' (1976) Threshold Hypothesis. This hypothesis infers that there might be a threshold level of language competence that bilingual students must achieve both in order to avoid cognitive deficits and to allow the potential benefits of being bilingual to come to the fore. This paper will report on preliminary findings investigating the relationship between language proficiency and mathematics performance for Gaeilgeoirí. Implications and applications for mathematics teaching and learning will be presented.

Introduction

Although many perceive English as the official and dominant language of Ireland, many other languages are used throughout the country on a daily basis. In the context of primary and second level education, two language options exist as mediums of instruction of the curricula, namely Gaeilge (Irish) and English. For children growing up in a Gaeltacht (Irish speaking area), Gaeilge is the dominant language of the community and the natural language through which communication and socialisation takes place. In turn, Gaeilge is the medium of instruction employed in the local primary and second level schools, which tend to have been established prior to or since the early twentieth century. Although a relatively new phenomenon (mid seventies), immersion education also exists in Ireland in which Gaeilge-medium schools have been established outside of Gaeltacht areas. These schools are known as Gaelscoileanna (primary level schools) and Gaelchólaistí (second level schools) and can be found in all counties throughout the Republic of Ireland. The rise in popularity of immersion education is significant with an estimated thirty three thousand students in attendance and this figure is increasing annually (Fás ar an nGaelscolaíocht sa Ghalltacht, 2005). By combining both the number of Gaeltacht students with immersion students it unveils a significant and increasing minority of our schools' population learning through the medium of Gaeilge; approximately forty eight thousand students in total (MacDonnacha *et al.*, 2005). What is of importance to this research project is that the majority of these students will face an impending transition to English medium education, either at second or third level. For these students (Gaeilgeoirí), the authors anticipate difficulties arising in the Science, Engineering and Technology (SET) subjects – with mathematics as the focus of study given its fundamental importance in all three areas. For Gaeilgeoirí in the transition, they will be confronted not only with learning new mathematics but also with the task of learning it and understanding it through the medium of English (Barwell, 2003).

Theoretical Framework

Research has demonstrated that language is related to thinking, learning and cognitive development (Stubbs, 1976). Misconceptions about how the brain stores language have lead to negative perceptions of bilingualism. The most prominent being that bilingualism may result in "cognitive overload" and thus disadvantage the learner (May, Hill & Tiakiwai, 2004). This narrow perception of the mind and its storage of language is described as the Separate Underlying Proficiency (SUP), which views the two languages being stored independently of one another (Baker, 2001; Cummins, 1980). Consequently an increase in one language will result in an imbalance and loss of the portion of the other language. However, this model is not an accurate reflection of the working mind. The Common Underlying Proficiency (CUP) is a more apt description of language construction within the mind (Cummins, 1980). Outwardly both languages are different in conversation. However, internally both languages are merged so as that they do not function independently of one another (Baker, 2001). Storage of both languages occurs together and this acts as a central processing unit that both languages contribute to, access and use (Baker & Prys Jones, 1998).

Therefore, given that both languages are dependent on one another, consideration of this needs to be taken into account when investigating bilinguals and their learning of and understanding of mathematics. One such theory that provides a framework for investigation is Cummins' (1976) Threshold Hypothesis. This theory states that the level of first and second language competence reached by a student determines if he/she will experience cognitive deficits or benefits from learning in a second language (Cummins, 1976/1979). This implies that there is a certain 'threshold' that one must reach in one's first and second language before the benefits of studying in a second language can develop. Therefore students who have learnt mathematics through the medium of Gaeilge but have not developed their language to a sufficient level may experience difficulties when transferring to learning through the medium of English. Likewise, students who have developed a proficient level of Gaeilge, but not in English, may be at risk of experiencing difficulties when learning mathematics through the medium of English. A more positive aspect is that those who have reached the appropriate 'threshold' in both Gaeilge and English should experience positive cognitive benefits in their mathematical learning.

Cummins (1979) also distinguishes between basic interpersonal communicative skills (BICS) and cognitive academic language proficiency (CALP). What is important to note here is that, while second language learners may pick up oral proficiency (BICS) in their new language in as little as two years, it may take up to seven years to acquire the decontextualised language skills (CALP) necessary to function successfully in a second language classroom (Cummins, 1979). This theory is furthered by his Interdependence Hypothesis which proposes that the greater the level of academic language proficiency developed in the first language the stronger the transfer of skills across to the second language (Cummins, 2000). Although these theories have led to controversial debates among academics, they have influenced educational policies in the USA and in the UK (Yushau & Bokhari, 2005).

The influence of language on mathematics learning and understanding

Language and communication are essential elements of teaching and learning mathematics, and this is evident from research carried out in bi/multilingual settings (Gorgorió & Planas, 2001). Mathematics itself is a type of formal language, referred to as the 'mathematics register' (Pimm, 1987). This register is conveyed through the use of natural language and each language has its own mathematics register. Mathematics is not 'language free' and due to its particular vocabulary, syntax and discourse it can cause problems for students learning it in a second language (Barton & Neville-Barton, 2003). There are conflicting views about the learning of mathematics in a second language at all levels of education. Some studies (immersion programmes) have found positive correlations with learning mathematics in a second language and academic achievement (e.g. Barwell, 2003; Bournot-Trite & Tellowitz, 2002; Clarkson, 1992; Cummins, 1979; Swain, 1996; Turnbull et al, 2000, Williams, 2002). While other studies (submersion programmes) put forward concerns that such pupils underachieve in mathematics (e.g. Adetula, 1990; Barton et al, 2005; Galligan, 1995; Gorgorió & Planas, 2001; Marsh et al, 2000; Secada, 1992; Adler & Setati, 2000).

More specifically, empirical studies investigating the relationship between language proficiency and mathematics performance have been instrumental in furthering this area of research. Prior to the early seventies, it was assumed that bilingualism had a negative impact on cognitive development and mathematical learning (Clarkson, 2007). Research investigating the cognitive effect of bilingualism on mathematical learning began in the early eighties and has progressed from there. In particular the work of Dawe (1983) and Clarkson (1992) was significant, with Cummins' (1976/'79) framework forming the theoretical basis of their research. Both Dawe (1983) and Clarkson (1992) concluded that bilingual mathematics students competent in both their languages performed significantly better in mathematics than bilingual students dominant in only one language, and better than their monolingual peers. They also found that mathematics students who were weak in both their languages performed poorly mathematically also. This research substantiates the theoretical idea of threshold levels of language competence and this is further supported by research carried out by Secada (1992) with bilinguals in America. More recent research carried out at second and third level education in New Zealand (Barton et al, 2005) with students for whom English is a second language concluded that these students experience a disadvantage of between ten and fifteen percent in mathematics as a result of language difficulties, which again reinforces the notion of the necessity of language competency in both languages.

Mathematical Performance and Language Competence

Subjects involved in the study:

The bilingual participants at second level were chosen using the following criteria:
They were required to have studied mathematics entirely through the medium of Gaeilge at primary level,
That they were currently studying mathematics through the medium of English at second level,
All subjects were in their first year of second level education.

Both subjects from Gaeltacht schools (16 in total) and Gaelscoileanna (21 in total) were used in the study, as well as a control group consisting of monolingual English students (49 in total).
At the transition from second to third level education, the bilingual subjects were selected if:
They had studied mathematics entirely through the medium of Gaeilge at primary and at second level education,
They were now studying mathematics through the medium of English at third level,
They were in their first year of third level education.

Once again, subjects from Gaeltacht schools (9) and Gaelchólaistí (6) participated in the study, as well as a monolingual control group consisting of six students who had learnt mathematics entirely through the medium of English at primary and second level education.

	Bilingual Group	**Monolingual Group** (English Control Group)	**Total Cohort**
Primary – Second Level (Transition 1)	Entire Group *(BG –T1)*: n = 37 Gaelscoil *(BGc –T1)*: n = 21 Gaeltacht *(BGt –T1)*: n = 16	n = 49 *(M –T1)*	n = 86 *(T –T1)*
Second– Third Level (Transition 2)	Entire Group *(BG –T2)*: n = 15 Gaelchólaiste *(BGc –T2)*: n = 6 Gaeltacht *(BGt –T2)*: n = 9	n = 6 (M –T2)	n = 21 (T –T2)

Table 1: *Description of participants at each transition in the investigation***.**

Test Instruments:

At the transition from primary to second level education the participants completed a mathematics word problem test in English and in Gaeilge (bilingual students only), and language competency tests in English and in Gaeilge (bilingual students only). Word problems can be effectively used in investigating language issues for mathematics learners in a second language (see Newman, 1977). The author constructed the test instrument and appropriate piloting took place. The English mathematics word problem test consisted of twelve questions, with a number of subparts in some of the questions. In total the participants had to provide thirty answers on having completed the test instrument. A parallel version of the test instrument was constructed in Gaeilge. Therefore, the number and type of questions asked were the same except that they were through Gaeilge. The word problems were constructed using standard mathematics textbooks for first year second level students in Ireland. The content and wording of the textbooks were analysed and applied in the construction of the word problems. The author also carried out readability tests on each of the word problems using the Flesch Reading Ease and Flesch-Kincaid Grade tests available in Microsoft Word (2001). All word problems had a readability level within the expected range for twelve to fourteen year olds.

The English competency test consisted of a standard cloze test available for administration to all sixth class primary school students in Ireland (MICRA-T, Level 4). Given that the participants had just transferred from primary to second level education, this competency test was deemed appropriate for their expected level of English language competency. The participants were required to fill in the missing word in each of the blank spaces in the passage of reading. In total, the participants were required to fill in twenty five words. Currently, no standard competency test in Gaeilge exists in Ireland. However, Aonad na Gaeilge at the University of Limerick has designed an internal competency test in accordance with the Council of Europe's Common European Framework of Reference for Language (CEF). The CEF scales provide a basis for schemes to assess language skills in an objective, practical, transparent and portable manner. The CEF aims to provide an internationally recognised, linguistically coherent framework for every aspect of language teaching and learning. Permission was granted to use this test instrument for this research project. The competency test provided by Aonad na Gaeilge consisted of sixty-five multiple-choice cloze questions. However, for the purpose of assessing first year, second level students' competency in Gaeilge only thirty of the cloze questions were used, as this was the expected level of language competency for this age group as advised by Aonad na Gaeilge.

At the transition from second to third level education the participants completed a mathematics word problem test in English, and language competency tests in English and in Gaeilge (bilingual students only). The English mathematics word problem test consisted of nineteen word problems, with a number of subparts in some of the questions. In total the participants had to provide twenty-nine answers on having completed the test instrument. Once again, the test instrument was constructed and piloted by the author before initiating data collection. Sixteen of the word problems were constructed using the PISA mathematical literacy framework (2006). The PISA (2006) assessment utilises real-world problems and everyday settings, thus challenging mathematics learners to apply the mathematical skills acquired in school in a less structured context and in one which relies heavily on independent decision making and application of appropriate mathematical knowledge. Each question was set in one of four situation types: personal, educational/occupational, public and scientific. Also, each question related predominantly to one of the PISA (2006) overarching ideas which were space and shape, change and relationships, quantity and finally scientific. Combinations of open-constructed, closed-constructed and multiple-choice response types were utilised. The remaining three questions on the mathematics word problem test consisted of cloze type questions (see Hater & Kane, 1975). The questions involved definitions or explanations of mathematical terminology employed in a standard mathematics lecture/tutorial. Several words were deleted at random from each explanation and the participants were required to fill in the missing word in each of the blank spaces provided. Clearly an understanding of the mathematical terminology employed was required in order to complete the explanations. The author also carried out readability tests on each of the word problems using the Flesch Readability test available in Microsoft Word (2001). All the word problems had a readability level within the expected range for eighteen year olds.

A standard English cloze competency test was sourced through the Cambridge Certificate of Proficiency in English (Cambridge Examinations Publishing, 2002), as advised by the Language Support Unit at the University of Limerick. In total, sixteen words were required to be filled in by the participants in order to complete the competency test. The proficiency test provided by Aonad na Gaeilge was used and it consisted of sixty-five multiple-choice cloze questions. All questions were included for the second to third level transition as the competency test is designed for people aged eighteen onwards and therefore it was expected that all participants should be able to complete the competency test.

Analysis

All the data collected was coded and imported into SPSS (version 13) for quantitative analysis. A technique devised by Clarkson (2007) was used to segregate the participants into language competency groups. In accordance with their score on the language competency test in English, the participants were selected as having relatively high, middle or low competency in English. Threshold scores for each of these levels were determined based on the scores obtained by the monolingual English groups at each transition. By rank ordering the scores obtained by the monolingual English groups, the two scores that divided each group into thirds were noted and then applied to the bilingual groups, resulting in three sub-groups at each transition. The median score for the competency test in Gaeilge was used in order to divide Gaeilgeoirí into relatively high or low competence groups in Gaeilge, at each transition (Clarkson, 2007).

	English Language Rank Scores	**Gaeilge Median Scores**
Primary – Second Level Transition	High: ≥ 18 Medium: 14-17 Low: ≤ 13 (Out of 25)	Median = 18 High: ≥ 18 Low: < 18 (Out of 30)
Second – Third Level Transition	High: ≥ 8 Medium: 5-7 Low: ≤ 4 (Out of 16)	Median = 51 High: ≥51 Low: < 51 (Out of 65)

Table 2: *Threshold scores for the construction of the language competency groups.*

Students were then categorised as relatively high competence in both languages (combination of high/high or high/middle); dominance in one language (combination of high/low); or relatively low competence in both languages (combination of low/low or middle/low). Each student was assigned to *only* one of these language competence groups. The relevant variables in each of the data sets were explored and tested for normality before applying Pearson's correlation test and assessing its significance.

	Categorisation	**Primary-Second Level**	**Second-Third Level**
High/High	High Gaeilge & High English Or High Gaeilge & Medium English	n = 18	n = 4
Low/Low	Low Gaeilge & Low English Or Low Gaeilge & Medium English	n = 9	n = 4
Dominant Gaeilge	High Gaeilge & Low English	n = 10	n = 4
Dominant English	Low Gaeilge & High English	n = 0	n = 3
Monolingual	All-English Schooling	n = 49	n = 6

Table 3: *Description of the language competency groups.*

Results and Discussion

The first concept explored is the relationship between mathematics performance and English language competency. When taking the entire group of participants at the primary to second level transition, it was found that mathematics performance (in English) and language competency in English was moderately correlated (Pearson's = 0.48) and this was significant at the 0.01 level. When looking at the two individual groups within this cohort similar findings were evident with both the monolingual group and bilingual group displaying modest but significant correlations between mathematics performance and English language competency (Pearson's = 0.518 and 0.411 respectively). At the transition to third level education, a stronger correlation (Pearson's = 0.685) between mathematics performance and language competence in English was evident for the entire group and this was significant at the 0.01 level. In particular, for the monolingual English group mathematics performance and English language competence was highly correlated at 0.909 (Pearson's), with significance at the 0.01 level again, while a strong relationship was also evident for Gaeilgeoirí (Pearson's =0.648). Therefore, it is apparent that mathematics performance in English is related to language competence in English for Gaeilgeoirí and for monolingual English students at both transitions, with greater importance at the transition to third level education.

	Pearson's Correlation	**Significance**	**Description**
Primary – Second Level	T-T1: r = 0.48 M-T1: r = 0.518 BG-T1: r = 0.411	$p < 0.01$ $p < 0.01$ $p < 0.05$	All are moderate correlations but are significant
Second –Third Level	T-T2: r = 0.685 M-T2: r = 0.909 BG-T2: r = 0.648	$p < 0.01$ $p < 0.01$ $p < 0.01$	Moderate correlation Very strong correlation Moderate correlation (All correlations are highly significant).

Table 4: *Correlations between mathematics performance (in English) and English language competency.*

Further analysis investigated the relationship between mathematics performance (in English) and language competency in Gaeilge for Gaeilgeoirí. This was particularly significant at the primary to second level transition where a strong relationship was evident for the all Gaeilgeoirí (Pearson's = 0.651). This group of Gaeilgeoirí can be segregated further in relation to the school type attended i.e. either a Gaeltacht school or a Gaelscoil. For the Gaeltacht group, mathematics performance in English was strongly related to Gaeilge competency with Pearson's correlation equal to 0.706 (significant at the 0.01 level). For the Gaelscoil group a moderate relationship is also evident (Pearson's = 0.605). However, these findings were not replicated at the transition to third level where moderate relationships were found not to be significant for either of the groups.

	Pearson's Correlation	Significance	Description
Primary – Second Level	BG-T1: $r = 0.651$ BGt-T1: $r = 0.706$ BGc-T1: $r = 0.605$	$p < 0.01$ $p < 0.01$ $p < 0.01$	Moderate correlation Strong correlation Moderate correlation (All correlations are highly significant.)
Second –Third Level	BG-T2: $r = 0.226$ BGt-T2: $r = 0.470$ BGc-T2: $r = 0.462$	$p > 0.05$ $p > 0.05$ $p > 0.05$	Weak to moderate correlations but they are not significant.

Table 5: *Correlations between mathematics performance (in English) and Gaeilge language competency.*

Also, Gaeilgeoirí at second level completed a mathematics test in Gaeilge and performance in this test is moderately correlated with the students' competency in Gaeilge with Pearson's correlation = 0.545 and significant at the 0.01 level. Gaeilgeoirí's performance in the mathematics word problem test in English was highly correlated and significant with their mathematics performance in the test through Gaeilge (Pearson's = 0.806). Clearly, there is a strong relationship between Gaeilgeoirí's mathematics performance in English and in Gaeilge and their competency in the Gaeilge language at the transition to second level education. This relationship is not apparent at the transition to third level.

A final analysis looks at the different language competency groups – high/high, low/low and dominant (either in Gaeilge or in English). Another group consisting of the monolingual English students is also included in this exploration. No competency group dominant in English exists at the transition from primary to second level. Those categorised as dominant at this transition were dominant in Gaeilge. At third level two dominant groups exist, those dominant in English and those dominant in Gaeilge, so five language competency groups exist at this transition.

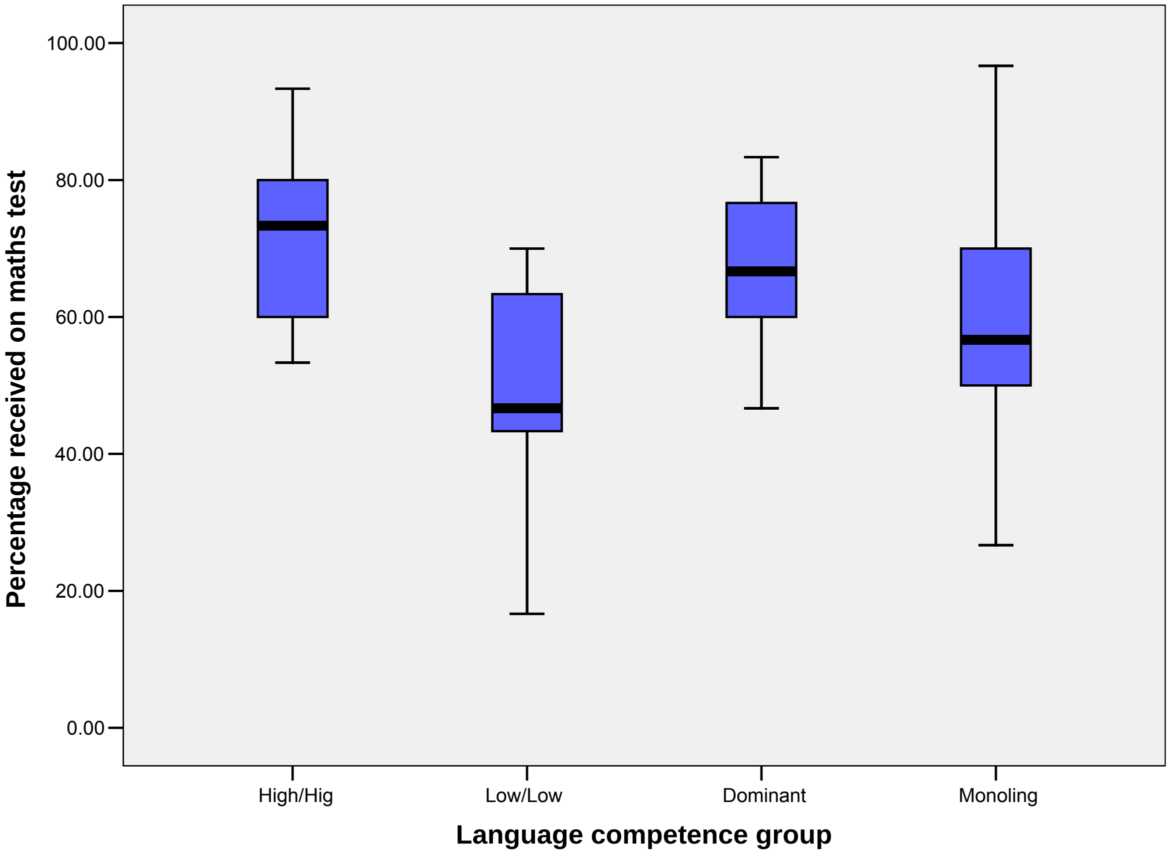

Figure 1: Comparison of language competency groups with mathematics performance (in English) at second level education.

From the above figure it is obvious that Gaeilgeoirí with relatively high competency in both languages perform mathematically better than Gaeilgeoirí dominant in one language (Gaeilge) and better than monolingual students. What is important to highlight here is that Gaeilgeoirí dominant in Gaeilge performed better than the monolingual students also, which is consistent with the correlations found between mathematics performance and language competence in Gaeilge. Given that this dominant group outperformed their monolingual peers it merits further investigation into the mathematics register through Gaeilge and whether this register and the Irish language facilitates Gaeilgeoirí's understanding of mathematics at this transition. The most at risk group consists of the Gaeilgeoirí with low competency in both languages, as all other groups outperformed this group mathematically.

Similar findings were revealed at the transition to third level education. Once again, bilingual students with a high level of competency in both languages outperformed their monolingual peers, and those dominant in one language. Equally, the bilingual students with low competency in both languages performed poorly in comparison to all other groups.

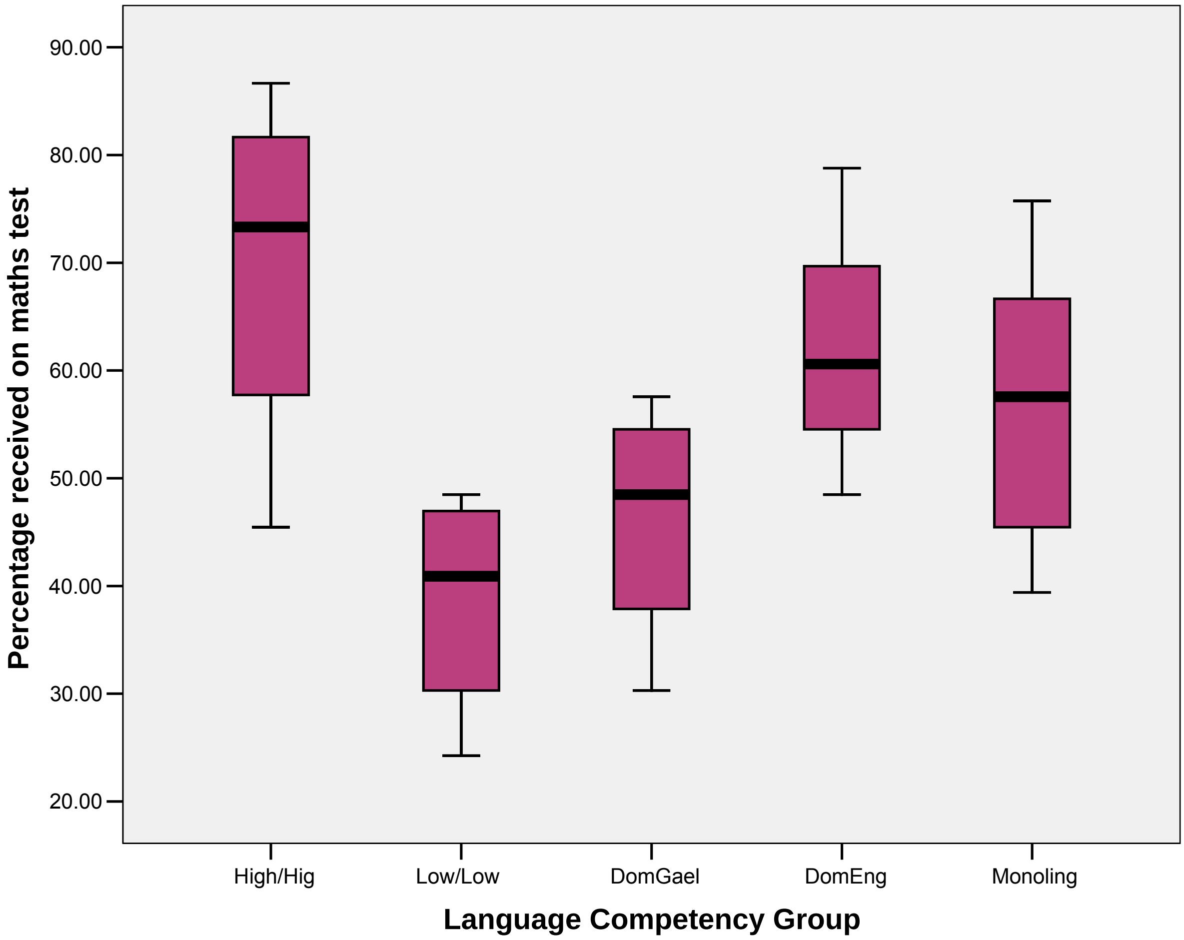

Figure 2: Comparison of language competency groups with mathematics performance (in English) at third level education.

Also evident from the above box plots is that the bilingual students' dominant in English performed mathematically better than their monolingual peers and better than bilingual students dominant in Gaeilge. This suggests that these students had not reached the threshold level necessary in English in order to reap the cognitive benefits from being bilingual evident from those with high competency in both languages. The monolingual group in turn were mathematically superior to bilingual students dominant in Gaeilge. This is consistent with the strong correlations found between English language competence and mathematics performance at third level.

Overall, these findings demonstrate that students' mathematical performance is related to their language competencies. For primary level Gaeilgeoirí in the transition to English medium mathematics education proficiency in Gaeilge is of particular significance. This is consistent with Cummins' Developmental Independence Hypothesis (1979) which proposes the greater the level of academic language competence in a student's first language will allow for a stronger transfer of skills across to the new language of instruction. Thus, Gaeilgeoirí with a high level of competence in Gaeilge performed well due to a strong transfer of mathematical skills across to English. For students in the transition to third level education a more significant relationship was found between English language competence and mathematics performance. This is perhaps due to the more decontextualised nature of the mathematics word problems utilised and their reliance on independent decision-making and the application of appropriate mathematical knowledge. However, the most significant overall finding is the support for Cummins' (1976) Threshold Hypothesis. In both transitions language competency groups were identified and those with a high competency in both languages outperformed their monolingual peers, those dominant in one language and those with low proficiency in both languages. Also, bilingual students displaying low competency in both languages were mathematically weak and lagged behind their peers. These results are consistent with the findings of Dawe (1983) and Clarkson (1992, 2007) who also draw on the work of Cummins.

Implications for teaching and learning

A number of implications and applications for the teaching and learning of mathematics for bilingual students can be concluded. These include:

- Mathematics performance is related to language competency. Therefore on entering second or third level education, bilingual students' competency in English and in Gaeilge should be assessed in order to identify the most at risk students in relation to mathematics.

- For Gaeilgeoirí in the transition from primary to second level education a significant relationship exists between their mathematics performance in English and their competence in Gaeilge. Teachers at primary level should ensure that bilingual students' mathematical skills in Gaeilge and academic language are developed to a high level in order to facilitate the transition to English medium mathematics education.

- For bilingual students entering third level education there is a high correlation between a student's mathematics performance in English and their competency in English. Therefore, it may be appropriate for second level mathematics teachers to introduce partial instruction through the medium of English in appropriate lessons so as to facilitate the development of bilingual students' competency in the English mathematics register.

- Collaboration between mathematics departments and language departments should be fostered so as to provide for the optimum development of mathematics competence in unity with language competence.

- Mathematics is not language free and mathematics educators at all levels need to incorporate the language aspects of mathematics into the teaching and learning of the subject matter.

Therefore, to conclude this research has demonstrated how mathematics performance is related to language competency for bilingual and monolingual students in Ireland. The language aspect of mathematics has long been ignored in the teaching and the learning of the subject matter. In order to cater for bilingual students in the transition to English medium education, appropriate teaching interventions need to be implemented so as to enhance the mathematics competence and the language competence of these students, and to allow the potential cognitive benefits of being bilingual to come to the fore.

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PISA Mathematics: What Should Teachers Know?

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One of the primary functions of the OECD's Programme for International Student Assessment (PISA) is to generate overall indicators of achievement and to provide ministries of education with advice on ways in which to improve their educational systems. Yet PISA 2003, which assessed the mathematics achievement of representative national samples of 15 year olds in 41 countries including Ireland in considerable detail, also has implications for teachers of mathematics. In addition to reflecting on the overall performance of students, including high- and low-achievers, teachers should examine ways in which aspects of the PISA approach to mathematics can be integrated into mathematics classes. It is recognised that this may represent a challenge to teachers also trying to prepare students for existing state examinations in mathematics.

In 2003, mathematics was the major assessment domain in the OECD Programme for International Student Assessment (PISA). Tests of mathematics were administered to representative national samples of 15-year olds in 41 countries, including Ireland, while students, their mathematics teachers and their principal teachers also completed questionnaires. Although international (OECD, 2004) and national (Cosgrove et al., 2005) reports on the study have been issued, there is a concern that teachers of mathematics in post-primary schools may not be familiar with the outcomes of the study, or its implications for teaching and learning mathematics.

This paper is divided into four sections. The first provides an overview of the PISA mathematics framework and shows some mathematics items that were used in the assessment. The second provides an overview of the main outcomes of PISA mathematics. It looks at overall performance as well as performance on the four PISA mathematics content scales. The third section draws comparisons between PISA mathematics and the Junior Certificate mathematics examination. The final section provides specific suggestions for integrating aspects of PISA mathematics in classroom teaching and learning activities.

The PISA Mathematics Framework

PISA mathematical literacy[1] is defined as 'an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen' (OECD, 2003, p. 24).

The definition and framework are heavily influenced by the realistic mathematics education (RME) movement, which stresses the importance of solving mathematical problems in real-world settings (e.g., Freudenthal, 1973, 1981). Central to this approach is the notion of mathematising. According to the PISA mathematics framework (OECD, 2003), mathematisation is a five stage process:

[1] PISA uses the term mathematical literacy to refer to mathematics ability/ performance. This report uses the terms 'mathematical literacy' and mathematics interchangeably.

1. starting with a problem situated in a real-world context;

2. organising the problem according to mathematical concepts;

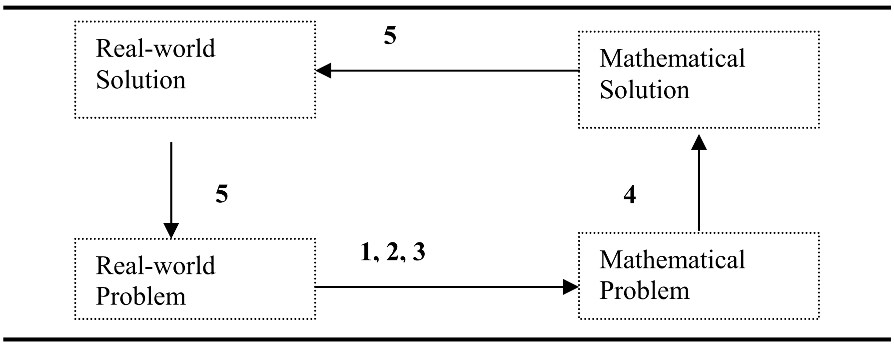
3. gradually ‘trimming away the reality’ by making assumptions about which features of the problem are important, and then generalising and formalising the problem;

4. solving the mathematical problem;

5. making sense of the mathematical solution in terms of the real situation.

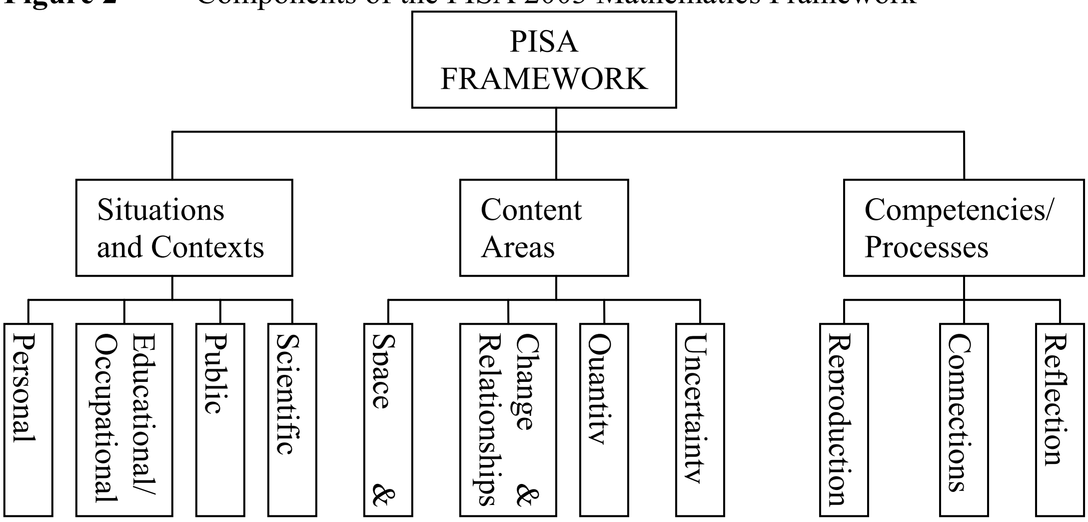
The process of mathematising is illustrated in Figure 1 (numbers indicate the dimensions of mathematisation described above).

Figure 1 The Mathematisation Cycle



The PISA mathematics framework has three dimensions: (i) situations and contexts; (ii) content; (iii) and competencies (Figure 2).

Figure 2 Components of the PISA 2003 Mathematics Framework



Mathematics Situations and Contexts

The ability to use and do mathematics in a variety of situations is considered an important part of mathematics education and the type of mathematics employed often depends on the situation in which the problem is presented. In PISA 2003, four categories of mathematical problem situations and contexts are used: personal, educational/occupational, public, and scientific. The situation is the part of the student’s world in which the problem arises. Context reflects the specific setting within that situation.

Mathematics Content Areas

PISA 2003 measured student performance in four areas of mathematics (also called 'overarching ideas'):

> *Space & Shape* – recognising and understanding geometric patterns and identifying such patterns in abstract and real-world representations;
> *Change & Relationships* – recognising relationships between variables and thinking in terms of and about relationships in a variety of forms including symbolic, algebraic, graphical, tabular, and geometric;
> *Quantity* – understanding relative size, recognising numerical patterns and using numbers to represent quantities and quantifiable attributes of real-world objects;
> *Uncertainty* – solving problems relating to data and chance, which correspond to statistics and probability in school mathematics curricula, respectively.

Mathematics Competencies/Processes

PISA identifies eight types of cognitive processes involved in mathematisation – reasoning; argumentation; communication; modelling; problem-posing and -solving; representation; using symbolic, formal and technical language and operations; and use of aids and tools. A mathematical task may involve one or more of these processes at various levels of complexity. In PISA, these processes are represented at different levels of complexity in three broad competency clusters: Reproduction, Connections, and Reflection. Key features of each competency cluster are described in Table 1.

Table 1 *The PISA Competency Clusters*

Reproduction Cluster	Connections Cluster	Reflection Cluster
Reproducing representations, definitions and facts	Integrating and connecting across content, situations and representations	Complex problem solving and posing
Interpreting simple, familiar representations	Non-routine problem solving, translation	Reflecting on, and gaining insight into,mathematics
Performing routine computations and procedures	Interpretation of problem situations and mathematical statements	Constructing original mathematical approaches
Solving routine problems	Using multiple well-defined methods	Communicating complex argument and complex reasoning
	Engaging in simple mathematical reasoning	Using multiple complex methods
		Making generalisations

Source: Adapted from OECD (2003), Figure 1.4, p.49

Classification of Items by Framework Components

Table 2 provides a breakdown of PISA 2003 items by situation, content area, and competency cluster. It can be seen that, whereas the four content areas are represented by similar proportion of items, the connections cluster is represented by a greater proportion of items than either the reproduction or reflection cluster. In line with PISA's emphasis on education for citizenship, there are proportionally more items classified as public than personal, educational/occupational or scientific.

Table 2 *Distribution of PISA 2003 Mathematics Items by Dimensions of the Mathematics Framework*

Dimension	Number of Items	Percent of Items
Content Area (Overarching Idea)		
Space & Shape	20	23.5
Change & Relationships	22	25.9
Quantity	23	27.1
Uncertainty	20	23.5
Total	**85**	**100.0**
Situation		
Personal	18	21.2
Educational/Occupational	21	24.7
Public	29	34.1
Scientific	17	20.0
Total	**85**	**100.0**
Competency Cluster (Process Category)		
Reproduction	26	30.6
Connections	40	47.1
Reflection	19	22.4
Total	**85**	**100.0**

PISA 2003 mathematics consisted of 54 units. Each unit consisted of a written description of a problem, associated graphics, and one or more items, giving a total of 85 items. However, individual students were required to answer only a proportion of those items, as the item pool was distributed over 13 over-lapping test booklets that were distributed to students at random.[1]

Each PISA mathematics item followed one of five formats: *traditional multiple-choice items,* in which the student selects a response from among several alternatives [20% of items]; *complex multiple-choice items,* in which the student chooses responses for a series of items (e.g., true/false statements) [13%]; c*losed-constructed response items,* in which the answer is given in numeric or other form, and can be scored against precisely-defined criteria [15%]; *short-response items,* in which the student writes a brief answer to a question, and for which there may be a range of possible correct responses [27%];

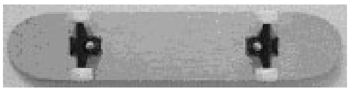
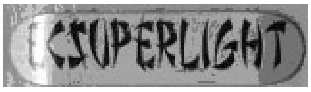
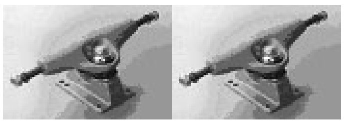
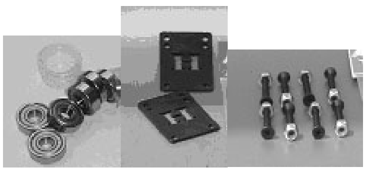
[1] Some of the test booklets also included reading literacy and science units as these domains were also assessed in PISA 2003.

and *open-constructed response items*, in which the student provides a longer written response. There is usually a broad range of possible correct responses. Unlike other item types, the scoring of these questions typically requires significant judgement on the part of trained markers [25%].

Two example items are given.[1] The answer keys accompanying the items illustrate how, for some, there was a single correct answer, while, for others, either partial or full credit was available. The proficiency levels corresponding to the questions will be described later in this paper.

Unit: "Skateboard" [Quantity]. This unit can be considered as presenting archetypal PISA tasks. The introductory scenario involves pictures; moreover, knowledge of the context may well be helpful, though not actually necessary, in addressing the problem. The first question, of Reproduction type, was fairly easy for students in Ireland (69% fully correct), as it was for OECD students in general (67% fully correct). An additional 8% of students in Ireland, and 11% on average across OECD countries had partially correct answers to this question. Students in Ireland did poorly on question 2 (30% correct), compared to the OECD average (46% correct). This is not surprising because the required enumeration algorithm is on the Leaving Certificate rather than the Junior Certificate course, and so would have been unknown to the majority of the group.

Eric is a great skateboard fan. He visits a shop called SKATERS to check some prices. At this shop you can buy a complete board. Or you can buy a deck, a set of 4 wheels, a set of 2 trucks and a set of hardware, and assemble your own board. The prices for the shop's products are:

Product	Price in zeds	
Complete skateboard	82 or 84	
Deck	40, 60 or 65	
One set of 4 Wheels	14 or 36	
One set of 2 Trucks	16	
One set of hardware (bearings, rubber pads, bolts and nuts)	10 or 20	

[1] Additional example items may be found at www.erc.ie/pisa

Question 1:
Eric wants to assemble his own skateboard. What is the minimum price and the maximum price in this shop for self-assembled skateboards?

(a) Minimum price: _______________ zeds.
(b) Maximum price: _______________ zeds.

Item type: Closed constructed response.
Key: Full credit: Both the minimum (80) and the maximum (137) are correct; partial credit: Only the minimum (80) is correct, or only the maximum (137) is correct; no credit: Other responses, missing.
Process: Reproduction. Interpret a simple table, find a simple strategy to come up with the maximum and minimum, and use of a routine addition procedure.

PISA Item Difficulty
Scale Score
463.7 (PC); 496.5 (FC)
Proficiency Level
2 (PC); 3 (FC)

Item Statistics	% OECD	% Ireland
Fully correct	66.7	69.0
Partially correct	10.6	8.2
Incorrect	18.0	20.8
Missing	4.7	2.0
Total	100	100

Question 2:
The shop offers three different decks, two different sets of wheels and two different sets of hardware. There is only one choice for a set of trucks.
How many different skateboards can Eric construct?

A 6
B 8
C 10
D 12

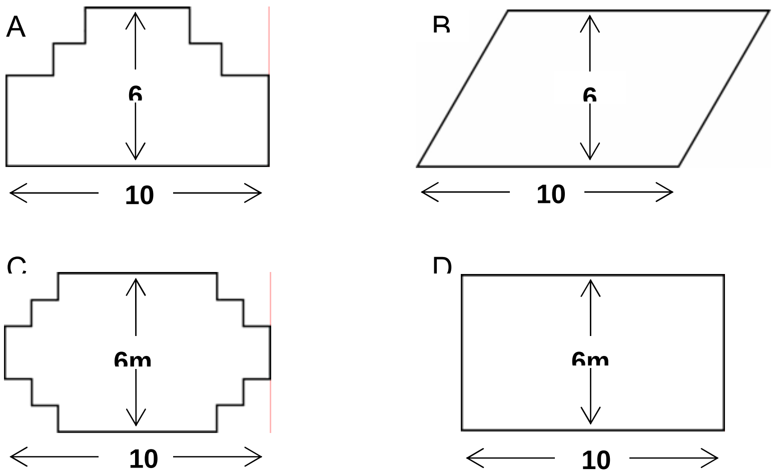
Item type: Traditional multiple choice.
Key: *Full credit*: D; *no credit*: Other responses, missing.
Process: *Reproduction.* Interpret a text in combination with a table; apply a simple enumeration algorithm accurately.

PISA Item Difficulty
Scale Score
569.7
Proficiency Level
4

Item Statistics	% OECD	% Ireland
Correct	45.5	30.2
Incorrect	50.0	66.9
Missing	4.5	2.9
Total	100	100

Unit: "Carpenter" [Shape & Space] This unit consisted of one item. It was a difficult item for students across OECD countries (20% achieved full credit), and particularly for students in Ireland (13%). It is a rare example of an item for which the formal study of traditional Euclidean geometry ("technical geometrical knowledge") – more emphasised in the syllabus of Junior and Leaving Certificate levels than in some other countries – might have proved helpful. In particular, such knowledge might have been helpful in identifying the fact that the "slant" sides of the non-rectangular parallelogram are greater than 6m in length; but few students made the required connections. However, skills of visualisation might have proved equally helpful, and these are not greatly featured in the syllabi.

A carpenter has 32 metres of timber and wants to make a border around a vegetable patch. He is considering the following designs for the vegetable patch.
Circle either "Yes" or "No" for each design to indicate whether the vegetable patch can be made with 32 metres of timber.



Vegetable patch design	Using this design, can the vegetable patch be made with 32 metres of timber?
Design A	Yes / No
Design B	Yes / No
Design C	Yes / No
Design D	Yes / No

Item type: Complex multiple choice.
Key: *Full credit*: Four correct (yes, no, yes, yes, in that order); *partial credit*: Three correct; *no credit*: Two or fewer correct; missing.
Process: *Connections.* Use geometrical insight and argumentation skills, and possibly some technical geometrical knowledge.

PISA Item Difficulty
Scale Score
687.3
Proficiency Level
6

Item Statistics	% OECD	% Ireland
Fully correct	20.0	13.0
Partially correct	30.8	30.9
Incorrect	46.8	54.6
Missing	2.5	1.6
Total	100	100

The Performance of Students in Ireland on PISA 2003 Mathematics

Performance on PISA mathematics was reported with respect to an overall scale (called the 'combined mathematics scale') and four content subscales (Space & Shape, Change & Relationships, Quantity, and Uncertainty). The OECD mean (average) and standard deviation on the overall scale was set at 500 and 100 respectively. These values varied slightly for the subscales (Change & Relationships – mean = 499, sd = 109; Space & Shape – mean = 496, sd = 110; Quantity – mean = 501, sd = 102; and Uncertainty – mean = 502, sd = 99).

Overall Performance on Combined Mathematics

Table 3 shows that the mean score for Ireland on the Combined Mathematics Scale was 503 points. This is not significantly different from the OECD country average of 500. In PISA 2003, students in Ireland achieved mean scores in reading literacy and science that were significantly higher than the corresponding OECD country average scores. This indicates that, relative to performance in reading and science (as measured by PISA), students in Ireland did less well in mathematics.

In PISA combined mathematics, 12 countries (including European countries Finland, the Netherlands, Belgium, Switzerland, and Iceland) had mean scores that are significantly higher than Ireland's (Table 3). Eight countries, including Denmark, Sweden, France and Germany, had mean scores that are not significantly different from Ireland. Norway, Poland and Hungary were among the countries with mean scores that are significantly lower than Ireland.

Table 3 *Countries with Mean Scores on the PISA Combined Mathematics Scale That Are Significantly Higher than, Not Significantly Different from, and Significantly Lower than Ireland's*

Mean Score Significantly Higher than Ireland	**Mean Score Not Significantly Different from Ireland**	**Mean Score Significantly Lower than Ireland**
Hong Kong-Ch (550, ▲)	Czech Rep (517,▲)	Norway (495, ▼)
Finland (544, ▲)	Denmark (514, ▲)	Luxembourg (493, ▼)
Korea (542, ▲)	France (511,▲)	Poland (490, ▼)
Netherlands (538, ▲)	Sweden (509, ▲)	Hungary (490, ▼)
Liechtenstein (536, ▲)	Austria (*506,* o)	Spain (485, ▼)
Japan (534, ▲)	Germany (*503,* o)	*Latvia* (483, ▼)
Canada (533, ▲)	[Ireland (503, o)]	United States (483, ▼)
Belgium (529, ▲)	Slovak Rep (*498, o*)	Russian Fed (468, ▼)
Macao-Ch 527, (▲)		Portugal (466, ▼)
Switzerland (527, ▲)		Italy (466, ▼)
Australia (524, ▲)		Greece (445, ▼)
New Zealand (524,▲)		Serbia and Monte (437, ▼)
Iceland (515,▲)		Turkey (423, ▼)
		Uruguay (422,▼)
		Thailand (417, ▼)
		Mexico (385, ▼)
		Indonesia (360, ▼)
		Tunisia (359, ▼)
		Brazil (356, ▼)

Non-OECD ('partner') countries in italics; (▲) = mean score above OECD average; (o) = mean score not significantly different from OECD average; (▼)= mean score significantly lower than OECD average

Performance of Higher and Lower Achievers on Combined Mathematics

The performance of higher and lower achievers in a country can be evaluated by considering the scores of students at the key benchmarks such as the 10th and 90th percentiles. In Ireland, students scoring at the 10th percentile on the combined mathematics scale achieved a score of 393, which is 34 points higher than the corresponding OECD country average. It is also higher than the scores of students at the same benchmark in some countries with mean scores similar to Ireland, including Germany (363) and Norway (376), suggesting a smaller 'tail' of low achievers in the Irish distribution. Students in Ireland at the 90th percentile achieved a score (614), which is lower (by 14 points) than the corresponding OECD country average, and lower than the scores of high achievers in some countries with similar mean scores to Ireland, including Germany (632) and Sweden (631). This suggests students in Ireland scoring at the 90th percentile in particular are underperforming relative to their counterparts at the same benchmark in other countries with similar overall performance. In general, the difference between high and low achievers in Ireland (221 points) is smaller than the OECD average difference (259), indicating a relatively narrow spread in achievement (a finding also observed when performance on the proficiency levels is considered).

Mathematics Proficiency Scales

The combined mathematics scale was divided into six levels of proficiency, each characterised by different levels of skills and knowledge. The difference between one level and the next is about 62 score points. The descriptions on Table 4 are based on analyses of the content and processes underlying items at each proficiency level.

Table 4 *Summary Descriptions of Proficiency Levels on the Combined Mathematics Scale, and Percentages of Irish and OECD Students Achieving Each Level*

Level	Summary Description
Level 6	Conceptualise, generalise, and utilise information based on investigations and modelling of complex problem situations; link different information sources and representations and flexibly translate among them; demonstrate advanced mathematical thinking and reasoning, and apply this insight along with a mastery of symbolic and formal mathematical operations and relationships to develop new approaches and strategies for attacking novel situations; formulate and precisely communicate actions and reflections regarding findings, interpretations, arguments, and the appropriateness of these to the original situations.
Level 5	Develop and work with models for complex situations, identifying constraints and specifying assumptions; select, compare, and evaluate appropriate problem-solving strategies for dealing with complex problems; work strategically using broad, well-developed thinking and reasoning skills, appropriate linked representations, symbolic and formal characterisations, and insight pertaining to these situations; and reflect on their actions and formulate and communicate their interpretations and reasoning.
Level 4	Work effectively with explicit models for complex concrete situations that may involve constraints or call for making assumptions; select and integrate different representations, including symbolic ones, linking them directly to aspects of real-world situations; utilise well-developed skills and reason flexibly, with some insight, in these contexts; and construct and communicate explanations based on own interpretations, arguments, and actions.

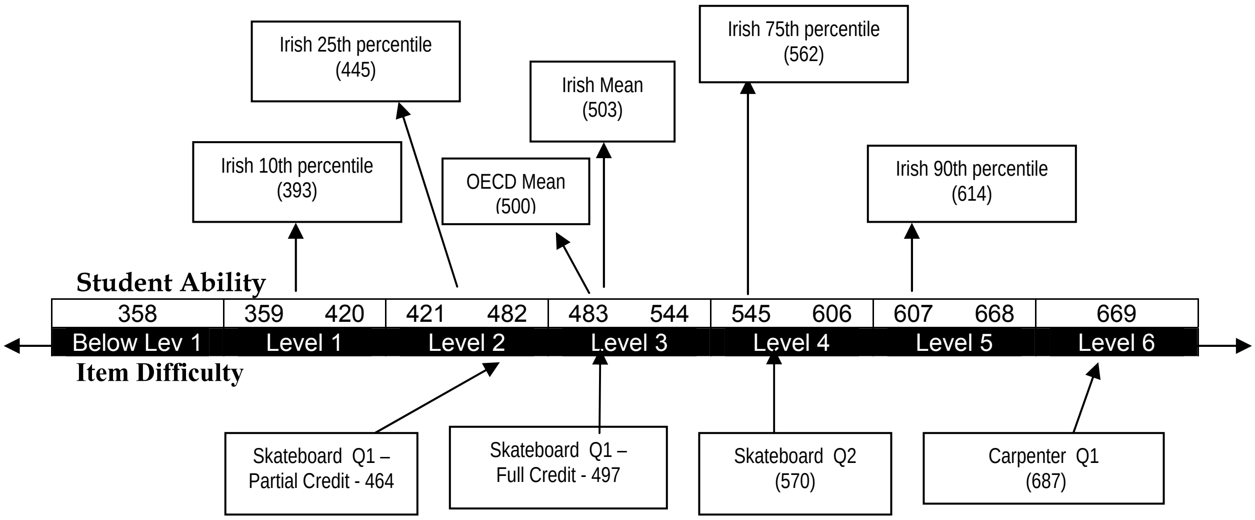
Level 3	Execute clearly described procedures, including those that require sequential decisions; select and apply simple problem-solving strategies; interpret and use representations based on different information sources and reason directly from them and develop short communications reporting interpretations, results and reasoning.
Level 2	Interpret and recognise situations in contexts that require no more than direct inference, extract relevant information from a single source and make use of a single representational mode; employ basic algorithms, formulae, procedures, or conventions, and demonstrate direct reasoning and make literal interpretations of the results.
Level 1	Complete tasks involving familiar contexts where all relevant information is present and the questions are clearly defined; identify information and carry out routine procedures according to direct instructions in explicit situations; and perform actions that are obvious and follow immediately from the given stimuli.
Below Level 1	Has less than .50 chance of responding correctly to Level 1 tasks. Mathematics skills not assessed by PISA.

Source: Cosgrove et al. (2005), Table 3.11.

The PISA proficiency levels were defined in such a way that all students at a given level are expected to respond correctly to at least half of the items they attempt at that level. Further, they are expected to respond correctly to fewer than one-half of items at higher levels, and more than one-half of items at lower levels. Level 6, the highest level, has no ceiling. This means that some high-achieving students have an ability that is higher than the most difficult PISA mathematics, and are likely to get most of the PISA mathematics items they attempt correct. On the other hand, students with a score below Level 1 are unlikely to succeed at even the easiest PISA mathematics items.

In addition to placing students at different points on the proficiency scales, PISA places individual items on the scale. Thus, the partial credit version of Question 1 on the Skateboard Unit, which was described earlier, is at Level 2, while the full-credit version is at Level 3. This indicates that Question 1 was relatively easy, with at least 62% of students scoring at Levels 3, 4, 5 and 6 expected to get full credit. The second Skateboard item was somewhat more difficult. This item is a Level 4 item. Hence, students scoring below Level 4 would be expected to find it challenging. The final item examined earlier, Question 1 in the Carpenter unit, is even more difficult as has a proficiency score of Level 6. This means that students at Level 6 have about a 62% chance of getting it correct, while students at lower levels would less likely to do well on it.

Figure 3 *The PISA 2003 Combined Mathematics Scale: Cut-points for Proficiency Levels, Scores of Students in Ireland at Key Markers, and Difficulties of Selected Items*



In Ireland, 11% of students scored at the highest mathematics proficiency levels (Levels 5 and 6 combined) whereas the corresponding OECD average was 15% (Table 5). This indicates that there are fewer higher-achieving students at these levels in Ireland than the average across OECD countries. Seventeen percent of students in Ireland scored at the lowest levels (Level 1 and below), compared to an OECD average of 21%. Hence, there are fewer very low achievers in Ireland than there are on average across OECD countries. This suggests that Ireland's moderate overall performance may be attributed to the comparatively low performance of high achievers rather than the low performance of low achievers.

Table 5 *Percentages of Students in Ireland, and OECD Average Percentages, Scoring at Each Proficiency Level on PISA Combined Mathematics*

Proficiency Level	Ireland	OECD Average
Level 6 (highest)	2.2	4.0
Level 5	9.1	10.6
Level 4	20.2	19.1
Level 3	28.0	23.7
Level 2	23.6	21.1
Level 1	12.1	13.2
Below Level 1 (lowest)	4.7	8.2
Totals	100.0	100.0

Performance on the Mathematics Content Areas

The performance of students in Ireland is significantly above the OECD average on the Change & Relationships and Uncertainty content scales, while Ireland's performance is significantly lower than the OECD average on the Space & Shape scale, and does not differ significantly from the OECD average on the Quantity scale (Table 6). Of the 29 OECD countries for which results were available, Ireland ranked 10th on the Uncertainty scale, 15th on the Change & Relationships subscale, 18th on the Quantity subscale, and 23rd on the Space & Shape subscale.

Table 6 *Mean Scores and Standard Deviations on the Mathematics Content Scales—Ireland and OECD*

Country	Space & Shape		Change & Relationships		Quantity		Uncertainty	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD
Ireland	476.2[a]	94.5	506.0[b]	87.5	501.7[c]	88.2	517.2[b]	88.8
OECD	496.3	110.1	498.8	109.3	500.7	102.3	502.0	98.6

a = significantly below the OECD average
b = significantly higher than the OECD average
c = not significantly different from OECD average

Comparing PISA and Junior Certificate Mathematics

Many of the objectives of the Junior Certificate mathematics syllabus are reflected in the PISA mathematics framework. For example, the Junior Certificate mathematics objectives of recalling mathematical facts and establishing competencies needed for mathematics activities (instrumental understanding) are consistent with the assumption underlying the PISA framework that, by age 15, students will have mastered basic mathematics skills. Further, the Junior Certificate objective of developing relational understanding is consistent with the PISA view that students need a conceptual understanding of procedures to know which to apply to solve a real-world mathematical problem.

Although many of the Junior Certificate objectives compare well with the aims of PISA, not all of them are assessed in the Junior Certificate examination. For example, PISA emphasises that real-world knowledge and skills, and therefore the ability to solve problems in novel contexts, is an important prerequisite for many of the items. However, the only Junior Certificate objective addressing this skill (analysis of information, including that presented in unfamiliar contexts) is not actually assessed in the Junior Certificate mathematics examination and therefore is likely to receive less emphasis in instruction than those objectives that are assessed. Other objectives that are not assessed in the Junior Certificate examination relate to the ability to create mathematics and development of an appreciation of mathematics. Yet these are consistent with PISA's emphasis on the importance of fostering an interest in and appreciation of mathematics valuable educational outcomes in themselves.

To acquire a better understanding of the links between PISA and Junior Certificate mathematics, curriculum experts in Ireland (all experienced teachers of mathematics) were asked to rate the expected familiarity of each PISA mathematics item for a typical third-year student. Each item received nine ratings, one for each of three aspects (concept, context of application, format) at each of three syllabus levels (Higher, Ordinary, Foundation). Ratings ranged from 1 ('not familiar') to 3 ('very familiar').

The concepts underlying approximately two-thirds of items were rated as being somewhat or very familiar to students at Higher and Ordinary levels while just under half of the items were rated in this way for students at Foundation level (Table 7). On the other hand, the contexts in which the mathematics problems were presented (usually real-world situations) and the item formats (often multiple-choice) were judged to be mostly unfamiliar to students in Ireland at all three syllabus levels.

Table 7 *PISA 2003 Mathematics Curriculum Familiarity Ratings, by Junior Certificate Level*

	Not familiar	*Somewhat familiar*	*Very familiar*
	%	%	%
Concept			
Higher	30.6	24.7	44.7
Ordinary	35.3	29.4	35.3
Foundation	51.8	25.9	22.4
Context			
Higher	65.9	22.4	11.8
Ordinary	70.6	20.0	9.4
Foundation	80.0	16.5	3.5
Format			
Higher	62.4	24.7	12.9
Ordinary	72.9	20.0	7.1
Foundation	83.5	14.1	2.4

Note. Ratings on these scales are made considering the typical third-year student at each syllabus level.

Performance of Junior Certificate Students on PISA

Mean scores on PISA can also be interpreted in terms of the PISA mathematics proficiency levels, described earlier. The mean score of Higher level students in Ireland on PISA was 563.0, which is at Level 4 on the overall proficiency scale. Ordinary level students had a mean score (469.1) which is at Level 2, and Foundation level students have a mean score (385.4) which is at Level 1.

Table 8 shows the percentages of students at each PISA proficiency level classified by the syllabus level at which they took the Junior Certificate mathematics examination. One third of Foundation level students scored below Level 1, indicating that they did not demonstrate even the most basic skills associated with PISA mathematics. Over one-fifth of students at Ordinary level are at or below Level 1, while less than half of students at Ordinary level are at Level 3 or higher. If one accepts the OECD (2004) specification of Level 2 as a basic minimum that students need to achieve to meet their future needs in education and the world of work, it is a matter of concern that relatively large proportions of students taking Ordinary and Foundation levels achieved below this benchmark.

Table 8 *Percentages of Students in Ireland at Each PISA Mathematics Proficiency Level, Classified by Junior Certificate Mathematics Examination Level (2002 and 2003).*

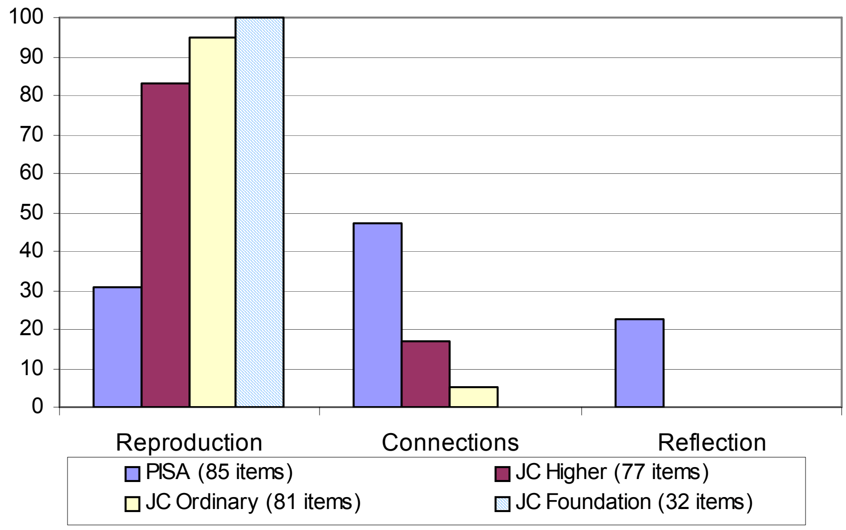
Syllabus Level	**% Below Level 1**	**% at Level 1**	**% at Level 2**	**% at Level 3**	**% at Level 4**	**% at Level 5**	**% at Level 6**	**Total**
Higher	0.3	1.2	9.0	28.8	35.8	19.7	5.2	100
Ordinary	4.1	17.8	36.2	30.4	9.9	1.5	0.1	100
Foundation	33.4	38.5	22.5	5.5	0.0	0.0	0.0	100

Number of students at Higher = 1651; Ordinary = 1941; Foundation = 265; Missing = 24.

Junior Certificate Mathematics through the Lens of PISA

A study by Close and Oldham (2005) examined the extent to which specific mathematics items on the 2003 Junior Certificate Mathematics Examination mapped onto the PISA mathematics framework. Here, we look at the categorisation of Junior Certificate items into PISA competency clusters. Figure 4 shows that over 80% of items in the Higher level examination, over 90% in the Ordinary level examination, and all of the items in the Foundation level examination fell into the PISA Reproduction cluster. This compares with 30% of PISA items that were categorised as Reproduction. The figure also indicates that, whereas over 50% of PISA items fell into the 'Connections' cluster, this was so for 17% of Higher level items, 5% of Ordinary level items, and no Foundation level items. Finally, although over 20% of PISA mathematics items were categorised as 'Reflect' (indicating that they called on higher-level thinking processes, including communicating the reasoning underlying the solution to a problem), none of the Junior Certificate mathematics items was rated in this way. This suggests that Junior Certificate students might have been unprepared for PISA items requiring reflection.

Figure 4 *Percentages of PISA and Junior Certificate Mathematics Examination Items), by PISA Competency Clusters, by Gender.*



Source: Close & Oldham, 2005, Figure 2.

Selected Student and school Characteristics Associated with Performance on PISA Mathematics

This section considers associations between selected variables and performance on PISA mathematics.

Student Gender

Male students significantly outperformed females on the combined mathematics scale in 21 of 29 OECD countries, including Ireland. The difference between males and females in Ireland was moderate (15 points), and slightly larger than the OECD country average (11). Iceland is the only country in which females significantly outperformed males.

Males in Ireland scored significantly higher than females on all four mathematics content scales (Table 9). The difference is greatest for the Space & Shape scale (26 points) and smallest for Quantity (9 points).

Table 9 *Mean Scores of Students in Ireland on the PISA Mathematics Content Scales, by Gender*

	% of Students	Space/Shape	Change/Rel.	Quantity	Uncertainty
Males	50.4	488.9	512.2	506.1	524.9
Females	49.6	463.4	499.6	497.2	509.4
All available	100.0	476.2	506.0	501.7	517.2

Grade Level

Participants in PISA in Ireland were spread over 4 grade levels – Second year (2.8%), Third year (60.9%), Fourth/Transition year (16.7%) and Fifth year (19.6%). Students in Fifth year achieved a mean score (515.5) that is significantly lower than the mean score of students in Fourth year (542.9). Students in Third year (492.3) outperformed students in Second year (406.8), but did less well than students in Fourth and Fifth years.

Student Anxiety about Mathematics

Students were asked to rate their anxiety about mathematics achievement by responding to statements such as ‘I get very nervous about doing mathematics problems’. As in the case of self-efficacy, students were categorised into low, medium, and high groups based on their aggregate responses across several statements. Students in the low anxiety group (i.e., those in the bottom third of the distribution of ‘anxiety’ scores) obtained the highest mean mathematics score (Table 10). The difference between students in the low and medium groups is moderate (34 points), while there is a large difference between the low and high groups (69 points). At the international level, an anxiety about mathematics composite measure was constructed, with an OECD mean of zero, and a standard deviation of zero. In all countries except Poland and Serbia, male students reported significantly lower levels of anxiety than female students. In Ireland, the difference (-0.27) is about the same as the OECD average difference (-0.25).

Table 10 *Mean Combined Mathematics Scores of Students in Ireland, by Level of Anxiety about Mathematics*

Level	Percent of Students	Mean Mathematics Score
Low	30.7	536.8
Medium	39.8	502.6
High	27.7	468.1
No response	1.8	459.9
All available cases	98.2	503.6

School Socioeconomic Status

In Ireland, schools were categorised according to whether or not they were in the Department of Education and Science Disadvantaged Area Scheme. Students in schools designated as disadvantaged achieved a mean score that was 35 points lower than the mean score of students in non-designated schools (Table 11).

Table 11 *Mean Combined Mathematics Scores of Students in Ireland Attending Designated Disadvantaged and Non-Designated Schools*

Disadvantaged Status	Percent of Students	Mean Mathematics Score
Designated	25.4	477.0
Non-designated	74.6	512.3
All cases	100.0	500.3

In each school, the percentage of 15-year old students who were entitled to the Junior Certificate fee waiver was weighted by the number of students in the school who took the Junior Certificate Examination in 2002 or 2003. Each student was then assigned the value of this variable for his or her school. Students attending schools with high proportions of fee-waiver recipients (i.e., those in the top third of the distribution) performed significantly less well on the combined mathematics scale than students attending schools with medium (middle-third) or low (bottom-third) proportions of recipients. The difference in mean achievement between students in high fee-waiver schools (i.e., schools serving mainly low SES students) and students in low fee-waiver schools (serving mainly high SES students) was large (60 points).

School Disciplinary Climate in Mathematics Classes

Students were asked to rate how often each of five events occurred during mathematics classes, including: 'There is noise and disorder', and 'Students don't listen to what the teacher says'. An overall measure of school disciplinary climate was formed by combining students' responses to such items, and averaging them at the school level. Each student was then assigned the disciplinary climate score corresponding to his/her school. Students in schools with high (positive) disciplinary climate scores significantly outperformed students in schools with medium and low disciplinary climate scores (Table 12).

Table 12 *Mean Combined Mathematics Scores of Students in Ireland, by School-level Disciplinary Climate in Mathematics Classes*

Disciplinary Climate	Percent of Students	Mean Mathematics Score
Low	29.2	482.0
Medium	40.4	504.6
High	28.3	525.7
No response	2.1	452.1
All cases	97.9	504.0

It might be noted that disciplinary climate in mathematics classes in Ireland was broadly similar to the OECD average in some respects, and marginally better in others. For example, 32% of students in Ireland reported that there was noise and disorder in mathematics classes, compared to an OECD average of 36%, while 25% said that 'the teacher had to wait a long time for students to quieten down', compared to an OECD average of 32%.

Implications for Teachers

The following are some possible implications for teachers of mathematics that arise from the PISA 2003 findings for mathematics.

- Emphasise a more interactive approach to teaching mathematics, in which students are engaged in discussing problems, both before they are solved, and afterwards. Discussion should focus on identifying the mathematics needed to solve a problem, and on communicating students' reasoning after the problem has been solved.
- Emphasise the full range of cognitive competencies (processes) during teaching. The over-emphasis on reproduction in classrooms and in examinations means that many students may not get an opportunity to apply higher-level competencies such as Connecting and Reflecting. It is likely that the application of these competencies by students at all levels of ability will result in greater conceptual understanding and more independence in solving problems.
- Implement a better balance of context-free questions and questions that are embedded in real-world contexts. Many of the questions in current textbooks and examination papers are context-free. While such items play an important role in developing basic mathematics skills, it is also important to provide students with opportunities to engage with real-world problems. Such engagement serves to make mathematics more relevant for them, and provides them with opportunities for developing a broader range of mathematical competencies.
- Emphasise more use of language in mathematics classes. A potential drawback of the PISA approach is the need for students to call on language skills (including reading and writing) as they engage with mathematics problems. Teachers can support these processes by engaging students more often in discussions about how to solve problems, and how the solutions of problems can be applied in real-world contexts.
- Help students to develop mathematical knowledge in the context of solving problems. This can be achieved in part by providing students with real-world mathematics problems and by discussing with them the mathematics involved and the ways in which the mathematics can be applied to other problems.

- Provide higher-achieving students with more challenges in mathematics. PISA 2003 suggests that higher-achieving students in Ireland could be challenged to a greater extent. Notwithstanding the requirement to prepare such students for the Junior Certificate Mathematics examination, it would be advantageous to challenge them to solve more complex PISA-style mathematics items which would require them to extract mathematical information from real-world problems.
- Transition year may provide an opportunity to engage students at all levels of ability in solving the types of real-world mathematics problems found in PISA.
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A Study of Third Level Students' Beliefs about Mathematics

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In 2006 and 2007, first year students in three third level institutions in Ireland were surveyed in order to gauge their attitudes to mathematics. In particular students were asked whether they believed that mathematical ability could be improved and were asked to rate their confidence in approaching mathematics. Dweck (1986) maintains that a student's theory of intelligence (as to whether intelligence is fixed or malleable) and confidence in his/her present ability combine to influence the student's behaviour when presented with an unfamiliar task. Bandura (1977 and 1993) postulated that self-confidence has a major influence on whether a person will attempt a task and with what determination and perseverance. As part of the survey the mathematical literacy of the students was also measured (Breen et al., 2007) and the data collected will be used to investigate both of these theories. Moreover, the responses of the students with regard to theories of intelligence are contrasted with the responses of mathematics teachers to similar questions as part of the TIMSS (Beaton et al, 1996) and PISA (Cosgrove et al., 2004) studies. Gender and age differences in confidence levels and theories of intelligence are also explored.

Introduction

For many years, third level Mathematics Departments in Ireland have been using diagnostic tests to identify 'at-risk' students and to construct support systems for these students. These diagnostic tests usually seek to identify any gaps in students' basic mathematical skills. The authors felt that more insight into the problems faced by students as they make the transition to third level education could be gained by a broader investigation of students' mathematical literacy and their mathematical attitudes or experiences. It was hoped that the data collected might help to identify ways in which these transition problems could be addressed. Most lecturers have encountered students who have very little belief in their ability to learn new mathematics, and this lack of confidence acts as an obstacle to progress. As part of this study, students were asked questions about the nature of mathematical ability and about their confidence in approaching the subject. We will investigate how these beliefs affect students' mathematical performance and perseverance.

Bandura (1977) introduced the term self-efficacy to describe a person's judgement of his/her own ability to successfully take part in a specific activity. He claims that self-efficacy influences the goals that people set for themselves, how much effort they expend, how long they persevere in the face of difficulties, and their resilience to failure (Bandura 1993, p 131). He claims that people with a strong sense of self-efficacy are more likely to view challenges in a positive light, and will recover quickly from setbacks. On the other hand, people with low self-efficacy will avoid challenges, and any failures will cause them to lose confidence. Hackett and Betz (1989) investigated the relationship between mathematical performance and mathematics self-efficacy. They studied over 200 students at a US university and found that performance and self-efficacy were correlated and that self-efficacy was a better predictor of whether a student would choose a mathematics-related major than mathematical performance. They had previously postulated that female students had unrealistically low levels of mathematics self-efficacy compared to male students but were unable to find evidence to support this theory.

Dweck (1986) claims that a student's goal-setting is influenced by his/her theory of intelligence. She claims that students who believe that intelligence is fixed tend to have performance goals, their aim is to receive positive feedback and to avoid negative judgements of their ability. Students who believe that intelligence is malleable, however, have learning goals and aim to increase their competence. She claims that these goals, along with confidence, have a strong influence on behaviour in the following way: students with learning goals will enjoy challenges and display perseverance whether their confidence is high or low; however, students with performance goals will behave in this way only if their confidence is high and otherwise will avoid challenges and will give up easily. Elliott and Dweck (1988) experimentally tested the latter hypothesis with 101 fifth-grade children by manipulating the children's beliefs about their current levels of ability and the relative values of the two goals. They report that the results obtained confirmed their expectations. Dweck (1986) also reported differences between males and females, she reported that girls (especially bright girls) have less confidence in their abilities and tend to avoid challenges.

Carmichael and Taylor (2005) conducted a study of 129 students to test the hypothesis that motivation was a key factor in determining the success of students on a tertiary preparatory mathematics course in Queensland. They found student beliefs on intelligence did not appear to influence their confidence or performance. Initial findings indicated that specific measures of confidence (that is, confidence in ability to succeed on a specific question) can predict student performance, but it was remarked that any such effect is far more complex than originally thought.

In a study of US college students, Sax (1992) found significant differences between the levels of confidence in mathematical ability between males and females. She followed the students for four years and found that mathematical confidence actually decreased over this time and that the decrease was larger for women. She also found that even after four years of college, students' mathematical confidence was most strongly predicted by second level experiences.

Attitudes of Irish mathematics teachers have been investigated as part of two international studies. In the 1995 TIMMS study (Beaton et al, 1996) and in the 2003 PISA study (Cosgrove et al 2004), teachers were asked whether they agreed with the statement 'Some students have a natural talent for mathematics and others do not'. In TIMMS it was found that 90% of students were taught by teachers who agreed with this statement and this was supported by the results of the PISA survey where 92.4% of teachers agreed or strongly agreed with the statement. Lyons et al., (2003) asked the ten teachers taking part in their intensive study of ten post-primary mathematics classes in Ireland the same question: all 10 agreed. However, they found that the students, especially the girls, taking part did not view innate ability as important as did the teachers. It was also observed in this study that the boys' assessment of their own ability was slightly higher than the girls, though this difference was not significant.

The Study

In February 2006 and 2007, first year students at IT Tralee (ITT), St Patrick's College, Drumcondra (SPD), and the National University of Ireland, Maynooth (NUIM) participated in a one hour survey which comprised of a PISA-style test and a questionnaire. The PISA-style test aimed to measure the students' mathematical literacy and the questionnaire was used to gather their opinions and experiences of mathematics and their mathematical education. The survey was carried out early in the second semester of the students' first year at third level and was anonymous.

Participants

In 2006, this survey was administered to first year students in the Engineering Department at ITT, the Arts faculty at NUIM and the Humanities and Education faculties of SPD. In 2007, the test was again administered to first year students in the Engineering Department at ITT, the Arts faculty at NUIM, and two groups of first year Science students at NUIM. In total, 316 students took part in the study. There were 60 students in the ITT group, 33 students in the SPD group, 131 students in the NUIM Arts group, and 81 students in the NUIM Science group. This Science group is split into separate mathematics classes called Standard (N=36) and Quantitative Methods (QM) (N=45). The main difference between these two classes is that the Standard group have mostly taken Higher Level (HL) mathematics at Leaving Certificate and the QM students have mostly taken Ordinary Level (OL) mathematics. The ITT students and the NUIM Science students are required to study mathematics in first year at third level while the SPD students and the NUIM Arts students have chosen to study mathematics. The full group was made up of 184 females (58.2%) and 132 males (41.8%). There were 29 mature students (9.2% of the total sample). 142 students (44.9%) took mathematics at HL at Leaving Certificate, 169 students (53.5%) took OL and 5 students did not report their Leaving Certificate level. Table 1 below shows the spread of Leaving Certificate results for the group. Nine students (2.8% of group) did not report their Leaving Certificate Mathematics grade. 52% of the female students and 36% of the male students studied mathematics at HL.

Table 1: Leaving Certificate Mathematics Grades

Grade	HA	HB	HC	HD	HE	OA	OB	OC	OD	OE
Number	15	54	58	13	1	62	66	28	9	1
% of Gp	4.7%	17.1%	18.4%	4.1%	.3%	19.6%	20.9%	8.9%	2.8%	.3%

Instrument

The PISA-style test used test items released from the PISA 2000 and PISA 2003 studies (OECD 2004a). The PISA studies aim to measure how well 15-year-old students are equipped to deal with mathematics in real-life situations as opposed to how well they have mastered particular curricula. PISA questions are classified into four content subdomains (Space and Shape, Quantity, Uncertainty, and Change and Relationships), and three competency clusters (Reproduction, Connections, and Reflection). Each item is assigned a difficulty level and students are awarded scores which reflect the difficulty of items they could answer. Thus student scores and item difficulties are measured on the same scale (OECD 2004b). In PISA 2003, scores were standardised so as to have a mean of 500 and a standard deviation of 100. Furthermore, six proficiency levels were identified in PISA 2003. Level 1 students succeed only on the most basic tasks whereas level 6 students are able to handle complex problems and have advanced reasoning skills. The PISA-style test used in this survey contained 13 questions to be completed in 30 minutes. The questions were spread across the four content subdomains, the three competency clusters, and the six proficiency levels in such a way as to make it comparable to a single PISA question-cluster. Item Response Theory was used to convert the raw scores (out of 13) to a score on the PISA scale and this score was then used to assign a proficiency level to each student (OECD 2004b). Table 2 below shows the percentage of students who participated in this study in each literacy level. For an analysis of how Irish students performed on PISA 2003 see Cosgrove et al., (2005). Breen at al., (2007a) describes in detail the PISA-style test administered and subsequent results for first year students at ITT, SPD and NUIM in 2006. Previously, Corcoran (2005) used a PISA-style test to study the mathematical literacy of a group of pre-service primary teachers.

Table 2: Mathematics Literacy Levels

Literacy Level	**Below Level 1**	**Level 1**	**Level 2**	**Level 3**	**Level 4**	**Level 5**	**Level 6**
Percentage	0.9	0.9	5.1	16.5	38.6	15.2	22.8

Once they had completed the PISA-style test, students were asked to fill in a questionnaire. The questions could be grouped into the following categories: second level experience of mathematics; third level experience of mathematics; attitudes to mathematics; study habits. Findings from the 2006 study relating to students' experiences of mathematics at second and third level are reported elsewhere (Breen et al., 2007b). This paper will be concerned with the students' responses to two questions relating to the nature of mathematical ability and their confidence in approaching mathematics.

Motivational Factors

Improving Mathematical Ability

Students were asked whether it is possible to improve natural mathematical ability. Of the 303 students who answered this question, 82% answered yes, 14% said no and 4% were unsure.

Of the students who commented further, 58% said that practice and hard work can help improve ability. About 10% expressed opinions like 'you either have it or you don't'. One student said 'Everyone has a limit to how much they understand. Maths requires a lot of understanding, if you don't have this initially it is hard to develop it'. It should be noted that 17% (N=55) of students did not offer any comments. As noted earlier, in TIMSS 1995, 90% of Irish second year students were taught by teachers who agreed that some students have a natural talent for mathematics while others do not (Beaton et al, 1996). In a survey of Irish mathematics teachers in schools participating in PISA 2003, more than 90% held a similar opinion, (Cosgrove et al, 2004). Moreover, Irish teachers interviewed by Lyons et al (2003) agreed. This illustrates a stark contrast in the views held by teachers and adult learners.

It is interesting to look at the comments in this section. Many of the 58% who think practice makes perfect say that you need to work hard, do lots of examples, or learn formulas in order to improve. Some typical comments were: ‘maths is about practice’; ‘do loads of questions’; ‘you can see trends in problems the more you repeat them’. This attitude seems to stem from the view of mathematics as a series of procedures rather than a series of concepts. It also reflects the study methods that many students employ. Not all students felt like this however, for example, one said ‘A deeper understanding of maths instead of mechanically memorising would improve natural skills’. Many of the responses expressed opinions similar to ‘you can always improve any skill’ or ‘if you have an open mind or challenge yourself then you can improve’. These echo findings of Carmichael and Taylor (2005), who concluded from their study of 129 students on a Tertiary Preparatory Mathematics Course, that most adult learners subscribe to an incremental view of intelligence (intelligence is malleable) rather than an entity view (that is, the interpretation of smartness as a static entity or fixed trait).

Dweck (1986) cites a study of bright junior high school students that showed there was a greater tendency for the girls surveyed to subscribe to an entity theory of intelligence than the boys. This is not the case here: theories of intelligence are independent of gender (p=0.222, Fisher Exact Test (FET)) with 83.1% of males and 81.6% of females subscribing to the idea of malleable intelligence. No age differences or differences between HL and OL students were found in the responses to this question. The view of intelligence held by students is not independent of the compulsory nature of their third level mathematics courses (p=0.038, FET). Of those who have chosen to study mathematics at third level 9.8% believe mathematical ability cannot be improved while 18.6% of those for whom third-level mathematics is compulsory share this belief.

Confidence

The students were asked whether they felt confident in approaching mathematics, they were asked to answer on a 5 point Likert scale. Of the 314 students who answered this question 123 (or 39%) said that they were confident or very confident. There was no significant difference between the students at the three different colleges. There is a difference (p=0.000, FET) however, when looking at the whole sample, grouped by Leaving Certificate Mathematics level, as shown below in Table 3 with HL students being more confident as might be expected. 58% of HL students said they were confident or very confident but only 24% of OL students did. There was also a difference in confidence between the students who had chosen to study mathematics and those for whom it is compulsory (p=0.001, FET). 48% of students who had chosen mathematics said that they were confident or very confident while only 29% of the compulsory group said this.

Table 3: Confidence and Leaving Certificate Level.

		Confidence in Approaching Mathematics					
		1	2	3	4	5	Total
LC Level	**Higher**	5	17	37	67	15	141
	Ordinary	28	48	52	39	1	168
Total		33	65	89	106	16	309

In the sample as a whole, 46% of males classed themselves as confident (choosing 4 or 5 on the Likert scale) compared with 34% of females which shows that there is a significant gender difference (p=0.03, Chi-squared). If LC Level is kept fixed, then gender and confidence are not independent (p=0.000, FET). Just over half (50.5%) of HL females said they were confident or very confident whereas 74% of HL males felt confident or very confident. To see if there was a gender difference in confidence within each Leaving Certificate grade a Cochran test was used. It was found that confidence and gender are not independent when Leaving Cert grade is kept fixed (p=0.000). As before, males were more likely to be confident than females. A 95% confidence interval for the common odds ratio is (0.193, 0.598). This means that the odds of a female being confident are no more than 60% of the odds for a male with the same Leaving Cert grade. For example, at OA1 grade, 77% of males rated themselves as confident but only 23% of females did. These results are similar to the findings of the PISA 2003 study (p.119 Cosgrove et al, 2005), where it was found that Irish male students were more confident about their mathematical abilities and also less anxious about mathematics than female students. Moreover, it echoes the results of a study of Carmichael and Taylor (2005) who also found that females consistently reported lower levels of confidence in relation to mathematics than males, although there were no significant differences in their prior knowledge or subsequent academic performances.

Surprisingly there was no significant difference between the proportions of mature and non-mature students rating themselves as confident when approaching mathematics, although a higher proportion of mature students felt confident or very confident (44.8% as opposed to 38.5% of the non-mature group). This contrasts with the results of Carmichael and Taylor (2005) who found that students for whom there had been a longer period of time since studying reported lower levels of confidence. Possible explanations may be that only 29 of the 316 students in the Irish study were mature or that not as much time had elapsed for the Irish students since last undertaking formal study in mathematics.

For the group as a whole, confidence and PISA score (categorised by literacy level attained) are not independent (p=0.000, FET). 53% of confident students achieved a mathematical literacy level of 5 or 6 but only 28% of the less confident students scored at these levels. However we have seen that HL students are more confident than OL students and it would be expected that HL students would perform better on the PISA test so HL and OL students were considered separately. If HL students are considered as a group then PISA score and confidence level are not independent (p=0.002, FET) with students with higher confidence levels doing better on the PISA test. For OL students PISA scores and confidence levels are independent (p=0.146, FET). It may be that OL students underestimate their mathematical ability.

Adaptive and Maladaptive Behaviour Patterns

Dweck (1986) focuses on psychological factors, other than ability, that characterise adaptive and maladaptive behaviour patterns in students, which in turn determine how effectively these individuals obtain and use skills. The adaptive pattern of achievement behaviour can be identified by the challenge-seeking behaviour and high persistence of those who display it, while maladaptive patterns are recognised by the challenge-avoidance and low persistence of its carriers in the face of difficulty. Dweck maintains that a child's tendency towards adaptive or maladaptive behaviour is intrinsically linked with the class of goal motivating his sense of achievement and that, furthermore, a child's theory of intelligence orients him towards a certain class of goal. In particular, children subscribing to an entity theory of intelligence, in which intelligence is believed to be fixed, tend towards performance goals (through which favourable judgements of their competence are sought) whereas those believing intelligence is malleable, or supporting an incremental view of intelligence, tend towards learning goals (by which they seek to increase their competence).

According to Dweck, for a child with performance goals, the entire choice and pursuit of a task revolves around the child's own concerns about his ability. If he is to obtain a positive judgement of his ability then he needs to be confident in his present ability at the outset in order to seek challenges on a certain task and persist when obstacles are encountered. However, if his confidence in his present ability is low then he will try to conceal his ability and protect it from unfavourable evaluation (see Figure 1). On the other hand, a child with learning goals will always choose challenging tasks that foster learning and is willing to risk displays of ignorance in order to acquire knowledge.

Figure 1: Achievement Behaviour Patterns (Dweck, 1986)

Theory of Intelligence	**Goal Orientation**	**Confidence in present Ability**	**Behaviour Pattern**
Entity Theory →	Performance Goal	If high →	Seeks challenge High persistence
		If low →	Avoids challenge Low persistence
Incremental Theory →	Learning Goal	If high →	Seeks challenge High persistence
		If low ↗	

In order to investigate this theory using the data collected for students in ITT, SPD and NUIM, students with an incremental theory of intelligence are deemed to be those who agreed that mathematical ability can be improved and those who disagreed are taken to subscribe to a fixed theory of intelligence. As before, students exhibiting high confidence in their own abilities are those who chose 4 or 5 on the Likert confidence scale, while the remainder are said to be of low confidence. Table 4 illustrates the division of participants into the four categories that result.

Table 4: Theories of Intelligence and Confidence

		Confidence in Present Ability		**Total**
		Low Confidence	High ***Confidence***	
Theory of Intelligence	Entity Theory	27 8.5%	15 4.7%	42 13.2%
	Incremental Theory	145 45.9%	103 32.6%	248 78.5%
Total		172 54.4%	118 37.3%	290 91.7%

The placement of a student in one of these categories is not independent of the level at which LC Mathematics was studied (p=0.000, FET): this is largely due to the confidence factor. Placement in a category is independent of gender (p=0.075, FET) and of 'maturity' (p=0.827, FET).

Transfer of Knowledge

Dweck (1986) goes on to state that one of the hallmarks of effective learning is the tendency to apply or transfer knowledge to novel tasks. In a 1985 study with Farrell, the relationship between children's goal orientation and transfer of learning was examined. The results showed that children with learning goals rather than performance goals (a) attained significantly higher scores on the transfer test (regardless of pretest performance) and (b) produced about 50% more work on their transfer test (suggesting a greater level of activity in the process).

The PISA-style test administered in the ITT, SPD and NUIM study represented an unfamiliar or novel task for the students in question and required the transfer of mathematical knowledge and skills learned previously. If Dweck's findings were to be replicated by this study, the students supporting an incremental theory of intelligence should score more highly on the test than those who believe intelligence is fixed. However, the theory of intelligence supported by students was found to be independent of the PISA level of literacy attained (p=0.215, FET). In fact, 38.2% of those who believe in the incremental nature of intelligence, and 42.9% of those who don't, performed at levels 5 or 6.

The PISA test presented students with 13 items or questions. One student attempted as few as 4 of these. Details on the number of questions attempted are given in Table 5.

Table 5: Number of Test Items Attempted

No. of items attempted	13	12	11	10	9	8	7	4
No. of students	176	84	25	20	7	1	2	1
% of Students	55.7	26.6	7.9	6.3	2.2	0.3	0.6	0.3

The number of questions attempted by students was taken as a measure of the level of activity they exerted on the test. To enable statistical tests to be carried out, the number of items attempted was divided into three categories: all 13 items, exactly 12 items, 11 or fewer. An analysis of the number of questions attempted together with the theory of intelligence supported by a student shows them to be independent (p=0.807, FET). Again the results obtained disagree with those of Dweck, with 61.9% of students who believe intelligence is fixed attempting all 13 items compared with 55.4% of students who agree with an incremental theory. Furthermore, comparing the number of level 6 questions (that is, the most difficult items) attempted by students with the theory of intelligence supported shows these too to be independent (p=0.341, FET) as is the students' performance on these level 6 items (p=0.880, FET). In both cases, the results again contradict Dweck's findings with students who believe intelligence is fixed being more inclined to attempt the more difficult questions and scoring more highly when they do so.

High or Low Persistence

The level of confidence reported by students was incorporated, in order to investigate whether the behaviour patterns suggested by Dweck (see Figure 1) were exhibited by this group of students. Dweck's categories will be labelled as follows in reporting the results: EL (entity theory of intelligence, low confidence), EH (entity theory, high confidence), IL (incremental theory of intelligence, low confidence), IH (incremental theory, high confidence).

The number of questions attempted by students on the PISA test was not independent of the latter categories (p=0.003, FET). In fact, the following pattern of behaviour was noted:

Table 6: Persistence Behaviour

Category	% of students attempting all 13 items
EH	80.0
EL	51.9
IH	68.9
IL	45.5

It is true to say that, within the entity theory of intelligence categories, students of lower confidence showed less persistence than those of higher confidence. However, these are not the least persistent group and Dweck's suggestion that students holding a view of intelligence as malleable will always persist in the face of obstacles is not borne out by the data collected here. In fact, confidence was seen to be the influential factor when determining a students' persistence as measured by number of questions attempted: these are not independent (p=0.001, FET). Likewise, the number of level 6 questions attempted by students was not independent of confidence displayed (p=0.009, FET) nor was the students' performance on level 6 questions (p=0.000, FET). Students with high confidence in their present ability tended to attempt more level 6 questions and performed better on them than students with low confidence. (The relationships between the students' theories of intelligence and these measures of persistence have been commented on in the previous section.) These results support Bandura's assertion that students with high confidence are more likely to attempt difficult problems, will display greater perseverance, and so ultimately will be more likely to succeed.

Another interesting observation is that the number of questions attempted is not independent of the level at which mathematics was studied for the Leaving Certificate (p=0.002, FET) with 64.8% of HL students and 49.1% of OL attempting all 13 items. In fact, a Cochran test provides some evidence of a relationship between confidence and the attempting of all test items, even when the students' Leaving Certificate grade is kept fixed (p=0.066). This would indicate that the relationship between confidence and perseverance on the PISA-style test cannot be explained purely in terms of previous mathematical performance.

Effect of Gender on Behaviour Patterns & Persistence

Although the appearance of students in the categories suggested by Dweck is independent of gender (as remarked above), splitting the group into males and females shows some differences in their category placements as illustrated by Table 7.

Table 7: Theories of Intelligence and Confidence by Gender

Category	No. & Percentage of Males	No. & Percentage of Females
EH	9 6.8%	6 3.3%
EL	10 7.6%	17 9.2%
IH	51 38.6%	52 28.3%
IL	52 39.4%	93 50.5%

Once again, this is largely due to the confidence factor and it has already been remarked that levels of confidence displayed are not independent of gender. Thus, the behaviours of males and females, in terms of their persistence on the PISA test, were investigated separately.

For the male group, the number of questions attempted (13, 12, 11 or less) is not independent of the categories inhabited (p=0.007, FET) (see Table 8 below), neither is the number of level 6 questions attempted (p=0.002, FET). While some relationship can be observed between the categories of Table 7 and a student's performance on the more difficult level 6 test items, this is not significant (p=0.067, FET).

The results are somewhat different for the female group with the categories inhabited in this case being independent of the number of questions attempted (p=0.521, FET), the number of level 6 questions attempted (p=0.931, FET) and performance on the level 6 questions (p=0.210, FET).

Table 8: Persistence Behaviour by Gender

Category	% of students attempting all 13 items	
	Males	**Females**
EH	77.8%	83.3%
EL	50%	52.9%
IH	72.5%	65.4%
IL	36.5%	50.5%

Again it can be seen, contrary to Dweck's assertion, that those students showing the least inclination to persevere with a difficult task are those who subscribe to an incremental theory of intelligence but have low confidence in their present abilities: this is more pronounced in the male group than the female group.

Separating the students into two subgroups based on whether or not they obtained an above average score on the PISA test (corresponding to a raw score of 9 or above), results in a grouping that is not independent of Dweck's suggested behaviour categories for males (p=0.001, FET) or for females (p=0.048, FET). Details for each of the four categories of the percentages of students who obtain above average PISA scores are shown in Table 9.

Table 9: Achievement by Gender

Category	**% of students obtaining above-average score**	
	Males	**Females**
EH	88.9%	100%
EL	50%	35.3%
IH	74.5%	51.9%
IL	42.3%	55.9%

In the case of the males, it can be seen that confidence is the influential factor with males who display higher confidence being more likely to obtain an above-average score (p=0.000, FET). This is not the case for females (p=0.876, FET) and indeed, the picture is a lot less clear in this case.

Dweck (1986) states that a number of previous studies have found that girls, and particularly bright girls, display greater tendency towards challenge avoidance and debilitation in the face of obstacles. (However, Elliott and Dweck (1988) did not report any observations on gender difference in their study of 57 girls and 44 boys.) This may go some way to explain the relatively poor performance of the confident females here who believe intelligence to be incremental in nature. In fact, Dweck (1986) suggests these differences in behaviour patterns between girls and boys may be an important factor in the discrepancies in mathematical achievement often reported between the sexes and may prevent females from choosing to study mathematics at more advanced levels.

As remarked earlier, some of the students participating in this study are required to study mathematics at third level while others have chosen to do so. For 69.4% of the males surveyed, mathematics courses are compulsory and the view of intelligence held by males is independent of the compulsory nature of their mathematics courses (p=0.475, FET). Only 30.2% of the females involved have not chosen to study mathematics and while it is also true in this case that the view of intelligence they hold is independent of the compulsory nature of their courses (p=0.073, FET), this result is less significant.

Discussion
It has long been accepted that factors other than ability influence whether students use and develop their skills effectively, and how they do so. The study reported here found the level of confidence incoming third-level students have in their mathematical ability to be an important factor in how they approach, persevere and perform on unfamiliar tasks. This follows Bandura's (1977, 1993) findings on self-efficacy. However, it should be noted that Bandura measured self-efficacy in terms of an individual's conviction of his ability to execute the behaviour required to succeed on a particular task, whereas the level of confidence measured here dealt with the student's confidence in approaching mathematics in general.

On the other hand, students' beliefs about the nature of mathematical ability do not seem to influence the behaviour patterns adopted by these students when presented with the task described. This runs contrary to the theory outlined by Dweck (1986). Perhaps this is due to the difference in age and level of previous academic achievement of the subjects: the participants of the ITT, SPD, NUIM study are essentially adult learners and, by their presence at third-level, can be assumed to be towards the high-achieving end of the general population for the age cohort to which they belong. Also the survey described here asked participants only to respond to a single question on the nature of mathematical intelligence and it is likely this was not sufficient to give a consistent, robust picture of the students' beliefs.

Alternatively, it may be that goal-orientation influences student learning behaviour as described by Dweck (1986) but that theory of intelligence does not play as important or integral a role as the model suggests.

Broadly speaking, there was no difference in the views held by female and male students on the nature of mathematical intelligence, but there were significant differences in the levels of confidence displayed. A full picture of the gender differences uncovered is much more complex.

Finally, it was found that the views of participants as to the nature of mathematical intelligence greatly contrasted with the views of Irish teachers (Beaton et al., 1996; Lyons et al., 2003; Cosgrave et al., 2004). There has been some concern expressed (Dweck, 1986; Lyons et al., 2003, p.270) that teachers' views of students' ability can actually affect students' performance, leading to a 'self-fulfilling' prophecy. Thus, the overwhelming agreement among students (82% of respondents) that mathematical ability can be improved is to be welcomed.

The authors hope to undertake a more comprehensive survey on third-level students' beliefs about the nature of mathematical intelligence and the level of confidence felt in approaching mathematical tasks and courses in order to investigate these issues further.

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Diagnostic Testing - An Evaluation 1998-2007

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Diagnostic tests to assess basic mathematical skills of students entering service mathematics courses are commonplace within third level institutions. Their purpose and use are varied: they can identify learner deficiencies, misconceptions and skill level in order to design and implement remedies; they give quick feedback to students on what is expected, and guidance for effective support. The collation and analysis of students' results on such tests can play an important part in 'identifying students at risk of failing because of their mathematical deficiencies, targeting remedial help, designing programmes and modules that take account of general levels of mathematical attainments, and removing unrealistic staff expectations', (Engineering Council, 2000) The author has been running a programme using diagnostic testing for the past ten years. This paper analyses the test data and looks at the effectiveness of the programme. It identifies key issues and areas for further study.

Introduction

There is a worldwide concern about the mathematical skills of students entering third level education. (See for example, Barry and Davis 1998, Brennan 1997, Gill 2006, Lawson 1997 and McGuinness 2004). The lack of basic skills is seen as a particular impediment for students in scientific, engineering and technical disciplines. These concerns were voiced in Ireland as far back as 1997 when, at a conference in Waterford Institute of Technology, Brennan (1997) described the gaps and misconceptions students had in topics, skills and notation when they entered college. A UK report *Measuring the Mathematics Problem* (Engineering Council, 2000) noted a serious, well established decline in students' basic mathematical skills and level of preparation for mathematics-based degree courses. This downward trend has also been shown in Irish third level (Gill 2006 and McGuinness 2004). There is an increasing range in the mathematical attainments and knowledge of students entering science and engineering degree programmes. A leading mathematics educator in the UK, Croft, described the situation succinctly:

> A long-term joint effort will be required by government, funding agencies, schools, universities and others. … [and] until such efforts yield an improvement in the situation, universities need to take steps to alleviate the problem within their own immediate sphere of influence. (Croft 2002)

Reports and reviews on methods of measuring basic skills in mathematics and methods of addressing the issues abound. Following on from recommendations from the above-mentioned UK report, many universities in the UK began using diagnostic testing. There was a survey of methods used in the UK, described in Diagnostic Testing in Mathematics (MSOR, 2003). Post secondary institutions in the USA have been utilizing placement tests for college entrance and the U.S. Department of Education (2005) reviewed literature around this topic. It reported that assessment tests are currently the most common requirement of students who wish to pursue college-level mathematics (US Department of Education, 2005, p17).

In Ireland, O'Donoghue (2004) proposed the mathematics problem at third level should be tackled as a multi-faceted issue that includes investigating mathematical shortcomings and deficiencies in incoming students, what pre-requisite mathematical knowledge and skills were required, and general numeracy and literacy deficiencies. These issues have given rise to research into the mathematical preparedness of students, how this is dealt with at the second-third level interface and implications for teaching mathematics at third level, particularly teaching mathematics as a service subject.

Background

Mathematics is a mandatory module in all years of civil engineering third level courses in the Institute of Technology sector. As there is generally no requirement for Higher level Leaving Certificate mathematics, first year is considered a consolidation year, covering similar content as the Leaving Certificate Higher level with an emphasis on application to engineering. To ensure that first year engineering students in the researcher's institute are equipped with the basic arithmetic and algebraic skills required for the remainder of the course, the first seven weeks are dedicated to what is called the 'Benchmarking Programme'. This programme is a result of a strategy introduced in 1997 by the Benchmarking Mathematics Network within the Schools of Engineering in Institutes of Technology that had the following objectives:

- To remedy deficiencies in Leaving Certificate mathematics knowledge of first year engineering students
- To achieve 70% of first year engineering students passing mathematics at first attempt
- To develop a common first year core syllabus

Benchmarking mathematics in the researcher's institute consists of an initial diagnostic test, followed by approximately six weeks of small group tutorials and then a second test on the same material. The topics covered in the programme have remained fairly constant over the years, sometimes with different emphasis on topics depending on students' needs.

The Measuring Instrument – the Diagnostic Test

The benchmarking programme has run for nine of the last ten years in the researcher's institute. For most of these years the diagnostic test (pre-test) and post-test were identical multiple-choice questions. Much care was taken to ensure that incorrect answers would identify misconceptions and systematic errors. Students were encouraged to answer a question only if they were sure, and to leave questions unanswered if unsure in order to identify areas they needed to work on during the programme. For the post-test negative marking was implemented to reduce guesswork. After the pre-test was corrected, papers were returned to students in a class to compare their own answers with correct ones, identifying strengths and weaknesses. All papers were then collected so that the same paper could be used for the post-test. Having the same questions has allowed for a measurement of improvement that is reliable.
The topics covered on the test and in the programme are

- Order of operations
- Numerical and algebraic fractions
- Scientific notation
- Calculation estimates
- Meaning of equations, simplification and factorization of polynomial expressions
- Transposition of formula
- Factorization of quadratic expressions and solving quadratic equations
- Linear equations and graphs $y = mx + c$
- Indices – application of laws

The diagnostic test (pre-test) and post-test have remained relatively consistent over the years in terms of the topics covered. After two years some questions on linear graphs were added, and in 2006 the number of questions was reduced and algebraic fractions were not included. This second change, although untimely for this study, was instigated to facilitate another group of first year engineering students. Even with these changes nearly all questions on the current paper are the same as before, with 16 of the 20 questions identical to previous papers.

Data Collection

Students' results for the pre-test, post-test and final mathematics scores were available. The final mathematics score is made up of an exam mark (60%) and continuous assessments, including projects, presentations and class tests (40%). In addition, students' Leaving Certificate mathematics grades for five of these years were also available. In total, there are 484 students in the study and the data collected is summarized in Table 1. There has been no sampling as all students with at least some data available were included.

Table 1: *Data collected for each of the years 1997 to 2006*

Year of Entry	LC grade	Pre-test	Post-test	Final Grade
1997	42	42	42	44
1998		44	42	39
1999		60	57	54
2000		43	43	44
2002	34	39	41	42
2003	80	62	59	65
2004	35	47	49	47
2005	26	37	39	40
2006	48	47	48	39
Total	265	421	420	417

Essentially the same measuring instrument has been used over the years with only minor changes. The format of the programme has also been consistent for the duration, and so all students have experienced very similar settings. However, there are time-dependent changes over which the researcher has no control. The most crucial will be that of students' development over the duration of the programme. Students may have actually known some the material, but had not recalled it for the first test. Hence the 'improvement' may not necessarily be a measure of how much a student has learnt during the programme.

The data was analysed to identify whether the programme has been of benefit for the students over these years, looking for correlations in improvements, post-test results and final grades.

Student Profile

The vast majority of students entering first year Civil Engineering have completed their Leaving Certificate. The main route for entry is through the Central Admissions Office (CAO) and is dependent on total Leaving Certificate points earned. For years 1998-06 Leaving Certificate cut-off points for entry into the course were available. For years 2002-06 median entry points were also available.

Figure 1: *Leaving Certificate Cut-off points and Median entry points*

It can be seen that even though the minimum requirements are quite low students are coming in with substantially higher points, given the median points for the years available. The last four years in particular have had median points of approximately 300, but the mathematical profiles over these four years vary greatly, as can be seen later in Table 4.

Results

Results for the three variables, pre-test, post-test and final maths scores, are consistent over the years. A general pattern emerges in that for most years the post-test has the highest average, and the pre-test and the final maths scores produce lower averages (See Figure 2). The higher than expected pre-test average in 2006 was due to the reduction in the number of questions and the omission of some more difficult questions about algebraic fractions. The surprisingly lower average for the post-test in this year is due to some very low marks and large dis-improvements.

Looking at all 484 students, there is an improvement over the duration of the programme. The pre-test had an average of 50.0% and the post-test had an average of 60.4%. This shows an increase in the average of 10.4%, which is statistically significant at the 5% level. An F-test for the variances for the two sets of data show there is no statistical difference ($p = 0.071$).

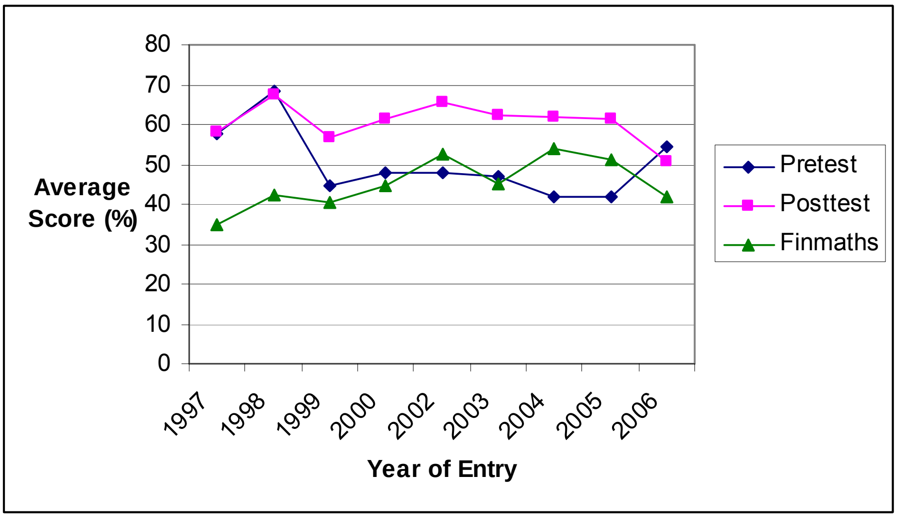

Figure 2: *Average Scores on Pre-test, Post-test and Final Maths Score*

Leaving Certificate mathematics grades have been collected for six of the nine years, 263 of the 474 total number of students. Of these 263 students only 34 have studied Leaving Certificate at Higher level.

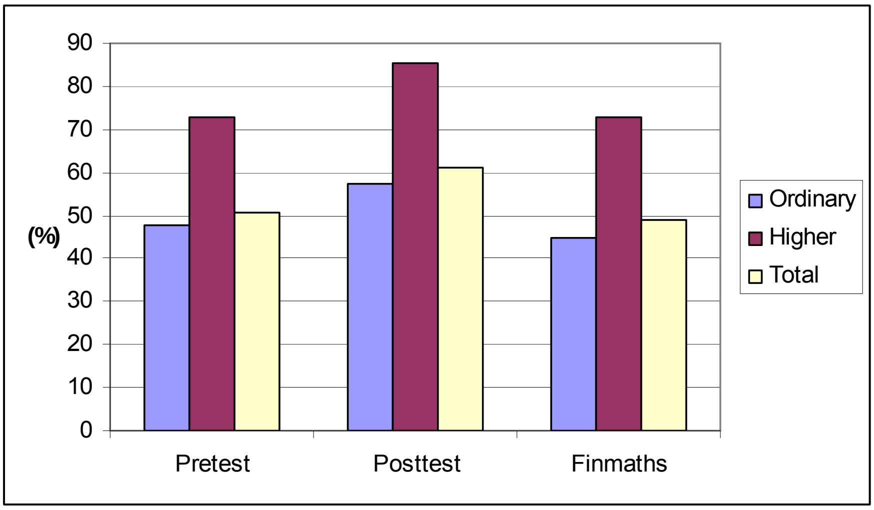

Figure 3: *Average Test Scores by Leaving Certificate Level Maths*

Figure 3 displays the average results for each of the three variables when divided into Ordinary and Higher level. Students with Higher level mathematics average score for the pre-test was 74%, almost 25% higher than the students who had studied Ordinary level mathematics. A telling fact is that these students also improved from the pre-test to the post-test by 11.5% on average, significantly more the 8.8% for Ordinary level.

The data must also be analysed case by case, as the programme was designed to improve each student's basic skills. Firstly, it is important to note that if students score very highly on the pre-test, then they must necessarily have only a small improvement over the duration of the programme, and it could possibly be a dis-improvement. For example a student could have scored 95% on the first and 92% on the second test, still only getting 2 questions incorrect.

There are a total of 379 students with both pre-test and post-test scores, and a new variable measuring the improvement of each student was created (Post-test – Pre-test).

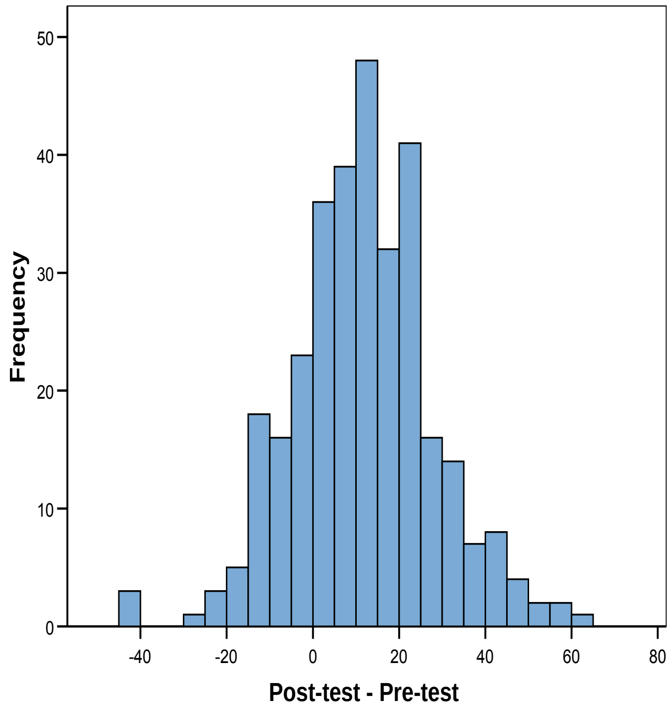

Figure 4: *Improvement of Scores over Duration of Programme*

Figure 4 shows that when looking at individual improvements, the distribution is approximately normal. The mean is 10.3%, and there is a large spread, with the range from -45 to 63% and standard deviation 16.64%. The interquartile range is 20%, indicating outliers.

Students below the average pre-test score of 50% made a significantly larger improvement of 17% compared to those above the average whose improvement was just 3% ($p<0.000$).

Those who did not improve
The profile of the students who did not improve needs to be examined more closely.
A total of 88 students showed a dis-improvement. The overall average decrease was 11.1%, but Table 2 shows there is no significant difference between those passing or failing.

Table 2: *Dis-improvement of students who Passed/Failed*

	N	Mean	Std. Deviation
Failed Maths	43	-11.5	10.3
Passed Maths	45	-10.6	8.0

For 58 of the 88 (66%) students Leaving Certificate maths grades are also available.
Figure 5 shows that for students who did not improve, 86% with Ordinary D and 68% students with Ordinary C did not pass first year mathematics, yet 75% of grade B go on

to pass. Of the two failing students with Ordinary A, one passed by compensation, and the other left the course. All students with Honours level passed the year.

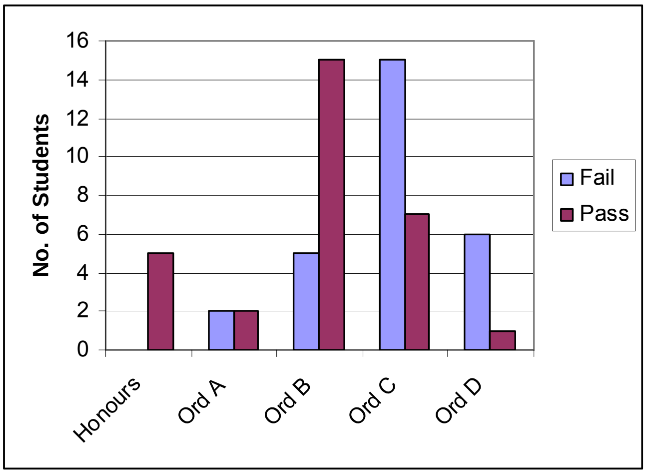

Figure 5: *Leaving Certificate profiles of those who dis-improved*

Prior Mathematics Profile

Focussing on just those students who previously had studied Ordinary level maths, we see firstly there is a marked difference across the grades as to whether or not a student completes the year.

Table 3: *Progression of Students entering with Ordinary level Maths*

	Sat final exam		Total	% not completing year
	Yes	No		
Ord A	18	2	20	10
Ord B	67	7	74	9
Ord C	71	18	89	20
Ord D	31	15	46	33
Total	218	45	263	17

Table 3 shows that two students, both from 2006, entering with an A in Ordinary Leaving Certificate left the course during the year. Otherwise, as the Leaving Certificate grade decreases, the non-completion rate increases, up to 33% of students with a D grade not completing the year.

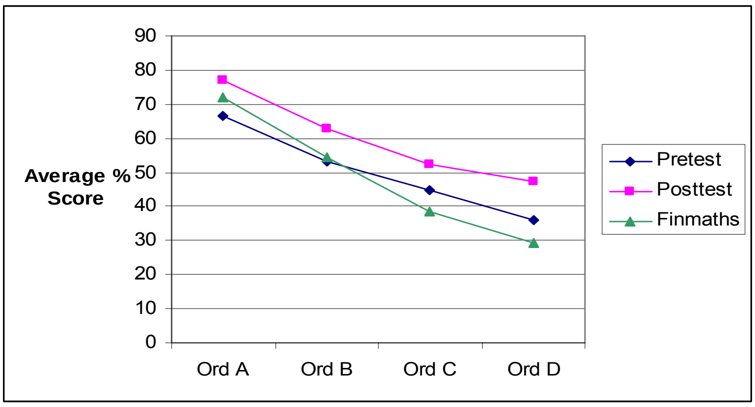

Figure 6: *Average Scores for Pre-test, Post-test, Final Maths For Ordinary Level*

Figure 6 shows a consistent decline across the grades for the three variables. The average final maths scores are significantly different for all grades. It reveals that the students most at risk of failing maths in first year are those students with a C or D grade as the average final maths score for these students is below 40%. We now focus on these students.

Table 4: *Percentage of Students entering with C or D in Ordinary level Leaving Certificate Mathematics*

Entry Year	Total No. Enrolled	No. of students with C or D	% of students with C or D
1997	45	27	60
2002	45	16	36
2003	85	47	55
2004	53	11	21
2005	44	11	25
2006	50	27	54

Table 4 gives a percentage breakdown of students with Ordinary C or D for the years where Leaving Certificate mathematics grades are available. It is notable that the group of students with C or D grades can often make up over 50% of a year group. When the students who did not complete the year are combined with those who failed the year we see from Table 5 and Figure 7 that 58% of grade Cs (52 of 89) and 80% of grade Ds (37 of 46) failing or not completing the year.

Table 5 : *Completion of First Year Mathematics for Students with C and D*

	C	**D**	**C and D**
Did not complete year or failed	52	37	89
Passed first year mathematics	37	9	46
Total	89	46	135

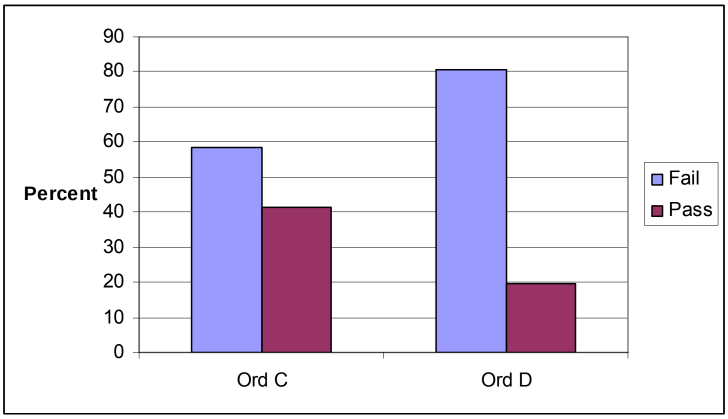

Figure 7: *Pass Rates for First Year Mathematics for Students with C and D*

Can any differences in performance be identified between students who entered with a C or D, and other students? Although their pre-test and post-test scores are significantly lower than for the rest, there is no significant difference in improvement from the pre-test to post-test.

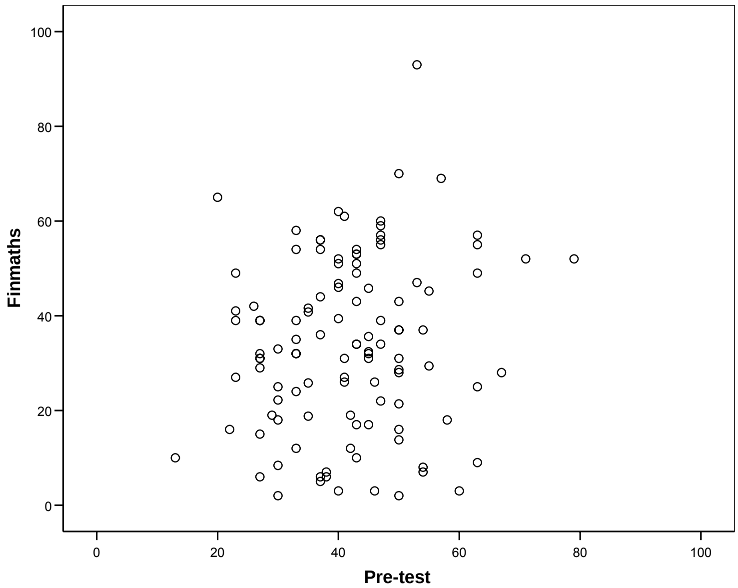

Figure 8: *Final Maths v Pre-test Scores Students entering with C or D in Ordinary level Leaving Certificate Mathematics*

Looking at the 94 students with ordinary C or D, who sat the pre-test and completed the year Figure 8 shows that while there is not a significant correlation between pre-test and final maths scores there is a very clear trend. Almost all of those who fail first year mathematics (40%) achieved less than 65% in the pre-test, the threshold stated in the original objectives of the programme.

Correlations and Predictions

There is a significant correlation between a student's pre-test score and post-test and final maths scores. But there is a stronger correlation between the post-test and final mathematics score, as shown in Table 6.

Table 6: *Summary of Pearson Correlation coefficients (r) for students who completed the year*

	Post-test	Final Maths
Pre-test	0.608 (n=370)	0.477 (n=369)
Post-test		0.689 (n=393)

Looking at the observed figures, 67.4% of students in the study passed, giving a 67.4% chance of predicting if a student selected at random would fail or pass. In order to see if any of the three variables, Leaving Certificate grade, Pre-test or Post-test, could predict whether or not a student will pass the year, a logistic regression analysis was undertaken, with result (Pass or Fail) as the dependent variable. The predictor variables were Leaving Certificate Grade, Pre-test score and Post-test score. The full model was significantly reliable ($\chi^2 = 76.0$, $p < 0.0005$), with 79.8% of students being correctly predicted as passing (or failing). Notably the pre-test was not a significant predictor of passing or failing, but Leaving Certificate and post-test scores were. A one stage decrease in Leaving Certificate grade (e.g. from Higher to Ordinary A, or from Ordinary C to Ordinary D) was associated with a 74% decrease in the odds of passing first year mathematics. A one percent increase in the post-test score was associated with a 5.1% increase in the odds of passing first year mathematics.

A further regression analysis was performed on all 231 students who did not reach 65% in the pre-test. A total of 117 students failed and 114 passed, giving a 50.6% chance of predicting if a student selected at random would fail or pass. The predictor variables were Improvement (Post-test – Pre-test) and Leaving Certificate Grade and the full model was significantly reliable ($\chi^2 = 45.0$, $p < 0.0005$). A one stage decrease in Leaving Certificate grade was associated with a 75% decrease in the odds of passing, and a one percent increase in a student's improvement was associated with a 4% increase in the chance of passing first year mathematics.

It can be deduced that for students entering with a C or D at Leaving Certificate who achieves less than 65% on the pre-test, their level of improvement may indicate their chances of passing first year mathematics.

Discussion

From the data analysis it is clear that students with Higher level Leaving Certificate mathematics or a grade A at Ordinary level have no problems in first year mathematics. A total of 83% of students with Ordinary level B successfully complete the year, and Figure 6 shows that these students have an average for all three pre-test, post-test and final maths score of over 50%. The average improvement for students with Higher level was 12.5%, for Ordinary level grade A it was 11.7% and for grade B it was 9.8%. This could indicate that they had just forgotten some techniques when sitting the pre-test.

The 183 students who scored less than 80% on the pre-test and passed the final year exam appear to have benefited significantly from the programme made an average improvement of almost 17%, substantially higher than the average for the total population. This could indicate that they may not have covered the material previously or that they benefited from the programme by being given the time to refresh and revise the material.

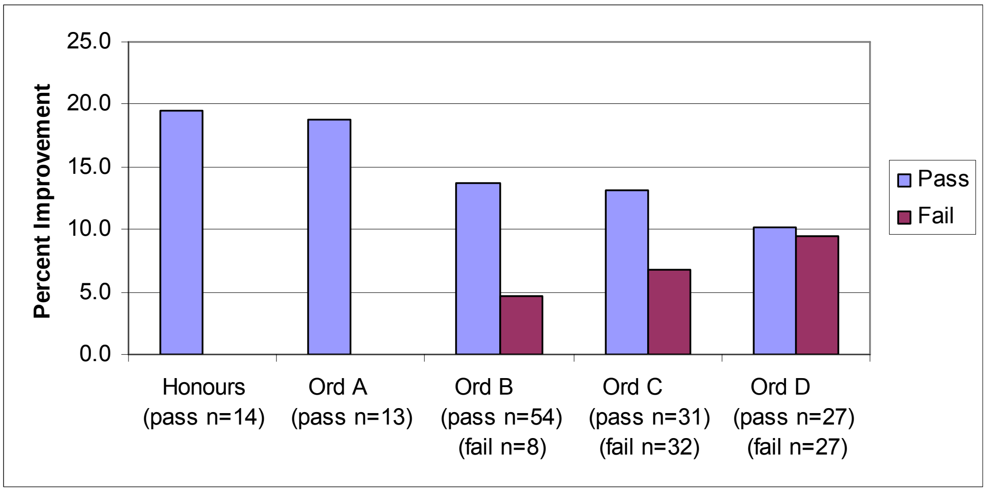

Figure 9: *Average Improvement for students scoring under 80% on pre-test and completing the year*

Yet the programme does not appear to work for all students. Recalling that the impetus for the programme was to remedy deficiencies in weaker students, it is in fact just these students for whom the programme appears to be less effective. A large majority of students entering with a C or D in Ordinary level mathematics, 58% of students with grade C and 80% of students with grade D, either do not complete the year or fail first year mathematics. But students' motivation and capability need to be factored in to the equation, and their improvement over the duration of the programme was shown to be a good predictor for success. In fact Figure 9 shows that students with C in Ordinary level mathematics who passed the year, made a significantly larger improvement than those who failed, possibly an indication of individual students' motivation. For those with a grade D in Ordinary level mathematics the improvement is larger, although not significant. This could indicate that students with grade D are not yet capable of reaching the level of understanding required for college mathematics.

Conclusion

None of the material in the benchmarking programme was new to the students. All the content would have been covered in the senior and junior cycles, and some as far back as the primary curriculum. Some procedures, for instance solving linear and quadratic equations, have been learnt well. Yet, for many students, even these procedures become difficult if presented in an unfamiliar format, or if they are required to solve a problem. Other skills, such as fraction manipulation, are very poorly executed. From the researcher's observations, many students with these problems find it extremely difficult to 'unlearn' and cover new methods within the year (Rogers, 1996, p210).

This brings the issue into a broader arena. Firstly, there appears to be a mismatch between students' performance in Ordinary level Leaving Certificate and first year engineering mathematics. The chief examiners' reports (for example SEC, 2001, p15) comment that there is a low level of attainment of basic skills. Yet students are able to pass these examinations. Weaknesses, by and large, relate to inadequate understanding of mathematical concepts and a consequent inability to apply familiar techniques in anything but the most familiar of contexts and presentations (SEC, 2005, p48).

The Ordinary level mathematics course was designed specifically for those planning to study scientific, business and technical disciplines and a grade D at this level is the minimum requirement for nearly all engineering courses in Institutes of Technology in Ireland. But from this investigation it appears that passing Ordinary level is not sufficient to succeed in first year engineering mathematics in the researcher's institute which is one of the Leaving Certificate's stated objectives (NCCA, 1992, p3). A grade A at Ordinary Level will ensure a pass and over 80% of grade B's pass, whereas a C or a D will almost guarantee a fail (See Figure 7). Gill (2006) found that even students with a grade A or B at Ordinary level were considered 'at-risk' of not passing first year mathematics in her institute. As Hourigan and O'Donoghue (2007, p462) state, it 'seems to defy all logic' that students passing the pre-requisite mathematics subject for first year mathematics would still be 'at-risk'.

For ten years, the researcher's institute has recognised the problems some students have when studying first year mathematics and has invested resources to address them. Yet even with these supports, the majority of students with lower grades in Ordinary level in Leaving Certificate do not pass first year mathematics. Further research needs to be undertaken to identify factors that affect these students in their endeavours in mathematics and college in general.

As Croft (2002) stated, these issues in mathematics education needs to be tackled jointly by higher education institutes, schools, and curriculum advisory agencies and lead by

Government. It is encouraging that the present review of Mathematics Education in post-primary schooling (NCCA, 2006) is acknowledging these and has begun the process of consultation with stakeholders.

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The Mathematical Deficiencies of Students Entering Third Level: An Item By Item Analysis of Student Diagnostic Tests

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The UL database, compiled at the Mathematics Learning Centre, was designed to shed light on what might be termed 'macro' level issues in relation to Service mathematics courses at UL e.g. numbers of 'at-risk' students among entering students, trends in performance in national examinations for entering students and the relative merits of Leaving Certificate grades as indicators of mathematical attainment and competency. However, it also affords the researchers an opportunity to address 'micro' level issues such as gaps in students' mathematical knowledge. This paper focuses on the mathematical deficiencies of entering students as uncovered by the UL diagnostic test in terms of skills and knowledge deficits.

Introduction

Students in Technological Mathematics 1 and Science Mathematics 1 in the University of Limerick (UL) are presented with a diagnostic test in their first mathematics lecture. The UL diagnostic test (sample given in appendix A) is a paper-based test comprising 40 open-ended questions with a single correct response. The test was designed for marking by hand so that one could investigate the specific errors that students make and identify where the gaps in student knowledge lie. An item by item analysis of 1050 scripts was undertaken. This paper reports on the item analysis of UL students' diagnostic tests, the purpose of which was to pinpoint the nature of the students' mathematical difficulties and where they occurred, content-wise, with a view to developing appropriate support mechanisms in the UL Mathematics Learning Centre.

Methodology

The methodology used by the author (OG) in this study was based on an implicit model of the State marking scheme for Junior Certificate mathematics examinations (of which the author (OG) had direct experience as a marker). This was augmented by the approach used in the Chief Examiner's reports (2000) to report to the government on the results of the State mathematics examinations. The authors were systematically looking for gross errors, errors in calculation, errors in logic, evidence of gaps in content knowledge, evidence of understanding, evidence of misunderstanding and general ability to solve problems (see, for example, Rees and Barr (1984)). As the students were provided with rough work areas, it was possible to determine why students were making the type of mistakes they were.

Performance on individual questions and typical errors

One way of investigating where students' gaps in knowledge are is to analyse students' scripts and explore their performance question by question. In this section the author looks at the success/failure rate of Leaving Certificate Ordinary and Higher Level students combined in selected questions of the diagnostic test (due to the length of the test, not all questions are covered in this paper). By knowing exactly what level they are at in various areas of mathematics one can better prepare for the mathematical demands of the term ahead and adapt as necessary.

Q2: Write down the value of $10 - 8 \div 2 + 9$

Question 2 was very poorly answered with 85.9% of students (89.2% Ordinary Level, 82.3% Higher Level) in the database getting this wrong. Knowledge of the order of operations i.e. the *BOMDAS* rule is required to get this right. The correct answer here is 15. Many students read the question from left to right as they would read a sentence and arrived at the incorrect answer of 10. In other words, instead of carrying out the division operation first ($8 \div 2$) then adding or subtracting, they did the following: $10 - 8 = 2 \div 2 = 1 + 9 = 10$. Another common incorrect answer for this particular question was $^2/_{11}$. This answer was arrived at by doing the following: $(10 - 8) \div (2 + 9)$. Again students ignored (or were just unaware of) the order of operations and added and subtracted before dividing. Some students arrived at the answer of 5.5 for the same reason: $10 - 8 + 9 = 11 \div 2 = 5.5$.

Q8: Write down the value of $8^{1/3}$.

Question 8 posed a few more problems with 36.6% of all students getting it wrong. 17% of students with Higher Level Leaving Certificate mathematics and over half (54.6%) of Ordinary Level Leaving Certificate mathematics answered this question incorrectly. Here they were asked to evaluate the third root of 8, the answer being 2. $^1/_{512}$ was a common incorrect answer given. Some thought that: $8^{1/3} = {}^1/_{8^3} = {}^1/_{8x8x8} = {}^1/_{512}$ Others wrote down the answer $^1/_{24}$. They started off as above but made the added mistake of assuming $8^3 = 8x3$. It seems these students are aware that there are rules to be learned for indices but are unfamiliar with what they are or how they should be applied to evaluate expressions. Some students read (or understood) the question to mean $8 \times {}^1/_3$, giving the answer $^8/_3$. It appears that these students were completely unfamiliar with the rules of indices, as they did not even recognise them here in this particular question. Some students wrote down $\sqrt[3]{8}$, which is correct, but they did not evaluate it. It is unclear if they felt this was a sufficient answer or just did not know how to finish it off.

Q9: Write down the value of $^1/_{2^{-4}}$.

50.9% of all students got this wrong (34.1% Higher Level, 66.3% Ordinary Level). The correct answer to this is 16 i.e. $^1/_{2^{-4}} = 2^4 = 2x2x2x2 = 16$. One common wrong answer given was $^1/_{16}$. Here, students ignored (or misunderstood) the negative power and just evaluated 2^4 instead. $^{-1}/_{16}$ was another common answer. Students evaluated the expression as above then added in the minus sign at the end, clearly unaware of the rules of indices. Some wrote down the answer ½. What they did was the following: $\frac{1}{2^{-4}} = \frac{1^4}{2} = \frac{1}{2}$. They were obviously aware of the rule that an index changes sign when it is brought above the line but did not know that the whole denominator should have been brought up. Most of the students who failed to give the correct answer simply ticked the *Don't know* box.

Q10: If $x = 10^2$ then write down the value of $\log x$.

Questions 10 and 11 were two of the most poorly answered questions on the test. 77.2% of all students (62.6% Higher Level, 90.7% Ordinary Level) failed to answer question 10 correctly. Logarithms are not on the Leaving Certificate Ordinary Level syllabus which is why so few in this group were able to answer this question (They were until recently on the Junior Certificate Higher Level syllabus so those who took the Higher Level paper at this level then took Ordinary Level at Leaving Certificate may have been the ones in this group who answered the question correctly). The answer here is 2. There was quite an assortment of answers given here, for example: 1, 10, $\log_2 10$, which appeared to the author to be guesswork more than anything as there was not enough rough work or calculations to show where they attained these answers. The most common incorrect answer was 100. Evidently, they calculated the value of x rather than $\log x$. Many ticked the *Don't know* box.

Q11: If $\log x = 5$ then write down the value of $\log (x^2)$.

Question 11 also assesses students' knowledge of Logarithms. Again, it was very poorly answered with 85.6% of all students unable to answer it accurately. As would be expected, like in question 10, 97.9% of Ordinary Level students were unable to answer it. More worryingly is the fact that 72.1% of Higher Level students got it wrong. These students should have covered this topic at a high level prior to arrival at university. This question is slightly more difficult than question 10. Students were unaware of the rule that $\log (a^b) = b \log a$. The most common incorrect answer given was 25. Students calculated 5^2 which showed they had no idea of the rules for Logarithms, much less, how to apply them. It is obvious from questions 10 and 11 that lecturers cannot assume that students have any knowledge of this topic and, if it is a requisite topic for a particular service mathematics module, it will need to be taught from scratch.

Q14: Solve for h: $V = \Pi r^2 h$

Question 14 is the first question in the algebra section of the test. All students who sit the Leaving Certificate examination must do an extensive amount of algebra, though not as much as in previous years. It is a core topic on both syllabuses. The correct answer is attained by dividing both sides by Πr^2 to get: $h = V/\Pi r^2$ Almost a quarter (24.8%) of the students in the database got this question wrong. Ordinary Level students had much greater difficulties here than those who took Higher Level Leaving Certificate mathematics (40.3% of Ordinary Level students got this wrong, compared with only 7.9% of Higher Level students). There were many combinations of incorrect answers given here, including: $h = \Pi r^2/V$ The correct answer has been inverted to arrive at this answer. In some cases the students subtracted V to get $h = \Pi r^2 - V$. It is clear from this answer that these students knew they had to isolate h but used the wrong operation to do so. A number of students gave the answer as $h = V/\Pi r^2 h$ The method/operation (i.e. division) was correct but they should not have divided by h as well. This suggests rote learning. The students knew they should divide but clearly did not understand if they failed to see that h cannot equal $V/\Pi r^2 h$ (unless $V/\Pi r^2 = h^2$ which can not be assumed here).

Q16: Solve the equation: $3(x + 2) - 24=0$.

The solution here is $x=6$. This appears to be a simple equation yet 13.9% of students (9.9% Higher Level, 17.5% Ordinary Level) either gave the wrong answer or ticked the *Don't know* box. Again, this is subject matter which would be taught in first year in secondary school. It would be assumed by lecturers that all first years in third level would be very proficient in this area. The high number of mature students in UL may explain how such a simple equation would not be solved correctly by everyone. For many mature students, it has been a number of years since they finished second level (if indeed, they *did* get a chance to finish) so, perhaps, they are merely out of practice and a refresher course may be sufficient to bring them up to the required level of proficiency. One would hope that this is the case! Various students left their answer as $3x - 18 = 0$.

It is uncertain if they were unable to proceed any further or if they felt this was a sufficient answer, in which case they did not understand the word "solve". x = -6 was another wrong answer that was frequently written down.

Q17: Solve for x: $x^2 + x - 6 = 0$

This topic is on both Junior and Leaving Certificate mathematics curricula, yet only 74.5% of all students solved this equation correctly. Over $^1/_3$ (36.2%) of Ordinary Level students failed to get this right as did 13.8% of Higher Level students. Some students factorised to get (x + 3) (x – 2) but did not go any further. It is unclear whether they did not know how, did not read the question accurately or did not know what "to solve" meant. Other students picked the wrong factors of –6 e.g. –3 and +2, thus ending up with the answer x = 3 or x = -2. This appeared to be the most common error. Some gave the answer x = 2 but failed to give x = -3 also. This appeared to the author (from observation of rough work) to be guesswork i.e. they just picked a value of x which *fitted in* to the equation:

$x = 2 \quad x^2 + x - 6 = 0.$

$2^2 + 2 - 6 = 0$

$4 + 2 - 6 = 0$

$0 = 0$

This is correct but is only half of the answer. This implies an incomplete understanding of the task at hand.

Q18: Solve the set of equations:

$2x + y = 7$

$x + 2y = 5$

Here the correct answer is x = 3 and y = 1. This topic is taught at Junior and Leaving Certificate level. Despite this, the results for this question were not what one would hope they would be. 36% of Ordinary Level and 17.4% of Higher Level students (27.1% overall) got this wrong or did not answer it at all. Many students had the correct method but made arithmetic errors resulting in the wrong answer which is not really a huge concern. Others, when multiplying the equation by –2, multiplied the x term and the constant but forgot to multiply the y value. Again this is a small mistake and does not really imply that the students have poor understanding. The reason for finding the solution to two simultaneous equations is that it tells us the point where the two lines intersect. Some students gave x = 3 as an answer but neglected to find the y value. A mistake like this does imply a dearth of understanding.

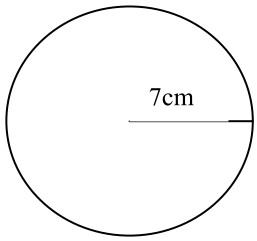
Q20: Solve for x: $3 - 6x < 21$.

Question 20 involves the solving of an inequality. The answer is x > -3. There appeared to be quite a bit of difficulty with this particular question. 60.6% of students overall (70.7% Ordinary Level, 49.6% Higher Level) got this wrong. x <3 was the most frequent incorrect answer given here. Students solved the inequality as they would an equation, neglecting to turn the inequality sign around. It is unclear if they have just forgotten to do so, in which case they just need reminding. On the other hand, if they are unaware that they should change the sign or do not see it as any different to an equation, then this is another matter entirely: the topic will need to be (re) taught. x=-3 is another example of the type of errors made. Here, the inequality sign was dropped and replaced with an equals sign. If they see these signs as interchangeable, they evidently do not comprehend the difference between the two.

Q21: Simplify: $^{1}/_{x-1} - {}^{2}/_{x+1}$.

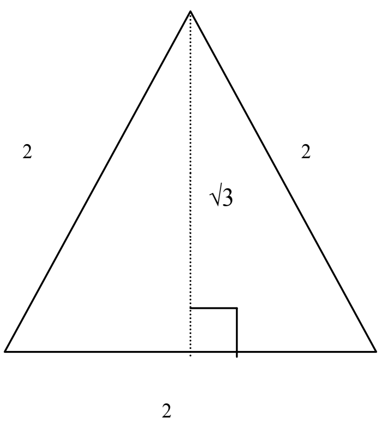
Students were asked to simplify the above expression for question 21. Answering here was very disappointing. 61% of all students (81.3% Ordinary Level, 38.8% Higher Level) were unable to answer this question correctly. For question 3 students had to add two fractions and most were able to. Therefore, if they were able to carry out that operation successfully, this question should be within their capabilities. Many students were successful to a point in that they managed to get -x+3 but then they dropped the denominator. This is a classic sign of rote learning: the students are obviously able to do this task but do not fully understand what they are doing. $^{-x-1}/_{(x-1)(x+1)}$ was another typical incorrect answer given. This was due to a slip in arithmetic (-2(x-1) = -2x-2), which is no great cause for concern.

Q23: Find the area of the circle. (Use $\pi \approx {}^{22}/_{7}$*)*



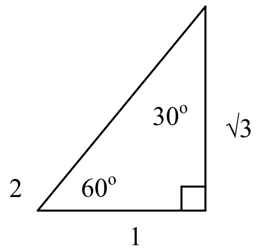
In question 23, the students were presented with a circle of radius 7cm and, given that $\pi \approx {}^{22}/_{7}$, they were asked to find the area of the circle. Mathematics tables were not permitted during the test so to answer this correctly, students would have to know that the formula for the area of a circle is πr^2.This is not unreasonable as this is a basic geometry topic, one which would be taught at Junior Certificate level. The correct answer is, of course, $154cm^2$. Over $^{1}/_{5}$(21.7% overall: 25.6 % Ordinary Level, 17.4% Higher Level) of students were incapable of answering this which is quite a high number considering the level of difficulty of the question. Many students used the wrong formula here. They used the formula for finding the circumference of a circle ($2\pi r$) instead and got the answer $44cm^2$. Many used the formula $2\pi r^2$ and got the answer 308. There were few who had trouble with substitution of values into the formulae even if they had the wrong one. Had students been permitted to use mathematics tables (which contain the formula), perhaps more would have obtained the right answer.

Q24: Calculate the area of the triangle



Here the area works out at $\sqrt{3}$ squared units. Once more, this question was not handled very well. Only 62.8% (47.9% Ordinary Level, 79.1% Higher Level) of all students in the database managed to answer this accurately. The same problem arose here as in the last question. Students were able to substitute into formulae but picked the wrong one. Some got the answer $2\sqrt{3}$ by assuming the formula was the base multiplied by the height. Another common answer was 3. Students had the correct formula: ½ base x perpendicular height and substituted to get $\frac{1}{2} . 2 . \sqrt{3}$ which is correct. But then they squared everything as they felt they needed to get rid of the square root sign and got the following: $\frac{1}{4} . 4 . 3 = 3$. Many stated that they did not know how to do the question and did not even attempt it.

Q26: *Write down sin 30^o as a fraction*



There are three questions in the Trigonometry section of the diagnostic test, question 26 being the first. A right-angled triangle is shown with lengths of all sides and size of angles given. Students are asked to write down sin 30^o as a fraction, the answer being ½. Almost $^1/_3$ (32.6%: 46.7% Ordinary Level, 17.4% Higher Level) of students got this wrong which is a lot considering it is a basic trigonometry problem, one that should have been covered by both Higher and Ordinary Level students for Leaving Certificate. $^{\sqrt{3}}/_2$ was a widespread answer given. This is in fact the sine of 60^o. The students were mixing up opposite and adjacent sides of the triangles. Other answers varied considerably which, more than likely, were just guesswork. Any combination of sides was given, from $^2/_{\sqrt{3}}$ to $^1/_{\sqrt{3}}$.

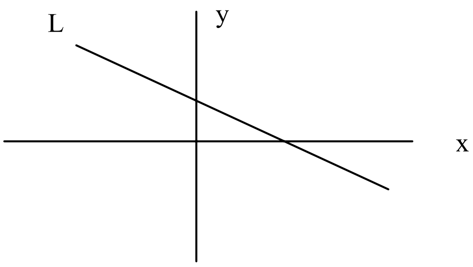
Q27: Find the value of $sin^2 66^o + cos^2 66^o$.

$sin^2x + cos^2x = 1$. Students need to know this to answer question 27. This topic is on the Higher Level Leaving Certificate mathematics syllabus only so it would not be expected of Ordinary Level students to know this. Not surprisingly, therefore, only 4.1% of Ordinary Level students got this right. These students may have covered the Higher Level course for some period of time before dropping back to Ordinary Level. What is surprising is that just over half (51.5%) of the Higher Level students got the right answer. This particular formula is on page 9 of the Log Tables. Students attempting a trigonometry question in the Leaving Certificate examination should be more than familiar with the formulae on this page. In fact, it is required that they are able to derive and apply the first 12 formulas on this page, which includes the one in question. There did not appear to be one common error made by students here. Most ticked the *Don't Know* box.

Q28: Write down 90^o in radians.

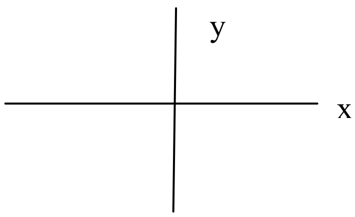
The answer here is $^{\pi}/_2$. This was poorly answered. 31.4% overall (11.3% Ordinary Level, 53.4% Higher Level) got this right. Ordinary Level students do not learn about radians which explains the low number of this group who got this right. Some wrote ½ radians down as an answer. They had the right idea but neglected to multiply it by π. Others guessed ¼ as the answer as 90^o is one quarter of 360^o. Most who did not score on this question did not attempt it.

Q30: State whether the line L has a positive or a negative slope.



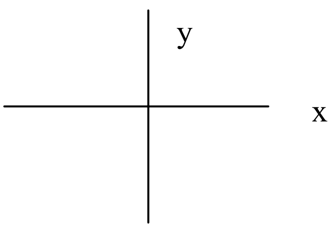
The problem here is to state whether L has a positive or negative slope, the answer being negative. Only 57.9%(54.5% Ordinary Level, 61.5% Higher Level) of students got the exact answer here. The graph of a function is read from left to right. One can see that the slope of the line is decreasing as it moves from left to right and, therefore, has a negative slope. Another way to come to this conclusion would be to pick two points on this line and work out the slope using the slope formula. Many students used this method effectively. The answer had to be either positive or negative so there was no variability in answering but it seemed that a lot of the answers given (correct or incorrect) were guesswork.

Q31: Sketch the line y=3x+2 on the diagram



For question 31, students are asked to sketch the line y=3x+2. To do this, they must pick two points on the line (usually they let x=0 and find the equivalent y value and vice versa), plot them and then draw the line. The word "line" is given in the question so students should know by this that it is a straight line they have to illustrate. This was handled very badly. Less than half (49.2%: 35.6% Ordinary Level, 64% Higher Level) got this right. Many made mistakes when finding points so their graphs came out wrong. Others calculated the correct coordinates but plotted them in the wrong place.

Q32: Sketch the curve $y=x^2+2$ on the diagram.



This time it is a U shaped curve. Again, the word "curve" is given in the question so it gives students a hint as to what the graph should appear like. The x^2 also indicates that it will be U shaped. This topic is taught at junior cycle level so all students coming from second level should have covered this quite extensively. Unfortunately this is not the case at all. An incredible 82.7% (91.6% Ordinary Level, 73.1% Higher Level) got this wrong. Those who did construct the graph accurately picked values of x then found the corresponding y coordinates and plotted them precisely. Many left their scripts blank here. There was wide variation in the types of erroneous curves that were presented, many having no idea that it should have been U shaped. Some did know that it should be of this form but were either very careless drawing it or did not know how to pick the coordinated pairs needed to graph it. This is one topic which should, perhaps, be taught from scratch. It cannot be assumed that students are competent in the area of graphing lines and quadratics on entry to service mathematics courses in UL.

Q33: If z is the complex number 1 + 2i find the modulus |z|.
Question 33 is the first of two questions in the Complex Numbers section of the paper. Both Higher and Ordinary Level students should cover this in the Leaving Certificate programme as it is on both syllabuses. This question and question 34 are both of Ordinary Level standard. Even so, only 36% (31% Ordinary Level, 41.4% Higher Level) gave the correct response. The most common error made here was where students gave 5 as the answer, forgetting to include the square root sign. Others mistook the modulus for the conjugate and wrote the answer down as 1-2i.

Q34: Simplify: $^{1+2i}/_{2-3i}$.
The correct answer here is $^{-4+7i}/_{13}$. 40.3% (31.5% Ordinary Level, 50% Higher Level) had the correct result for this particular question. Many students had the correct method here and did multiply by the complex conjugate of the denominator but made arithmetic errors and ended up with the wrong answer. Some forgot that $i^2 = -1$ and many did not attempt it at all.

Q36: If y = x sin x find $^{dy}/_{dx}$
Question 36 demands use of the product rule. The correct answer works out to be $^{dy}/_{dx}$ = xcosx + sinx. Although Ordinary Level students must learn how to recognise and apply the product rule, they do not cover the differentiation of trigonometric functions so it is not really surprising that only 2.2% of Ordinary Level students gave the correct response here. What is astounding is that only 39.9% of Higher Level students worked out the right answer. This is no harder than the previous question. If question 35 is within their capabilities, then so too should question 36. One must bear in mind though that the formula for the product rule is given in the Mathematics Tables which they were not permitted to use. It was apparent that the Ordinary Level students were guessing here. The most common error made by the Higher Level cohort was they differentiated sin x to get –cos x. So their answer was given as $^{dy}/_{dx}$ = -xcosx + sinx. Again, the absence of Mathematics Tables was probably a contributing factor here.

Q37: If $y = e^{-2x}$ *find* $^{dy}/_{dx}$.
Here $^{dy}/_{dx} = -2\, e^{-2x}$. Ordinary Level students would not have covered this in Leaving Certificate. Only 3.5% of these students got this right which is to be expected. Just over half (54.4%) of the Higher Level students got it right. Once more, the formula for calculating such a problem is given in the mathematics tables so more of these students

would be expected to have calculated it precisely had they been permitted to use them. Typical incorrect answers given were: ${}^{dy}/_{dx} = -2\,e^{\,x}$ (they multiplied the exponential by the x coefficient but neglected to leave the index as –2x), ${}^{dy}/_{dx} = -2x\,e^{-2x}$(they multiplied the exponential by the index instead of the differential of the index) and ${}^{dy}/_{dx} = -2x\,e^{-3x}$ (here they multiplied the exponential by the index, then subtracted 1x from the power. This is the method they used for question 35 and assumed it was the same here).

Q38: Evaluate $\int (x^2 + 2x + 3)\,dx$.

The answer is ${}^{x^3}/_3 + x^2 + 3x + c$. Integration is only covered on the Higher Level syllabus so students who sit the Ordinary Level paper would not be likely to answer this question correctly, unless they did the Higher Level course up to a certain point then dropped back to Ordinary Level or waited until the examination and just decided to take the Ordinary Level paper on the day. 1.2% of the Ordinary Level students appear to have fallen into this category, as this is the number who got the right answer. Only 37.1% of Higher Level students got this right. This is quite a low number but, fortunately, Science mathematics 2 and Technological mathematics 2 teach Integration from the beginning and do not assume any prior knowledge. Many students neglected to add on the +c at the end of their answer. This suggests that while they know the correct method of integration, they do not fully understand its relevance when they make a mistake like this. Some added one to the index but neglected to divide each term by the new power to get the following answer: x^3+2x^2+3x. Some students differentiated instead of integrated to get 2x+2 but most of these were Ordinary Level students.

Discussion

In this paper, the authors looked at student performance on the UL diagnostic test in an attempt to better comprehend the mathematical challenge faced by students at the beginning of their third level education in Service mathematics courses.

The students in Technological mathematics 1 and Science mathematics 1 are presented with a diagnostic test on arrival at their first mathematics lecture in UL. This does cause some anxiety for students since they are not pre-warned about the test so this must be taken into account when scrutinising their replies. Use of calculators and mathematical tables was prohibited. This immediately presented problems in the geometry section of the test where students were unable to recall the correct formulae for the area of a circle and of a triangle since they were unable to consult the mathematical tables. Again for questions 36 and 37 (differentiation), many students could not recall and apply the product rule or differentiate exponential or Trigonometric functions without the aid of the mathematical tables. This suggests an over reliance on the mathematical tables but in any university or Leaving Certificate mathematics examination, students are allowed use these tables so perhaps they chose not to learn rules they did not really need to. Still, these topics must be taught from scratch. Integration is taught from the beginning in Science and Technological mathematics 2 so, at this stage, students' level of preparedness in this area is of no great concern.

The item analysis showed that logarithms need to be re-taught since students do not have a good understanding of this topic. It is evident that the rules of indices have been covered at second level but they also need to be revised, as students are not proficient at applying them. Scientific Notation is not usually covered in these Service mathematics courses but the students need to understand this topic for using calculators so it should be revised.

Students' algebraic skills, arguably the most important skills in mathematics, need a lot of work. Manipulation and addition of algebraic terms are areas that are important not only in mathematics but physics and chemistry also to name but a few other disciplines. Responses to questions 14 and 21 reinforce this fact. Inequalities need to be re-taught from the beginning because it is evident from their scripts that this topic is not well understood. Too many students displayed an unsatisfactory understanding of this subject matter.

Trigonometry is another important area in which students need to be educated. Students with Ordinary Level Leaving Certificate mathematics have not covered this topic to a sufficiently high standard for what is required in Science mathematics 1 as quite an advanced level is required almost immediately after they start. In Technological mathematics 1 however, trigonometry is taught from scratch and thus students are well catered for in this regard.

In the area of graphing lines and quadratics, it cannot be assumed that the students are adept in such a crucial area as students' scripts show. These topics need to be taught again as the responses to questions 31 and 32 demonstrate.

Less than 40% of students coming to UL from second level are skilled in the domain of Complex Numbers. It appears to be a topic that many students leave out/avoid in the Leaving Certificate examination. From the authors' own experience of teaching in the Mathematics Learning Centre it is not uncommon to be asked the meaning of *i*, something many would assume common knowledge. Again, it is just one topic that must be taught from an elementary level in these Service mathematics courses.

The general outcome of the item analysis is consistent with recent Chief Examiner's reports highlighting similar deficiencies and concerns in the State mathematics examinations, junior and senior (Chief Examiner's report, 2000).

Analysis of replies does not just show us where students are fallible: it is important that this information be used to revise syllabuses or create support mechanisms that will bring students up to the standard required by the university. The diagnostic test is considered by the author and the mathematics department in UL to be a good measure of entering students' mathematical preparedness for Service mathematics courses at UL and the author is aware through her contacts with colleagues at meetings, conferences and seminars that they concur with this view. This is not surprising given that the diagnostic test was constructed in a particular way based on the Ordinary Leaving Certificate mathematics syllabus and the SEFI core syllabus for engineers (Barry and Steele, 1993). Description of the diagnostic test and its construction is available in Murphy (2002).

Summary

In this paper the authors have investigated, through an item analysis of the UL students' diagnostic test scripts, where some of the gaps in students' mathematical content lie. Calculus, Algebra and Trigonometry are just some of the topics where students are not as proficient as one might assume. This can and does cause some serious problems for students starting off on a degree course. Not only do they have to keep up with general studies (mathematics included), they also have to find time to catch up too. It is not always practical or, indeed, possible to revise and dilute syllabuses so support mechanisms are now in place to fill these gaps in the form of support tutorials and other tailored support services at the UL Mathematics Learning Centre.

The item analysis provides important information for front-line practitioners teaching first year Service mathematics courses at UL in the first instance but also for other practitioners in similar situations in Irish universities.

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Appendix A: Sample of UL Diagnostic Test

ARITHMETIC Q1 - Q13

ROUGH WORK

1. Work out -(-8) + (-3)

   Ans ____________________ ☐ Don't know

2. Write down the value of $10-8\div 2+9$

   Ans ____________________ ☐ Don't know

3. Work out: $\frac{1}{2}-\frac{1}{3}$

   Ans ____________________ ☐ Don't know

4. Find the mean of the numbers 12, 14, 10

   Ans ____________________ ☐ Don't know

5. Work out: $\frac{2}{3}\times\frac{4}{5}$

   Ans ____________________ ☐ Don't know

6. Find 25% of 500

   Ans ____________________ ☐ Don't know

7. Write down the value of 3^4.

   Ans ____________________ ☐ Don't know

8. Write down the value of $8^{1/3}$.

   Ans ____________________ ☐ Don't know

9. Write down the value of $\frac{1}{2^{-4}}$.

   Ans ____________________ ☐ Don't know

10. If $x=10^2$ then write down the value of $\log x$.

    Ans ____________________ ☐ Don't know

11. If $\log x = 5$ then write down the value of $\log(x^2)$.

    Ans ____________________ ☐ Don't know

The Influence of Affective Variables on Learning Mathematics – Tackling the Mathematics Problem

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The 'Mathematics Problem' is particularly well documented in the U.K. e.g. Smith (2004), and in more recent years in Ireland. At the University of Limerick, the author is part of a research team investigating this so-called 'Mathematics Problem'. Gill (2006) and O'Donoghue (1999) looked at the global 'Mathematics Problem' and examined it in an Irish context by investigating university mathematics courses at the University of Limerick. Murphy (2002) meanwhile focussed her work on the secondary school experience and its contribution to under-preparedness among third-level entrants. In this paper, the author inquires into Ireland's 'Mathematics Problem' during the transitional phase from secondary school mathematics to university mathematics with a particular concern for how students' attitudes, beliefs, self-concepts, conceptions of mathematics and approaches to learning affects their mathematical performance. A preliminary quantitative study was undertaken at third level to investigate the influence of affective variables on students' mathematics learning in the transition process. Through critical analysis of the data, relevant findings are highlighted and recommendations made for future research.

Introduction

The importance of Mathematics Education in Ireland is reflected in the NCCA's Review of Mathematics in Post-Primary Education – Report on the Consultation (2006:6) statement that it is significant "in the development of logical thinking and problem-solving skills, as well as its importance as a foundation for other subjects, especially the science and technology subjects." It is of particular importance at senior cycle given its requirement for admission to third-level courses.

There are concerns about a lack of mathematical preparedness among third-level entrants. The most recent Chief Examiner's Report at Leaving Certificate (2005: 72) made significant comments for both Higher and Ordinary Level mathematics. At Higher level Leaving Certificate "candidates conceptual understanding of the mathematics they have studied is inferior to that which one would hope for and expect at this level." For Ordinary level Leaving Certificate, the Chief Examiner commented that students possessed poor foundation skills, inadequate understanding of mathematical concepts and under-developed problem-solving and decision-making skills. Gill and O'Donoghue (2006) also made important findings among first year university students completing service mathematics courses at University of Limerick (U.L). Through the use of diagnostic testing they found the Leaving Certificate Ordinary level mathematics syllabus to be inadequate preparation for service maths courses at U.L. and although Higher level students scored more highly on the test, this did not say much about their mathematical preparedness because the test is set at Ordinary level Leaving Certificate standard.

The author believes, along with much of the literature (Klinger, 2007; Hoyles et al, 2001; Thompson, 1992; Ramsden, 1992) that affective variables strongly influence both the learning and teaching of mathematics. Their importance is highlighted in Atkins and Helm (1993) claim that affective components are as important as the content itself. Research internationally has identified affective problems students encounter in relation to mathematics learning e.g. Owens et al., 1998; McLeod, 1992. Many adults exhibit some degree of anxiety when confronted with mathematical tasks, they have a vague concept of what mathematics is really about and they lack confidence in their own mathematical abilities (Klinger, 2007). This creates an even bigger ‘gap’ when students reach university mathematics.

The purpose of this paper is to investigate the extent of the “mathematics problem” in an Irish context by examining the impact of affective variables such as attitudes, beliefs, emotions, self-concept, self-efficacy etc. on students making the transition from secondary school mathematics to university mathematics. Data was collected from three service mathematics courses (Engineering, Science and Technology Maths one) at the University of Limerick. The author presents a preliminary analysis of this exploratory research carried out in the academic year 2005/2006.

Role of Affective Factors in Mathematics Learning

Research on affect and mathematical problem solving highlights in detail the “pervasive influence of affective factors on both students and teachers from kindergarten to the university” (Greer and Verschaffel, 1990:14). The author believes such issues must be addressed in tackling the “mathematics problem”.

McLeod (1992) divides affect into three dimensions:

- Beliefs
- Attitudes
- Emotions.

These affective domains along with self-concept, a key aspect of the exploratory research, will be briefly discussed in relation to their role in students’ mathematics learning.

Attitudes

Attitudes are “internal states that influence personal action choices” (Good & Brophy, 1990:132). In relation to mathematics, Neale (1969: 632) defined attitude to mathematics as “an aggregated measure of a liking or disliking of mathematics, a tendency to engage in or avoid mathematical activities, a belief that one is good or bad at mathematics, and a belief that mathematics is useful or useless.” Much of the literature emphasises that attitudes are based on past experiences (Crawford et al., 1994; Schoenfeld, 1989; Fishbein and Azjen, 1975). The literature also suggests that a correlation exists between attitudes’ and achievement. Richardson and Suinn (1972) developed the Mathematics Anxiety Rating Scale (MARS), which suggests that high levels of mathematics anxiety appear to interfere with achievement in mathematics. A study by Ma and Kishor (1997) on assessing the relationship between attitude toward mathematics and achievement in mathematics, showed that the overall mean effect size was 0.12, statistically significant but not strong enough for educational practice. Ajzen and Fishbein’s (1980) “theory of reasoned action” highlights the relationship between attitudes and achievement. This theory examines attitudes and subjective norms i.e. beliefs concerning the expectations of parents, brothers, sisters and teachers.

Differences do exist among researchers as to how strongly correlated attitude to mathematics and achievement are, for example, Fraser and Butts (1991) found little correlation between achievement and attitudes. Perhaps taking Tocci and Englehard’s (1991) advice and placing more emphasis on determining how attitudes are developed would be more beneficial in tackling the problem of negative attitudes towards mathematics.

Beliefs
The literature suggests that attitudes and beliefs are interlinked (Rokeach, 1976; Fishbein and Ajzen, 1975). Beliefs towards mathematics are central in the development of attitudinal and emotional responses to the subject (McLeod, 1992). Beliefs about individual competence in mathematics are closely tied to confidence and self-concept (Reyes, 1984). The following statement by Mason and Scrivani (2004:154) highlights the strong influence beliefs about mathematics have on students, "beliefs about math shape students' behaviour and since they are powerful, can often produce negative consequences".

Clearly students' beliefs about knowledge in mathematics play an important role in their performance and achievement. It is commonly felt among researchers that beliefs can be resistant to change. Rokeach (1976) explains that the more central a belief, the more difficult it is to change. Preventing and dealing with negative beliefs becomes an obvious task. Schoenfeld (1988) suggests, "curriculum needs to change to encourage beliefs whereby students don't believe mathematics problems should be solved routinely in five minutes" (Cited by McLeod, 1992: 579). The teacher also has a large part to play here. Teacher's beliefs in their ability to teach mathematics affect the learner. Several researchers identify a correlation between teachers' beliefs in their abilities to teach and student performance e.g. Buehl & Alexander (2001) and Thompson (1992).

Emotions
The importance of the learner's emotions cannot be underestimated. They can affect our motivation, beliefs and attitude towards learning. Unlike beliefs and attitudes, McLeod claims "emotions may change rapidly" (1992: 578). Laurie Buxton (1981:13) explains how "mathematics is commonly seen as a study based on reason, with the emotions rarely engaged". Buxton (1981) created a model based on Skemp's theory (1979) showing how mathematics and emotions interact. He refers to goals being set either by others, or us and how our emotions are generated in response to our progress towards this goal. Problems in achieving a goal may result in feelings of frustration or perhaps disappointment. On the other hand, approaching a goal may afford feelings of confidence and elation.

Strong emotions, predominantly negative ones, are linked with mathematics (Larcombe, 1985). George Mandler's (1985) work focuses on the role of emotions. McLeod (1988) claims that his theory is especially appropriate for describing students' emotional reactions to mathematical problem solving. Mandler explains that when a student is given a mathematical task, he/she produces an action sequence to complete the task. If the student experiences an interruption whereby he/she can't finish the task, the student normally experiences arousal in the nervous system e.g. muscular tension, increased heart rate. The individual also uses cognitive processes to evaluate the interruption that is interpreted as satisfaction, frustration or some other emotion. Emotions can have various effects on the learner. "People who don't know what math is don't know what math isn't" (Tobias, 1978:25). Here, Tobias is explaining how a fear of mathematics may lead to people avoiding all manner of data and to feel uncomfortable working with things.

Self-Concept
Research suggests self-concept is another area that can influence learning. According to Drew and Watkins (1998: 175) "self-concept is a psychological construct which refers to the cluster of ideas and attitudes an individual holds about himself." The role of self-concept in mathematics learning is explored by researchers such as McLeod (1992) and Gourgey (1982) although it is my opinion that more research is needed in the area to understand why students' feel the way they do about their own mathematical ability. Reyes (1984) says mathematics self-concept is a perception of personal ability to learn and perform tasks in mathematics. Again McLeod (1992) believes mathematical self-concept can be linked with achievement in mathematics.

The relationship between conceptions of mathematics and approaches to learning
Two other key areas in mathematics learning, conceptions of mathematics and approaches to learning are addressed in this paper. Findings from the exploratory research about the relationship between the two are discussed later in the paper. Based on Marton's (1988) framework, Crawford et al.'s (1994) work demonstrates this relationship between conceptions of mathematics and approaches to learning.

A survey questionnaire on student's conceptions of mathematics and approaches to learning mathematics was administered to approximately 300 first year mathematics students at the University of Sydney. Structured interviews took place with 12 selected students. Crawford et al (1994: 343) made an important finding, "students' conceptions of mathematics are formed by their approaches to learning it and also form their approaches". Based on the results of this study they used Marton's (1988) framework to create their own.

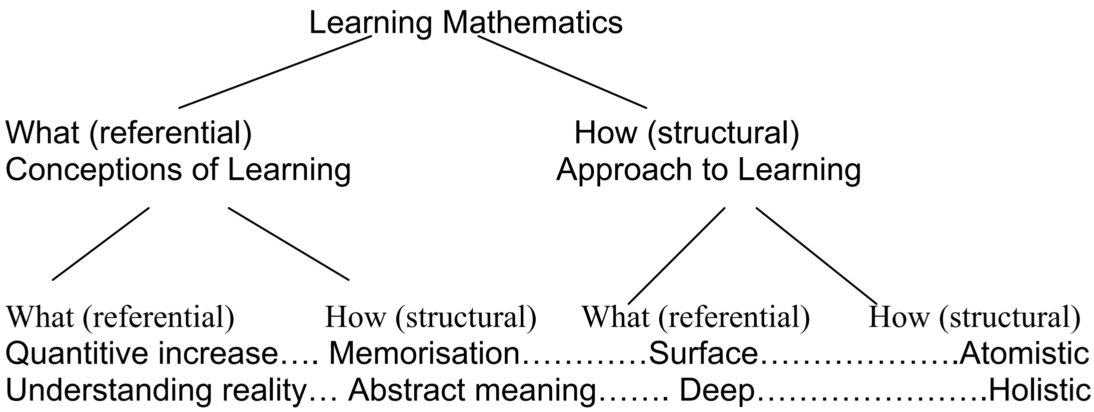

Figure 1.1 Relationship Between the Conception of Mathematics and Approach to Learning Mathematics.

The above figure shows that both the "what" of learning (conception) and the "how" of learning (approach) can themselves be broken into referential and structural components i.e. analysed in terms of what is focussed on and how the focus is achieved. The conception can be further analysed by reporting on what people focus on when looking at conceptions of learning e.g. quantitative increase in the amount of knowledge or a changed understanding of reality. How this can be achieved may be by memorisation or by abstract meaning. The approach, on the other hand, can be analysed in terms of what is focussed on i.e. adopting either a surface or deep approach to learning and how it is achieved; an atomistic or a holistic approach.

Crawford et al (1994) believe that conceptions can be described referentially i.e. from being about numbers, rules and formulae to it being about a complex logical system. How students approach learning mathematics is achieved in a fragmented or cohesive way. Similarly, approaches can be described referentially, either to reproduce or understand information, and structurally, where strategies vary from rote memorisation to understanding where the theory applies.

The transition from secondary school mathematics to university mathematics

While the dominant focus of this paper is to examine the influence of affective factors on mathematics learning, the impact on learning of the transition from secondary to university mathematics cannot be disregarded. Not only is making the transition a difficult time for students, both academically and socially (e.g. Kantanis, 2000; Pargetter et al., 1999) but evidence suggests that there is a significant 'gap' between secondary school mathematics and university mathematics (e.g. Kayander and Lovric, 2005; Hoyles et al., 2001; Anderson, 1996).

According to Kayander and Lovric (2005: 149), " the transition in mathematics is by far the most serious and the most problematic". This view is supported by Pargetter et al. (1998) who believe many students making the transition from secondary school to university experience difficulties. Researchers such as Thwaites (1972) and Kayander and Lovric (2005), agree that students lack basic concepts, knowledge and skills needed at university level. Hoyles et al. (2001: 833) identified three main problem areas in the conceptual gap between school and university mathematics.

- Lack of mathematical thinking (i.e. the ability to think abstractly or logically and to do proofs),
- Weak calculational competence,
- The students' lack of 'spirit' i.e. lack of motivation and perseverance**.**

Student's Approaches to Learning

Students' approaches to learning are an important aspect of the transition to university. Anthony (2000) says students' conceptions of learning have an onward effect on the way they approach their studies and in turn affects the quality of their learning.

The type of approach to learning that student's adopt is a strong deciding factor on whether students transition to university is successful or not. Cano (2005: 206) says approaches to learning reflect "learner's ideas or conceptions of learning, how they experience and define their learning situation, the strategies they use to learn and the motivation underlying their conduct". Deep-level versus surface-level, reproduction versus comprehension and relational versus instrumental understanding are vitally important concepts according to many researchers (Biggs, 1993 and Marton and Saljo, 1976). Marton and Saljo's (1976) work focussed on students approaches to learning and they identified two processes, deep-level and surface-level. In surface approach learning, the main focus is reproduction of knowledge. Biggs (1993:6) describes the 'surface approach' in the Student's Approaches to Learning (SAL) framework as a "guiding principle or intention that is extrinsic to the real purpose of the task". Deep-level learning on the other hand aims for comprehension.

Ramsden (1992:45) believes "surface approaches are uniformly disastrous for learning". His Lancaster study (1983) where university students were interviewed, established that surface approaches were linked to negative attitudes on studying. He made an important finding with regards to university students' learning. Those who use deep approaches adapt better to higher education demands and are most committed to studying.

Ramsden (1992: 83) reported that there are other factors that contribute to learning other than the students' approach. He proposes a model of student learning in context.

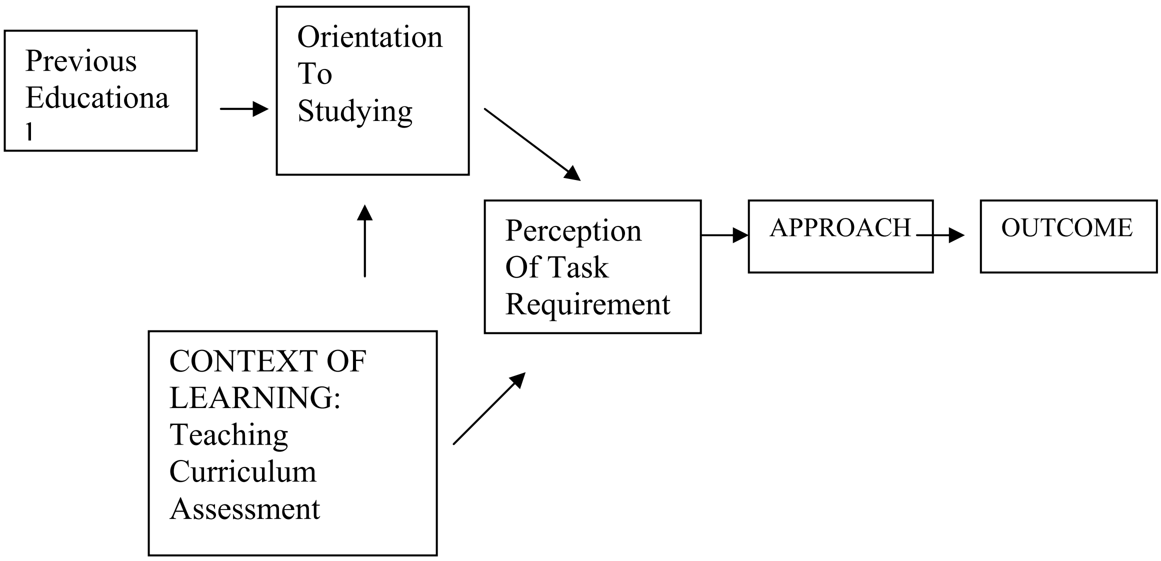

Figure 1.2 Ramsden's (1992) Student Learning in Context

Ramsden says student's previous experiences influences their orientation to study. The context in which learning takes place e.g. teaching, curriculum and assessment contributes to their study and perceptions of the demands of the task. This in turn determines their approach to learning and finally their learning outcome.

Anderson (1996) also investigated instrumental and relational understanding among mathematics undergraduates. He believes students making the transition to undergraduate mathematics in the UK, have become heavily reliant on the instrumental approach. This according to Anderson (1996:813) hinders students' learning of mathematics

Exploratory Research
The aim of the exploratory research was to establish and examine the extent to which affective variables influence students' mathematics learning in the transition from secondary school mathematics to university mathematics. Relevant issues and findings will thus be addressed by the author in subsequent research.

Methodology
Poulson and Wallace (2003) claim the methodology is justified in relation to the firmed up research questions. It is the intention of the author to deepen the understanding of the phenomenon in question through the systematic process of collecting information and analysing data. The research in the exploratory phase incorporates quantitative methodologies. The data consisted of the coded response of 607 questionnaires returned from first year students at the University of Limerick. Questionnaires were analysed statistically using SPSS software (Version 13). The data was analysed using firstly graphical representations such as frequency tables and bar charts for categorical data as well as cross tabulations, histograms and box plots for analysis of continuous data. Spearman's Rank correlation was used to check for correlations in Aiken's (1974) E and V Scales. Kolmogorov-Smirnov was used to test the data for normality in Gourgey's (1982) Mathematical Self-Concept Scale. The latter was also chosen because of the large sample size.

Research Sample
The author needed to work with students making the transition from secondary school mathematics to university mathematics and who are studying service mathematics courses at the University of Limerick. Three groups were chosen: Engineering Mathematics 1, Science Mathematics 1 and Technological Maths 1. The large sample size and various groups within in the sample allowed for much diversity in ability. Most courses require an Ordinary Level B3 in mathematics or a D3 at Higher Level will suffice. Those studying engineering generally have more interest in mathematics than some courses in the science and technological service mathematics streams. This is not true for all groups e.g. Physical Education students who have selected mathematics as their academic subject with the view to becoming a secondary mathematics teacher.

Data Collection
Development of the Research Instrument: A questionnaire for third-level students was designed and implemented using Foddy's (1993) 'TAP' paradigm i.e. the topic should be properly defined so that each respondent clearly understands what is being talked about; questions should be applicable to each respondent and have a specified perspective. Keeping this in mind a draft questionnaire was developed that is fit for this purpose.

Drafting the questionnaire: Kulm (1980) regards the Likert scale as the most widely used self-report procedure as regards measurement of attitudes. They are widely used in education and according to Cohen et al. (2000) they offer the researcher freedom to use measurements with opinion, quantity and quality. Therefore the author felt it both appropriate and useful to adapt studies that used Likert scales to measure attitude, belief, self-concept, conceptions of mathematics and approaches to learning. The author's questionnaire consisted of 78 statements based on these attitudinal scales. They include Aiken's (1974) 'Two Scales of Attitude Towards Mathematics (Enjoyment and Value)', Schoenfeld's (1989) 'Beliefs about Mathematics', Gourgey's (1982) 'Mathematical Self-Concept', Crawford et al.'s (1998) 'Conceptions of Mathematics' and Biggs et al.'s (2001) 'Revised two-Factor Study Process Questionnaire'. They were adapted for use as considered necessary.

Piloting the Research Instrument: The questionnaire was piloted with six Leaving Certificate students and six mathematics teachers in May 2005. The purpose of this was to ensure wording and length were appropriate. The author spoke to each participant following their completion of the questionnaire, discussing the content, wording of questions and the length of the questionnaire. Subsequently any necessary changes were made to produce the final research instrument.

Final Research Instrument: The questionnaire comprises of the following five scales, Aiken's (1974) 'Two Scales of Attitude Towards Mathematics (Enjoyment and Value)', Schoenfeld's (1989) 'Beliefs about Mathematics', Gourgey's (1982) 'Mathematical Self-Concept', Crawford et al.'s (1998) 'Conceptions of Mathematics' and Biggs et al.'s (2001) 'Revised two-Factor Study Process Questionnaire'. The questionnaire was divided into 3 sections. The first section contained information about the student and their background. Section A and Section B were statements based on a five point Likert Scale. Strong feelings could be indicated on either side of the scale and there was an option for respondents who were unsure of statements (i.e. 1= strongly disagree, 2= disagree, 3=unsure, 4=agree, 5=strongly disagree).

Data Analysis

The data (both categorical and continuous) resulting from the questionnaire at third-level was analysed using the statistical package of SPSS (Version 13). Questions were coded for later analysis. All questions were given a unique code number and responses were entered into SPSS using these codes. Missing data were also coded so as to ensure that no question in particular was answered with a significantly lower frequency than other questions. Initial frequency checks were carried out on all variables to detect any coding errors in the data. The findings are discussed below.

Discussion

The focus of this paper is on affective components when learning. Klinger (2004) highlights the need for challenging and changing students' negative perceptions of mathematics. The author wished to consider the extent of such perceptions of maths during the transition phase from secondary school mathematics to university mathematics.

The first area of concern is the *attitude* of students. Aiken's (1974) 'Two Scales of Attitude Toward Mathematics' was used to assess students' attitudes and as the name suggests comprises of two parts: enjoyment of mathematics (E scale) and value of mathematics (V scale). According to Aiken (1974: 70) "the E scale is more highly related to measures of mathematical ability and interest…" Out of 607 respondents, only 3 scored the total maximum 55 points (0.5%) in relation to enjoyment of mathematics. The minimum total E Scale score was 13 (0.2%). The mean score obtained was 37.6, which shows relatively high enjoyment levels of mathematics across the sample. Aiken (1974: 70) claims, "The V scale is more highly correlated with measures of verbal and general-scholastic ability." The V scale consists of 10 statements to E scale's 11. Therefore the highest possible total score is 50. The highest percentage of respondents scored 40 (9.3%) followed closely by 8.8% who scored a high 43. The minimum score by one respondent was 14. A mean score of 39.3 indicates students' appear to view mathematics as valuable.

Further analysis of the scales using Spearman's Rank correlation gave a clearer view of which statements correlated highly or not with enjoyment and value of mathematics. All

correlations were statistically significant ($p < 0.01$) although not all items on both scales correlated very strongly with item scores. Strong correlations were found for most item scores and total scores on the E scale. The lowest correlation, r = .623, was between item 1 of the scale ("I enjoy going beyond the assigned work and trying to solve new problems in mathematics") and total E score. The highest correlation, r = .845, was between item 6 and total E score ("I have always enjoyed studying mathematics at school").
In relation to the V scale, the item-total correlations were not as high as those for the E scale. This is consistent with Aiken's findings. The lowest correlation, r = .484, was between item 1 ("Mathematics has contributed greatly to science and other fields of knowledge") and total V score. The highest correlation, r = .650, was between item 7 ("Mathematics is not important in everyday life") and total V score.

The next area of concern was the students' *beliefs* about mathematics using Schoenfeld's (1989) scale. A low score was assigned to negative responses and a high score to positive responses. Therefore the higher the mean score the stronger the beliefs about mathematics. The mean score was 20.7 out of a possible high score of 30. The scores ranged from 10 out of 30 (one subject) to 29 out of 30 (two subjects). The spread of scores may be attributed to the prior experience the students have had. This often determines how students behave e.g. the amount of time and effort students are willing to invest in a mathematics problem (Schoenfeld, 1989).

There was a relatively positive response to beliefs about mathematics although there was some evidence of reliance on procedural knowledge. Table 1 (below) indicates that almost all questions yielded positive responses, in particular item 35 ("Math problems can be done correctly in only one way."). The mean score for this item was 4.1 with 35.6% and 46.3% of respondents strongly disagreeing or disagreeing with the statement.

		Everything important about maths is already known by mathematicians	In maths you can be creative and discover things by yourself	Maths problems can be done correctly in one way	Real maths Problems can be solved by commonsense instead of the maths rules you learn at school	To solve maths problems you have to be taught the right procedure, oryou cannot do anything	The best way to do well in maths is to memorise all the formulas
N	Valid	607	606	606	605	607	607
	Missing	0	1	1	2	0	0
Mean		3.6326	3.3548	4.0924	3.3521	2.8830	3.3624
Median		4.0000	4.0000	4.0000	3.0000	3.0000	4.0000
Std. Deviation		1.14687	1.04819	.90071	1.02130	1.20416	1.08718

Table 1 *Means, Medians and Standard Deviations on Beliefs about Mathematics*

The statement **"**To solve maths problems you have to be taught the right procedure, or you cannot do anything" showed the lowest mean (2.9). This response would seem to indicate that students consider procedural knowledge of utmost importance, a situation described as problematic by researchers such as Biggs (1993), Dweck (1986) and Marton and Saljo (1976).

As discussed earlier *mathematical self-concept* can affect student's learning.

> "Mathematical self-concept is defined as beliefs, feelings or attitudes regarding one's ability to understand or perform in situations involving mathematics." (Gourgey, 1982:3)

Items in the questionnaire, based on Gourgey's scale, were worded both positively and negatively i.e. scoring on negatively worded items was reversed so that a high score would indicate a favourable mathematical self-concept. Possible scores range from a low of 12 and a high of 60. The mean of the mathematical self-concept scale for the sample of 600 students (7 missing) was 40.6, which is a positive finding and perhaps slightly higher than anticipated. Further statistical analysis of the data in future work may explain why this is so. Scores ranged from 16 to 56.

The author wished to briefly compare gender differences with mathematical self-concept. Boxplots for both males and females did detect outliers that may have skewed the data slightly (See Figure 1.3). This analysis is based on median scores since the outliers present affect the mean. Hence, the median is a more reliable measure of centrality. Median scores for males (42) were just slightly higher than that of the females (41) despite a much larger male sample size (435 males in comparison to 165 females).

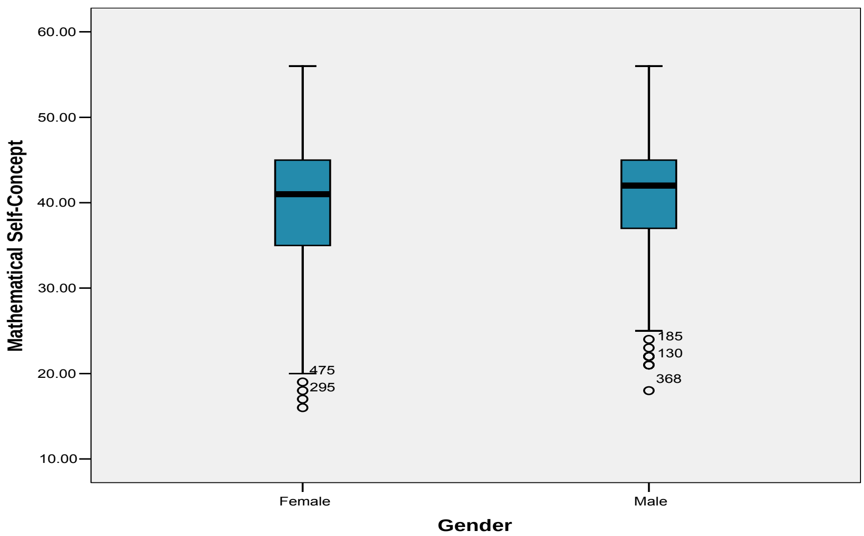

Figure 1.3 Boxplots for Total Mathematical Self-Concept Scores in Relation to Gender

Crawford et al.'s (1998) '*Conceptions of mathematics'* scale was incorporated into the questionnaire to determine if students are either fragmented or cohesive learners. It was found that the mean for the Cohesive Conception scale (3.5) was substantially higher than the mean for the Fragmented Conception scale (2.7). This was a positive finding as it indicates that the students tended to lean towards being cohesive learners although not completely rejecting fragmented statements. This is surprising given other findings by Crawford et al. (1994), *The Chief Examiner' Report, 2005,* and the *NCCA Review of Mathematics in Post - Primary Education – Report on the Consultation, 2006.* A low mean score however for the statement, “the subject of mathematics deals with numbers, figures and formulae” suggests a reliance on rules and procedures indicating an existence of rote learning.

As mentioned earlier a relationship exits between conceptions of mathematics and *approaches to learning*. Findings from Biggs et al.’s (2001) two-factor Study Process Questionnaire (See Appendix 1, Section B) addressing deep and surface approaches to learning correlated with the findings from Crawford et al.’s (1994) scale i.e. deep approaches to learning correlates with cohesive learning and surface approaches to learning correlates with fragmented learning.

The questionnaire also includes subscales where students motive and strategy for learning can be calculated by adding the relevant item scales. Table 2 below describes Biggs et al.’s (2001) R-SPQ-2F dimensions, motives and strategies.

	Surface	Deep
Motive	Fear of failure	Intrinsic interest
Strategy	Narrow target, rote learn	Maximise meaning

Table 2 *Revised Study Process Questionnaire: Dimensions, motives and strategies*
-+
Surface learners are usually motivated by a fear of failure and employ rote learning strategies. Deep learners tend to be intrinsically motivated to learn and wish to maximise their meaning and understanding of a subject. Scores ranged from 1 (“never or rarely true of me”) to 5 (“always or almost always true of me”).

The highest possible score on both scales was 50. The higher the score on the deep learning scale the better as this indicates a positive response to the deep approach statements and suggests that students favour comprehension rather than reproduction of knowledge. This can be linked to earlier findings where students seemed to lean towards cohesive learning. High surface scale scores however would show students were surface learners and aimed for reproduction of knowledge rather than aiming to understand the information. Scores ranged from 10 to 49 on the deep approach to learning scale, and from 10 to 42 on the surface approach to learning scale. The mean for the deep scale was 29.8 in comparison to the surface scale mean of 24.3. It should be also noted that 18 responses were missing from the deep scale and 6 from the surface scale. Students’ opinion on deep approaches to learning seemed quite positive and the relatively low surface scale scores are also a positive finding. When the author examined the subscales however there is evidence of rote-learning and procedural knowledge. For example, students’ scored a mean of 13.4 out of possible 25 on the strategy for surface approach learning scale indicating that rote learning is a prominent strategy employed by surface learners in this sample.

Conclusion

Problems facing mathematics education often referred to as the “mathematics problem” have been identified and researched to a certain extent in Ireland. However, little research has been done on the influence on affective variables on the learning and indeed the teaching of mathematics. The majority of students face a difficult transition from secondary school mathematics to university mathematics. Based on the study carried out thus far by the author and by researchers in other countries, it is clear that attitudes, beliefs, emotions, mathematical self-concept, conceptions of mathematics and approaches to learning mathematics are crucial areas in the learning of mathematics and needs attention in an Irish context. While the findings have not been all negative, both literature and particularly studies by Gill (2006) have shown the struggle students in Service Mathematics courses endure. Further research in the form of statistical analysis and qualitative studies will shed more light on the issues at hand.

It is recognised that affective factors impact on the mathematical preparedness of both Higher and Ordinary level mathematical students as they make the transition from secondary school mathematics to higher education. Technical and mathematics specific issues are well recognised e.g. The Chief Examiner's Report (2005) commented on candidates only performing well in routine and well-rehearsed problems, with particularly inadequate understanding of mathematical concepts at ordinary level, but the impact of affective factors is not nearly so well appreciated or understood.

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Reflections of University Students on the Leaving Certificate Mathematics Experience

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This study presents the reflections of eleven university students on their mathematics education at the Leaving Certificate Level. These BA students were about to complete their first year of Mathematical Studies in University College Dublin. From this perspective they described, in individual interviews, their experiences with Leaving Certificate Mathematics. Specifically they were asked to address various aspects of their relationship with the subject, including their decision to take Higher or Ordinary Level at secondary school, and their motivation for studying mathematics at university. This paper presents an analysis of the student responses, and discusses issues relating to the transition from second- to third-level mathematics.

Introduction

Almost all second level students who sit the Leaving Certificate (LC) State Examinations (18-19 years old) in Ireland take mathematics as one of their six or seven subjects. LC Mathematics is offered at three levels: Foundation Level (FL), Ordinary Level (OL), and Higher Level (HL), with average uptake each year of 11%, 71%, and 18% respectively (Chief Examiner's Report, 2005). While the total number of students studying mathematics to the LC level is very positive - approximately 55,000 each year - the low uptake of HL Mathematics is cause for concern.

The HL Mathematics Syllabus states that one aim of the course is to "… equip mathematical "specialists" – the students who will pursue advanced mathematics courses" (p. 8). Indeed, in addition to achieving the pre-requisite number of CAO points, a number of third level undergraduate degree programmes in Ireland have, as a minimum entry requirement, a grade C or B or higher in HL Mathematics. For example, in University College Dublin, such pre-requisites are required for entry to undergraduate degree programmes in areas such as Actuarial Science, Economics and Finance, Engineering, Mathematical Science, and Theoretical Physics. Thus the low uptake of HL Mathematics has a direct impact on the number of students eligible to study courses such as these at university. It may also be worth noting at this point, that it is from 'mathematical-based' fields such as these that highly trained graduates are in demand for the growth and sustainability of Ireland's knowledge-based economy (SFI, 2005).
However, even among those who successfully complete the HL Mathematics course, there are two further causes for concern. Anecdotally, it seems that of the relatively small number of eligible students, few are choosing to embark on mathematics degree programmes, and of those who do, it seems that after the first or second year of undergraduate study, a number choose not to major in mathematics. As university mathematics lecturers, this is an issue that we, in particular, are keen to better understand.

We acknowledge that HL Mathematics is not for everyone, and neither is the study of mathematics at the third level. However, even among those students who may have the *potential* to take HL Mathematics, and/or study mathematics at the third level, we conjecture that there is some unnecessary 'leakage' from the 'Mathematics Pipeline' Pre-HL and Post-HL Mathematics Examination. Through the reflections of eleven university students, about to complete their first year of *Mathematical Studies* in

University College Dublin, we attempt to explore the reasons for these 'leaks' in order to shed some light on the retention issues involved.

Research Context And Methods

The *Bachelor of Arts (BA) in Mathematical Studies* is a three-year honours degree programme offered by University College Dublin (UCD), Ireland. Students choosing to study *Mathematical Studies* as a first year subject are advised that they should have achieved an A, B, or C in HL Mathematics, or an A in OL Mathematics, or equivalent. In their first year they are required to take 4 mathematics modules: *Calculus I*, and *Combinatorics & Number Theory* in Semester 1, *Calculus II,* and *Linear Algebra* in Semester 2.

In the academic year 2004-2005, 55 students chose *Mathematical Studies* as a first-year subject. The first author was the lecturer for both *Calculus I* and *II*. Each module consisted of 24 one-hour lectures, and 5 one-hour workshops. Since most of the class had just completed HL or OL Mathematics, the Calculus modules had two main aims: firstly, to further develop the students' procedural techniques; and secondly, to encourage students to develop a more conceptual understanding of the subject. The first aim was particularly relevant for those students with an OL Mathematics background – for example, unlike their HL Mathematics counterparts they would not have seen any integration techniques. Workshops were introduced specifically to address the second aim of encouraging students' conceptual development of the concepts of Calculus. Using various questioning techniques, many of which are described by Mason (1998) and Watson & Mason (2002), the lecturer designed worksheets aimed at encouraging students to explore the relevant concepts of Calculus, in a supportive, small-group, workshop environment.

In an attempt to gain an insight into how some of these students managed the transition from second to third level mathematics, interviews were conducted with eleven student volunteers from the class in April 2005 - just before they sat their final examinations. Each participant met individually with research assistant Brigena Doherty, and the interviews, which were tape-recorded, lasted on average twenty-five minutes. Questions were prepared in advance under three headings: 'LC Mathematics', 'first year *Mathematical Studies'*, and 'Calculus workshops'. Firstly, students were questioned on why they had chosen HL or OL Mathematics at LC, whether they had enjoyed the course, their study habits, and their overall experience of LC Mathematics. Secondly, they were asked to discuss areas such as: their motivation for studying mathematics at university; their expectations and level of enjoyment of the course; their study habits; and, whether their feelings and thoughts about mathematics had changed over the course of the year. Finally in the third part of the interview, each student was asked about specific aspects, and their overall experience, of the Calculus workshops.

Analysis of the interview data is a work-in-progress. Data from the first part of the interviews with the eleven students have been analysed and a remarkable degree of consistency in responses has been found. The students' responses from the second and third parts of the interview are currently being analysed and while certain themes relating to the transition to third level mathematics are emerging, we are reluctant to present them until we have the opportunity to explore them further. However, from our initial analysis of the data it is becoming clear that some students are coping better with the transition than others.

By coincidence, the eleven participants broke almost evenly into two groups: those who had taken the OL Mathematics Examination at LC (Anna, Ava, Caoimhe, Jessica, and Lauren), and those who had taken the HL Mathematics Examination (Katie, Lucy, Luke, Molly, Peter, and Roisin). This gave the authors an invaluable opportunity, to not only explore how students who had successfully completed HL Mathematics managed the transition to third level mathematics, but also how high-achieving OL Mathematics students managed the same. As a very simplistic measure of how successfully our eleven participants navigated the transition, rating them using their final marks in the first year *Mathematical Studies* examinations from the highest to the lowest, yields the following list: Roisin (HL), Katie (HL), Jessica (OL), and Molly (HL) all achieved first class honours (70% or higher); Ava (OL), Caoimhe (OL), Peter (HL), and Lucy (HL) all achieved second class honours (60-69%); Luke (HL), Anna (OL), and Lauren (OL) all passed (with scores in the range 45-59%). From this list we make two initial observations: firstly, based on their scores our participants seem to represent a good cross-section of the class; and secondly, not only did all five OL Mathematics students successfully pass their first year *Mathematical Studies* examinations, three achieved second class honours or higher. Therefore the participation of these students in our study gave us the chance to explore why five students, who voluntarily chose to study mathematics at university and successfully passed their first year examinations, had not taken the HL Mathematics Examination.

Our "Results and Discussion" section below will be organised into four parts. In Parts 1 and 2, we present results from the first part of the interview data. Specifically, in Part 1, we present and discuss the reasons our OL Mathematics students gave for "dropping-down" from HL to OL Mathematics. In Part 2, using our findings from the data, along with the Chief Examiner's Report on HL Mathematics (2005), the Review of Mathematics in Post-Primary Education (2005) conducted by the National Council for Curriculum and Assessment (NCCA) in Ireland, the research findings of Lyons, Lynch, Close, Sheerin, and Boland (2003), along with those of Boaler (1997, 1999, 2002), and Boaler and Greeno (2000), we attempt to piece together the mathematical profile of a student who has just completed LC Mathematics, and examine how aspects of this profile may have a negative impact on his/her transition to third level mathematics. Given that our analysis of the entire data is not yet complete we do not want to discuss general issues arising for students in the transition from LC to third level mathematics. Therefore in Parts 3 and 4 respectively, we present two 'transition case studies' – one based on one of our HL Mathematics students, and one based on one of our OL Mathematics students.

Results And Discussion

Part 1: "Dropping-down" to OL Mathematics

Of our eleven participants, Anna, Ava, Caoimhe, Jessica, and Lauren had taken the OL Mathematics Examination. Apart from Jessica, all had started out taking HL Mathematics, but "dropped-down" to OL Mathematics at various stages. Anna, Caoimhe, and Lauren switched to OL Mathematics within the first year, while Ava switched after she failed HL Mathematics in the "Mock Exams". The option of taking HL Mathematics at the Junior Certificate (JC) Level had not been available to Jessica, which in turn meant that she had to take OL Mathematics for the Leaving Certificate. Jessica will be the focus of our case study in Part 4.

Anna and Caoimhe both said they left HL Mathematics because they found the course too time-consuming, and felt their other subjects were being neglected.

Caoimhe: It was just too much time spent on maths and not enough on other subjects, and that's the only reason I dropped-down.

Anna: For the Leaving Cert. it [mathematics] was very time-consuming, kind of. It clashed with a lot of other subjects time-wise.

Anna's decision to move to OL Mathematics was a strategic one – she felt she had to 'weigh up' the time spent on HL Mathematics against the grade, and hence the number of CAO points, she might receive for it:

Anna: But then for the Leaving Cert. you kind of weigh up the pros and cons of the points and everything. So then you have the chance of failing it [HL Mathematics], and the chance of getting a good A [in OL Mathematics], so I went with the good A.

She didn't regret her decision - studying OL Mathematics was a lot less time-consuming:

Anna: You know like, it took 10 minutes to do the homework instead of an hour.

While Anna admitted that she had found HL Mathematics "kind of difficult", at no stage did Caoimhe state that she had difficulties in the course before dropping-down - the time-commitment seemed to be her main concern. She was very passionate about mathematics in general, and now that she was in university, enjoyed the freedom of being able to spend a lot more time studying the subject:

Caoimhe: I love maths. It would be the only thing I would want to go on with. I'd never actually say I didn't want to do maths. It just doesn't happen. Like it really, really doesn't happen.

Caoimhe: It was only because it was my Leaving Cert. Like I spend ten times more time on maths now in college than I would on anything else, but it's only because I feel it alright to do.

She noted that in her school, students taking the HL Mathematics course were required to take two extra hours of classes each week, and these clashed with her sporting activities. This additional time commitment played a role in her switching to OL Mathematics.

Ava actually completed the HL Mathematics syllabus, but switched to the OL class after failing the "Mock Exam" in mathematics. Although not citing the time commitment as a factor in her decision to leave HL Mathematics, she did comment on it:

Ava: It was compulsory to do two extra classes to finish the [*HL*] course, so that shows how fast we actually had to go through it like.

One might argue that perhaps the reason these OL Mathematics students found the HL Mathematics course so time-consuming was because they simply had to spend more time trying to understand the material, than some of their successful HL Mathematics classmates. While this may be the case, it is worth noting that Katie, a HL Mathematics student who proclaimed to have 'loved' mathematics at school, also brought up the issue of time in relation to HL Mathematics:

Katie: I think if you are doing Higher Level maths for the Leaving Cert. you generally have a liking for it 'cause it's a lot of work compared to the rest of the other subjects.

Interviewer: Overall how would you describe your experience of maths at Leaving Cert. level?

Katie: I really liked it. It was very time-consuming compared to every other subject. I mean I definitely spent 50% of my time on maths and 50% on everything else. But, em, I think everyone found that. But I loved maths at Leaving Cert so I didn't mind.

Discussion

Our students' responses indicate that there is a significant time-inequity between studying HL and OL Mathematics, and indeed between HL Mathematics and other LC subjects. We find the case of Caoimhe particularly worrying. She clearly loved mathematics at school and university, and her grade in first year *Mathematical Studies* suggests that she is a capable mathematics student. Yet, she made a strategic (and perhaps sensible!) decision to drop HL Mathematics and focus on getting CAO points instead. Had she been excluded from enrolling in *Mathematical Studies* because she hadn't taken HL Mathematics, then perhaps she would have been lost to mathematics for good. (Incidentally, Caoimhe will graduate this year with a first class honours *B.A. in Mathematical Studies* and hopes to train as a mathematics teacher.)

It is worth noting that OL Mathematics didn't seem to provide a challenge for our five students.

> Lauren: And then when I dropped to Ordinary Level - there is such a difference between the two of them like, it's unreal like. Honours is so hard and then Ordinary is just like a walk in the park.

Taking the example of Caoimhe again - she talked of how she was 'so bored' with OL Mathematics when she repeated the LC, and explained that she gave grinds to two fellow students, and taught the class for a week when the teacher was ill. Three of our OL Mathematics students repeated their LC, and we note that this year could have provided them with the opportunity to learn more mathematics had the LC Mathematics courses and syllabi been structured differently.

Why would a student persist with HL Mathematics? Well, certainly if it was a prerequisite for his/her third level degree programme, or perhaps if the student needed close to the maximum number of 600 CAO points. Katie suggests that one needs to have a 'liking for it', but is this the case for example in HL English, which is taken by approximately 60% of the LC population? The NCCA Review (2005) suggests that low uptake of HL Mathematics can be attributed to the 'perceived difficulty' and the ''elitist' status' (p.10) that the subject can sometimes have in schools and among students. None of our OL Mathematics students voluntarily expressed regret about leaving HL Mathematics, and when Lauren was specifically questioned on this point she summed it up as follows:

> Lauren: Oh God no, because you always hear people going on that 'Oh, honours maths is just so hard, like, for the Leaving Cert', so I don't really mind like. I don't really care.

Part 2: The Mathematical Profile of a 'Typical' LC Mathematics Student?

> If I had to reduce all of educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly. (Ausubel, Novak & Hanaesian, 1978, p. iv)

If a first year university lecturer is to be effective in supporting her students in the transition from second to third level mathematics, then it is absolutely essential that she first make an attempt to 'ascertain what they know'. Given a class of first-year students, most of whom have just taken OL or HL Mathematics, what can she say about what they know?

Based on the findings of studies in second level mathematics undertaken in schools in the UK and US, Jo Boaler (2002) distinguishes between 'mathematical knowledge' and 'mathematical capability'. She proposes that:

> … notions of mathematical capability expand beyond knowledge, to include mathematical practices and the identities these encourage. Further that different classroom practices

> encourage students to develop different relationships with the discipline of mathematics that impact their capability in profound ways. (p. 113)

Boaler (1999) undertook a three-year case study in the mathematics classrooms of two secondary schools in the UK. In one school, mathematics was taught using a 'traditional' textbook-based approach where students observed and practised algorithms and procedures. In the other school, a more exploratory, project-based approach to the teaching of mathematics was taken. Although students in both schools developed similar degrees of satisfaction with the subject, their views about mathematics differed greatly. Those from the 'traditional' school formed the view that mathematics was made up of rules and formulas that needed to be memorised. Not surprisingly many of these students felt that 'school mathematics was incompatible with thought' (p. 263). While these students worked hard, and practised many procedures and textbook exercises, when asked, most were unable to apply their mathematics to open or 'real-world' problems. Those from the 'exploratory' classroom experienced no such difficulties, and out-performed the 'traditional students' in the GCSE Mathematics Examination. Boaler (2002) suggests that the way these students had learned mathematics – the practices in which they had engaged in the classroom – had actually 'shaped the forms of knowledge produced' (p. 114).

A further study of 48 students attending Advanced Placement (AP) Calculus classes from six Californian high-schools was conducted by Boaler and Greeno (2000). In four of the high-schools, Calculus was taught in the 'traditional' fashion – methods were explained by the teacher at the board and students practised similar problems. In the other two schools, the teachers encouraged a more collaborative and discussion-based approach to learning. The findings in relation to the 'traditional' Calculus classes are perhaps not surprising:

> For students to be successful in such classes, they needed to both assume the role of a received knower and develop identities that were compatible with a procedure-driven figured world. (p. 183)

Of the 32 students attending the didactic classes, eighteen said they enjoyed mathematics and all gave reasons that suggested they were happy to be 'received knowers' – they enjoyed using a technique to produce a right answer, without having to offer an opinion or be creative. Of the fourteen successful mathematics students who said they disliked mathematics, most gave reasons that suggested they wanted to know the 'why' rather than the 'how', and be given the chance to express their creativity. Boaler and Greeno (2000) suggest that for these students, the subject of mathematics was incompatible with their emerging identities as creative individuals, who wanted the opportunity to discuss and debate ideas. The sixteen students in the two non-didactic classrooms had more positive identifications with mathematics, and for many this was because they viewed it as a subject where they could be creative and thoughtful, and develop understanding.

In summary then, if we are to really ascertain what our LC Mathematics students know, it is not enough to determine their mathematical knowledge by looking at the OL and HL Mathematics syllabi. Rather we should try to ascertain their 'mathematical capability'. In order to do this we need to explore the practices in which they have engaged in the mathematics classroom, and attempt to understand the identities that they have formed with the discipline as a result.

There is overwhelming evidence to suggest that the majority of Irish LC Mathematics students have experiences in the classroom similar to those students in the studies of

Boaler (1997, 1999, 2002), and Boaler and Greeno (2000), who attended the 'traditional' classrooms in the UK and US. Lyons et al. (2003) conducted a study on the teaching and learning of mathematics at the Junior Certificate Level in ten Irish schools. Teacher demonstration and student practice of problems were the two predominant pedagogical practices in the classrooms observed. Extracts are presented (chapter 4), from a top stream and a bottom stream class in two of the schools, which clearly illustrate the pedagogical approach of 'drill and practice' of a method.

The NCCA Review (2005) notes the reliance on the textbook for teaching mathematics in Irish second level schools, and the emphasis placed on preparation for the examinations:

> As evidenced by inspection visits, teaching is highly dependent on the class textbook (which tends to reinforce the 'drill and practice' style) and the examinations, and there is frequently a very close relationship between these two. (p. 21)

Our findings support those of Lyons et al. (2003) and the NCCA Review (2005). When asked to describe their study habits in LC Mathematics, all eleven students said they practised problems and/or did past examination papers. In their own words:

Caoimhe:	I just took a page and I wrote every single formula that had to be used on a page and I learned every single one. And then just exam papers, exam papers, exam papers.
Jessica:	Eh, exam questions mostly. Just keep on going over exam questions I suppose.
Lauren:	Just exam papers, going through them all the time like.
Luke:	By doing past papers. … *I didn't do much learning* - it was just kind of working out sums.
Lucy:	Mostly just doing kind of examples and questions from different books and stuff just to see how the kind of, eh, techniques worked. That was mostly it. I'd say I was lucky in that I kind of got to grasp with the idea behind them fairly quickly so that it didn't take me too long to try and understand the maths – more learning just how to work them out. So it was just practice more than anything else I studied with.
Katie:	Em, with just practise, just going through problems. It's definitely, *I think it's the only way you can learn maths* or [*inaudible*]. I think kind of learning a method and then just going away – I think you kind of have to put it to practice a few times before you get the hang of it.
Peter:	It was more kind of practise, going through different styles of questions and practising them.

From the most recent 2005 Chief Examiner's Report on LC Mathematics, we get a very clear picture of the effect that the 'drill and practice' approach to learning is having on students' ability to tackle even slightly unfamiliar problems, or problems that require some conceptual understanding of the topic involved. On students' weaknesses in the 2005 OL Mathematics Examination, the report states:

> Weaknesses, by and large, relate to inadequate understanding of mathematical concepts and a consequent inability to apply familiar techniques in anything but the most familiar of contexts and presentations. (p. 49)

The analysis of the 2005 HL Mathematics Examination is perhaps a lot more worrying:

> As identified in previous Chief Examiners' Reports, candidates' conceptual understanding of the mathematics they have studied is inferior to that which one would hope for and expect at this level. Whereas procedural competence continues to be adequate, any question that requires the candidates to display a good understanding of the concepts underlying these procedures causes unwarranted levels of difficulty. (p. 72)
>
> … examiners have been commenting on a noticeable decline in the capacity of candidates to engage with problems that are not of a routine and well-rehearsed type. Whereas within

> the relatively recent past it was common to see two, three, or more attempts at a problem with which a candidate was struggling, it is now more common to see the work abandoned after the first attempt fails to yield success. Also more prevalent is the inclination of quite good candidates to not even attempt parts of questions that do not resemble well rehearsed examples from class. (p. 72)

In relation to candidates' strengths the report states that 'strong performance was most evident in procedural questions where a definite sequence of familiar steps was required'. (p. 72) Overall the Chief Examiner's findings overwhelmingly support those of Boaler (1999, 2002) and Boaler and Greeno (2000) in relation to the students in these UK and US studies who were taught in the 'traditional', didactic mathematics classrooms.

Based on the findings of the studies and reports described above, if a student has taken either OL or HL Mathematics, then it is may be the case that they will have:

- been taught using a 'drill and practice', textbook based approach, with emphasis placed on preparation for final examinations (Lyons et al., 2003; NCCA, 2005);
- mastered procedural techniques, and be able to answer questions where a definite sequence of familiar steps is required (Chief Examiner's Report, 2005);
- difficulties in applying their mathematical knowledge in anything but the most familiar situations (Boaler, 1999; Chief Examiner's Report, 2005);
- a poor conceptual understanding of relevant mathematical topics (Chief Examiner's Report, 2005 - in relation to HL students);
- formed a view of their role in the mathematics classroom as that of 'received knowers', taking little responsibility for creative or independent thought (Boaler and Greeno, 2002; Chief Examiner's Report, 2005).

Many of the disaffected students from the 'traditional' classrooms in Boaler and Greeno's study (2000) rejected mathematics because the classroom practices they had engaged in resulted in their viewing further study of the subject as incompatible with their emerging identities as creative individuals. It is not unreasonable to suggest that successful HL Mathematics students may be 'turned-off' the subject for the same reasons. However if a student chooses to study the subject at third level based on their LC experiences and are expecting 'more of the same', then they may well end up being disappointed. As Boaler and Greeno (2002) point out:

> Students who choose mathematics as their main field of study, based upon the idea that the subject is structured, certain, and nonnegotiable, may encounter significant problems as the mathematics they learn at university becomes more advanced. (p. 196)

The first year mathematics lecturer, who has ascertained what her students know, and now must 'teach them accordingly', may indeed have an uphill struggle on her hands. The HL Mathematics Syllabus claims that it is designed for students who will 'pursue advanced mathematics courses' (p. 8), but it is this the reality? The cognitive transition that these students face in third level advanced mathematics courses is certainly not a trivial one. As Tall (1991) clearly articulates:

> The move from elementary to advanced mathematical thinking involves a significant transition: that from *describing* to *defining*, from *convincing* to *proving* in a logical manner based on those definitions. (p. 20)

The challenge of supporting a student, who has experienced a didactical, 'traditional' approach to mathematics at second level, in his/her transition to third level advanced mathematics, is the focus of an emerging body of research internationally, and we are keen to further explore this issue in the Irish context.

Finally, to give the reader a flavour of how mathematical experiences at second level, can influence a student's transition to third level mathematics we present the transition experiences of one OL and one HL student to third level mathematics. Any word or phrase in italics is the student's own.

Part 3: The transition from HL to third level Mathematics – a case study.
Katie loved maths at the Leaving Cert. level. She liked the whole challenge, kind of the problem solving aspect of maths and felt she had an ability for it. Katie studied HL Mathematics by practising problems. *Em, with just practice, just going through problems. It's definitely, I think it's the way you can really learn maths….* Her teacher would assign problems from a workbook that students had the option of using – *it just had loads of problems in it, and there wasn't any explanation or anything, just a book of problems.*

Katie enrolled in an Engineering programme the previous year, but didn't like it, and left. She chose Arts in UCD the next year, because she didn't really know what she wanted to do. Within the BA programme she decided to study mathematics because she liked it so much at school. Initially she joined the four-year Honours BSc Mathematics class, but decided to switch to Mathematical Studies at Christmas, in order to give herself more time to focus on one of her other subjects, which she had now decided to try and take as a single major for her degree.

When asked if she felt that first Mathematical Studies had changed the way she felt about mathematics she had the following response: *I don't think it has kept the kind of passion that, maybe passion is a very strong word but you know what I mean, that it did for the Leaving Cert.* When questioned on her study habits in mathematics at university, she stated: *I pretty much like learn the method, or know the method, and practise.* At one stage in the interview she described the Calculus modules as being very Leaving Cert, and at later stage, described them as being all Leaving Cert. Was she correct – was Calculus more of the same from school? Or was she just, perhaps subconsciously, choosing to engage with the parts she identified as being mathematics?

Her response to how she found the workshop tasks suggests that perhaps this is the case. Her overall opinion of the workshops was that: *I thought they were good because they made you look at it differently and they kind of, made you kind of see why you were doing things. They weren't just applying methods or anything. But I actually found that quite difficult to be honest. Getting the theory, getting my head around it.* When questioned about the 'Example Generation Tasks' (Watson and Mason, 2002) she commented: *They were good intentions. I found them incredibly difficult, but like they definitely ... the idea, I think the idea of trying to come up with examples is a good idea because, yeah, just to get you understanding their ... but I found it quite difficult.* She admitted that: *I found them really difficult, and like occasionally came up with one myself, but to be honest it was mainly with the help of the tutor.* On one of the other tasks she commented: *… they're the same as the examples they just, they just ... they didn't throw me because like it didn't throw me off what I was thinking already. It just ... I found it difficult to get that kind of deeper grasp of it.* The interviewer asked if the task had 'rocked the boat'. *Oh it did, but it didn't rock the boat in the sense that in ... I now am confused about applying methods or anything. But just ... it's given me, it's made me a bit more insecure about maths maybe. ... I was so confident at Leaving Cert*

Level and now, well that would have made me a bit more insecure. Katie didn't think she would ever become a mathematician: *I think because it's a lot more theory and I'd be much more into the kind of, just the problem solving and the kind of getting a problem and challenge it and work it out. Whereas I wouldn't be so much interested in all the kind of theory and theorems. … I just don't like them as much.*

Part 4: The transition from OL to third level Mathematics – a case study.

Like Katie, Jessica achieved overall first class honours in her first year *Mathematical Studies* examinations. In Calculus, Katie's score was higher that Jessica's by 2.5% - an unremarkable fact until you realise that Jessica didn't take HL Mathematics at the JC Level, never mind, at the LC Level. Her mathematics teacher left in second year and they didn't have a teacher until after Halloween of third year, *and our class was just wild to begin with so we fell really far behind.* As a result, she wasn't *allowed* to take the HL Mathematics Examinations at the JC Level.

She loved OL Mathematics at the LC Level. *I think 'cause I could do it. I found pass maths easy so it just was something I knew I could do like, I wasn't worried about failing or anything.* She studied mainly by doing questions from past examination papers: *Eh, exam questions mostly. Just keep on going over exam questions I suppose.* Jessica hadn't done any trigonometry, and she felt that this caught up with her at university: *Oh I didn't do, when we were doing our Leaving Cert, em, our teacher was like, she wanted us to have, to get an A. She didn't care about us learning. […] But we never did trigonometry or nothing so that sort of came back this year. She was a good teacher and all, just a lot of things were left out and there were a lot of basics I didn't get to learn because of the Junior Cert. and all.*

She chose *Mathematical Studies* because she *really liked* mathematics. When asked about the 'Example Generation Tasks' in the workshops, Jessica had the following to say: *They were good because they made you think logically, like you know, it wasn't based on problem sheet questions. You were actually seeing yourself why things were done.* Sometimes she'd be *really stuck* and one of the tutors would talk her through it *and then I'd get it myself and I'd be all up like, because I wouldn't exactly be the best person in the class at maths. … You know when you are working yourself and you're seeing yourself, and it's your thoughts that are going into it, so it's easier for you to remember because it's not like you were given something completely strange, it's how you see it like, and when you see it right yourself, you'll remember it better.*

Had first Mathematical Studies changed the way she felt about mathematics?: *Yeah. I think I've changed my opinion since the start of it because at the start I thought it would be boring. And you know like, Einstein up the front teaching you like … but it's more, I don't know, you think more logically, it just sort of broadens your mind like, it's really good. I like it a lot.* She feels more challenged: *It's not something you get the first time off, you have to keep on going back on it. I like it because first it'll make you a bit scared and then you try another go, but then, when you actually learn it you feel really good because you're like "Oh my God I know how to do like whatever".* At the time of the interview she felt scared because *I don't know what to expect but I'm sort of, you know like, excited too maybe to see how I get on with it.*

Does Jessica think of herself as a mathematician? *Em, [long pause] I think I could think like a mathematician. I might not know as much as a mathematician, I'm not as smart* *enough to be a mathematician but I think I'll be able to once I have all my knowledge, that I'll be able to apply myself like, I'll look for sort of reasoning … I think it's how you apply yourself to something, you know, and how you think about things. It's like actually more like how you apply yourself to the world and all like, 'cause it's all like based around maths.*

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The Traditional Versus The Progressive Style Of Teaching

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The aim of this paper is to examine, through the theories of some educational psychologists, the teacher teaching, that is, how the teacher approaches teaching with regard to what is referred to as the traditional approach and the progressive or modern approach. Are both approaches very different? Can they be married together in some workable format? Which of the two, if either, stands on its own as the perfect model? Do the two approaches go hand-in-hand to give that ideal framework for teaching? Having posed these questions, the paper continues by tracing some of the accounts of the traditional and the progressive teacher. It further cites some up to date evaluations of both approaches in relation to mathematics education. It concludes by finding there is no 'perfect model', but certain approaches lead to a relational understanding of motivation and engagement.

The traditional teacher and the progressive teacher

The traditional teacher and teaching is something of the past or is it? Are we are now looking at the progressive teacher with progressive teaching methodologies? Two very different pictures, or so, it would seem. What is a traditional teacher? What is a progressive teacher?

A very broad, off the cuff answer to the question of the traditional teacher is that, until recent times the teacher stood at the top of the classroom, passed on information to the students, who sat in straight rows of desks learning by rote, doing the given homework and not asking questions. A progressive teacher, on the other hand, is a facilitator, a teacher who 'throws out' the idea or concept and allows the student to investigate for her/himself and draw her/his own conclusion while the teacher facilitates the work undertaken.

As both a teacher and parent, Rachel Pinder considered these very general throwaway remarks.[1] When education is discussed she found that words such as exploratory, investigative, child-centred, experienced-based and modern methods are all very much in use on the progressive teaching front, while at the same time other people are discussing a return to the traditional methods which were much more formal and unyielding, yet considered to be sound. Within all of this discussion of education and what it should be about, what is the role of the teacher? Is it to help children to learn from their experience or to pass on knowledge? Considering the best method of teaching mathematics for example, Pinder poses the question as to whether children should learn tables by rote in the traditional mode or if they should be helped to understand facts about number before memorising tables, using methodologies, which are of the progressive mode.[2]

[1] Rachel Pinder, Why Don't Teachers Teach Like They Used To? (London: Hilary Shipman, 1987), 1

[2] Ibid, 2

[3] *Encylopedia Britannica*, CD 2000, 10th ed (U.S.A.: Encyclopedia Britannica Inc., 2000)

What is traditional teaching?

The earliest traditional teaching ideas were based on the theory of mental disciplines in which the reason for teaching was to train the mind and develop the intellect. In classical times the ideal was to be trained in a restricted number of subjects - grammar, logic, rhetoric, arithmetic, geometry, music and astronomy. The intellectual authority of the teacher reigned supreme and learning took place by imitation and memorising. The theory of mental discipline is associated with Aristotle's *faculty psychology* in which he maintained that the mind is made up of different faculties each of which are independent of the other and certain subjects trained certain faculties. In a mathematical context for example, geometry trained the faculty of reason. Methodologies used in imparting information have more recently been termed *chalk and talk*. The teacher is didactic, engaging in whole class teaching and carrying out practical work when applicable in the different subject areas so that very little is demanded from the student in the way of initiative or independent thought. The student is expected to have a good memory, and be capable of regurgitating what has been learnt. Under such circumstances, many students will be well behaved and try to please, but this is not, some would argue, what education should be about. The traditional teaching approach may have come about at one time due to very large class sizes and therefore as a means of crowd control. The format of traditional teaching was believed to be of benefit to the student as s/he was engaged in a system of education, which was fixed in content and aim, method and discipline.[1]

What is progressive teaching?

How new or modern is progressive teaching? If children are to learn from their own experience, how far back can we date what are today known as modern methodologies? It is interesting to note that one of the earliest references to the modern idea of *learning by doing* can be dated back to the 4th-century B.C. when Plato was quoted as saying:

> Enforced learning will not stay in the mind. So avoid compulsion and let your child's lessons take the form of play.[2]

Francis Bacon, a philosopher and educational reformer of the 17th-century, based his teaching methodologies on observation and deduction, which have a lot in common with the modern methods known as 'discovery' or 'investigative'.[3] His philosophy was that truth is not derived from authority and that knowledge is the fruit of experience.

Another 17th-century teacher, John Amos Comenius, placed a great emphasis on *learning by doing*. A student engaged in activity is an essential ingredient to the whole process of education:

> Craftsmen do not hold their apprentices down to theories; they put them to work without delay so that they may learn to forge metal by forging, to carve by carving, to paint by painting, to leap by leaping. Therefore in schools let the pupils learn to write by writing, to

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[4] Elizabeth Lawrence, *The Origins and Growth of Modern Education*, (London: Penguin, 1970), cited by Rachel Pinder, Why Don't Teachers Teach Like They Used To? (London: Hilary Shipman, 1987), 5

[3] Joan Simon, *Education and Society in Tudor England,* (Cambridge University Press, 1960) cited by Rachel Pinder, Why Don't Teachers Teach Like They Used To? (London: Hilary Shipman, 1987), 5

> speak by speaking, to sing by singing, to reason by reasoning, and so on, so that schools may simply be workshops in which work is done eagerly.[1]

Jean Piaget held Comenius in high regard in the field of developmental psychology and in the whole area of progressive education. In the 18th-century the philosopher Rousseau, noted for his progressive ideas in modern education wrote in his book *Emile* that:

> It is only by movement that we acquire an understanding of things which are separate from ourselves and it is only by our own movement that we acquire the idea of distance.[2]

Taking this a step further Rousseau applied his observations of children's movements to the teaching of mathematics:

> Anything that brings unconstrained movement of the body comes easily to children. There are therefore many ways of interesting them in measuring, noticing and estimating distances. There is a very tall cherry tree; how shall we gather the cherries? Will the ladder in the barn be long enough? There is a very wide stream, how are we going to get across? Will one of the planks in the yard span it from side to side? We would like to fish in the moats of the chateau from our windows; what ought to be the length of our line? I want to make a swing attached to two trees; will a rope 12 feet in length be sufficient for this? I'm told that our room in the next house will be 25 feet square; do you think this will suit? Is it larger than our present room? We are extremely hungry; there are two villages; which one of the two can we reach soonest and so eat? And so on.[3]

Therefore, Emile was not going to solve mathematical problems written down in black and white from textbooks. He was, first and foremost, going to have to work with everyday real life mathematical situations and come to his own mathematical understanding of the problem at hand before moving onto black and white textbook calculations. What is interesting about this approach is that in recent years it has become the modern, up to date view in the teaching of mathematics.

Pestalozzi, who was influenced by Rousseau, was also a believer in the fact that children should be actively involved in education. He was of the opinion that school should suit the needs of the child and not the other way around.[4] It was Herbart who, having visited Pestalozzi and worked with him for a period time at one of his schools, developed Pestallozi's ideas of using psychology in education into a theory of learning in which he described how students deal with acquiring large quantities of information, which may be presented to them too quickly. Such teaching, in the traditional mode can create a sense of fear and as a result of this fear there is a lack of self-confidence within the student. In Herbart's opinion, this is not education. Teaching should be aimed at the level of the student so that the student feels comfortable with the task in hand. The teacher should begin with the concrete knowledge already established and work through to the new concept to be introduced. For Herbart the role of the teacher was of utmost importance. He appreciated that the large class sizes of the time meant that teachers had to be quite strict and rigid in their classroom practice but he did everything he could to

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[1] Ibid, 3, 9

[2] Ibid, 12

[3] Ibid

[4] *Encylopedia Britannica*, CD 2000, 10th ed (U.S.A.: Encyclopedia Britannica Inc., 2000)

try to help teachers realise how students learn and to highlight the disadvantages of such classroom practices at that time.[1]

It would seem that 1828 was the first time the word *progressive* was used in relation to education when Albertine Necker de Saussure wrote and published *Progressive Education*. She was very like Herbart in her thinking and was also critical of the teaching practices of the time, where facts were taught to be learned off by heart and to be examined.[2]

Later in 1875, Francis W. Parker attacked the mechanical teachings of the traditional schools. He put great emphasis on what he termed quality teaching, which included activity, creativity, an understanding of the individual and the development of the personality.

More recently, the educationalist Maria Montessori wrote almost 80 years ago:

> We do not stop to think that the child who does not do, does not know how to do…education is a natural process spontaneously carried out by the human individual, and is acquired not by listening to words but by experiences upon the environment.[3]

Montessori also shared the view of Pestallozi that the environment of the school should adapt to the needs of the child. Her theory of education was based on the fact that the senses of the child are separately trained at the different stages of development using natural materials and methodologies, which arouse the interest of the child at that particular stage. What is important is that the child is allowed to arrive at her/his natural acquisition of the basic skills of everyday living. As a natural progression from this, the child moves on to the academic materials and methodologies to acquire reading, writing, language and arithmetic skills. The whole process of education is not about the quantity of knowledge a child can have rather the quality of that knowledge and the natural desire to learn, grow and develop.[4]

Montessori's ideas were on a par with those of the psychologist Piaget. First-hand experience for the young child, as a result of teaching methodologies which are concrete and active, is fundamental. Having to deal with the abstract or formal mental operations is not within the ability of the child before the age of twelve years.

Jerome Bruner, an American psychologist in the 20th-century, summarised the progressive approach to teaching and learning very clearly in three stages, namely *enactive, iconic* and *symbolic*.[5] *Enactive* is teaching/learning through activity, *iconic* takes us through the senses and summarising things and finally, *symbolic* is the abstract,

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[1] Rachel Pinder, Why Don't Teachers Teach Like They Used To? (London: Hilary Shipman, 1987), 18

[2] Ibid, 20

[3] Paula Polk Lillard, *Montessori-A Modern Approach*, (New York: Schocken Books, 1972) 49

[13] Ibid, 71

[5] Jerome Bruner, Toward a theory of Instruction, cited by Rachel Pinder, Why Don't Teachers Teach Like They Used To? (London: Hilary Shipman, 1987), 88

which is the representation in words or language. Ignoring the first two stages and working solely at the symbolic stage may not be the best approach to teaching any subject unless the learner has a well-developed symbolic system.[1]

All of the progressive educational ideas referred to have such a long history and are not as new or as modern as we would like to think.

Traditional or Progressive Mathematics

Mathematics is a practical subject. However, it is not usually seen in that light. Due to the way it has been taught in the past, it is very often regarded as having topics, which were of absolutely no value to the student in her/his daily life. Yet artists, musicians, writers, builders, cooks to name but a few professions all use mathematics, which they have learned at school, in their working lives, some without even realising it. The problem that arises is that the connection between mathematics and everyday living was not always made in the classroom. Mathematics was considered a highly academic subject, rather than a practical subject, aimed at those students who were intellectual.

Yet, Jean Piaget wrote of mathematics:
In the area of logico-mathematical structures children have real understanding only of what they invent themselves and each time we try to teach them something too quickly, we keep them from reinventing it themselves. Thus there is no reason to try to accelerate this development too much: the time which seems to be wasted in personal investigation is really gained in the construction of methods.[2]

In 1982, the Cockroft Committee reported:

> Practical work is essential throughout the primary years if the primary curriculum is to be developed…it is, though, necessary to realise at the outset that such work requires a considerable amount of time. However, provided that the practical work is properly structured with a wide variety of experience and clear stages of progression, and this is followed up by the teacher by means of questions and discussion, this time is well spent.[3]

Understanding through concrete experience is vital for the young mathematical learner. Written exercises of calculations involving addition, subtraction, multiplication and division where the child has to get the right answer are not conducive to a mathematical understanding. The child needs to play with, feel, and see concrete materials which are different in shape, size and colour, then be able to manipulate them, put them together and take them apart to see what is happening. The use of the word *play* in relation to mathematics teaching and learning may sound somewhat absurd. To some it may even sound like a bad word in relation to education. However, play and mathematics together are in fact constructive, as it is through play that the child can develop mathematically; s/he can try out things and come to her/his own understanding. Understanding allows the child to feel competent. As stated in the Cockroft Report, organisation, guidance and support from the teacher during such play or activity is very important.

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[1] Ibid

[2] Jean Piaget, *The Growth of Logical Thinking* cited from Rachel Pinder, *Why Don't* Teachers Teach Like They Used To? (London: Hilary Shipman, 1987), 76

[3] Cockroft Committee, *Mathematics Counts* cited from Rachel Pinder, *Why Don't Teachers Teach Like They Used To?* (London: Hilary Shipman, 1987), 76

The downside to such activity organisation is the reality of large class sizes and time constraints, as the teacher cannot physically get to every child in the group when needed. What may happen then is that the teacher has to tell the child the answer before s/he has time to work through it for her/himself. No understanding has taken place and the child may be left to feel inadequate which can have long-term damaging effects for future mathematics classes. The *I can't do it* factor may take over.

According to Professor Richard Skemp, there are two types of mathematical understanding namely *relational* and *instrumental*.[1] *Relational* understanding is based on the whole area of practical exploration in order to reach a formula, which can then be applied to different situations. *Instrumental* understanding is based on learning by rule and ensuring that all corresponding rules are taught. Instrumental learning is very much dependent on short-term memory and getting the right answer. In relation to teaching, instrumental understanding would appear to be a result of the traditional style of teaching while relational understanding would appear to be a result of the progressive style. Can instrumental understanding go hand-in-hand with relational understanding? It is suggested that the overall approach to teaching and understanding at primary school level is towards relational understanding, while at secondary level the approach is towards instrumental understanding as long courses had to be covered and good results in examinations were expected from the students. Now however the approach to teaching mathematics at secondary level has just started to shift more towards relational understanding, especially in the last two years.

The TIMMS study of 1995 showed that Junior Certificate students in Ireland rated second place in the world as being able to remember formulae and procedures with no real understanding of application.[2] The report also pointed out that students in Ireland were at the bottom of the list when it came to practical handson mathematical experience and how mathematics is used in the real world.

The Department of Education and Science in Ireland was concerned that the Junior Certificate Mathematics Syllabus did not provide the student with enough practical mathematical experience or the teacher with enough material and guidelines to introduce active methodologies and student participation in the classroom. Therefore, the syllabus was revised fully to accommodate the introduction of active methodologies; to reduce content where necessary and to encourage teachers to make mathematics user-friendlier, more approachable, more applicable in everyday living and more enjoyable for all concerned especially the student.[3]

That is not to say that the traditional approaches to mathematics teaching that have existed should be eliminated altogether. Rather that the revised syllabus would allow mathematics teachers, as a professional body, to be open to change, to be innovative, working with old materials already available and new materials being made available, adding to them with new methodologies and taking mathematics that step further by

[1] Richard Skemp, *Mathematics Teaching* cited from Rachel Pinder, *Why Don't Teachers Teach Like They Used To?* (London: Hilary Shipman, 1987), 82

[2] TIMMS 1995

[3] Department of Education and Science, *Junior Certificate Mathematics Syllabus*, (Government Publications, 2000)

exploring it from different angles with their students and applying mathematics to everyday life where possible.

It is not a question of traditional mathematics versus modern mathematics. Learning mathematics is not just about getting the correct answer to the question or getting the desired or optimum number of correct ticks in a copybook. Learning mathematics is about the student gaining an understanding, a relational understanding. Instrumental understanding of mathematics does not appear to be enough.

Conclusion

Optimisation of learning is the focus of educational psychology. Educational psychologists, some of whom have been considered throughout this paper, are specialists in teaching and learning, in researching teaching methods and teacher training.[1] Optimisation of learning is considered by many to be the result of an effect on teaching style or styles. Yet it appears that given the numerous studies that have taken place, and will continue to take place, in relation to teaching styles, there are no results to confirm that optimisation of learning is linked to just one style in particular.[2] Various educational psychologists over the centuries have promoted the traditional style of teaching and the progressive style of teaching. Certain words have been attached to each of the styles in order to distinguish on from the other. 'Rote-learning', 'teacher-centred', 'exam-centred', 'passive pupils', 'rigid' are words that have become associated with the traditional style of teaching and teacher. Overall it is an approach which has dominated the teaching scene for many years and viewed as bad practice and ineffective. 'Pupil-centred', 'small group', 'task-based learning', 'active pupils', 'creative' and 'teacher-flexibility' conjure up the image of the progressive teacher and teaching style. An approach considered to be somewhat modern, positive, reinforcing and effective. An oversight by many practicing teachers today perhaps, is that the so-called modern approach is not as modern as one would like to think.

Can teaching be traditional and progressive at the same time? Are the two very separate identities or can they be married together in some format? Which of the two, if any, stand on their own as the perfect model? Do the two approaches go hand-in-hand to produce the perfect model or framework of teaching? Traditional teaching styles and progressive teaching styles are two separate entities, neither of which is the perfect model, whether on a stand-alone basis or united. It would appear that there is no such thing as the perfect model. However, for education to take place and to optimise the learning potential of the student, teaching styles can possibly be both traditional and progressive at the same time once they are properly structured, developed and followed through in a meaningful way. As a result, a student should be able to achieve her/his goals given that the teacher has allowed for and accommodated the different levels of understanding within her/his class through demonstration, conversation, sharing and negotiating.

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[1] Rita Atkinson et al, *Hilgard's Introduction to Psychology*, 13th ed., (U.S. A.: Harcourt College, 2000), 25

[2] http://www.ped.gu.se/biorn/phgraph/papers/lomun.pdf Lo Mun Ling, *"Progressive" versus "Traditional" teaching styles – What really matters?* (Hong Kong: University of Hong Kong, 2000) last updated April 2000; last accessed December 2001; 1

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"Put Out into Deep Water and Pay out Your Nets for a Catch": Lessons learned from a pilot study in Mathematics Lesson Study

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This paper reports on a pilot study of a single cycle of Japanese Lesson Study (LS) in which six student teachers participated during 'home' TP in September 2006 at the beginning of their third year of the B Ed course. The goal of the research was to investigate Lesson Study as a process of mathematics teacher development. During the first week of TP students met with me in groups of three to collaboratively prepare a 'research' mathematics lesson. Later in the week, two students taught one 'research' lesson each; one on shape and space to fifth class and one on length to fourth class. These lessons were video-recorded. At a subsequent meeting in week two each group analysed the lesson it had prepared, with a focus on children's responses to the teaching activity. Following the analysis, the remaining two students in each team, individually taught a new version of the 'research' lesson, which was also video recorded and analysed. Like the fishermen in the gospel, these student teachers were amazed at the data about children's learning available to them when they 'put out into the deep'. By concentrating their planning and observation skills on children's words and actions in relation to the material they wished to teach, the link between lesson planning, classroom practice, and improved outcomes for pupils became less problematic.

Introduction

There is consensus among interested parties that mathematics education in Ireland is ripe for improvement. Various remedies have been suggested to complement the initiatives already in place. Among these is the adoption of Japanese Lesson Study as a means of mathematics teacher development potentially suitable to the Irish educational scene (Sloane, 2005). My use of LS as way of learning to teach mathematics grew out of an earlier research project into how Irish student teachers' mathematics subject knowledge influences teaching (Corcoran, 2005a; 2007). In that instance individual students were observed and videotaped teaching a mathematics lesson during spring teaching practice in their second year of the B. Ed programme.[1] The Knowledge Quartet (KQ) was used as an analytic tool for these lessons and as a framework for providing feedback and initiating discussion on lessons observed with the student teachers involved (Rowland, Huckstep and Thwaites, 2005). Indications were that a more collaborative approach to preparation of and reflection on lessons would be valuable and a third year B Ed elective module on Learning to Teach Mathematics Using Lesson Study was conceived and implemented in the academic year 2006-2007.

Theoretical Framework

Lesson study as a form of teacher development is integral to the Japanese education system both for practising and pre-service teachers (Peterson, 2005). Essentially, the LS cycle consists of planning time which includes an investigation of teaching materials, a 'research lesson' taught by one teacher and observed by the group and a reflective post-lesson colloquium. This may or may not be followed by further 'improved' research lessons[2]. Various researchers have articulated different theoretical frameworks for LS. It has been described as *ethnographic enquiry* (Matoba, 2006), as an application of

[1] I am thankful to the Education Department of St Patrick's College who facilitated this research by releasing me from supervisory duties during TP and the Research Committee who partially funded the research.

[2] Second iterations of the research lesson are more typical of U.S. than of Japanese lesson study.

cultural, historical social activity theory (Fang and Lee, 2006) and it is sometimes known as 'learning study' (Ling, Yan and Man, 2005). My stance as researcher and as teacher educator at the formation of a LS group with my student teachers has greatest saliencies with the Situated Cognition Theory of learning, and so I locate the LS intervention in a theory of social practice, which conceptualises learning as legitimate peripheral participation (Lave and Wenger, 1991), and my students and me as a learning community (Jaworski, 2007). The LS cycle offers participants an interface between a *situated, social* and *distributed* activity in a dynamic setting, which is much more than an opportunity to plan lessons collaboratively (Ling, 2006). Rather, "membership in a community of practice translates into an identity as a form of competence" (Wenger, 1998, p. 153). Choosing to participate in the LS pilot project was an act of identity work[1] on the part of each student.

Data consist of field notes from each of the LS meetings, student teachers' lesson plans, a video recording of three research lessons, artefacts produced by the children during the lessons, field notes from a reflection meeting to review the first lesson and plan the second and third lessons which were taught on the same day by the remaining two students. The three students, whom I have called Áine, Bríd and Cáit, each wrote a personal evaluation of the lesson she taught and an evaluation of the LS process, which is included in the data set.

There are multiple perspectives from which analysis of this LS pilot project is possible. Using a cognitive lens, learning might be expected to have occurred for individual students[2] within the LS group context by the incremental acquisition of knowledge of mathematics and knowledge of how to teach mathematics. Such a claim would negate an understanding of knowledge as "cognition observed in everyday practice [which] is distributed – stretched over, not divided among – mind, body, activity and culturally organized settings (which include other actors)" (Lave, 1988, p.1). However, by adopting a situated cognition lens (Boaler, 2002) evidence of learning, which takes place in LS groups is by definition likely to be situated, social and distributed and in such a setting concepts associated with *belonging* to a community of practice such as *mutual engagement*, *joint enterprise* and *shared repertoire* become both *the* marks and measures of *identity development* Further, identity develops through legitimate peripheral participation (Lave and Wenger, 1991) and evidence of *engagement, imagination and alignment* become the yardsticks of success.

Analysis of the Lesson Study cycle

Preliminary findings from the pilot study indicate that there are at least two strands of analysis worth pursuing. First, the dialectical nature of the learning of persons in activity presents a rich tapestry of interactions and interpretations of how that learning occurs. Second, there is a rich harvest of learning about development of mathematics knowledge for teaching to be gleaned from a study of the research lessons. The Knowledge Quartet (KQ) is an excellent tool for this analysis (See Rowland, this volume). For the purposes of this paper, I shall confine my analysis to the foundation

[1] See Wenger, 1998; Boaler 2002; Hodgen, 2003; Sfard and Prusak, 2005 and Mendick, 2006 for various elaborations on "doing" mathematics as building identity.

[2] The term 'student' in this paper refers to student teachers (known in the UK as trainees) while the term 'pupil' refers to the school children. When the student teacher is in charge of a class, she is referred to as 'teacher' to distinguish her from other student teachers observing her.

unit of the KQ and to certain contingency moments in the three lessons taught by one LS group.

Participation as Identity

The LS research group was conceived within but distinct from a number of communities of practice. Its focus is *inquiry into pupils' responses to mathematics lessons* and it is developmental in intent. Analysis can be at a number of levels. At a superficial level it could be claimed that there was full participation for the duration of the project. All students met me in their own personal time for periods of upwards of three hours on two occasions during the home TP. In all cases there was some travel involved and sometimes students borrowed the family car to drive more than twenty miles for a meeting. In each case the student was happy to allow herself to be videotaped and cooperated in assuring school staffs and parents of the value of the research and her part in it.

The first LS meeting introduced the knowledge quartet as an analytical tool for discussing the teaching of mathematics lessons (Rowland et al., 2005). The KQ framework was intended to build community by becoming part of the "shared repertoire of ways of doing things" (Wenger, 1998, pp. 82-84). The potential of LS as a means of bridging the gap between planning and pupil outcomes was discussed and the possibilities for developing teaching inherent in observing students' responses were explored. The group decided to try teaching lessons on shape and space because they considered it extremely challenging to know how to address what they perceived as a 'difficult' mathematics topic, largely because they considered their own understanding of geometry to be deficient. This is not surprising as findings from a recent PISA study indicating that Irish 15 year-olds do not score highly on shape on space items (Cosgrove, Shiel, Sofroniou, Zastrutzki, and Shortt, 2004) were corroborated among a cohort of pre-service teachers (Corcoran, 2005b). Their *identity* as teachers of mathematics generally, and of non-number lessons in particular might be described as *marginal* (Wenger, 1998, p.166) or fragile.

When the three students met with me for our first LS meeting it might be considered that students were enjoying an opportunity to plan a lesson together; engagement levels were high within the group. But the essence of LS is not collaborative planning. Rather its essence is the focus which teachers place on studying pupils' responses to the "showcase[1]" lesson, so that future teaching goals may be met more successfully. A recent report by Department of Education and Science Inspectors posited that "most schools are unclear about the link between whole-school planning, classroom practice, and improved outcomes for pupils" (Government of Ireland, 2006a, p. 82). The lack of clarity about links between planning, practice and pupil outcomes is even more apparent among student teachers (Gov. of I., 2006b, p. 32). A mark of *participation* in the LS group was the manner in which the three participants engaged in planning their research lessons *imaginatively* and in *alignment* with the primary mathematics curriculum (Gov. of I., 1999a, b) and good mathematics teaching. This last measure of participation - *alignment* with current good practice - proved most difficult for the students as the following analysis of individual lessons suggests. Further discussion of participation in LS arising from the lessons observed will follow the KQ analysis.

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[1] The terms 'research' lesson and 'showcase' lesson are interchangeable. Both are approximate translations from the Japanese. Practitioners in Japan favour 'showcase' lesson as nearer in meaning than 'research' lesson, commonly used in the USA.

Foundation Knowledge

Convention dictates that a lesson plan begins with the proposed teaching objectives[1] yet student teachers have little experience of how children will respond to the material they present or to their actions as teacher. It is quite novel for these students to focus on possible children's responses as learning outcomes and could only be achieved by the LS group when the problem perceived by the students as more pressing had been solved i.e., how to work on 2-D shapes in the absence of resources not found in schools. (e.g. geoboards). [2] All were agreed that children need to be given shapes to manipulate if they are to learn to reason about properties of shapes. This is an example of *foundation* knowledge in the KQ, related to the student teachers' beliefs about the teaching of mathematics. Traditional approaches to geometry emphasised memorisation of definitions, whereas modern approaches emphasise "handling and manipulating shapes" (Gov. of I., 1999b, p. 25). This change in approach creates a stumbling block for many student teachers whose confidence in their own ability to reason mathematically is curtailed by their early experiences of learning mathematics (Corcoran, 2005b). At the planning meeting, Áine admitted to having difficulty with shapes and this sharing of her identity as mathematics teacher who is unsure of her ground, was a crucial step in engaging the group in wanting to address mathematics teaching issues differently. We worked as a group, for forty minutes, on the notion of *class inclusion* of shapes and how children might be encouraged to think of sets of shapes in terms of how they are the same and how they are different. This knowledge belongs in the *foundation* unit of the KQ as *overt subject knowledge,* and may be that to which Áine was referring. But teachers also need to know about how geometry is learned. Student teachers are introduced during their mathematics education courses to the van Hiele model of geometric thought (Crowley, 1987). The first three van Hiele levels[3] are appropriate to primary school children and children in fourth, fifth and sixth classes[4] might be expected to be working at the second and third level. Five sequential phases of learning: a) inquiry, b) directed orientation, c) explication, d) free orientation and e) integration arise from the van Hiele model and have implications for planning geometry lessons (ibid. pp.5-6). It would be difficult for a teacher to plan for the more advanced phases of learning at levels 1 and 2 if she herself were still at level 0. Each of the three lessons reported here deal with issues of equivalence-how things can be different yet be the same (Level 1: Analysis) - in different, but also quite similar ways.

Bríd's Research Lesson

Bríd's teaching objectives for a co-educational fifth class of twenty seven children were that the children would be enabled to –
draw a variety of quadrilaterals,
discuss some properties of quadrilaterals and
identify a quadrilateral from its properties.

Her *use of terminology* is suspect here, since she seems to be confusing the group of shapes known as quadrilaterals – critical attribute: "having or formed by four sides"- with special cases of quadrilaterals having further delineating properties.

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[1] Much of the Primary Mathematics Curriculum is set out as teaching objectives for each class group.

[2] Use of cut-out paper shapes is advised. Dot paper is a good alternative to geoboards.

[3] Level 0 (basic): Visualisation; Level 1: Analysis; Level 2: Informal Deduction

[4] Children in 4th, 5th & 6th classes are typically aged 9-10years, 10 - 11 years, & 11-12 years respectively.

Descriptive Synopsis

Bríd issued a challenge. "I want you to draw four sided shapes. I can draw six. Can you draw more than six?" Individual children were invited to the board to draw their different four sided shapes. Bríd established that all the shapes belonged to the set of shapes known as quadrilaterals. The class was then asked to "think about and discuss in pairs, then write how you would describe to a blind person what a quadrilateral is". In the next sentence she changed the task from one of describing the critical attributes of a quadrilateral to that of describing noncritical attributes i.e., properties of particular quadrilaterals. "Choose one of those shapes and describe it to a blind person". Children were then invited to choose one shape on the board and while working in pairs, to describe it in writing by listing its properties. The lesson was summarised by inviting children to name properties of quadrilaterals, which Bríd recorded on a poster.

Analysis of Research Lesson One

Bríd's lesson offers many instances of each of the elements of the KQ and immediately after the lesson I mentioned a particular example of the *foundation* unit of the quartet. Among the contributory codes to this unit is *identifying errors.* Having elicited from the class the similar properties of two related shapes a child was asked to distinguish between a square and a rectangle.

> Megan: In a square they're all equal and in a rectangle the top and bottom are longer.
> Bríd affirmed: Well done! Very good. Did you hear what Megan said? She figured out that there were four lines. There was a top one and a bottom one and they were parallel. The two side lines are parallel.
> (These words are accompanied by making an imaginary shape with her fore arms).
> Will it be the same for a square? In a rectangle the top line and the bottom lines are longer, in a square all the sides are the same.

Bríd considered she dealt well with children's talk and was unaware of how she reinforced "inflexible images" (Zevenbergen, Dole and Wright, 2004, p. 311) based on length of side and orientation of shape by her words and gestures. She had focussed on the inquiry/information or first phase of learning (Crowley, 1987), while unaware of the need for complete concept images on the part of the teacher (Hershkowitz, Bruckheimer and Vinner, 1987), in order to help children to build concept images successfully. She later reflected that:

> I wasn't sure how my lesson on 2-D shapes would go and so before the video session I had to do a lot of thinking and planning around all the possibilities that could arise and this was of great benefit to me during the actual lesson as I was able to deal better with the children's responses. Having watched the DVD, I saw areas that I could have dealt with during the lesson that I will now plan for in the future…It was of real benefit to discuss the lesson with others. (Written reflection on LS process)

Bríd's planning was imaginative in her use of the blackboard and in how she organised the children to work in pairs. She was open and encouraging to children's ideas and garnered a great deal of information as to which level of geometric thought the children had achieved, while unsure which direction their thinking should take next.

Discussion of Lesson One

Theoretical underpinning for the pedagogy of shape and space, based on van Hiele's levels of geometric thought, highlights the role of teaching, rather than maturation in moving children from one level to another. It advocates that the teacher provide a wide variety of examples of shapes in different sizes and orientations so that children may begin to reason logically and make deductions about shapes within classes. During the LS session to reflect on Bríd's lesson, we discussed the DVD of the lesson using the KQ framework. It was a source of regret that the full LS team had not been present in the classroom to observe the extent of children's efforts to articulate understandings of 2-D and 3-D shapes, using language and gesture at the beginning of the lesson. (See Appendix 1 for transcript). While analysis of video-tape has potential for teachers to learn about how children learn mathematics (Andrews, Hatch and Sayers, 2005) in this instance, I was convinced of the value of first hand observation of children by fellow teachers, as is the norm in Japanese LS. There are at least two reasons for this: first, the scope of a single video lens is limited by the 'eye' of the person holding it, (Roschelle, 2000) and second, the collation of multiple accounts of observers noticing different aspects of the teaching/learning episode will help to build a more complex and more useful picture of learning in a mathematics classroom. If planning is to be more successfully linked to classroom outcomes such a complex picture should constitute part of a teacher's *foundation* knowledge, which includes knowledge of how mathematics is learned.

Directed Orientation

During the post-lesson reflective meeting the LS group discussed 'directed orientation' activities arising from the DVD of Bríd's lesson. Signs of attempts to design and implement these types of activities were evident in the second and third lesson in the LS cycle. Designing particular activities to aid analysis of shapes belongs also to the *foundation* unit of the KQ. It can be linked to many of the contributory codes, *adherence to text books, awareness of purpose, overt subject knowledge, theoretical underpinning* and *use of terminology.* The KQ is particularly valuable in the LS setting because an episode labelled with any of these codes can lead to discussion of the aims of teaching and when pursued in conjunction with observation of children's responses during a lesson becomes a powerful site for learning.

Áine's Research Lesson

Áine's lesson for fourth class in a middle-class area of a neighbouring town was similar in many respects to Bríd's lesson. The influence of the post lesson reflection meeting was in evidence in subtle ways. Áine avoided any mention of 3-D shapes. Her teaching objectives were that children would be enabled to:

identify octagons in the environment,

identify properties of an octagon,

complete a property chart on the 2-D shapes studied so far.

Descriptive Synopsis

Children were invited to draw a quadrilateral. Individual children were invited to show their shapes to the class while teacher initiated a question and answer session to revise properties of shapes already taught. The only four sided figures she acknowledged were rectangles, squares, parallelograms or rhombuses. Lists of properties were revised but there was little sense of discovery of similarities and differences between shapes. A parallelogram was defined as a rectangle stretched or pushed out of shape. Similarly a rhombus was described as a square pushed out of shape and children were shown that opposite angles were equal, but not that a square, rectangle and rhombus all meet the requirements for a parallelogram. Áine asked the class to identify shapes with five or more sides. She reminded them of two kinds of pentagon, a regular, with all sides equal, which she identified on a chart and an irregular pentagon, which was not described. She then distributed paper circles and demonstrated how they were to be folded into eighths then flattened out and made into an octagon by joining the points where the fold lines met the circumference. Children were then asked to fold back these lines to make a shape like the face of the clock (octagonal). Áine drew lines and helped children fold as needed.

A *contingency* moment arose when two children found their shapes were different from the others.

Áine:	Look at your shape. How many sides has it got?
Children:	One, two….Eight (*in chorus)*
Áine:	Eight sides. Ok look at it. How many corners has it got? Lámha suas.
Child:	Mine has six [sides]. …
Child:	Miss I've only six too.
Áine:	Ok. It's 'cos ye folded them different. What's a shape with six sides called anyway?
Child:	I know, a hexagon.
Áine:	A hexagon. Good man Sean. Well done. Did everyone hear that? A shape with six sides is called a hexagon. Ok. Here, Cian, would you fold Mark's for him so that he has eight sides please.

No investigation was made as to how the different folding had resulted in a different shape. It emerged later in the lesson that Áine did not consider that the above exchange constituted 'doing' the hexagon, so when "shapes we know" were being listed, Áine refused to count the hexagon! The properties of the octagon were then rehearsed.

Áine: How many sides? How many corners? Any right angles? How many pairs of parallel lines?

She invited Joseph to the board to identify a "just one pair of parallel lines" on the octagon she had drawn there and he pointed first to a diagonal.

Áine prompted: On the outline of the shape…which two lines will never meet on it?

Joseph pointed to two points at either end of a single line and said: "Those two". Áine gestured by swinging her arms in tandem as she said:

Yeh, but parallel lines are two lines that'll never meet.

There were no further suggestions from Joseph. So she then asked Mary to show him a pair of parallel lines. Mary pointed to a pair of lines, opposite each other, but not parallel.

Áine: Hang on now for a second. If this line goes on and on up there and this one goes on and on up there will they meet?

Mary: Yeh.

Áine: So they're not parallel. So it's not this one.

Mary then pointed to the adjoining line which was parallel to the first one she had identified.

This *contingency* moment could offer a powerful insight for LS participants. Brisk, teacher-led, rehearsal of lists of properties of shapes cannot be taken as an indication of children's understanding of the concepts involved. Bobis, Mulligan, Lowrie and Taplin. (2004, p.106) warn teachers against using a "restricted set of prototypes" such that children learn how squares, rectangles, rhombuses and parallelograms are all different to the exclusion of their commonalities. Áine appeared to conduct much of the lesson out of such a restricted set of prototypes, where emphasis on the language of properties of shapes took precedence over experience and understanding leading to concept formation.

The value of contingency opportunities

There had been evidence of contingency teaching also in an earlier exchange:

Áine: Are there any sets of parallel lines [in an octagon]?
Children: Yes
Áine: Yes. How many pairs of parallel lines are there? Count them and see. Now this one is a tricky one. Lámha suas…if you get it. How many pairs of parallel lines? Susan?
Susan: Eight.
Áine: How many **pairs** of parallel lines though? It has eight sides altogether. How many's in a pair?
Child: Four
Áine: How many's in a pair?
Child: Oh. Two
Áine: Two. Ok. So if there was…if you had eight pairs how many sides would it need to have?
Child: Two.
Áine: Listen right. If it had eight **pairs** of parallel lines how many sides would it need to have? Does anybody know? If it had eight pairs of parallel lines how many sides would it need, Jane ?
Jane: Sixteen.
Áine: Good girl. 'Cos 8 multiplied by 2 is sixteen. Ok? How many pairs of parallel lines are there actually on the octagon?

Áine deviated from her agenda to pursue the number of "pairs of parallel lines" but it is worth querying the relevance of this line of questioning where close observation of Joseph's response later in the lesson indicates that he hadn't yet integrated for himself the relationship between parallel lines. Jane may not have developed that concept either despite her ability to give teacher the answer she wanted, eventually. These two episodes can both be coded as *contingency* and their value in LS terms stems from the opportunity for participants, teacher of the lesson and her student colleagues observing, to re-examine the teaching objectives for the lesson and reflect on how they might be more specifically framed and implemented to achieve the 'directed orientation' she sought to facilitate for her pupils.

Cáit's Research Lesson

Cáit's lesson for sixth class was entirely about tangrams. Her teaching objectives were that student be enabled to
create their own tangram pieces
explore how the area of polygons are related
come to a basic understanding of area, using tangrams, and without using formulae.

This may well have been a real attempt to provide children with actual 2-D shapes to handle. She distributed sheets bearing a photocopy of a Chinese tangram. Children were asked to identify the names of shapes and cut them out. Her teaching approach had a phase 1 (inquiry/information) design to encourage analysis and informal deduction, but was based on a more primitive level of geometric thought where teacher herself dealt in "a restricted set of prototypes" (Bobis et al., 2004):

Cáit: Ok, Mark Brady, can you tell me one [of the shapes that you see in front of you]?
Mark: ehm, two triangles.
Cáit: You see two triangles. Declan, can you tell me another one?
Declan: A rhombus

Cáit:	A rrr...rhombus? Is it a rhombus?...Oh, that's a diamond yeh, or a rhombus yeh.
Child:	No. There's four triangles.
Cáit:	Yeh, there's four triangles…
Child:	Six triangles…cause look…
Cáit:	There is a rhombus.
Child:	Look. One, two three, four, there's five triangles.
Cáit:	There's five triangles. There are…What other shape do you see Mary?
Mary:	A parallelogram.
Cáit:	A parallogram. Very good. And there's a…
Child:	Square.
Cáit:	It's not a rhombus, it's a square or a diamond turned on its side. Good boy.

Lesson Analysis

Cáit knew there were seven shapes in the tangram but her approach to children's listing of the shapes they see is open and accepting of all offers. She hesitates but accepts the square being called a rhombus, which indicates she has some sense of class inclusion, but her calling the rhombus/square "a diamond" and later "a diamond turned on its side" points to the most basic level of geometric thought on the part of the teacher. Mary sees a parallelogram, for which she is commended and then another child sees "a square", which Cáit assures the class is "not a rhombus" and we have here an opportunity to explore commonalities and differences between the two quadrilaterals. But this conversation didn't take place. Cáit had children cut out paper versions of the Chinese puzzle, try to assemble the pieces, then after one boy succeeded invited them to take away the pieces and try at home. Her comment on his success indicates that she sees mathematics as a difficult and competitive enterprise.

Cáit: Anyone get it? It's tricky, isn't it?
You don't know how to do it? Just keep on trying…
(And later) Oh, what's that? We have a winner. Jack has made a square.

Cáit then, used the tangrams for another purpose. By a series of closed questions she led the whole class to establish the area of the individual pieces and the puzzle as a whole, by using the square piece as "one square unit". When this had been recorded in copybooks exactly as Cáit directed, the children were invited to try to make other shapes with the tangram pieces according to photocopied sheets she distributed. Questions for discussion arise around Cáit's beliefs about how mathematics is learned, *foundation knowledge,* as evidenced by the following episode:

Cáit:	Tracing your small square into your copybook and beside it, I want you to write "one square unit". One square unit. What do you think one square unit is…going to be? What did you just draw in?
Child:	A square.
Cáit:	A square…so it's going to be the…? What's always in square units? In maths what do you always have in square units?
Child:	Area

Not alone is mathematics sometimes "tricky" and competitive but it is also often teacher- centred and "always" rule-bound. The lesson gave children opportunities to manipulate shapes in what might be very loosely called a 'problem-solving' situation and the tasks certainly engaged them. Cáit occasionally prompted a child to be more exploratory:

Cáit: You can flip them over as well, don't forget.

and many of them went home keen to reassemble the tangram square, but if Cáit planned this lesson as "directed orientation" to further children's development of analysis of shapes, there was little evidence of enhanced development of spatial thinking in the discourse of the lesson.
The KQ was useful in identifying *foundation* and *contingency* moments in Cáit's teaching and her *awareness of purpose* and *choice of representation* from the **foundation** and **transformation** elements respectively are matters for LS discussion. Cáit chose to link shape and space to measurement which was a good idea but differed from the original LS group plan. A late night text from Cáit on the previous evening to the lesson observed may be an insight into her own motivations and anxieties:

> Sorry it's so late but would it be ok if my lesson tomorrow is *just*[1] all working with tangrams? Making the different shapes and that ... 21:33 (21/09/06)

From this I conclude that Cáit was aware that she was deviating from the LS agenda and was anxious about doing so. It highlights choices and constraints facing participants in LS groups. It is possible that participants may not be aware of a need to change or improve their own practice. Neither Áine nor Cáit opted to take the Learning to Teach Mathematics Using Lesson Study elective module, in the following semester. Both had unusually strong mathematics qualifications on entry to the course. Perhaps they weren't ready at that point to confront or risk changing an already comfortable identity as people who are 'good at maths'. Trying to research identity like this points to just how much of a "slippery reality" one is trying to capture in interpreting data of this nature and also raises the question for me of just which stories to tell and the ethics of "storying" individual students (Povey and Angier, 2006) from a series of isolated incidents gleaned about a complex persona which is itself one of many selves (Mendick, 2006). Identity is multifaceted and always in the making.

Discussion of LS

There was no opportunity to hold a second post-lesson reflection meeting, so lessons two and three were never discussed by the group. This is a big limitation to the pilot study. A learning outcome for me from this research is that LS needs lots of dedicated time from all the participants which is scheduled at the outset (Lewis, 2002). Áine's evaluation of the process included personal learning about her own strengths and weaknesses and a regret that it hadn't taken place in Dublin during spring TP which she perceived as having "meant greater access to resources etc". This could indicate that she was more interested in the performance of a lesson than the process of LS. Bríd would like to have seen the 'same' lesson adapted and re-taught. This could indicate that she too was interested in "polishing the stone" (Lewis, Perry and Murata, 2006). Cáit didn't express an opinion. While she travelled a considerable distance to participate in the two meetings and was an active participant each time, could be intimated that she engaged with LS in a nominally rather than dynamically peripheral sense. Wenger's explication of legitimate peripheral participation as being one of a number of trajectory (1998, pp.153-155) implies movement and development with participation as engagement and learning or engagement as non-participation. Cáit may have considered the price of mutual engagement too high to warrant further involvement. Perhaps her exposure to

[1] Italics are mine. I interpret this to mean that Cáit believed that she should prepare a more appropriate or 'aligned' lesson but for some reason felt unable to do so.

the LS cycle was too short, since time constraints did not allow for the group reflection on her lesson.

Measures of Participation

Three indicators of degrees of participation in a community of practice are offered by Wenger (1998): *engagement, imagination* and *alignment*. These marks of shared practice contribute to *meaning making* by participants in the community and are a sign of *learning* among them and a source of *identity* for them, all of which arises from participation. Joining the LS pilot research community was an act of engagement. The different ways in which each student sought to facilitate "hands-on" experiences of learning about properties of 2-D shapes are all examples of imaginative participation. Buríd used dot paper. Áine was quick to see the potential of individual white boards. Cáit introduced cut-out tangrams. In the LS context, use of *imagination*, to design creative mathematics lessons, to observe and respond to pupils' cues in lessons and to assimilate and accommodate the KQ framework in order to give, receive and respond to constructive analysis of teaching are all relevant to identity work. Jaworski (2007) qualifies Wenger's *alignment* as *critical*. This is a weight bearing word, by which she means "looking critically at that practice while aligning with it" (p. 6). In the present arena which is the Irish mathematics curriculum *critical alignment* is particularly challenging. So is the adaptation of teaching ideas from textbooks. It is in the *alignment* of their lesson plans and pedagogical practices with what they perceive to be good mathematics teaching that the student teachers in this study are most challenged. Each sought to address the second and third of van Hiele's levels of geometric thought by planning for discovery/recognition of properties of 2-D shapes and discussing these with the class. However it is by the conversion of these worthy aspirations into carefully sequenced and fruitful pedagogical activities that children will be enabled to build on previous experiences and expand and clarify their emerging views about the structures of shapes they have observed. Such ability to facilitate phase two learning -"directed orientation", followed deliberately by phases three, four and five (Crowley, 1987, p.5) – requires secure foundation knowledge of geometry on the part of the teacher. Individual reflection on elements of the foundation knowledge may well enhance learning on the part of the student teacher, but the opportunity to observe children's actions and reactions to different episodes in the lesson, to explore their use or non-use of particular affordances or prompts provided by the teacher, to test teaching objectives against learning outcomes instead of the other way round, to discuss these with equally committed colleagues in a structured environment using a "shared repertoire" has potential over time to achieve alignment with good mathematics teaching.

Conclusion

Engagement with attempts to mediate the curriculum in imaginative ways can only result in the generation of other and potentially better teaching and learning episodes. This claim is of necessity a mere aspiration until further research is conducted. A tentative hope is expressed that further investigation of LS cycles will result in sufficient commonalities in findings so as to lead to an expansion and hence greater understanding of what it means to learn to teach mathematics in primary schools. Findings from this very small study are that participation in LS provides an opportunity to student teachers (and potentially to all teachers) to build mathematics subject knowledge by building identity as LS practitioners and to learn about the effects of alternative modes of teaching on pupils because of participation in such a community of practice. This form of professional development is not geared towards devising the "best lessons" since there can never be a "best lesson" but towards building a community of practice, which through the engaged and imaginative participation of its members seeks a meaningful understanding of the primary mathematics curriculum. It can be achieved by collaborative participation in LS, by studying children during mathematics lessons, by optimising the use of the available supporting documents and organising classrooms to maximise the development of mathematics process skills

Returning to the title of the paper:

> And when they had done this, they netted such a huge number of fish that their nets began to tear, so they signalled to their companions in the other boat to come and help them; when they came, they filled two boats to sinking point (Luke 5:v.8).

Such was the magnitude of the catch which resulted from fishing in deep waters that Peter needed help to bring it ashore. The metaphor of a bountiful yield, best harvested by a group of people suggests the riches in learning about teaching available to a community of practitioners mutually engaged in the activity of Lesson Study.

Appendix 1

This episode is an example of Bríd's capacity for *contingency* because she deviated from her quadrilateral agenda to pursue the meaning of the term "2-D". I offer it as an example of fruitful material for LS.

Bríd:	Yesterday… We were grouping shapes. What sort of shapes were we grouping? Anyone remember? There was a special name on them. Matt?
Matt:	2-D shapes.
Bríd:	2-D shapes. Very good. Well done. (*Writes "2-D shapes" as a heading on the black board)* Can anybody explain what '2-D' stands for? Matt explained it for us yesterday. Was anyone listening to him? Eoin?
Eoin:	Two dimensional shapes.
Bríd:	2-dimensional shapes… I still don't know what that means, though. What does 'dimensional' mean? Frank?
Frank:	It means a shape…like the back of the shape would go like…It's …it's 3-D
Bríd:	If you could see the back of it, it becomes…?
Frank:	Dimensional
Bríd :	It becomes…what did you say first?
Frank:	3-D.
Bríd:	3-D. So what's the difference between 3-D and 2-D? John?
John:	If the… if the…sort...it goes back…its 3-D and it go forward.
Bríd:	Right… You're a little bit confused. Can anyone clear it up for him, a little bit better? Vincent, you try?
Vincent: (*indistinct)*...2-D you see the front and 3-D you can see the back as well.	
Bríd :	Ok. So 2-D you see the front of it. 3-D you see the back of it. There's abetter way to describe it. Joe, I'd say you have it.
Joe:	On 2-D the shapes are flat and on 3-D they're … bulky.
Bríd:	Bulky…?
Joe:	They're…(gesturing forward with hands)
Bríd :	I know what you're saying. Have you a different word for it?
Joe:	They're full.
Bríd:	They're full. Ok. I get ya… Leo, I know you know it. Will you tell us? Are you listening to him now 'cos he knows it very well?
Leo:	2-D are flat on the page and 3-D look like they stand out on the page.
Bríd :	And they can stand out on the stage…and when you're looking at 2-D shapes what are you only looking at? (*She makes a vertical line in the air in front of her…)*

Child:	The front.
Bríd:	You're looking at the front. And you're looking at..? (She repeats the vertical line in the air and holds her finger poised at the top to draw an imaginary horizontal line)
Child:	Up…
Bríd:	If you're looking at 2-D shapes your looking at up and …what are you looking at…?
Child:	Across
Bríd:	Up and across. Ok? So you're looking up and across. And if you're looking at 3-D shapes you're looking at the back as well. Ok? Anyway…today we're going to…

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Exploring Pedagogy in Early Years Mathematics for Children in the First Year of Primary School in Ireland: A Survey of Teachers' Views

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There are now established pedagogical norms to guide early childhood teachers in the teaching of early childhood mathematics (e.g. Clements, Sarama & DiBiase, 2004; Gifford, 2004; Australian Association of Mathematics Teachers, 2006). With regard to the development and learning of mathematics, and numeracy in particular, it is now generally accepted that certain experiences and practices are necessary to ensure that all children have access to what Perry and Dockett (2004) term powerful mathematical ideas. However, we know very little about teachers' views in relation to the value and use of particular pedagogical strategies in developing young children's mathematical skills, understandings and attitudes. Information about JI teachers (n=266) opinions, beliefs and self-reported pedagogy related to the teaching of mathematics in the first year of school in Ireland was gathered by questionnaire survey distributed to a random sample of schools in the Republic of Ireland. In this paper findings related to aspects of teachers' views on a range of pedagogical strategies are discussed and implications are drawn for pre-service and in-service education.

Background

Literature from the early childhood field defines a range of general pedagogical strategies that are considered particularly appropriate and effective in promoting young children's learning in early education settings. With regard to the development and learning of mathematics, and numeracy in particular, it is now generally accepted that certain experiences and practices are necessary to ensure that all children have access to what Perry and Dockett (in press, p.12) term 'powerful mathematical ideas'. They argue that each of these ideas is begun and needs to be nurtured in the early years. The ideas referred to by these authors comprise of mathematization, connections, argumentation, number sense and mental computation, algebraic reasoning, spatial and geometrical thinking and data and probability sense. In a discussion of the teaching of mathematics to 3-, 4- and 5- year old children, Ginsburg et al. (2005, p. 176) describe young children as *eager* mathematicians. They make this claim based on evidence that young children deal with mathematical ideas in everyday play (even if their thought is qualitatively different to that of adults), they are curious about the subject and they can learn complex mathematics when it is taught. Similarly, Perry and Dockett (in press, p. 53) argue that young children are capable of dealing with great complexity in their mathematical learning. They also argue that teachers are capable of dealing with great complexity in their facilitation of children's learning. They suggest that teachers can harmoniously link these complexities

> if [teachers] build relationships with the children in their class, ascertain what mathematics they know and how they can use it to solve realistic problems. Using this and the children's interests as a basis, teachers can plan challenging and complex experiences for their young children, with the aim of helping them reach their potential in mathematics learning.

However, we know very little about teachers' views in relation to the value and use of particular pedagogical strategies in developing young children's mathematical ideas, skills, understandings and attitudes (Ginsburg & Goldberg, 2004; Ginsburg et al., 2005). The purpose of the research reported here was to gather information about JI teachers' opinions, beliefs and self-reported pedagogy as it relates to the teaching of mathematics in the first year of school in Ireland. This paper reports on teachers' attitudes to a range of pedagogical strategies generally considered appropriate in relation to early childhood mathematics.

Introduction

In the United States the National Association for the Education of Young Children (NAEYC) and The National Council of Teaching in Mathematics (NCTM) have spoken with one voice in respect to the assertion that 'high-quality, challenging and accessible mathematics education for 3- to 6-year-old children is a vital foundation for future mathematics learning' (2002, p.1).

In the Irish context the mathematics education of children aged from four years in primary schools is delineated in The *Primary School Curriculum: Mathematics* (Government of Ireland, 1999a). This is a detailed statement of content in the form of skills and concepts to be acquired and learning objectives to be achieved. In common with the curriculum statements of many other countries the learning intentions for young children at school are organised around the usual strands that include Number, Shape and Space, Data, Algebra and Measure and detailed learning objectives are listed for each of these. The *Primary School Curriculum: Mathematics: Teacher Guidelines* (Government of Ireland, 1999b) accompanies the curriculum statement and is described as '... an aid and resource for teachers and schools as they encounter the curriculum and begin to implement its recommendations.' (p. 66). The guidelines seek to explore a wide range of approaches and methodologies that develop the new emphases and give expression to new thinking on teaching and learning. They also explicate the content of the curriculum. In addition, they include what are described as detailed exemplars and sample lessons that demonstrate the newer approaches. In *The Primary School Curriculum: Introduction* (Government of Ireland, 1999c), it is claimed that the curriculum incorporates current educational thinking and the most innovative and effective pedagogical practice. It was heralded as a curriculum that set out clearly not only *what* the child should learn but *how* the child should learn most effectively (National Council for Curriculum and Assessment (NCCA), 2000). For instance, the curriculum emphasises the importance of the children's own experiences as key reference points in learning mathematics. It advocates that children be encouraged to use a range of forms of recording mathematical activity including, as and when appropriate, the traditional written algorithms. The key role of discussion is emphasised and the necessity for children to work with materials individually and in small groups is outlined.

It was envisaged that the implementation of the revised curriculum was a process that would take a number of years to complete. It was intended that teachers would be involved in planning for implementation at school level and in professional development programmes on each of the areas of learning, including mathematics. To date, nine years after the introduction of the curriculum to schools, teachers have received just two days in-service development in relation to each of the curriculum areas, including mathematics. In addition, schools have also been involved in some whole school planning, sometimes with the assistance of an advisor from the Primary Curriculum Support Programme. However, these advisors were not selected for their particular expertise in mathematics or for their expertise in early years pedagogy, but were generally teachers who had obtained some additional qualifications and who had some experience in giving voluntary in-service courses of short duration to teachers.

Ideas about pedagogy arising from socio-cultural theory

Theoretical developments of recent years in our understandings of learning have influenced what eminent theorists in early childhood education refer to as 'a theoretical seachange' (Anning et al., 2004). This change saw '… individualistic developmental explanations of learning and development replaced by theories that foreground the cultural and socially constructed nature of learning.' (p. 1). Anning et al. point out that the professional language of early childhood education is now replete with terminology and understandings from socio-cultural perspectives and these have now come to form part of its knowledge base. The roots of socio-cultural perspectives are to be found in the writings of the Russian psychologist Vygotsky (1978, 1986) who argued that children are cultural beings, living in particular communities, at particular times and living and constructing a particular history. Socio-cultural perspectives take into account the social, historical and cultural dimensions of everyday activities and seek to better understand children taking each of these dimensions into account.

Leading theorists in the field have encouraged and urged early childhood educators in general towards a socio-cultural approach in their practices (e.g. Anning et al., 2004). Such an approach conceives of effective practice as practice that is built on the construct of the learner as active, and as an equal partner in any transaction. In a socio-cultural approach the learner is foregrounded and adult and child learners are seen as situated in particular social, cultural and historical contexts. Learning is constrained/limited by the beliefs, artefacts and practices of the particular context in which learning is taking place. It is marked by a pro-active pedagogical approach in which the teacher promotes learning through active engagement with the learner; interactions that occur between learners are seen as critically important for learning; knowledge is understood to be co-constructed between learners; and the context in which learning is taking place is central. The relationships that mediate learning are seen as an important focus for evaluation of quality, and collaboration between the child and peers is valued as well as that which occurs between child and adults. Dialogue is important from a socio-cultural perspective and conversations with, and between, children are viewed as occurring in joint activity contexts that promote dialogic enquiry and knowledge building. Thus the role of the teacher is seen as central since it is the teacher who enables the learning to take place by pro-actively engaging with the learner, the curriculum and the learning context.

Recommendations on pedagogy in early years mathematics

It has been observed that we know almost nothing about the early teaching of mathematics and science (Ginsberg & Goldberg, 2004). Gifford (2004) provides one possible explanation with her analysis that early childhood mathematics research has focused extensively on children's competence, but not on pedagogy. The Committee on Early Childhood Pedagogy for the United States Research Council described pedagogy as referring broadly to '…the deliberate process of cultivating development.' (Bowman et al., 2001, p. 182) From the committee's perspective, pedagogy has three basic components; (1) curriculum, or the content of what is taught; (2) the methodology or the way in which the teaching is done; and (3) techniques for socialising children in the repertoire of cognitive and affective skills required for successful functioning in society that education is designed to promote. The report describes teaching strategies or methods used in implementing the curriculum as the arranged interactions of people and materials planned and used by teachers. It adopts Siraj-Blatchford's (1998) description of these strategies as inclusive of the teacher's role, teaching styles and instructional techniques.

In seeking to get to the core of the socio-cultural perspective, Bowman et al.'s (2001) analysis is that a pedagogy coherent with this view of learning is one that it is not ultimately about free play, instruction or placing the child in a carefully chosen stimulating environment, but rather they assert that the critical factor is a high degree of direct adult engagement and guidance in the process of construction of learning. They further suggest that there is no one best pedagogy for early childhood but that many teaching strategies work. More specifically,

> Both direct instruction and child –initiated instruction, teaching through play, teaching through structured activity and engagement with older peers and with computers are effective pedagogical devices. (p. 231)

Sensitivity to individual children's current competence is recognised by them as a key factor in pedagogy in early childhood. Building on this implication of an interactive pedagogy, Ginsburg et al. (2005, p. 176) characterise the adults role in early childhood mathematics education for 3-, 4- and 5-year old children as one of providing what they term 'strong adult guidance'. However, they also caution that this involves more than free play or a push-down curriculum and they are clear that what is required is that teachers change the way they teach to ensure that the pedagogy used is appropriate for this age-group. Their vision for early childhood mathematics education involves three elements; the guidance of the adult, the introduction of challenging mathematics and the development of children's natural interest in mathematics. They suggest that one of the fundamental requirements for teachers must be the ability to develop appropriate pedagogy for young children.

In England, the authors of a major longitudinal study of effective pedagogy in early childhood (Siraj-Blatchford et al., 2002) suggest that specifying pedagogy for the early childhood field may be more important than specifying curriculum. Children have already formed a view of mathematics when they enter school and they bring their knowledge and understandings with them into school (Dunphy, 2006). Teachers must take this experience and learning into account but this can be challenging in terms of pedagogy. In this respect, Aubrey (2003) described the demands on teachers of the Foundation Stage Curriculum in England (the play-based curriculum for 3-5 year old children) and she characterised the challenges to teachers in respect of early mathematics pedagogy as follows

> [it] lies in knowing how to plan and structure the curriculum to take account of and extent this rich knowledge by close attention to the balance of whole-class, group and individual teaching, as well as child-initiated adult-directed or adult-supported activity…a wide range of teaching strategies will be required to motivate, support and extend appropriately. (p. 50)

Recognising the need for pedagogical guidance for early childhood mathematics teaching, in the United States a range of experts including mathematicians, mathematics educators, researchers, curriculum developers, teachers and policy makers came together to agree the Standards for Early Childhood Mathematics Education (Clements, Sarama & DiBiase, 2004). Seventeen research-based standards were developed, eight of which pertain directly to pedagogy. In essence these recommendations present a vision for practice in relation to teaching mathematics to children in the age range 2 to 8 years. Specifically, these eight focus on the following key issues

- The range and type of learning experiences (Recommendation 6 and 7)
- Teaching techniques and strategies (Recommendation 8)
- The role of various types of technology (Recommendation 9)
- The child's perspective (Recommendation 10)

- The development of conceptual knowledge alongside the development of skills (Recommendation 11)
- Appropriate integrated and formative assessment (Recommendation 12)

In Australia too, there have been substantial efforts to guide teachers in relation to pedagogy. The Australian Association of Mathematics Teachers and Early Childhood Australia have recently issued a joint statement regarding pedagogic recommendations to early childhood educators in respect of mathematics teaching (Australian Association of Mathematics Teachers & Early Childhood Australia, 2006). This joint position paper specifies sixteen recommendations which deal with the range of aspects of pedagogy including interactions and communication, planning, resources, assessment, building on children's experiences, key learning and the role of language.

In seems, then, that in a number of developed countries there has been a concern to ensure that early childhood teachers are fully aware of issues regarding desirable pedagogy for early childhood mathematics. We have seen above how this concern has resulted in specific instances in the issuing of strong recommendations for pedagogy that are based on research and expert practice.

Teachers' Attitudes

Researching teachers' thinking is a relatively new area of research. Bennett, Wood and Rogers (1997) observed that research endeavour in this area was somewhat bedevilled, initially, by confusion regarding the extent to which various terms used could be considered to be related, analogous, synonymous or encapsulating. The terms disputed include those such as attitudes, beliefs, views or theories. Bennett *et al.* above report the emergence of a consensus that where these terms are used interchangeably, it is important to state this at the outset. For the purposes of this article I am equating the terms *attitude* and *belief.* The term attitude was the term used in the teacher-questionnaire since I felt that this was the term *belief* might be problematic. The term *attitude* was the one that I felt most closely relayed to teachers my area of interest.

Gipps et al. (1999) examined the beliefs or 'implicit theories' (p. 123) of Year 6 primary teachers' in 32 schools in Northeast England and concluded that teachers were not working from any particular model of teaching and learning but rather a complex mix of models. Teachers interviewed in their study articulated the use of a range of pedagogical strategies in assessing children's learning. Measuring teachers' attitudes/beliefs is important since research has shown that there is a complex relationship between these and teachers practice in the classroom. In the United States, a recent study that examined the relationship between the beliefs and intentions of 71 early childhood teachers indicated that beliefs were predictive of intentions (Wilcox-Herzog & Ward, 2004). In England, Bennett et al. (1997) researched pedagogy in relation to play and the findings indicated that the teachers' child-centred and constructivist theories of play and learning were consistent with the roles they took on in relation to children's play. They developed a model of infant teachers' practice that involved teachers theories; general orientation to teaching; teacher intentions and the constraints that affected teachers actual practice in the classroom. Context then is also important when considering pedagogy.

In the United States, Ginsburg et al. (2005) investigated the beliefs of a small number of preschool and kindergarten teachers who were participating in in-service education in early childhood mathematics education. The researchers found evidence that some of the teachers did not want to engage in active teaching, but held a more laissez-faire belief that they should have little involvement in helping children learn. The researchers observed that others appeared to consider that they should dominate, and even domineer, children's mathematical learning and those teachers appeared to the researchers to be uncomfortable with the unpredictability of letting children play a substantial role in their own learning. Ginsburg et al. also reported a polarisation of teachers in relation to their beliefs concerning children's interest in and ability to learn mathematical ideas. In seeking to account for these findings it is important to note that many of the teachers in that study displayed serious misunderstandings of mathematical content. It could be argued that because early childhood education in primary schools in Ireland is undertaken by primary teachers, all of whom have comprehensive knowledge of the kind defined by Shulman (1986) as *subject-content knowledge* and *pedagogical–content knowledge*, their attitudes to important aspectsof pedagogy are worth probing.

Key Research Questions

Arising from the review above the following research questions are stated:

- What are JI teachers' views in relation to the value of particular pedagogical practices in early childhood mathematics education?
- Do JI teachers generally appear to hold attitudes to teaching mathematics that are consistent with what we know about young children's unique styles of learning?

Design of the Study

Method

The overall study uses a mixed methods design. During this initial stage of the research a national sample of 346 schools (c. 460 JI teachers) were surveyed using a questionnaire. This phase of the research was carried out from February 2007 to August 2007 and the questionnaires were distributed at the beginning of May. During the follow-up phase some of the key issues in relation to early years' mathematics pedagogy will be further explored with a small sample of teachers (c. 10) using a semi-structured interview methodology (2008). The findings from these two elements will be triangulated in order to establish levels of internal validity and to strengthen reliability (Cohen et al., 2000).

Design, development and piloting of the questionnaire

The items for the questionnaire were developed after a review of the literature to determine the key issues that should be addressed in the survey. Statements related to attitudes and teaching approach were developed to match recommendations on desirable pedagogical practices (e.g. Clements et al., 2004; Gifford, 2004; Australian Association of Mathematics Teachers & Early Childhood Australia, 2006; Perry & Dockett, 2006). The questionnaire was designed in six sections: Biographical information; Attitudes to a range of mathematical teaching strategies; Teaching approaches; Challenges; Exploratory questions inviting an extended answer Findings related to teachers' attitudes form the basis for the paper reported here (See Appendix 1 for details of the relevant sections of the questionnaire).

A small number of teachers in a Junior School known to me assisted me in piloting the questionnaire. They felt very strongly that respondents should be asked about whether they used a textbook/ workbook with the children since they argued that this would greatly influence the responses of the teachers. For this reason the question 'Do you use a maths textbook/workbook with junior infants?' was inserted in Section 1. Questions found by the teachers to be problematic in terms of wording were revised and alternative wording for these was discussed and agreed. For example, on the basis of our discussions Question 5 in Section 2

"Attention to ensuring that all children understand the particular use of language in mathematics is essential in JI classes"
was amended to

"Ensuring children's understanding of the particular use of language in mathematics is an essential part of teaching mathematics in JI classes"

The research sample
I examined a list of 3,289 primary schools taken from the Department of Education and Science Website (DES, 2007). Since the population of interest was teachers of children in their first year of school I deleted any schools where there appeared to be no junior infant classes covering the regular curriculum for the first year at school. All schools with 19 or less pupils were also removed on the grounds that it was unlikely that there were any more than just a few junior infant children and consequently unlikely that the teacher was using a very specific and differentiated pedagogy with these children. At this stage there were 3013 schools on the list. The file was then re-organised by school size and by gender in order to ensure that there was no inherent bias in the sample. A systematic random sample of 377 schools (sampling interval of 8) was generated. It included a variety of school types and locations and involved single-sex, co-educational, junior and full vertical schools.

The distribution sample
There were 377 schools in the random sample. As far as it was possible to tell from the information on the DES website, these schools were most likely to have children in junior infants i.e. in their first year of school. All of these got at least one questionnaire on the assumption that each of these schools had at least one teacher of junior infant children. The list of schools was then examined to identify the schools which, based on enrolment numbers were likely to have two or more junior infant classes. The average class size of 24 (INTO, Mar 07) provided a rough indicator of number of teachers and thus of whether the school had one stream of classes or more. The list was further scrutinised to identify Junior Schools that could be expected to have double the quantity of Junior Infant classes. Finally the list was examined in order to identify schools that were likely to have designated disadvantaged status and thus smaller child-teacher ration. These schools were also sent additional questionnaires, the number of which depended on all of the factors identified here. This process added a further 116 questionnaires to the original list of 377.

The above account illustrates the difficulties of working with generic lists of schools and the difficulty of targeting specific populations of interest using a random sample as an element of the research design. A major challenge for researchers in the Republic of Ireland in selecting samples of teachers teaching children in JI is that it is very difficult to determine with any great accuracy when working with the lists of schools on the Department of Education and Science website which schools have children in JI.

Ethical considerations
It is advisable for questionnaire surveys to adopt an ethical stance (Denscombe, 1998). Ethical considerations were taken into account and ethical clearance for the study was obtained from the Research Committee of Saint Patrick's College.

Participants
Information related to both the participants and the response rate is displayed in Table 1.

Table 1: *Participants*

Schools initially contacted	377
Returned unknown at the address	2
Schools which indicated that they did not meet requirements	28
Schools which declined to participate	1
Schools participating (max)	346
Total number of questionnaires dispatched	467
Total number of possible respondents	< 460
Recruited participants	266
Response rate	58%

The response rate was very encouraging especially in the light of Denscombe's (1998) observation that it is not uncommon to get a response rate of 15% in questionnaire surveys. Based on the figures in Table 1, it could be argued that the true response rate is somewhere between 58% and 77% since it is not known exactly how many schools were eligible to participate on the basis of having children in JI nor do we know how many JI teachers were in each of the participating schools.

Findings and Discussion
Teacher and school profiles
The respondents were overwhelmingly female (94%) illustrating the extent to which teaching children in the first year of school in Ireland is predominately a female endeavour. Over a quarter of the teachers in the survey sample had less than five years teaching experience at any level in primary schools, and a further quarter had less than 10 years teaching experience.
Information related to teachers' early childhood teaching experience is displayed in Tables 2 below.

Table 2: Experience teaching infants

Experience teaching infants	Less than I year	Less than 5 years	5 to 8 years	> 9 years
Percentage of respondents	10%	35%	22%	33%

Table 3 below shows the range of class sizes. The majority of children are in classes of greater than 23. Only about 8% of teachers reported that they had a classroom assistant. In Ireland schools designated 'disadvantaged' enjoy lower than average pupil-teacher ratios and many of them enjoy a maximum class size of 15.

Table 3: Class sizes

Class size	<15	>16 but <22	>23
Percentage of respondents	15%	30%	55%

It is also important to note that 47% of teachers were working with either one (20%), two (14%) or three (13%) other classes alongside the junior infant class. About one-fifth (22%) of teachers classified their schools as having 'disadvantaged' status. The vast majority of teachers used a maths textbook or workbook (96%) with their junior infant children. This is in keeping with previous findings (Murphy, 2004) of widespread use of worksheets and textbooks in senior infant classes i.e. the second year of primary school (which generally caters for children in the age range 5- to 6-years of age).

Teacher attitudes/beliefs

Teachers were asked about their attitudes to a range of issues related to mathematics pedagogy in early childhood. These were assessed with 23 statements (See Appendix 1). For each of these belief statements teachers were asked to rate their level of agreement on a 5-point Likert Agreement Scale. Below the findings are discussed using three reporting themes: Strategies that focus on engaging children in mathematics; Strategies that focus on language and discussion; and Strategies that focus on promoting conceptual development.

Strategies that engage children in mathematics A number of statements relate to the theme of engaging children in mathematics. Table 4 presents the findings in relation to each of these.

Table 4: Attitudes to strategies that engage children in mathematics

	Strongly Agree	Agree	Undecided	Disagree	Strongly Disagree
2 Engaging in play/a playful approach assists young children's development of mathematical ideas.	79	21	0	0	0
3 Young children's interests, concerns and everyday activities should be exploited when developing different aspects of mathematics	69	28	2	1	0
1 Children's informal knowledge about aspects of mathematics should be taken into account in school.	44	47	7	1	1
15 Small-group teacher-led activity is essential for developing mathematics with young children.	43	52	2	2	1
6 It is necessary for the teacher to help children to make mathematical connections (e.g. between mathematics and everyday life).	37	52	5	5	1
17 Assessment is a critical aspect of providing appropriately challenging mathematical activity for young children.	30	57	6	7	0
7 It is important to find out *what* aspects of mathematics interest children and the reasons for this interest.	13	49	26	12	0
20 Workbooks and worksheets are essential in learning and teaching mathematics in JI.	10	41	12	32	5
14 The teacher's task is to follow the mathematics curriculum in a systematic structured way.	8	53	11	23	5
10 Whole class teacher-initiated activity is the most important aspect of teaching mathematics in JI classes.	4	28	18	44	6
11 Children in JI can set their own goals/tasks in mathematical activity.	3	32	25	34	6

Note: Statements 6 and 11 were stated negatively in the original questionnaire.

The table shows teachers reactions to each of the statements. It is clear that many of the statements received high levels of agreement, particularly if the percentages that relate to 'strongly agree' and 'agree' are combined.

Building on the interests and concerns arising from children's daily lives is a recurrent theme in the literature related to pedagogy in early childhood (e.g. Bowman et al., 2001). Almost all (97%) of the teachers in my survey sample were of the opinion that children's interests, concerns and everyday activities should be exploited when developing different aspects of mathematics. Similarly, teachers were generally very much in agreement (90%) that informal knowledge should be taken into account in school. However when asked about the importance of ascertaining the aspects of mathematics which are of interest to children and the possible reasons behind this interest, only about 60% of the survey sample agree that it was important. The remainder were undecided (26%), or did not consider this important (12%).

These findings may indicate some confusion or varying interpretations of the term 'informal' as used in the context of learning. Alternatively they may indicate that teachers are operating from the basis that they can make assumptions, on a general level, about children's interests and concerns and that specific interests and experiences are not of significance for early childhood mathematics learning/teaching. The latter interpretation gives rise to some concern in the light of Clements' (2004) assertion that the identification of children's rich experiential knowledge to an explicit level is a 'crucial' task for the teacher (p. 54). He argues that it is essential for teachers to draw on their own specific knowledge about what will be meaningful and engaging to their particular children individually and collectively (p. 59).

Only one-third of teachers have confidence that children can play a central role in determining aspects of their own learning. Almost 40% of teachers held the opinion that young children cannot set their own goals /tasks in mathematical activity and a further 25% were undecided about whether they could or not. We can conclude, then, that about two thirds of the teachers surveyed do not consider that child-regulated activity is an important aspect of the mathematics curriculum. However, in her synthesis of empirical and psychology perspectives, Golbeck (2001) suggests that the child's ability to regulate activity is critical. Her analysis strongly indicates that learning environments that militate against child-regulation of learning may undermine learning and development.

While about 60% of teachers agreed that the teacher's task is to follow the statutory mathematics curriculum in a systematic and structured way, 28% did not agree that this was case and about 10% were undecided. This suggests that a sizeable minority of teachers feel that there is more to take into account in enabling young children's mathematical education than that which is contained in the mathematics curriculum guidelines. It may be that teachers are relying on professional knowledge to make judgements about mathematics teaching or it may be that some other external guidance, besides that of the curriculum, is playing a role in shaping the curriculum. My findings suggest that workbooks and sheets exercise considerable influence in determining the curriculum for at least half of the teachers surveyed since that proportion believed that workbooks and worksheets were essential in learning and teaching mathematics in Junior Infants. Almost 40% disagreed that they were essential and about 10% were undecided. Almost all teachers (96%) reported using textbooks or workbooks with their junior infant children. Why then are almost half of the JI teachers using a resource that they don't believe is essential for children's learning? It is possible that while these teachers don't see the textbooks and worksheets as essential for engaging children in mathematics learning, they may see them in some circumstances, for example in consecutive or multi-classes as essential for the management of teaching. An analysis of variance was performed to examine whether teachers in multi-class situations were more likely to think these materials essential than teachers in single-class situations. The analysis revealed no significant difference.

Perhaps JI teachers are simply conveying in their attitudes above, the extent to which they are seeking to accommodate two different approaches to pedagogy and curriculum. Or perhaps these teachers have found an accommodation between a teacher-centred approach and a more child-centred approach. Close scrutiny of actual practice is necessary to test these assumptions.

Half of the teachers surveyed did not agree that whole class teacher-initiated activity was the most important aspect of teaching mathematics with junior infants. Just under one-fifth (18%) of teachers were undecided and the remaining 30% agreed. Overwhelmingly (94%), teachers were of the opinion that small-group teacher-led activity is essential for developing mathematics with young children.

The vast majority of teachers (90%) felt that it was necessary for the teacher to help children to make mathematical connections thus indicating that teachers in the sample believed in strong adult guidance. This is in line with the type of approach advocated by Ginsburg et al. (2005). Assessment was seen by most of the survey teachers (87%) as an important part of their teaching role.

All of the teachers were in agreement that children's engagement in play or a playful approach assisted their learning in the area of mathematics. This is interesting in light of Murphy's (2004) finding that senior infant pupils in Irish classrooms were given limited opportunities to be involved in the types of play-based activities and practices recommended in the curriculum guidelines. He argued that '…teachers' assumptions about play and learning differ from those upon which the curriculum is constructed.' (p. 256) It may be that teachers of the very youngest children in school i.e. junior infants, are more open to play-based approaches than their colleagues teaching the slightly older senior infants. Or perhaps there has been some change in attitudes amongst infant teachers in the intervening years since Murphy's study in 2004. In the intervening four years in-service education on the primary curriculum has continued and the effects of the new methodologies may now be more apparent as a result of related developments in teachers' thinking. Also, during this period, there has been some discussion and debate amongst teachers about developments in early years education generally (see, for example, INTO 2006), and perhaps this is contributing to developments in ideas about pedagogy.

Strategies that focus on language and discussion A number of statements relate to the theme of the role of language and discussion in early childhood mathematics teaching. Table 5 presents the findings in relation to each of these.

Table 5: Attitudes to strategies that focus on language and discussion

		Strongly Agree	Agree	Undecided	Disagree	Strongly Disagree
4	Children talking about mathematics with the teacher helps develop their understanding.	70	30	0	0	0
5	Ensuring children's understanding of the particular use of language in mathematics is an essential part of teaching mathematics in JI classes.	70	24	5	1	0
19	A great deal of mathematical understanding can be developed during informal activity/discussion.	53	44	2	1	0
21	Children in JI should report verbally on their mathematical activity.	30	65	3	2	0
8	It is particularly helpful for children in JI to talk with other children about mathematics in order to deepen their understandings.	29	51	7	10	3
12	Children's own methods of recording and their own symbols are particularly important in the early stages of recording mathematical ideas and processes.	13	51	18	15	3
18	Justifying mathematical ideas and making mathematical arguments are important at the early stages of learning mathematics.	9	43	21	22	4
23	Children's ability to use conventional symbols is important in assessing their mathematical ability.	4	37	24	31	4
13	It is important that children represent their mathematics through the use of conventional symbols.	2	43	18	33	4

Please note: Statements 8, 12 and 18 were all stated negatively in the original questionnaire.

All of the teachers agreed with the statement that children talking mathematics with the teacher helps develop children mathematical understandings. Almost all of them (97%) agreed that a great deal of mathematical learning is developed during informal activity and discussion. All but 6% were of the opinion that it was desirable to try to ensure children's understanding of the particular use of language in mathematics and that this was an essential part of teaching mathematics to young children. This is a particularly important issue since a major challenge facing schools in Ireland is the integration and education of young children for whom the language of instruction is not their first language. We know that it is essential for all children to develop sufficient language so that they can understand their peers and their teachers, and to present their explanations in discussion in the classroom (Perry & Dockett, 2006). Four fifths of teachers in the sample survey believed that it was helpful for children to talk with other children about mathematics in order to deepen their understandings. These findings taken together suggest that in the case of the teachers in the survey sample there is widespread agreement on the role of talk, both between children and their teacher and between children themselves, in developing mathematical understandings.

The picture presented by teachers in the survey sample however, contrasts somewhat with the that painted by Murphy (2004, p. 253) who found that, in practice, for children in senior infants '…interactive discussion at the pupil-teacher and at the pupil-pupil level did not seem to have been implemented to any significant degree.' Again, perhaps the differences may be accounted for by reference to the passage of time and intervening developments. Alternatively, there is the possibility that there may be a considerable gap between teachers' beliefs and the actual methodologies which they implement in the classroom. The vast majority of teachers (94%) felt that children in junior infants should report verbally on their mathematical activity. Perry and Dockett (2006) suggest that since argumentation forms the basis of later mathematical proof '…it is important for us to realise the early genesis of this process' (p. 21). When asked about the importance of developing children's ability to justify and argue in the context of mathematics, one quarter of teachers though this was unimportant and a further one fifth were undecided. Discussion can encourage these and other higher (mathematical) processes and metacognitive skills, as children engage in reflection, prediction, questioning and analysis (Bowman et al., 2001).

About two thirds (64%) of teachers believed that children's own methods of recording and their own symbols are important during the period in which they are in the junior infant class. However the remaining one third do not consider them important. Just under a half of the teachers in the survey sample (46%) believe that it is important that children in Junior Infants represent their mathematics through the use of conventional symbols. Indeed, almost 40% of them believe that children's ability to use such symbols is important in assessing their mathematical ability. However, research has demonstrated that young children often develop their own systems of mathematical symbols that make sense to them and all teachers need to be aware of and become familiar with these (Hughes, 1986; Munn, 1997; Worthington & Carruthers, 2003).
Strategies that focus on the promotion of conceptual development A number of statements relate to the theme of promoting conceptual development. Table 6 presents the findings in relation to each of these.

Table 6: Attitudes to strategies that focus on promoting conceptual development

		Strongly Agree	Agree	Undecided	Disagree	Strongly Disagree
9	The investigation and presentation of their own mathematical solutions to everyday problems enables children to develop mathematically.	41	54	4	1	0
22	Children in JI should be encouraged to engage in the mental manipulation of mathematical ideas.	26	62	7	5	0
16	Open-ended activity (where there is no one solution/right answer) is essential for promoting mathematical understanding.	20	46	23	11	0

It is generally recommended that teachers should help children develop strong relationships between concepts and skills (e.g. Bowman et al, 2001; Clements et al., 2002). Research indicates that these should be developed together and indeed, learning skills before developing understanding can lead to learning difficulties (Clements, 2004). One way in which children can be enabled to develop these relationships is through being presented with interesting, realistic and challenging problems (Atkinson, 1992). The vast majority of teachers (94%) agree that the investigation by children of everyday problems and the presentation of their own mathematical solutions to these problems enables children to develop mathematically. Almost one third of teachers (35%) do not believe that open-ended activity, where there is no one right answer, is essential for promoting mathematical understanding. Just fewer than 90% of teachers agree that children should be encouraged to engage in the mental manipulation of mathematical ideas, i.e. thinking about action. This ability is important since it is this facility that will free children from dependence on manipulation of concrete materials and free them also from the here-and-now to consider 'what-if' scenarios, thus opening the way for abstract thinking.

Conclusion

In summary, teachers of JI are generally young and female. While about half of them may not see textbooks or workbooks as essential, nevertheless almost all of them use them for mathematics teaching/learning. For some of them the provision of opportunities for children to develop higher mathematical processes, for example, argumentation and problem solving, is not something that they are convinced about. A substantial proportion of them are unconvinced also about the importance of young children's own methods of recording mathematics. In the one third of classrooms in which teachers do not believe that children should engage in open-ended problem solving, children are likely to have restricted opportunities to play with mathematical ideas, to make connections with their existing knowledge and to engage in higher-order mathematical processes such as analysis and synthesis. Similarly, about one third of teachers appear to be unconvinced about the need to take into account children's prior experiences and their interests. When this is considered alongside the finding that most teachers use workbooks and texts to structure the teaching then it is likely that the activities that children experience are what 'artificial' (Aubrey, 2003, p. 49). She considered this to be a particularly worrying feature of the provision for young children in schools in England and my findings certainly indicate that 'artifical' rather than real activities, i.e. ones based on children's expressed interests and concerns in the context of the children's lives and focused on their daily experiences, are considered adequate by JI teachers in primary schools in Ireland.

From an ideological perspective, the JI teachers in the study sample certainly appear to endorse the central role of play and of talk in young children's learning. From a theoretical perspective the teachers' views appear to be coherent with the socio-cultural emphasis on the desirability of adopting a range of interactive and pro-active teaching strategies. From the point of view of the recommendations for practice we saw above how in relation to most of these, the majority of teachers appear to be of the opinion that these are important. These then are the ways that the teachers in the study would like to work and the extent to which these are reflected in their teaching intentions will be the subject of a later paper.

There is a number of implications for pre-service and in-service teacher education. Specifically, arising from my findings there is a need to work with student-teachers and teachers to enable them to:

- Explore the arguments related to why we must take account of children's everyday mathematics and why we must engage and build on children's existing understandings
- Interrogate terms such as 'informal' and open-ended activity' as they apply to early childhood mathematics
- Analyse processes such as the development of the skills of argumentation and justification
- Consider why it is necessary to enable and encourage children to sometimes set their own goals and tasks
- Investigate children's own methods of recording mathematics and how the transition to conventional recording can be best facilitated.

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Appendix 1
Section 2: Your attitudes to a range of mathematical teaching strategies
Read each statement below and indicate the extent to which you agree or disagree with it.

		Strongly Agree	Agree	Undecided	Disagree	Strongly Disagree
1	Children's informal knowledge about aspects of mathematics should be taken into account in school.	O	O	O	O	O
2	Engaging in play/a playful approach assists young children's development of mathematical ideas.	O	O	O	O	O
3	Young children's interests, concerns and everyday activities should be exploited when developing different aspects of mathematics	O	O	O	O	O
4	Children talking about mathematics with the teacher helps develop their understanding.	O	O	O	O	O
5	Ensuring children's understanding of the particular use of language in mathematics is an essential part of teaching mathematics in JI classes.	O	O	O	O	O
6	It is not necessary for the teacher to help children make mathematical connections (e.g. between mathematics and everyday life).	O	O	O	O	O
7	It is important to find out *what* aspects of mathematics interest children and the reasons for this interest.	O	O	O	O	O
8	It is not particularly helpful for children in JI to talk with other children about mathematics in order to deepen their understandings.	O	O	O	O	O
9	The investigation and presentation of their own mathematical solutions to everyday problems enables children to develop mathematically.	O	O	O	O	O
10	Whole class teacher-initiated activity is the most important aspect of teaching mathematics in JI classes.	O	O	O	O	O
11	Children in JI cannot set their own goals/tasks in mathematical activity.	O	O	O	O	O
12	Children's own methods of recording and their own symbols are not particularly important in the early stages of recording mathematical ideas and processes.	O	O	O	O	O
13	It is important that children represent their mathematics through the use of conventional symbols.	O	O	O	O	O
14	The teacher's task is to follow the mathematics curriculum in a systematic structured way.	O	O	O	O	O
15	Small-group teacher-led activity is essential for developing mathematics with young children.	O	O	O	O	O
16	Open-ended activity (where there is no one solution/right answer) is essential for promoting mathematical understanding.	O	O	O	O	O
17	Assessment is a critical aspect of providing appropriately challenging mathematical activity for young children.	O	O	O	O	O
18	Justifying mathematical ideas and making mathematical arguments are not important at the early stages of learning mathematics.	O	O	O	O	O
19	A great deal of mathematical understanding can be developed during informal activity/discussion.	O	O	O	O	O
20	Workbooks and worksheets are essential in learning and teaching mathematics in JI.	O	O	O	O	O
21	Children in JI should report verbally on their mathematical activity.	O	O	O	O	O
22	Children in JI should be encouraged to engage in the mental manipulation of mathematical ideas.	O	O	O	O	O
23	Children's ability to use conventional symbols is important in assessing their mathematical ability.	O	O	O	O	O

Here We Go Again! : Researching the role of teacher attitudes toward maths as they embark on a theoretically based early numeracy intervention programme.

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There is a significant body of literature which deals with the professional development expectations of early-career teachers. This paper explores the role of teacher attitudes to a highly specific theoretically-based early numeracy intervention which has proved highly efficacious across a wide range of children, and also evaluates the reverse role in which the efficacy of an intervention program itself impacts on teacher attitudes. The research outcomes suggest significant interplay between these two dimensions, indicating the need to ensure adequate preparation of teachers as key stakeholders in any curricular intervention as well as underlining the necessity of assessing the quality of proposed interventions in order to safeguard against 'operator failure' as opposed to 'process failure'. Additionally the researchers indicate important dimensions which apply specifically to theoretically based interventions and the need for teachers to thoroughly understand both the methodology and theory of such interventions as absolute prerequisites for successful outcomes in schools.

1. Introduction

This study was undertaken in association with the Maths Recovery Intervention programme which is currently being rolled out to approximately 350 schools throughout the Irish Primary School system as a pro-active professional enhancement initiative of the Department of Education and Science to address critical professional up-skilling needs of teachers engaged in learning support in mathematics education. The focus of the study was to elicit teachers' personal theories of teaching, learning, and assessment at the point of entry to the programme and to use this information to inform the delivery of the intervention and to provide base-line data which will be used comparatively to measure change in teachers' practical, pedagogical and content knowledge regarding Mathematics education as the programme evolves.

2. Theoretical Framework

Research into teachers' thinking is divided by Clark and Peterson (1986) into three categories: "teachers' planning", "teachers' interactive thoughts and decisions", and "teachers' theories and beliefs". The literature reviewed here concentrates on the last-mentioned category, that of teachers' theories and beliefs, and specifically attempts to explore teachers' "personal theories" (Wilson & Cole, 1992) and cognitions about the teaching of mathematics.

2.1` Teachers' Personal Theories and Beliefs about Teaching

Since Lortie (1975) pointed to the need to explore the role of prior experiences in the process of teacher learning, a significant body of research on the topic has developed. Now it is quite widely accepted that teacher education has an important but secondary impact on teachers' thinking, the primary influences being life, school and career experiences prior to entering formal programmes of teacher education (Aitken & Mildon, 1991; Calderhead & Robson, 1991; Clark, 1988; Feiman-Nemser McDiarmid, Melnick, & Parker, 1988; Florio-Ruane & Lensmire 1990; Goodman, 1988; Grossman, 1989; Guyton & McIntyre, 1990; Lortie, 1975; McDaniel, 1991; Tabachnick & Zeichner, 1991).

Experiences of school and family leave many teachers with memories and beliefs about the nature of teaching and learning (Book & Freeman, 1986; Crow, 1987; Feiman-Nemser & Buchmann, 1986; Wright & Tuska, 1968; Sugrue, 1997). It has been shown that these beliefs interact with the content and pedagogy of teacher education programmes and influence what and how teachers learn (Anderson and Bird, 1995; Borko, Eisenhart, Brown, et. al., 1992; Calderhead 1995; Calderhead and Robson 1991; Kagan, 1992; Pajares 1993; Wubbels 1992). In an attempt to document, understand and influence this phenomenon, researchers have raised questions about the nature of the beliefs that teachers hold (Book, Byers, & Freeman, 1983; Clandinin, 1985; Feiman-Nemser & Buchmann, 1986; Hollingsworth, 1989; Zeichner, 1980; Zeichner & Tabachnick, 1981), how they came to hold these beliefs (Bullough, 1990; Grossman, 1989; Grossman & Richert, 1988; Lortie, 1975; Measor, 1985) and how these beliefs might be modified to enable their professional development as teachers (Berliner, 1986; Carter & Doyle, 1987; Clark & Lampert, 1986; Doyle, 1977, 1990; Grossman, 1989;).

Beliefs or personal theories of teaching are constructed by individuals learning within their socio-cultural context and begin to form at a very early age. The earlier a belief is incorporated into the belief system the more likely it is that it will persevere. By adulthood, changes in beliefs are less frequent (Pajares, 1993). Beliefs serve as filters which screen new information, ultimately determining which elements are accepted and integrated into the professionals' knowledge base. So, normally, new knowledge will only be accepted in as far as it is congruent with teachers' pre-existing conceptions about teaching. A direct consequence of this position is that only congruent information will be integrated into the knowledge structure or, in other words, no true learning is likely to occur (Bereiter, 1985, 1990).

Hence, if teachers' beliefs are as influential on classroom practice and as resistant to change as research suggests (e.g. Bird, Anderson, & Swidler, 1993; Grossman, Wilson, & Shulman, 1989; Powell, 1992; Wubbels, 1992) and if beliefs are often misguided or inappropriate in the classroom context (Cole & Knowles, 1993; Knowles & Holt-Reynolds, 1991), then it is imperative that teacher educators investigate the beliefs that teachers hold as they embark on professional development programmes. Furthermore, if belief systems do influence classroom teaching, then it is also important that teachers be made aware of this relationship and be provided with opportunities to identify and explore their beliefs through critical reflection. In the absence of such an opportunity, research suggests (Goodman, 1988; Guillaume & Rudney, 1993; Kagan, 1992; Korthagen, 1988; McIntyre, 1993) that teachers are likely to revert to practices they recall from their own school days and merely reinforce established patterns of behaviour. Hence, research advocates the need to establish more challenging programmes that will impact on teachers' thinking and which will encompass their personal experiences prior to and during their professional development programmes.

2.2 Teachers' Personal Theories and Beliefs about Mathematics

Over the last twenty five years many studies documenting the impact of teacher beliefs on mathematics teaching have been published (Raymond, 1997). These beliefs or tacit understandings which inform teachers' belief systems generally have also been found to influence the teaching of mathematics specifically (Ernest, 1989). Researchers have found that teachers' personal beliefs and theories about mathematics and mathematics education influence significantly their teaching practices (Kagan, 1992; Pajares, 1993).

Links have been found between teachers' beliefs and their pedagogical approaches to mathematics (Peterson, et. al, 1989). Links have been found between teachers' beliefs about subject matter and the approaches they take to instructional practice (Borko, 1992). Furthermore, links have been found between beliefs about student ability and the pedagogical choices of mathematics teachers. In addition, the nature of classroom assessments has been shown to be influenced by teachers' beliefs about mathematics (Nathan & Koedinger, 2000). Researchers have also found that beliefs about mathematics content are significantly affected by teachers' own experiences of mathematics in school (Cooney, Shealy & Arvold, 1998; Raymond, 1997).

Raymond (1997) integrated the findings of psychological studies (e.g. Fazio, 1986) and related research to develop a visual model of relationships between mathematics beliefs and teachers' pedagogical practices as follows:

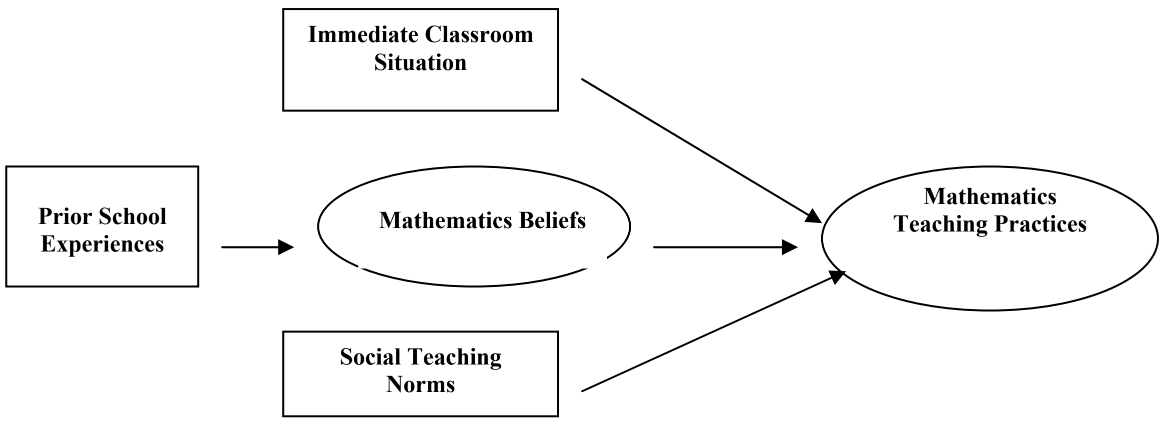

Figure 1. *A model of the relationships between mathematics beliefs and teaching practices, Raymond (1997)*[1]

In the model presented above, Raymond (1997) drew on prevailing definitions of beliefs (Cobb, 1986; Hart, 1989; Schoenfeld, 1985) and categorizations of mathematics beliefs (Lester, Garofalo & Kroll, 1989; McLeod, 1989). She defined mathematics beliefs as "personal judgments about mathematics formulated from experiences in mathematics, including beliefs about the nature of mathematics, learning mathematics and teaching mathematics."

Ernest (1989) stated that the components of mathematics teachers' beliefs are their:

- View or conception of the nature of mathematics
- Model or view of the nature of mathematics teaching

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[1] " **Prior school experiences:** as a student, prior teachers, teacher preparation program, prior teaching experiences
Immediate classroom situation: the students, the mathematics topic at hand, time constraints
Social teaching norms: the school setting, the curriculum, co-teachers, parents of students
Mathematics Beliefs: about the nature of mathematics, learning mathematics, teaching mathematics
Mathematics teaching practice: Mathematical tasks, discourse, environment, evaluation" Raymond (1997)

- Model or view of the process of learning mathematics.

Ernest states that "such views form the basis of the philosophy of mathematics, although some teachers' views may not have been elaborated into fully articulated philosophies" (Ernest, 1989). He outlines three broad philosophies of mathematics as follows:

> First of all, there is the instrumentalist view that mathematics is an accumulation of facts, rules and skills to be used in the pursuance of some external end. Thus mathematics is a set of unrelated but utilitarian rules and facts.
>
> Secondly, there is the Platonist view of mathematics as a static but unified body of certain knowledge. Mathematics is discovered, not created.
>
> Thirdly, there is the problem solving view of mathematics as a dynamic, continually expanding field of human creation and invention, a cultural product. Mathematics is a process of enquiry and coming to know, not a finished product, for its results remain open to revision. (ibid)

Research on the uptake of professional development programmes in mathematics education has also yielded interesting findings. Studies have shown that individual teachers respond differently to professional development programmes depending in part on the congruence between their beliefs and the nature of the reform programmes themselves (Wilson *et al.* 1996; Spillane & Jennings, 1997). Hence teachers' beliefs can facilitate or inhibit curriculum reform (Burkhardt, Fraser & Ridgway, 1990; Koehler & Grouws, 1992; Sosniak, Ethington & Varelas, 1991; Handal & Herrington, 2003).

Ernest (1989) stated that unlike the superficial changes required to upgrade the skills of a machine operative to more sophisticated equipment, a shift to a problem solving approach in mathematics, for example, requires more fundamental and deep-seated change.

> It depends fundamentally on the teacher's system of beliefs, and in particular, on the teacher's conception of the nature of mathematics and mental models of teaching and learning mathematics. Teaching reforms cannot take place unless teachers' deeply held beliefs about mathematics and its teaching and learning change. (ibid)

While some teachers resist change and meet the impetus for reform with equal and opposite force (Fullan, 1993), others adapt a more *á la carte* approach and integrate the more easily assimilated approaches (such as the use of manipulatives as an aid to teaching mathematics to young children), while ignoring the practices which challenge deeply held beliefs (Windschilt, 2002). Hargreaves (1994) has described this process of embracing change superficially as engaging in "safe simulations" which enable a surface engagement with the curriculum reform without changing the cultural norms of the classroom or altering fundamental beliefs. This is also evident in constructivist reforms where Windschilt (2002) found that some teachers adopted a narrow interpretation of constructivism, equating it to students structuring their own learning and perceiving student activity to be the equivalent of student learning (Prawat, 1992).

Hence, despite numerous reform efforts, research into teachers' pedagogical practices reveal that many still continue to teach mathematics as they have in the past (Sparks & Hirsh, 2006; Stigler & Hiebert, 1997). For curriculum reform initiatives to be successful, therefore, teachers must not only broaden their mathematical knowledge and competencies (Battista, 1994) but also challenge their prevailing attitudes and beliefs about the nature of mathematics (Sirotnik, 1999; Soder, 1999).

3. The Context and Purpose of the Study

This study set out to identify the personal theories of teachers embarking on the Maths Recovery intervention programme regarding the teaching, learning and assessment of mathematics. This paper documents the prior beliefs held by participants in the Maths Recovery intervention programme and forms part of an ongoing, longitudinal study which aims to document belief change over the course of the Maths Recovery intervention. Hence, this research study set out to explore the following questions:

1. What experiences of maths learning do teachers participating in the Maths Recovery Intervention Programme recollect from their schooling?
2. What personal theories of teaching, learning and assessment are held by the teachers embarking on the Maths Recovery Intervention Programme?
3. What beliefs do these teachers hold regarding the causes of pupil learning difficulties in early number?

The focus of the Maths Recovery programme is on early number knowledge and strategies. The authors considered it important to get an insight into the teachers' views and knowledge in relation to difficulties encountered in early number.

4 The Field of Study

This study was conducted during the initial training sessions of the Maths Recovery Intervention Programme. Thirty two teachers participated in the study. A qualitative instrument (see Figure 2) containing five open-ended prompts was used to elicit the personal memories of mathematics learning as well as the personal theories of teaching, learning and assessment of the participating teachers.

Talk about your experiences and memories of learning maths.......

I understand the teaching of maths to be essentially about........

I believe pupils learn maths best when......

I believe pupil learning in maths is best assessed when....

I believe that learning difficulties in early number stem from.....

Figure 2: *Qualitative instrument used in study:*

The researchers were cognisant of the ethical issues adverted to in the literature (Bogdan & Biklen, 1992; LeCompte & Preissle, 1993) regarding consent, vulnerability of participants and confidentiality. The researchers were anxious to ensure that potential informants did not feel any pressure or obligation to participate in this research. A general, verbal invitation to participate in the study was issued to all teachers involved in the professional development programme. An assurance of confidentiality was given and a commitment made that the data generated by the study would be used for research purposes only.

5. Approach to Data Analysis

A grounded theory approach was used for the analysis of data generated by this study. Grounded theory is a systematic method of qualitative data analysis leading to the discovery of theory from data (Glaser and Strauss, 1966). In this method, data collection, analysis and eventual theory stand in close relationship with one another. A researcher does not begin with pre-conceived notions but lets the theory emerge as the study progresses and the data are analysed. Grounded theory aims to organise the many ideas which emerge from systematic data analysis and to generate theory which is tested through further recursive analysis. The activities of collecting data, analysing data, and writing up the research often occur contemporaneously. The process is both reflexive and recursive. Analysis begins with the researcher holding "conversations" with the data and thinking about any regularities or patterns in the database that relate to the research questions. Moving toward theory involves coding, memoing, and diagramming procedures that depend on "constant comparison" (Ibid.).

The data generated by this study were entered into the NVivo qualitative data analysis program. Both researchers read transcripts of the data and independently created units of analysis by ascribing codes to the data (Miles & Huberman, 1984). The initial codes, referred to as 'nodes' in the NVivo program, were descriptive and derived responsively from the data. Nodes were kept as discrete as possible. Through a process of discussion, iteration and reiteration, some codes were modified. It was necessary to go through the data set on many occasions to ensure consistency, refinement, modification and exhaustiveness of coding. Through this process frequencies and patterns were identified within the data. At the next stage, the researchers reviewed the codes used to see if they naturally fell into clusters and to create overarching codes for these clusters. The Node Explorer function in NVivo enabled grouping and organising of nodes into hierarchical structures called Tree Nodes and their descendants, Child Nodes. The NVivo program was used to facilitate access to the original data files to check the data for background and contextual information.

6. Presentation of Data

6.1 Teachers' Personal Experiences of Learning Mathematics

Many teachers indicated that they had enjoyed and had a facility for mathematics while in school. "I learned maths quickly and always enjoyed the subject" (AG). In particular their positive recollections focussed on enjoying the challenge of problem solving and "getting the right answer" (DH).

> I loved maths. I loved the challenge of problem solving. We did mental arithmetic daily for ten minutes. This helped keep us focused. (MNC)
>
> I loved learning maths. It was my favourite subject. I found it challenging when faced with a new topic. (MOF)

Interestingly, many recalled the approaches to mathematics education taken by their teachers rather than the content of the mathematics classes themselves. Many spoke of the emphasis placed on rote learning and, in particular, on the learning of tables. Some highlighted the importance of the disposition and competence of the teacher in their experience of learning mathematics.

> It depended on the teacher as far as my memory goes back. (MMM)

Others drew distinctions between the types of mathematics teaching and learning they experienced at primary and post primary levels, recalling in particular the emotional environment in which Mathematics education took place.

> Maths in Primary school was not an enjoyable experience. It was taught to me with the threat of corporal punishment hanging over every wrong answer. I simply hated it. I loved maths in Secondary school where a pleasant learning environment allowed me to flourish. (LOD)

> I remember standing around the room with the class. We were only allowed to sit when we had answered two or three questions correctly. (JOC)

It was noticeable that the teachers involved in this study appeared to process their memories of mathematics teaching and learning through the lens of their understandings of current mathematical teaching theory and practice. Some were critical of the lack of availability of concrete materials and absence of relevance to their lives, a failure to allow discussion in class and negative experience around failure.

> There was an emphasis on arriving at the right answer above all else….This reinforced the idea that some people were 'good' at Maths and others were not. (DH)

6.2 Personal Theories of Mathematics Education

The personal theories of mathematics education elicited by the study fell into three broad thematic categories: affective dimensions of teaching, the content of mathematics education and the methodologies employed in the teaching of mathematics.

6.2.1 Affective Dimensions of Mathematics Teaching Teachers placed a high priority on enabling students to develop positive attitudes to mathematics and confidence in undertaking mathematics assignments as well as encouraging enjoyment and a love of the subject.

> making the child see that a math lesson can be fun and they can succeed.(JOC)

> …. [developing] a love for the subject in the children and a sense of confidence in figuring out realistic little mathematical challenges…. To encourage children to have a go! (EOC)

There was an acceptance of the importance of the teacher's role in this process and the need for the teacher to have a positive disposition towards mathematics and to provide opportunities where pupils are "allowed to investigate" (ML). They placed considerable emphasis on making mathematics relevant to the lives of their students.

> The teaching of maths is essentially about helping the child as best you can to gain a grasp of the topic you are teaching and to show him/her how it is relevant to their life. (MOF)

6.2.2 The Content of Mathematics Education When referencing the content of the mathematics education there was a clear priority ascribed to the teaching of number. This prioritisation was linked to the belief that the acquisition of number skills was a necessary prerequisite for pupils to engage in other curricular strands.

> Number is core – all other strands are dependent on pupils acquiring number skills. (MOS)

Teachers spoke of their attempts to help pupils to "de-code the language of numeracy" (LOD) and to use these skills to "aid daily living" (LG). They highlighted the importance of "teaching the properties of number" (DC), "number operations and connections" (POC), the "relationships in number" (GG) and "how numbers break down and can be related to one another" (AB). An emphasis was placed on:

> trying to help children to attach meaning to number and to understand the various processes that they will experience. Teaching and learning maths tasks is a step-by-step process. A particular step needs to be mastered before moving on to the next. (DB)

As has been shown above, within the area of number there was an emphasis on developing pupil understanding of the properties of number, number concepts and place value,. By way of contrast to this, priority was also given to "basic skills" in the application of number. Both aspects were perceived as leading to the development of pupil ability to do calculation.

6.2.3 The Methodology of Mathematics Education

A clear priority was assigned to the importance of problem-solving approaches within the teaching of Mathematics. Teachers spoke about the need to develop "thought processes to work out problems" (SNC), of "giving pupils the understanding of concepts so that they can figure out the problem" (SG), of "helping pupils to be able to tackle problems and look for solutions" (ML) and of "showing that there are several ways of solving something" (DH).

References to the methodology of mathematics education also favoured the application of mathematics to real life situations and ensuring that pupils "can deal with socially essential maths" (MOL). The importance of "relating maths to the child's experience" (CC) was also highlighted.

> I understand the teaching of maths to be essentially about helping pupils to de-code the language of numeracy, use it to aid daily living and grow to enjoy the pleasure it can give. (LOD)

There was also emphasis on encouraging questioning and discussion among the pupils as well as the development of mathematical language and an understanding of order and pattern.

6.3 Personal Theories of Mathematics Learning

Teachers' personal theories of mathematics learning elicited by the study fell into two thematic categories: affective dimensions of learning and the methodologies which facilitate the learning of mathematics.

6.3.1 Affective Dimensions of Mathematics Learning. Teachers believed that pupil learning was facilitated by their efforts to make the mathematics lesson a positive and enjoyable experience, where pupils "are encouraged" (MNB) and "feel confident about what they are doing and do not feel threatened by the risk of failure" (DH). They emphasised the importance of facilitating pupil dialogue and discussion and they believed that pupils learn best when "they are having fun, having success, are not under pressure and have time to talk and think" (MOL).

They believed that learning was also enhanced when mathematics was made relevant to the lives of the pupils, when they "can relate to it personally [and] it is based in the reality of their experience" (ML). The teacher's openness and professionalism were also regarded as essential.

6.3.2 Methodologies which Facilitate the Learning of Mathematics. The informants in this study believed that giving pupils access to concrete materials enhanced their mathematical learning and stated that pupils learn best when they are

> actively engaged using concrete materials, when they are clear about what they have to do and when children are supported by an adult in a positive environment" (KK)
>
> [when] concrete materials are used and when there is ample time given to review and consolidate" (MOS).

They also considered it important that pupils "have a chance to 'experiment' with maths rules" (JOC) and that "they can see connections between new learning and what has gone before" (GH) and can "see how maths is relevant to their own environment and experiences and not just a question in a book" (MOF). A range of other methodologies was considered beneficial, including group activities and problem solving.

6.4 Personal Theories of Assessment of Mathematics Learning

The responses of teachers to the request to identify best practice in assessing pupil learning in mathematics spanned a broad range of techniques including teacher observation, formative and summative testing, as well as students' mathematical activity in the classroom and in their homes.

6.4.1 Formative Assessment Teacher informants in this study prioritised ongoing formative, informal and diagnostic forms of assessment over summative and standardised types of assessment.

They expressed the belief that pupil learning in mathematics is best assessed on a regular basis, where it is developmentally appropriate, and where the testing is varied:

> it is done on a regular basis, in a consistent manner that it provides the basis for the next step in learning and highlights the gaps in the understanding of different areas (EOC)
>
> they can apply their knowledge to a variety of situations (ML)
>
> children are tested on learning and taken from their own stage of development. When assessment is used to influence teaching and learning (SOS)

6.4.2 Summative Assessment Teacher opinion was more divided about the merits of summative assessment. Some of the teachers regarded summative assessment as necessary but were concerned about the potential negative impact on pupils, "I find standardised test are very daunting for some children" (JOC). However others felt that the comparative data yielded by such tests were valuable, "standardised tests are necessary to keep an eye on progress relating to average age group" (AG)

6.4.3 Affective considerations in Mathematics Assessment There was a belief that assessments were most accurate when the pupils were relaxed, motivated and tested on more than one occasion. Teachers believed that the ideal context for assessment was when "the child is comfortable, at ease with an adult, in a supportive environment, using clear unambiguous language and familiar materials" (KK) and when "the child feels confident in their ability" (MOF)

6.5 Personal Theories Regarding the Origins of Learning Difficulties in Early Number

Participating teachers' beliefs regarding the causes of learning difficulties in early number were categorised into three main themes – factors associated with the pupils themselves; factors relating to the teaching they received; and factors relating to the wider environment.

6.5.1 Pupil-Related Factors In the pupil-related causes, there was an emphasis on the impact of lack of number experience and understanding as well as difficulty in following a pattern and sorting and matching. Student communication and language skills were considered vital, as were concentration and confidence. Thus the key problems were identifies as:

> The pupil's [lack of] ability (MOS)
> Difficulties with language assimilation (DB)
> Lack of confidence (KK) "The 'I can't do it' syndrome" MOL
> General learning difficulties or specific problems with maths concepts (DF)
> Lack of vocabularly to represent experience (GG)

6.5.2 Teaching- Related Factors The role of teachers was regarded as contributing to learning difficulties when there was an inappropriately rapid progression from concrete to abstract work, "moving from concrete to abstract too quickly" (MOS). Concern was also expressed about the overuse of textbooks and a mismatch between teaching and learning styles, "maths being workbook-based rather than learning-based. Filling in boxed in workbooks does not mean they understand the concepts" (MNB). Lack of opportunities for pupils to engage in thoughtful activities was also believed to be a contributory factor, "children not given an opportunity to think, over reliance on written work. (MM)

6.5.3 Environmental Factors Some respondents cited environmental factors deemed instrumental in impeding student learning, including poor parental involvement and disadvantaged backgrounds that were not conducive to students succeeding at school and mentioned:

> Lack of exposure, stimulation, conversation, discussion in infancy and early childhood (ML)
>
> Sparse language from home circumstances leading to difficulties in understanding the language of maths. Negative attitude towards maths from home (particularly for girls). Inadequate use of concrete materials until understanding is real. Lack of early intervention support. Difficulties in focus and concentration on the part of children. Disadvantaged schools, gaps in attendance in school on a regular basis. (EOC)

7. Discussion of Findings

With regard to the specific findings of this study, it is noteworthy that teachers' recollections of their own experiences of mathematics education in school focused on mathematics pedagogy rather than subject content knowledge. In addition, it would appear that teachers filtered their recollections of their own maths education through the lens of their current understandings of the theory and practice of mathematics education.

The data yielded by the study regarding the teachers' personal theories of teaching, learning and assessing mathematics displayed a clear emphasis on the pedagogy rather than the content of mathematics education. Teachers believed that concrete materials played a vital if not a predominant role in the development of mathematics understanding of their pupils. The limitations of this focus on concrete materials without a pedagogical rationale for their use should not be underestimated. That teachers appeared to assume that mathematics content can be learned by pupils being exposed to concrete materials may be redolent of the "safe simulations" adverted to by Hargreaves (1994) where teachers make superficial, cosmetic changes to their teaching practices in the classroom without fully understanding the underlying principles and rationale for the reform changes.

As has been highlighted in the international research, clear and consistent patterns of professional development activity on the part of teachers are difficult to identify as most teachers engage in what has been described as an episodic, kaleidoscopic (Skilbeck & Connell, 2003) patchwork quilt of topics (McCrae et al., 2001). However, it is interesting to note that many of the curriculum reforms that the teachers in this study had experienced previously were introduced during the implementation phase of the Revised Primary School curriculum and tended to focus on pedagogy and curriculum rather than teacher subject knowledge. Thus, whether teachers have a sufficiently deep level of knowledge of mathematics subject matter to enable them to engage fully with the constructivist curriculum in general and the Maths Recovery Intervention Programme in particular needs to be investigated further and borne in mind as the Maths Intervention Programme develops.

Teachers' espoused beliefs about the importance of problem solving approaches and social relevance; their resistance to text-book driven teaching and their encouragement of experimentation and exploration on the part of the pupils, however, point optimistically to their openness to constructivist, inquiry-orientated approaches to mathematics education.

The findings of this study showed that teachers ascribed priority status to the teaching of number as a fundamental element of mathematics education. This is not entirely surprising given that the teachers participating in the study are engaged in learning support where a predominant focus on number can be justified.

Finally, this study found that participating teachers' personal theories of teaching, learning and assessing mathematics, highlighted the importance of affective dimensions of mathematics learning and teaching such as student attitude and confidence, teacher disposition towards mathematics and efforts to make mathematics relevant to the lives of the students

Although very much an initial investigation of some of the key themes about teachers' beliefs regarding mathematics education prior to the commencement of a professional enhancement programme, this study has yielded some interesting early findings which merit further consideration and will be the subject of subsequent research as the Maths Recovery Programme is implemented for the first time in the Irish Primary School system. In particular the study has:

- Secured initial data about the beliefs regarding the teaching, learning and assessment of maths of a cohort of practising teachers which can be used to benchmark the evolution of these beliefs as a targeted professional development programme is delivered.

- Raised the awareness of the participating teachers of their own memories and experiences of learning mathematics which can be used as the subject of further recursive and reflexive analysis by the teachers and may be the subject of further study by the researchers.

- Developed a data set about teachers' beliefs regarding learning difficulties in one specific element of the primary school maths curriculum with a view to subjecting these beliefs to an intensive examination as the professional development programme advances.

8. Conclusion

Elmore and Burney (1997, p 1) have stated: “there is a growing consensus among educational reformers that professional development for teachers and administrators lies at the centre of all educational reform and instructional improvement”. One needs also to caution, however, that educational change takes place slowly (Eltis & Mobray, 1997), with fundamental changes in instructional practices requiring much time to become established (Snow-Renner & Lauer, 2005). This study has found that dialoguing with teachers to discern the breadth, depth and nature of their prior beliefs and experiences is an important dimension of developing a conversation which respects the prior learning and experiences of the teachers and provides an opportunity for the organic evolution of curriculum reform.

The data generated by this study provide a worthwhile foundation for the ongoing review and analysis of the evolution of teachers’ beliefs about mathematics education while engaged in a professional development programme. The data gathered in this study will be enriched and expanded longitudinally through ongoing interviews and classroom observations of teachers’ classroom practices.

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The Challenges Facing Pre-service Education: Addressing the Issue of Mathematics Subject Matter Knowledge among Prospective Primary Teachers

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Research has indicated that it is not necessary to study mathematics to degree level to teach mathematics effectively at primary level (Greaney et al, 1999; Cooney, 1999). Consensus exists however that graduating teachers do require a deep and rich knowledge of mathematics to a certain level in order to facilitate optimum mathematics teaching and pupils learning (An Roinn Oideachas agus Eolaíochta, 2002).
Internationally dissatisfaction exists regarding the standard of mathematics subject matter knowledge evident among both qualified and prospective primary teachers (Ma, 1999, Farmer et al, 2003). Concern is also evident within the Irish context, particularly within the Colleges of Education (Wall, 2001; An Roinn Oideachas agus Eolaíochta, 2002; Corcoran, 2005).The author (MH) aimed, through a two cycle action research methodology, to identify the existing level of mathematics subject knowledge among prospective teachers in one Irish College of Education in an effort to address weaknesses and develop 'deep' subject knowledge among participants through the development and implementation of a custom-built intervention. After a brief presentation of the characteristics of the research study, this paper will focus on the reconnaissance stage of the first cycle of the action research project.

1. Introduction

A mathematics lecturer responding to the N.C.C.A. (2006: 29) consultation document declared that "…students need their best teachers at a young age, teachers who really know what they are doing and really understand the simplicity of what they are doing. Once confidence is in place at a young age, I think the other issues…will right themselves". One must ask the question: what is required to develop such teachers? As well as more generic knowledge such as general pedagogical knowledge, Schulman (1986) proposed that in order to be able to teach any subject effectively teachers require three categories of subject knowledge: content knowledge, subject specific pedagogical content knowledge and curriculum knowledge. For the purposes of this study, the authors' focus in on 'mathematics content knowledge', which is often referred to as 'mathematics subject matter knowledge' (Rowland et al, 2005). Mathematics subject matter knowledge 'refers to the amount and organisation of knowledge per se in the mind of teachers' (Schulman, 1986: 9). This type of knowledge includes the facts and concepts of a discipline, its organizing frameworks, and the ways in which propositional knowledge has been generated and established (An Roinn Oideachas agus Eolaíochta, 2002). Corcoran (2005,b) refers to subject matter knowledge as 'mathematical literacy'.

2. Mathematics Subject Matter Knowledge: A Must!

It is only "In the past two decades teachers' knowledge of mathematics has become an object of concern" (Hill et al, 2004: 11). Research suggests the principal reason for this was that many nations especially the U.S. and U.K., following dissatisfaction with their pupils' relatively poor mathematical performance in international comparative studies when compared to their Eastern peers, were eager to identify the 'causes' of this unsatisfactory scenario (Wall, 2001). Consensus existed among policy makers that "…no curriculum teaches itself" (Ball et al, 2005: 14) and that improved subject matter knowledge among teachers would facilitate enhanced learning among pupils. Such thinking resulted in increased status and attention being assigned to the issue of teachers' mathematics subject matter knowledge (Wall, 2001; Goulding, 2003).

Nowadays, it is fair to say that "…there is general agreement that teachers' personal knowledge of mathematical content to be taught is the cornerstone of teaching for proficiency" (American Federation of Teachers (A.F.T.), 2005: 1). Ball et al (2005) suggest that the nature of a teacher's subject matter knowledge affects his/her ability to make apt decisions regarding the most appropriate instructional materials, presentation, emphasis, and sequence of instruction. The UK's Training and Development Agency for Schools (T.D.A.) (2006) asserts that a teacher requires a high level of knowledge and understanding in order be able to confidently and effectively develop pupils' mathematical knowledge and understanding. This assertion receives support from various studies which report a positive correlation between teachers' mathematics subject matter preparation and their effectiveness in the classroom, which in some cases was measured through pupil achievement (The Interstate New Teacher Assessment and Support Consortium (I.N.T.A.S.C.), 1995; Tirosh et al, 1998; An Roinn Oideachas agus Eolaíochta, 2002; Hodgen, 2003; Ball et al, 2005; Hill et al, 2005).

3. How much Mathematics Subject Matter Knowledge is enough?

While consensus exists that practicing primary teachers require 'deep' and 'rich' mathematics subject matter knowledge, it is essential at this stage to resolve what constitutes 'appropriate' mathematics knowledge for teaching. Ball (1990) challenges the assumptions that mathematical concepts and procedures addressed at primary level are easy. The question remains as to "…exactly what and how much mathematics they need to know and be able to do…" (A.F.T., 2005: 1).

It is now recognised internationally that subject matter knowledge beyond a certain 'threshold' is not associated with greater pupil achievement i.e. primary teachers do not need to study mathematics to degree level (Greaney et al, 1999; Burke, 2000; An Roinn Oideachas agus Eolaíochta, 2002; Goulding, 2003). This finding does not suggest that a teacher's knowledge of mathematics is irrelevant to the quality of mathematics teaching and learning. While in the past, there was a perception from some quarters that "…elementary teachers need very little …" mathematics subject matter knowledge (Rowland et al, 2005: 256) i.e. that is was sufficient for teachers to be able to do anything required of pupils (a 'minimalist' view), this position is challenged by the argument that teachers require more than 'learner knowledge' given that pupils can ask questions that extend beyond the formal curriculum (Prestage and Perks, 1999; Ball et al, 2005). The minimalist view also assumes that any well-educated adult possesses the subject matter knowledge required to teach at primary level. This reflects the belief that 'He who knows mathematics, knows how to teach it' (Boero et al, 1996). Ball et al (2005) propose that while teachers need to be able to use reliable algorithms i.e. demonstrate 'common' mathematics subject matter knowledge, procedural knowledge alone is insufficient for teaching.

Corcoran (2005 (b)) suggests that a certain kind of mathematics subject matter knowledge is needed to teach the subject effectively at primary level, additional to that required by those pursuing other mathematically intensive careers e.g. accountants (Ball et al, 2005). Hodgen (2003: 104) expresses a similar view referring to "…the need for primary teachers to know mathematics differently". Ball et al (2005) refer to this knowledge as 'specialised'. Hill et al (2005: 373) suggest that the 'specialised' mathematical knowledge required for the work of teaching is vast given that this 'work' includes

> explaining terms and concepts to students, interpreting students' statements and solutions, judging and correcting textbook treatments of particular topics, using representations accurately in the classroom, and providing students with examples of mathematics concepts, algorithms and proofs.

To be able to meet the aforementioned demands, a teacher must possess conceptual understanding of the various mathematical concepts and procedures as well as recognising and understanding the interconnections between them (Kessel and Ma, 2000; Conference Board of the Mathematical Sciences (C.B.M.S.), 2001; Ball et al, 2005). This belief receives support from Schulman (1986) who highlighted that subject matter knowledge required for teaching included "…both facts and concepts in a domain but also why facts and concepts are true and how knowledge is generated and structured in the discipline" (Hill et al, 2005: 376).

On the other hand, 'weak' mathematics subject matter knowledge is associated with less competent mathematics teaching (An Roinn Oideachas agus Eolaíochta, 2002). The coping strategies utilised by such teachers include avoiding topics altogether, overdependence on the text, limitation of interaction and a focus on rules and procedures as isolated facts (An Roinn Oideachas agus Eolaíochta, 2002; Barber and Heal, 2003). In such contexts pupils must depend on memorization rather than understanding which in turn leads to the "…failure to lay the groundwork for future development of student understanding" (Leavy and O' Loughlin, 2006: 54).

4. The 'Health' of Elementary Teachers' Mathematics Subject Matter Knowledge

4.1 International Findings

Internationally, the volume of research exploring the nature of both qualified and prospective elementary teachers' mathematics subject matter knowledge i.e. what they actually do know, has grown in line with increased status given to the issue (Wall, 2001; Thwaites et al, 2005). There is overall consensus supported by the reports that all is not well (Tirosh et al, 1998; Hodgen, 2003; Tsang and Rowland, 2005). Characteristics of elementary teachers who were deemed to have 'substandard' mathematics subject matter knowledge (e.g. UK, US, Hong Kong) include a dependence on rule-bound knowledge, shortcomings in both procedural and relational understanding of concepts, and ignorance to connections and gaps in knowledge (Ball, 1990; Ma, 1999; Rowland et al, 2005; Ball et al, 2005). Such dissatisfaction has resulted in the initiation of an accountability movement within the U.S. and U.K, which in turn led to the introduction of standards (e.g. Qualified Teacher Standards (QTS, UK)) which were rigorously tested at various levels (e.g. numeracy skills test for licensing purposes) (Wall, 2001; Rowland et al, 2005). Such standards had direct implications for the pre-service education in the relevant education systems (I.N.T.A.S.C., 1995; Rowland et al, 2005; Ball et al, 2005; Tsang and Rowland, 2005; T.D.A., 2006). In the UK for example, teacher training colleges are required to "…monitor trainee teachers' progress, give them feedback, review and meet their individual needs, and encourage them to take responsibility for their own development" (T.D.A., 2006: 76).

4.2 The Issue in the Irish Context: A Focus on Prospective Teachers

It is understandable that one may have the impression that there is little concern regarding the mathematics subject matter knowledge of Irish prospective primary teachers, given the sparse amount of research on the phenomenon. Wall (2001) and Corcoran (2005) both support this claim stating that of teachers' subject matter knowledge has not been a source of obvious concern in Ireland to date. The fact that the issue has been ignored is reflected within education policy documents. In reality however, the research which has been carried out by a number of individual researchers within the various Colleges of Education (Wall, 2001; Corcoran, 2005; 2005 (b); Oldham, 2005; Leavy and O' Loughlin, 2006) testifies that the mathematics subject matter knowledge that Irish student teachers bring to teacher education is largely unsatisfactory reflecting the characteristics of their international peers. Unlike the UK and US systems, the mathematics subject matter knowledge 'required' by Irish primary teachers is quite limited (Wall, 2001; Corcoran, 2005 (b)). Once prospective teachers achieve the minimum entry requirement i.e. D3 at Ordinary/Higher Level Leaving Certificate Mathematics, they are not obliged to provide any further evidence of their mathematics subject matter knowledge (Corcoran, 2005).

4.3 Potential Causes of the Problem

Ball et al (2005) suggest that the phenomenon of inadequate mathematics subject matter knowledge among prospective elementary teachers internationally should not come as a surprise, given that they are products of the mathematics education systems that are deemed unsatisfactory. Accordingly the C.B.M.S. (2001) suggests that gaps in prospective teachers' mathematical backgrounds are systemic rather than personal failings as their only mathematical experiences equate mathematical strength with computational proficiency.

A similar scenario exists within the Irish system. While this specific issue has not received the publicity warranted, concern regarding substandard mathematical skills evident among Leaving Certificate students generally has been escalating for some time (N.C.C.A., 2005). Consensus is now widespread among students, practitioner and professional groups alike that pre-tertiary mathematics education in its present form is short-changing those who wish to pursue further education (Oldham, 2001; N.C.C.A., 2006). Regardless of level of study or grade achieved, many such students are deemed 'at risk' or under-prepared' on entry to tertiary level courses as they lack basic understanding of concepts in fundamental areas (Murphy, 2002; Lyons et al, 2003; N.C.C.A., 2006). Consensus exists that the nature of predominant classroom practices, especially at the senior cycle of second level education i.e. exam-led, teacher-led didactic approach focusing on rules and procedures which are likely to be examined, is not conducive to the provision of a high quality mathematics education which develops conceptual understanding among students (Murphy, 2002; Lyons et al, 2003; N.C.C.A., 2006; Hourigan and O' Donoghue, 2007). Such a pre-tertiary mathematics experience which prioritises memorisation over connection-making is not conducive to the development of 'deep and rich' mathematics subject matter knowledge among prospective primary teachers (Corcoran, 2005; N.C.C.A., 2005; Hourigan and O' Donoghue, 2007).

The above findings call into question the validity of the minimum mathematics entry requirement provision as a suitable indicator of adequate subject matter knowledge for prospective primary school teachers (See section 4.2). While there have been calls within the Irish context for this requirement to be raised (An Roinn Oideachas agus Eolaíochta, 2002), the authors are not convinced that such adjustments alone are sufficient to address the phenomenon in question given the dissatisfaction which exists regarding the narrow range of mathematical skills among Leaving Certificate students regardless of the grade (Oldham 2005; N.C.C.A., 2006).

5. Motivation for the Study: Response to the Issue

Into the future, the scenario looks more positive from a pre-tertiary perspective. There is much hope to be gained from the initiation by the N.C.C.A. (2006) of the first root and branch review for almost forty years with a view to making future radical changes to the nature of the post-primary mathematics education (e.g. content, methods and assessment) which would facilitate the provision of a continuum of mathematics learning over the years in formal education e.g. ensure a smooth transition from primary to post-primary and post-primary to tertiary mathematics education (N.C.C.A., 2005; 2006). Undoubtedly for proposals to become a reality, support for teachers at all levels is essential. The authors are hopeful that the envisaged reform will facilitate the 'tackling of the problem where it arises' thus facilitating future prospective teachers to enter pre-service education possessing 'deep' mathematics subject matter knowledge (N.C.C.A., 2006). In the interim, however, there is no doubt that the nature of the predominant pre-tertiary mathematics experience is exacerbating the demands placed on all tertiary mathematics educators including pre-service educators.

This research study was the author's (MH) reaction to the phenomenon in question within her working environment, a College of Education. The author's decision to explicitly address the issue resulted from encountering increasing number of instances of substandard mathematics subject matter knowledge in her daily work with student-teachers (mathematics pedagogy course, teaching practice). In many cases within the teaching practice context, it was clear that poor mathematics subject matter knowledge directly impacted on student-teachers' perception of or attitude to the subject as well as their ability to transform the knowledge appropriately to facilitate pupil learning. Informal conversations also enlightened the author regarding the effects of substandard subject matter knowledge on student-teachers' attitudes and feelings (Nitko, 2001). While all of the aforementioned incidences represent mere anecdotal evidence, their repeated nature resulted in the author making a firm commitment to further explore this real but somewhat 'silent' issue (Elliot, 1991). The motivation for this particular research study comes from the authors' deep-seated values in relation to improving the preparedness of prospective primary teachers which in turn will affect the standard of mathematics teaching future generations of pupils will experience. Aware that if the phenomenon was left unchallenged, a vicious cycle of shallow knowledge and negative attitudes was likely to persist, it was considered essential to set in motion a process to address the issue (C.B.M.S., 2001; N.C.C.A., 2006; Hourigan and O' Donoghue, 2007).

6. The Context of this Study

While Oldham (2005) recommends that the mathematics subject matter knowledge of some entrants to primary teacher education needs enhancement, unfortunately in reality the "…subject matter preparation of teachers is rarely the focus of any phase of teacher education" (Ball, 1990: 465). Despite some changes in recent times, it is still the case for many Irish prospective primary teachers, including those attending the College of Education within this study, that the sole form of preparation for teaching mathematics is the mathematics pedagogy course. As these courses are expected, within limited time constraints, to provide student-teachers with the necessary knowledge to teach mathematics at all primary class levels, it is not surprising that finding the time to explicitly address student-teachers' mathematics subject matter knowledge proves problematic (Wall, 2001; An Roinn Oideachas agus Eolaíochta, 2002; Corcoran, 2005; Leavy and O' Loughlin, 2006). Consequently within these courses it is often taken for granted that the mathematical subject matter knowledge relating to the various concepts and procedures was addressed 'somewhere else' e.g. pre-tertiary mathematics. In such contexts no distinction is generally made between knowledge of content and knowledge of how to teach it (Ball, 1990; Wall, 2001; C.B.M.S., 2001; Rowland et al, 2005). Unfortunately in light of the reported nature of many student teachers' pre-tertiary mathematics experience (See section 4.3), such assumptions are unfounded and have serious implications (Oldham, 2005; N.C.C.A., 2006; Hourigan and O' Donoghue, 2007). While the authors initially considered the possibility of extending existing mathematics education courses in order to facilitate the desirable alterations, this option was subsequently deemed unfeasible given students' heavy timetables and workloads. Instead it was decided that some 'extra' provision outside the mainstream provision would be initiated in a bid to meet the identified needs of prospective primary teachers within the College of Education.

The author (MH) endeavoured to develop an initiative to address the phenomenon of substandard mathematics subject matter knowledge, which would reflect the practices of universities worldwide who have set up initiatives aimed at remediation. In line with best practice, the broad objectives for this research would be achieved through a sequence of events moving from 'Diagnosis' to 'Prescription' to 'Aftercare' (Murphy, 2002). This approach would consist of a systematic process of collecting and analysing data regarding the nature of the participating student teachers' mathematics subject matter knowledge which would increase the authors' understanding of phenomenon. Such insight would subsequently facilitate the development of a purpose-built intervention in a bid to provide prospective teachers with an opportunity to develop their mathematics subject matter knowledge of the concepts and procedures they will teach (Murphy, 2002; Edwards, 2003).

6.1 The Sample

While the author was aware that many pre-service providers initiate the process of a diagnosis and remediation programme from entrance to course onwards in order to develop instructional practices based on a sound knowledge base, this was not practical in economic terms in this study (C.B.M.S., 2001; Goulding, 2003; Rowland et al, 2005; Corcoran, 2005 (b)). The sampling technique utilized for the study was purposive i.e. a particular cohort of students within the wider student-teacher population was targeted (Cohen et al, 2000; Mertens, 2005). After discussion with colleagues, the sample selected for the purposes of this study was the cohort of second year prospective teachers. As their mathematics pedagogy course during the spring semester addressed mathematics issues directly related to the cohorts' subsequent senior teaching practice placement, it was suggested that these student-teachers would be optimally motivated to partake in the study as it would be perceived to be relevant to their immediate needs (Rowland et al, 1999). In light of these facts, the authors believed that this particular group of students would prove to be information-rich for the purposes of the present study (Mertens, 2005).

6.2 Methodology and Methods

In line with the pragmatic approach adopted by the authors that no one paradigm ensures a perfect grasp of the 'truth', the methodology and methods were determined on the basis of a 'fitness for purpose' criterion or 'What works?' (Cohen et al, 2000;

Mertens, 2005). Once the author had a clear insight into the objectives of the study, selecting the optimum methodology to 'fit' the problem at hand was a straightforward process. As the author (MH) intended to carry out on-the job professional enquiry in order to gain understanding of a problematic situation and subsequently address issues and problems arising within this context, the author selected an action research approach as the most appropriate research methodology (Colleran, 2002; Opie, 2004; Mertens, 2005). The research study falls into two identifiable cycles of diagnosis, intervention implementation and evaluation (i.e. Cycle 1 or Preliminary study and Cycle 2 or Main study). The first cycle began in February 2006 and concluded in May 2006 with a selected cohort of prospective teachers. The lessons from the implementation and evaluation of cycle 1 provided an improved service during cycle 2. The second cycle, which built upon the learning of the preliminary phase, commenced in February 2007 with the subsequent cohort of student teachers.

The author selected Elliot's (1991) model of action research on the grounds that this model promoted a smooth transition between the stages and cycles. This model facilitated the development of an 'initial idea'. This statement of what the researcher wished to improve was based on reflection of both the review of relevant research and experiences i.e. to address the issue of substandard mathematics subject matter knowledge among prospective teachers (Cohen et al, 2000; Colleran, 2002). The reconnaissance stage facilitated the quantification of the extent to which substandard subject matter knowledge posed a problem for pre-service teachers within the College of Education in question (Murphy, 2002). The investigation process was further focused through the development of appropriate testable hypotheses and research questions. Subsequently it was necessary to select, develop and administer the data collection methods deemed most compatible with the research questions bearing in mind the study's context and participants e.g. time available, population size. The findings of the previous stage facilitated the design of an appropriate general plan i.e. the development and implementation of a suitable intervention as well as the evaluation of its effects. Analysis of the evidence from various data collection methods facilitated the researcher in drawing conclusions regarding the effectiveness of the initiative in meeting its proposed goals and in making the necessary modifications to a subsequent cycle (Elliot, 1991).

Throughout both cycles, a multi-method approach was adopted i.e. both qualitative and quantitative strategies were used to address research questions in an effort to secure broader and better results through triangulation (Mertens, 2005). The methods employed serve to add to the researchers' knowledge of the phenomenon (Cohen et al, 2000). The various stages of the action research and the corresponding data collection methods used are summarized in Appendix 1.

6.3 Ethical Considerations
The study meets the ethical requirements of MIC and University of Limerick Ethics Committees. Prior to and during the implementation of the research study it was necessary to fulfill a number of ethical obligations. Initially it was necessary to formally seek consent and support from management and colleagues within the educational setting to pursue the research and access to the selected sample group. Subsequently in order to achieve the respect and trust of the potential participants within the study, it was essential to ensure that their dignity, privacy and interests were respected at all times. Measures utilized included the presentation of information (both orally and in written

form (information sheet)) to the entire cohort of prospective participants regarding the nature of the initiative (e.g. voluntary participation), its purposes and procedures as well as the potential benefits of participation prior to the initiation of each cycle. Students were advised that they could withdraw from the study without penalty at any stage of the study. While within the preliminary study confidentiality was assured to all participants, the promise of anonymity was viable for participants within the main study as project identification codes were developed for each participant. Throughout the initiative, arrangements were made to provide feedback (in various forms) as a matter of form or on request. Prior to the commencement of the respective cycles of action research participants were requested to give their consent for their personal data to be utilized for initiative and research purposes (Cohen et al, 2000).

7. Cycle 1- Reconnaissance Stage: The Process of Determining if Concerns are Warranted?

The reconnaissance stage of the initial cycle strove to acquire evidence of the actual nature of the mathematics subject matter knowledge among the participating prospective primary teachers for the purposes of providing feedback to the participants themselves as well as informing the development of an intervention to address the issue of inadequate subject knowledge (Nitko, 2001; Murphy, 2002; Corcoran, 2005).

7.1 The Data Collection Method

Given the weaknesses of the pre-tertiary system and consequently the Leaving Certificate mathematics grades as a predictor of ‘preparedness’, it was necessary to objectively assess student-teachers’ existing levels of subject knowledge (Nitko, 2001). The author decided to administer a mathematical pre-test. The purpose of the test was to determine the extent to which student-teachers have an acceptable grasp of the essential mathematical subject matter knowledge required to teach at primary level which in turn would facilitate decision making on a number of levels (See hypotheses/research questions in Appendix 2) (Tsang and Rowland, 2005).

Decisions regarding the testing procedure were made after due consideration of practices and procedures within a variety of third level institutions internationally (Murphy, 2002; Edwards, 2003; Learning and Teaching Support Network (L.T.S.N.), 2003; Corcoran, 2005; Mertens, 2005; N.C.C.A., 2006). As part of this process, the author analysed a number of measurement instruments previously utilized to gauge student teachers mathematic subject matter knowledge e.g. Sigma-T, SKIMA in a bid to evaluate their fitness of purpose within the present study. The author wished that the test instrument would reflect the mathematics subject matter knowledge required to fulfill their professional obligations i.e. reflecting the learning outcomes of Revised mathematics curriculum to the highest level (An Roinn Oideachas agus Eolaíochta, 2002; Nitko, 2001). Because the various instruments were not developed exclusively to reflect the Irish Mathematics Curriculum (1999), various mismatches in both content and thinking skills tested meant that none were deemed fit for this study’s requirements (Wall, 2001; Corcoran, 2005). Subsequently the authors felt that the development of a purpose-built test was necessary (Cohen et al, 2000).

7.1.1 Development of Pre-test Instrument

The author (MH), with the support of a colleague, set about creating an appropriate paper-based assessment instrument tailored to the needs of the study (Nitko, 2001; L.T.S.N., 2003). A criterion-referenced test was selected as the most appropriate tool within the present study i.e. it facilitated the testing of participants' ability to demonstrate desirable mathematical concepts and procedures (Cohen et al, 2000; Wall, 2001).

In terms of specific topics and knowledge to be measured, given that the Curriculum and its accompanying Teachers Guidelines are the only official documents which specify the mathematics that primary teachers need to know, it was decided that the test items would represent the breadth and depth of the Revised mathematics curriculum i.e. reflect curriculum objectives for each strand (i.e. Number, Shape & Space, Measures, Algebra and Data) and mathematical skills (i.e. Applying & Problem Solving, Integrating and Connecting, Reasoning, Implementing, Understanding and recall) promoted within the curriculum at the highest level (N.C.C.A., 1999; Wall, 2001; Nitko, 2001; Cohen et al, 2000; Corcoran, 2005). The development of appropriate test items was facilitated by referring to curriculum documents, existing measures as well as textbook activities (Wall, 2001; Ball et al, 2005). The subject knowledge required within the instrument did not, for the most part, extend beyond the 6th class mathematics curriculum objectives. In total the test instrument consisted of 41 'completion' items, most of which were closed-response tasks requiring participants to construct their own answers (in the rough work section), and independently place the final answer in the space provided (Nitko, 2001). A small minority of the test items were 'response-choice' items requiring participants to explain/justify the response deemed correct (Cohen et al, 2000; Nitko, 2001). The format selected facilitated the author in gaining insight into participants' level of understanding as well as the nature of their misunderstandings (Wall, 2001). The instrument does not, however, demonstrate the traditional characteristics of a criterion-referenced test (i.e. mastery determined by performance on 3 items per objective) (Nitko, 2001). While the development of additional items for the curriculum objectives was possible, such a test would be extremely off-putting for potential participants given the voluntary nature of participation and the unreasonable amount of time required to complete such an instrument (Wall, 2001).

The authors were aware that this mathematics test focuses primarily on 'common' content knowledge or knowledge that many adults would be able to demonstrate i.e. working through computations and solving problems (Goulding, 2002). There were a few exceptions within the tests, where 'specialised' knowledge e.g. item requiring the justification or rejection of the statement *'A square is a special type of rectangle- True or False? Explain'*. The author is not advocating that "If you can "do" these items, you can teach them" (Ball, 1990: 462). Although at face value it could be argued that high performance in the test does not guarantee that participants possess the subject matter knowledge required to teach mathematics, participating student teachers were requested to reflect as to whether they possessed the conceptual knowledge associated with each object i.e. if they could explain why (Hill et al, 2004). Therefore it was intended that the pre-test would act as a 'self-audit' for participants to make them aware of the mathematics concepts and procedures required for the purposes of teaching which in turn would facilitate them in identifying weaknesses in both their 'common' and 'specialised' mathematics subject matter knowledge (Goulding, 2003).

7.1.2 Ensuring the Reliability and Validity

The 'content validity' of the test i.e. the extent to which the assessment instrument is representative of the mathematics subject knowledge that primary teachers require was assured through the distribution of the draft instrument to a 'jury of experts (i.e.

colleagues and practicing primary teachers of the senior classes requesting them to review the 'fitness-for- purpose of the instrument and provide feedback (Mertens, 2005). Piloting of the test on a group of student teachers (N=89) facilitated feedback regarding issues such as clarity of directions, ambiguity of wording as well as completion times etc. (Cohen et al, 2000; Wall, 2001; Mertens, 2005).

7.1.3 Data Collection: Administration of Pre-test
Prior to test administration, the author ensured that all members of the cohort were made aware of the initiative generally and the assessment process (See section 6.3). During the information session details provided included the purpose and content of the test instrument as well as scoring criteria and subsequent processes (e.g. feedback, use of results) (Nitko, 2001; L.T.S.N., 2003). While it was possible to provide student-teachers with reading lists to facilitate preparation, the author wished the results to reflect what mathematics subject knowledge the students had at their fingertips (Cohen et al, 2000; Wall, 2001; Murphy, 2002).

The test was administered on the second week of semester in a bid to ensure that the test results reflected the subject knowledge achieved through pre-tertiary mathematics experiences. The test was administered in independent timeslots i.e. outside the students' mathematics pedagogy sessions, which meant that participants had to sacrifice some of their personal time in order to partake. The test was initially administered in large venues on Wednesday, February 15th (Week 2) at 1p.m. and 3p.m.). Although 'on paper' all student-teachers within the cohort were in a position to attend at least one of the two times selected, it soon became apparent that this was not the case. In order to facilitate all interested students to take the test, two extra testing sessions were subsequently organised for Week 3 (Tuesday, February 21st at 2p.m. and Thursday, February 23rd at 10a.m.). Students were informed of this opportunity via both announcement and notice.

The same conditions i.e. instructions regarding test completion and time allocated were provided for each testing session to ensure the reliability of findings. Prior to taking the pre-test, participants were required to complete a 'consent form' stating that they had received adequate information and that they permitted the author to obtain and use their personal information (e.g. test results, Leaving Certificate grade) for the purposes of the initiative and research (Cohen et al, 2000; Wall, 2001). They were also requested to indicate contact information for feedback purposes. In all, 163 undergraduate student-teachers attended and completed the pre-test i.e. approximately one third of the total cohort of second year students invited to participate in the initiative.

7.1.4 Test Correction, Data analysis and Use of Findings
Reflecting common practice within the numerate sector, the author selected an 'all or nothing' marking scheme i.e. correct answer = 1 mark; incorrect answer= 0 marks, partially completed item = 0 marks (Cohen et al, 2000; Murphy, 2002). The quantitative data available as a result of the pre-test process ranged from nominal (correct/incorrect) to ordinal (LC grades, level of mathematical study) to ratio data (total scores, scores in subsections). Data analysis carried out on the relevant data, using the tools of SPSS Version 14, included descriptive statistical measures as well as inferential statistics. Further analysis of the collected data was facilitated through the comparison of subgroups within the student population. The author sought insight into the apparent relationship between student teachers' level of mathematics study to date and their

performance within the pre-test (Wall, 2001; Murphy, 2002). While the author also acquired additional qualitative information into the nature of student-teachers deficits i.e. through error analysis, the nature and outcomes of this process will not be addressed explicitly here (Wall, 2001; Nitko, 2001).

The data analysis processes, both quantitative and qualitative, facilitated the author to gain insight into the needs of the participants (areas of strength and weakness, nature of misconceptions), which would prove invaluable in informing and guiding decisions regarding the nature of the optimum support structures (e.g. content, process and instructional decisions) (Wall, 2001; Murphy, 2002; Nitko, 2001; L.T.S.N., 2003). Participants also received individualised feedback regarding their performance in the pre-test within three weeks of sitting the diagnostic test i.e. on Monday, March 6th (Week 5) by e-mail (or phone if so indicated) in the form of raw scores for each test item (1/0), and as the number of correct answers as a function of the total responses in each section (strand) (e.g. number score/16) and the whole test (score /41) (Cohen et al, 2000). The author chose not to provide participants with a specific 'cut-off point' to guide their decisions regarding intervention participation, given the limited nature of this single indicator of their mathematics subject matter knowledge. While the preliminary test feedback was numerical, due to the pressures of large numbers and time, each participant was invited to view their script thus facilitating them to self-assess their strengths, misunderstandings and needs. Unfortunately few participants availed of this service. It was not viable to 'return' the tests as the author planed to administer the same instrument (with perhaps minor modifications) to subsequent cohorts in order to develop a data base of student performance. The feedback mechanism facilitated participants to evaluate their existing level of mathematical readiness and subsequently make informed decisions (L.T.S.N., 2003).

7.2 Findings

7.2.1Profiling the Populations' Strengths and Weaknesses

The initial profiling of 'strengths and 'weaknesses' made the author as well as the participants' themselves aware of levels of understanding within various concepts tested (Elliot, 1991; Wall, 2001). The average score for the test was 26.46 out of a possible 41 or 65% correct. While the modal score was 31, participants' scores ranged from 6 to 39. It is interesting that no student achieved full marks in the test. The test scores were categorized for the purposes of analysis i.e. 0-21; 21-30 and over 31. The author found that while almost one fifth (17.8%) of the population achieved a score of 20 or less (i.e. the lowest band), 33.1% of the population performed at the top band (i.e. 30-39). Almost half of the pre-test respondents performed in the middle band i.e. 49.1%.

Subsequently participants' performance within the various subsections was analysed in order to gain further into the strengths and weaknesses of the participant population. The average score for the 'Number' section, the Measures section and Algebra section were 63%, 61% and 61% respectively, while the average scores for the other two test subsections namely 'Shape and Space' and 'Data' were 67% and 76% respectively. While informative, no attempt was made to compare the populations' performance within the subsections, given the possibility of variability in difficulty between the subsections. Further information was required to facilitate the analysis of populations' specific strengths and weaknesses (Wall, 2001).

The population's performance on individual test items was also gauged. Percentages of mastery suggested that a number of individual items posed little difficulty for the participants, demonstrating that these students possessed adequate 'common knowledge' of the concepts and procedures in question. For example while almost all members of the population (over 90%) successfully completed items requiring them to add and subtract decimals and find the Highest Common Factor of two numbers, a majority of them (approximately 75-90%) demonstrated an understanding of place value, implemented the procedures to calculate the perimeter and mean, utilised knowledge of the angles within a circle/triangle to label specific angles, and solved word problems within real world contexts requiring an understanding of ratios, percentages, value for money, time, and directed numbers. The author was acutely aware, however, that a high level of mastery on specific items was not an assurance of conceptual knowledge among the population.

An informal analysis of students' rough work during correction suggested that basic fact errors or 'slips' or an inability to recall facts and definitions e.g. meaning of 'mode' or prime number or formulas and rules e.g. the angle with a circle $=360^{o}$ were the source of many 'errors'. Also a number of items on which a high proportion of the population demonstrated inadequate 'common' subject matter knowledge were an immediate source of concern in light of the importance of these fundamental concepts within the curriculum and that conceptual knowledge of these concepts and procedures would have facilitated item solution. For example while only 63.8% successfully completed an item requiring the subtraction of mixed fractions (requiring decomposition), just over half (51.5%) of the population successfully found the area of a shaded shape. Further items caused substantial difficulty for many participants, with fewer than half of the participants demonstrating the 'common' subject matter knowledge required to solve them. While 41.1% of participants successfully found the 'product' of two decimal numbers, 46% demonstrated the ability to divide by a decimal. An item requiring students to demonstrate an understanding of the connections within mathematics i.e. to order a list of fractions, decimals and percentages proved too demanding for 57.7% of the population (i.e. 42.3% correct). A further source of unease resulted from the fact that a number of items requiring higher order skills such as connection making and reasoning proved problematic for all except two fifths of the population e.g. finding the area of an irregular shape (44.2% correct), solving problems involving percentage profit (39.9%) and speed (38.7).

The small numbers of participating student-teachers demonstrating 'common' subject matter knowledge on items requesting the knowledge and understanding of concepts and procedures such as the Lowest common multiple (31.3%), division involving fractions (27.6%) and fractions within a story problem context (22.1%) is a reasonable source of concern in light of the demands of the Revised Curriculum (1999) (N.C.C.A., 1999). While in many cases these students may have forgotten the 'algorithm' or the 'rule', their inability to work through the solution suggests that they lack an understanding of the concepts required to make the concept associations and connections required. Overall the results demonstrate that while participants demonstrate high levels of proficiency in items which can be solved through the utilisation of recall and procedural knowledge in relatively context free or simple contexts, many participants demonstrated limited conceptual understanding when completing items requiring connection making and/or reasoning. In light of these findings it is questionable as to whether such students possess 'specialised' subject knowledge. Support for this belief comes from the fact that the test item which explicitly assessed the participants' 'specialised' mathematics subject matter knowledge i.e. the properties of 2-d shapes was poorly answered, with only 24.7% of the participants' able to explain the relationship.

7.2.2 Level of Mathematics Experience as a predictor of 'Common' Mathematics Knowledge

The authors were also interested to investigate the characteristics of the participants using the pre-test i.e. their previous mathematics experience in a bid to explore the ability of such factors to predict participants' 'common' mathematics subject matter knowledge as gauged by the pre-test. Of the 159 student-teachers whose Leaving grade could be reliably verified, 66% studied the Leaving Certificate mathematics course at Ordinary level (OL), while the remaining 34% had taken Higher Level (HL). Further insight into the Leaving Certificate performance of the pre-test population i.e. % of population who achieved various grades can be attained from the table below.

Table 1: Participants' Leaving Cert Grades

Grade	Ordinary Level	Higher Level
A (A1/A2)	30.8%	3.1%
B (B1/B2/B3)	27%	17%
C (C1/C2/C3)	6.9%	9.4%
D (D1/D2/D3)	1.3%	4.4%

While the fact that the entire spectrum of Leaving Certificate grades is represented in the group who participated in the pre-test may mean that many of these student-teachers are conscientious and wish to take the opportunity to gain insight into present level of subject matter knowledge. This response may also suggest that these student-teachers, regardless of their achievements in mathematics, lack confidence in their personal knowledge and understanding of mathematical concepts and procedures required for teaching the subject. Analysis of the relationship between Leaving Certificate and pre-test performance, as measured by Spearman's rank order co-efficient, uncovered that there was a strong positive and significant correlation [$r = .657$, $n = 159$, $p<0.01$] between the two factors. Cross-tabulation of these factors provided some interesting findings. Having taught this course the author was not surprised to find that that the two OL 'D' students achieved scores of 15 and 21 (out of 41). The poor performance in this pre-test among students who have achieved grades far exceeding the minimum requirements, however, was a source of disquiet. Approximately half of the OL 'C' students (i.e. 5 of the 11 students) and 23% of the OL 'B' students (10 of the 43 students) achieved a score of less than 20. Among student-teachers who achieved an 'A' in the OL (N= 49), 6% performed at a similar level. These findings further question the reliability of the Leaving Certificate grades as a valid predictor of student-teachers' ability to demonstrate mathematical concepts and procedures required to present the Revised Primary Curriculum (See section 4.3).

Exploration of the relationship between the populations' level of mathematics study and their pre-test scores provides insight into previous proposals that studying mathematics to degree level is not essential for primary teachers (See Section 3). While there was a significant positive correlation between the two factors [Spearman's rank order correlation: $r = .447$, $n = 163$, $r<0.01$], analysis, through cross-tabulation, found that while higher proportions of the prospective teachers studying mathematics beyond

Leaving Certificate demonstrated high levels of ‘common’ subject matter knowledge mastery, as gauged by the pre-test, equivalent subject knowledge was demonstrated by a number of participants who had not studied mathematics beyond Leaving Certificate. Therefore, while 81.3% of ‘major’ students (studying mathematics to degree level- 13 students) achieved a score exceeding 30, 63.2% of ‘minor students (studied mathematics as an academic subject in 1st year of their preservice course- 12 students) and 23% of ‘Leaving Certificate’ students (i.e. 20 students) demonstrated similar levels of proficiency.

8. Conclusion: Overall Contribution of the Pre-test within the Initiative

The reconnaissance process provided insight into the nature of the phenomenon of mathematics subject matter knowledge within one Irish College of Education. The authors believe that the high response rate suggests that many of the participants had existing personal concerns regarding their mathematics subject matter knowledge.

While this test does not begin to reflect the demands placed on teachers’ mathematical subject matter knowledge when responding to pupils answers/queries, many prospective teachers are demonstrating substantial difficulties. Pre-test findings highlight a significant mismatch between the fundamental ‘common’ subject matter knowledge required to teach at primary level and that demonstrated by a proportion of the population within the pre-test. The above levels of mastery reflect previous findings both nationally and internationally that a proportion of prospective teachers demonstrate thin knowledge, generally relying on rules and procedures as opposed to conceptual understanding of mathematical concepts and procedures (Ball, 1990; Ma, 1999; Wall, 2001; Corcoran, 2005; Rowland et al, 2005; Ball et al, 2005). It is no surprise in light of the reported nature of prospective teachers’ pre-tertiary mathematics experiences (See section 4.3) that a substantial proportion of the population despite having satisfied minimum entry requirements demonstrated weak ‘common’ subject knowledge of a number of the concepts and procedures required for teaching.

The author felt that despite the limitations of the pre-test instrument used, the reconnaissance stage of the initial cycle of action research within this study fulfilled a number of extremely important functions. As well as providing valuable information on the characteristics and needs of the population, which facilitates informed decisions regarding the nature of optimum support structures, the pre-test acted as a device to “surface and challenge” prospective teachers’ awareness of and desire to develop existing levels of subject matter knowledge (Goulding, 2002: 2).

Appendices:

Appendix 1: *Stages of Action Research Cycles and Methods Used*

Cycle 1:

o **Identify Initial Idea**: Address the issue of substandard mathematics subject matter knowledge among prospective primary teachers

o **Reconnaissance:** Collection of data regarding the phenomenon (M*athematical pre-testing*)

o **Devise general plan:** Develop needs-led intervention

o **Implement action step:** Pilot implementation of the intervention programme i.e. Professional mathematics Programme o **Monitor implementation and effects:** Monitor and evaluate implementation stage and effects (*Usage Statistics, Reflective Diary, Mathematical post-test, Survey*)

Cycle 2:

o **Reconnaissance:** Explain any difficulties, failures and effects through pilot case report and collect additional data regarding the phenomenon from the new cohort (*Mathematical pre-testing, Pre-survey*)

o **Amended plan:** Amend plan on the basis of pilot discoveries

o **Implement next action step:** Implement the main stage of the intervention i.e. Professional mathematics programme

o **Monitor implementation and effects:** Monitor main stage of intervention implementation and effects, explaining difficulties, developments and findings through case study report, making suggestions for future cycles of the process (*Usage Statistics, Reflective Diary, Mathematical post-test, Post-Survey, Interview, Observation of Teaching*)

Appendix 2: *Cycle 1: Reconnaissance Stage Hypotheses and Research Questions*

o Pre-tertiary experiences do not provide student teachers participants with the mathematics subject knowledge deemed essential to teach mathematics effectively (as indicated by the pre-test)

o The entry requirement in mathematics within the Leaving Certificate for entry to teaching (O/H D3) does not guarantee that student teachers possess adequate mathematics subject knowledge

o There is a relationship between student teachers' mathematics Leaving Certificate performance (level and grade) and their existing levels of mathematics subject knowledge as indicated by the pre-test

o There is a relationship between student teachers' level of mathematics study and their existing levels of mathematics subject knowledge as indicated by the pre-test

o What types of mathematical gaps and misconceptions were evident among participants?

o Were there any particular content areas and/or mathematical skills where participants demonstrated substantial difficulties?

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Effective Teachers of Mathematics: Insights from an Australian Research Project

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The effective teaching of mathematics is a concern of teachers and researchers alike. This paper provides insights into what the practices of effective teaching might look like, drawing on findings from the Early Numeracy Research Project, a major, three year project undertaken in Victoria, Australia.. Assessment data revealed marked differences between classes in growth in students' understandings which we attributed to the teachers. To gain insights into the practices of effective teachers, six case studies of highly effective teachers at the Prep (first year of school) to Grade 2 level were conducted. From this research, commonalities as well as differences emerged. The paper describes the methodology used, identifies the 25 themes identified as practices and attributes of these highly effective teachers, and illustrates a selection of these. It is also suggested that these 25 themes can potentially inform teaching at other levels and can be of use in work with inservice and preservice teachers.

Introduction

In working with preservice teachers I stress that it is they who make the decisions as to what they will do and how they will do it when teaching a mathematics lesson. That is, for example, they make the decisions of what words to use when introducing a task, what interactions to have with individual children, or which children's responses to focus on, and exactly how to utilise these during lesson share time. Even when teachers plan with others or follow a lesson from a published resource, they each make what might be called the fine-grain decisions of teaching. Research conducted by Boaler (2003) indicated that "it is not the fact that students work in groups, or listen to the teacher, that is important, it is how they work in groups, what the teacher says, and how the students respond [that is important]" (p. 6). Likewise, in reflecting on a range of research studies, Brown (1999) noted that "quality of teaching is more important than class organisation. … it's not whether it's whole-class, small group or individual teaching but rather what you teach and how you interact mathematically with children which seems to count" (p. 7). This suggests that when following a lesson procedure, or perhaps even utilising a quality game, the work of teaching can look very different, and thus a common structure cannot guarantee a highly effective mathematics lesson. Individual teachers each play a key role in determining the nuances of each lesson and therefore in contributing to their children's growth in mathematical understanding.

In reflecting on teaching, and the fine grain decisions made by teachers, it is worthwhile to consider what is known about effective teachers of mathematics. This paper provides insights into how effective teaching might look in an early years classroom by sharing findings from intensive case studies of six highly effective teachers. These teachers were participating in a large scale numeracy project, focusing on the first three years of school, and they were chosen because of the particularly impressive growth in their students' understanding of mathematics, as revealed by a one-on-one interview. It is believed that descriptions of the practices of these particularly effective teachers that arose from these case studies provide a framework that can be useful for teachers examining their practice and for all those involved in mathematics teacher education at preservice and inservice levels. It is also suggested that the practices of highly effective teachers identified within the ENRP may have applicability to levels beyond the early years. Similar research in Grades 3-12 (and possibly beyond) well may yield many elements in common with those identified within the ENRP.

The practices of highly effective teachers

The teaching of mathematics is complex. In addition, what constitutes good teaching is controversial (Frank, Kazemi, & Battey, 2007). However, in regard to mathematics, there is widespread concern for good teaching practice (e.g., Australian Association of Mathematics Teachers, 2006; National Council for of Teachers of Mathematics).

Whether from experiences as a teacher or student, those involved in teaching and teacher education would most likely have formed views on the characteristics of effective teachers of mathematics. For example, a study by Wilson, Cooney, and Stinson (2005) indicates that teachers have views of what constitutes good mathematics teaching, with nine secondary teachers revealing they believe it includes

- connecting mathematics between topics and to the real world;
- helping students "see" or visualise the mathematics through the use of computers or calculators, drawings, or concrete materials;
- assessing student understanding (portrayed within lessons as frequent and quick evaluation to inform instruction);
- refraining from telling;
- engaging and motivating students;
- managing effectively; and
- reflecting on teaching.

My own experiences with preservice teachers show that they also hold beliefs about characteristics of highly effective teachers of numeracy. For example, one group when asked included: using open-ended tasks, focusing on the process rather then product, exploring how children work out responses, encouraging questioning, being aware of students' needs and capabilities, and being positive and approachable.

While it is good to reflect on and articulate such beliefs, it is recommended that these beliefs are then considered in light of findings from research studies of the practices of effective teachers of numeracy as such comparison may extend or refine one's perceptions of the practices of an effective teacher.

For many years researchers have sought to describe teacher behaviours that correlate positively with growth in student achievement. For example, as early as the 1970s and 1980s, the so-called *process-product* research sought to describe behaviours that correlated positively with student achievement (See, e.g., Brophy & Good, 1986; Wilson et al., 2005). Wilson et al. believe that within these studies, effective teaching was primarily about what teachers did as an activity in itself rather than as an activity based on student thinking.

More recent research has provided a range of further insights into the practices of effective teachers.

In a major study of effective primary school mathematics teaching in the United Kingdom, Askew, Brown, Rhodes, Johnson and Wiliam (1997) studied the practices of ninety teachers, for whom student test results indicated varying levels of effectiveness. Using mean class gains on a test involving aspects of the number system, computation, and problem solving, aurally administered by teachers with children writing their answers, teachers were grouped into broad categories according to effectiveness. Insights into the practices of effective teachers came primarily from focus schools and case studies of individual teachers, six of whom were identified as highly effective.

The highly effective teachers

- connected different ideas of mathematics and different representations of each idea by means of a variety of words, symbols and diagrams;
- encouraged students to describe their methods and their reasoning, and used these descriptions as a way of developing understanding through establishing and emphasising connections;
- emphasised the importance of using whatever mental, written or electronic methods are most efficient for the problem at hand; and
- particularly emphasised the development of mental skills.
- The researchers also identified the importance of teacher beliefs and understandings of the mathematical and pedagogical purposes of classroom practices.

It is of interest to note that the less effective teachers gave priority to pupils acquiring standard arithmetical methods to solve problems in preference to students considering alternative, more efficient ways. Lower numeracy gains also resulted from teachers who gave priority to the use of practical equipment or any other method children felt comfortable with and delayed the introduction of more abstract ideas until they felt a child was ready for them.

Brown, Askew, Baker, Denvir, and Millett (1998, p. 373) noted that international observational studies "seem to show some agreement on some of the aspects of teacher quality which correlate with attainment. These included

- -the use of higher order questions, statements and tasks which require thought rather than practice;
- -emphasis on establishing, through dialogue, meanings and connections between different mathematical ideas and contexts;
- -collaborative problem solving in class and small group settings; and
- -more autonomy for students to develop and discuss their own methods and ideas".

More recently, Muir (2007) summarised characteristics of effective numeracy teachers as they

- -maintained a focus on and taught for conceptual understanding of important mathematical ideas;
- -used a variety of teaching approaches which foster connections between both different areas of mathematics and previous mathematical experiences;
- -encouraged purposeful discussion through the use of question types to probe and challenge children's thinking and reasoning and encouraging children to explain their mathematical thinking; and
- -possessed knowledge and awareness of conceptual connections between the areas which they taught of the primary mathematics curriculum and confidence in their own knowledge of mathematics (p. 513).

-Self-report data from a study of effective teachers of Length in the first year of school (Sullivan & McDonough, 2002), indicate that being effective in the teaching of Length at this level included that the teachers

- -had a clear vision of the mathematical experiences needed;
- -were able to engage the students; and
- -were prepared to probe the thinking and understanding of the children.

The more recent studies cited here suggest effective teaching includes consideration of student thinking. Perhaps this is reflected in the National Council of Teachers of Mathematics (2000) statement that "Effective mathematics teaching requires understanding what students know and need to learn and then challenging them to learn it well" (p. 16). The six intensive case studies discussed in this paper provide much support for the findings of others as discussed above, while also adding new insights to the discussion.

The Early Numeracy Research Project

The case studies discussed in this paper took place within the Early Numeracy Research Project (ENRP)[1] conducted from 1999-2001. The ENRP investigated mathematics teaching and learning in the first three years of school, and involved Prep to Grade 2 teachers and children in 35 project ("trial") schools and 35 control ("reference") schools.

The three key components within the ENRP were the following:

- the development of a research-based framework of "growth points" in young children's mathematical learning (in Number, Measurement and Space);
- a 40-minute, one-on-one interview, used by all teachers to assess aspects of the mathematical knowledge of all children at the beginning and end of the school year (February/March and November respectively); and
- extensive professional development at central, regional and school levels, for all teachers, coordinators, and principals.

The ENRP framework of growth points in young children's mathematical learning encompassed nine mathematical domains within three strands: Number (Counting, Place value, Addition and subtraction strategies, Multiplication and division strategies); Measurement (Length, Mass, Time); and Space (Properties of shape, Visualisation and orientation).

Within each domain typically five or six growth points, developed by drawing and building on previous research, were stated with brief descriptors for each. For example, there are five growth points in the domain of Length as shown in Figure 1.

0. Not apparent
   No apparent awareness of the attribute of length and its descriptive language.
1. Awareness of the attribute of length and use of descriptive language
   Awareness of the attribute of length and its descriptive language.
2. Comparing, ordering, & matching with the attribute of length
   Compares, orders, & matches objects by length.
3. Quantifying length accurately, using units and attending to measurement principles
   Uses uniform units appropriately, assigning number and unit to the measure.
4. Choosing standard units for estimating and measuring length, with accuracy
   Uses standard units for estimating and measuring length, with accuracy.
5. Applying knowledge, skills and concepts of length
   Can solve a range of problems involving key concepts of length.

Figure 1. ENRP Growth Points for Length

Growth points, sometimes thought of as key "stepping stones" in children's developing understandings, are big ideas. There is also growth between them. Typically, in reference schools, children took nearly one year to move one growth point in Length.

The ENRP Growth Points informed the creation of assessment items for the one-on-one interview titled the Early Numeracy Interview, and the recording, scoring and subsequent analysis. The interview is interactive and hands-on, and involves about 60 tasks. It has been described as being like a "Choose your own adventure" experience as at any one time within a domain, and as instructed by the interview script, the interviewer continues with questions in that domain for as long as the child is successful, or if the child is unsuccessful the interviewer moves to a detour task so that further insights can be gained into the child's difficulty, or moves to the next domain. With the focus on solving problems, the children's responses provide teachers with rich insights into student understandings and strategy use.

To demonstrate the nature of the interview, two Length tasks, designed to give insights into children's understandings related to growth points 1, 2, and 3, are provided (Figure 2).

The String and the Stick
Drop the string and the skewer onto the table.
a) By just looking (without touching), which is longer: the string or the stick?
b) How could you check? *(touching is fine now)*
c) So, . . . , which is longer?
The Straw and the Paper Clips
Get the straw and show the child the eight paper clips.
Here are some paper clips. Here is a straw.
a) Measure how long the straw is with the paper clips. . . . (*If child hesitates*) Use the paper clips to measure the straw.
b) What did you find? *(no prompting)*
If correct number is given (e.g., 4), but no units, ask "4 what?"

Figure 2. Two Length tasks from the Early Numeracy Interview

Interviews were conducted by the classroom teachers, who were trained in all aspects of interviewing and recording. The processes for assuring reliability of scoring and coding are outlined in Rowley and Horne (2000).

The data from the ENRP arise from intensive interviews with large numbers of children, with trained interviewers, and experienced coders, with double data entry, and using a framework for learning based on interpretation of research. It is argued that these data provide a reliable measure of learning, and a further perspective on previous research on the ways that teachers influence student learning.

Identifying highly effective ENRP teachers

While the three key components of the ENRP, as listed above, informed, involved, and potentially empowered the project teachers, it was the teachers, professional learning teams, and schools who ultimately made the decisions of whether and how the information and experiences provided within the project would impact upon their classroom practice. The approach taken was not to provide teachers with a recipe, but rather with rich ingredients that they might choose in meeting the needs of individual children, the mathematics and the teaching context. In this way, the ENRP teachers were viewed as co-researchers and professionals who would use their own judgment. Thus, the practices of the highly effective teachers were not determined by the researchers and could not be anticipated.

The key criterion for selection of highly effective teachers was student growth. This was ascertained from interview results for the first two years of the project showing children's mathematical growth across the nine ENRP framework domains. The six case study teachers with high student growth were chosen to represent a cross section of grades with one teacher from each of Grade Prep (first year of school in Victoria), Grade Prep/1, Grade 1, Grade 1/2, and Grade 2. One highly effective teacher of Prep children from predominantly non-English speaking backgrounds was selected also for study. The case study teachers had taught within the ENRP and at the same level for the three years of the project, and represented a cross-section of situations such as school location and socio-economic profile. Visits to other ENRP classrooms suggest there were also other highly effective teachers of numeracy in the ENRP but who were not chosen for study because of the limits on number of teachers, grade level, years in the project et cetera.

Investigating the practices of highly effective teachers of numeracy

Being mindful of the need to avoid spurious conclusions, the case study methodology incorporated corroboratory and alternative sources of data (LeCompte & Goetz, 1982). The six teachers were studied intensively through use of the following data sources:

- five lesson observations by teams of two researchers (three consecutive days in the middle of the school year, and two consecutive days a couple of months later), incorporating detailed observer field notes, photographs of lessons and collection of artefacts (e.g., worksheets, student work samples, lesson plans);
- teacher interviews following the lessons (audiotaped and transcribed) to discuss the teacher intentions for the lesson, and what transpired;
- teacher questionnaires completed through the duration of the project; and
- teacher responses to other relevant questions and tasks posed to them.

Many steps were involved in the case study data collection and analysis, beginning with detailed observer notes of each lesson using a laptop to record as much as possible of what was said and what happened, without interpretation. After each of the lesson observations by two observer/researchers, independent analysis of the lesson was carried out according to a lesson observation and analysis guide made up of nine broad categories agreed upon by the team. These were Mathematical focus; Features of tasks; Materials, Tools and representations; Adaptions/connections/links; Organisational style(s), Teaching approaches; Learning community and classroom interaction; Expectations; Reflection; and Assessment methods. Our aim was to describe the practice of demonstrably effective teachers and to look ultimately for common themes, but not to judge.

Following the first three lessons each observer/researcher team produced a summary statement for the teacher they had observed. These statements were shared verbally with the research team at a meeting in which two critical friends, not involved in the research, then provided feedback on the kinds of themes they were hearing. This process occurred again after all five lessons had been observed for each teacher.

This observation and analysis process produced a list of themes, which were then cross-referenced again with the data collected to ensure there was evidence of them in action.

The practices of highly effective teachers

From this process a list of 25 practices and attributes of effective teachers evolved. It was agreed to list common elements where evidence was available for at least four of the six teachers. These themes are provided in Figure 3 under the nine categories of the

ENRP lesson observation and analysis guide, along with the extra category that emerged from the data, that is, Personal attributes of the teacher.

	Effective early numeracy teachers…
Mathematical focus	focus on important mathematical ideas make the mathematical focus clear to the children
Features of tasks	structure purposeful tasks that enable different possibilities, strategies and products to emerge choose tasks that engage children and maintain involvement
Materials, tools and representations	use a range of materials/representations/contexts for the same concept
Adaptions/ connections/ links	use teachable moments as they occur make connections to mathematical ideas from previous lessons or experiences
Organisational style(s), teaching approaches	engage and focus children's mathematical thinking through an introductory, whole group activity choose from a variety of individual and group structures and teacher roles within the major part of the lesson
Learning community and classroom interaction	use a range of question types to probe and challenge children's thinking and reasoning hold back from telling children everything encourage children to explain their mathematical thinking/ideas encourage children to listen and evaluate others' mathematical thinking/ideas, and help with methods and understanding listen attentively to individual children build on children's mathematical ideas and strategies
Expectations	have high but realistic mathematical expectations of all children promote and value effort, persistence and concentration
Reflection	draw out key mathematical ideas during and/or towards the end of the lesson after the lesson, reflect on children's responses and learning, together with activities and lesson content
Assessment methods	collect data by observation and/or listening to children, taking notes as appropriate use a variety of assessment methods modify planning as a result of assessment
Personal attributes of the teacher	believe that mathematics learning can and should be enjoyable are confident in their own knowledge of mathematics at the level they are teaching show pride and pleasure in individuals' success

Figure 3. Common themes emerging from six individual ENRP case studies

To illustrate practices of these teachers, excerpts from a classroom vignette, teacher interview statements and observer/researcher summaries are shared below.

Illustrating effective practices: A sample lesson

To provide some illustration of practices of effective teachers, one lesson taught by the Prep teacher (Ms Prep) is described below by drawing on the lesson observation field notes and the post lesson interview. It should be noted that no two of the six highly effective teachers taught in exactly the same way, and that any one of the 25 characteristics might be enacted in different ways.

While reading this lesson outline you may wish to reflect upon which characteristics of effective teaching as identified in the ENRP (Figure 3) you believe are evident.

In this lesson, Ms Prep wanted to introduce the children to repeating patterns. She stated that "they had noticed numbers on the number chart but we hadn't done anything with other patterns at all". Ms Prep introduced the lesson through the reading of a short book called Patterns that included reference to patterns in leaves, sand, and wings of butterflies. She then brought out a bag of "shopping" containing gifts for family members who were having birthdays that month. The first item she took from the bag was a Telly Tubbies plate that attracted the interest of these five to six year olds! One child noticed there was a pattern on the plate. The teacher asked, "How many colours are in the pattern?" She also stated, "Around the plate is a repeating pattern".

In turn, a patterned handtowel, and patterned sheets of wrapping paper were shown to the children. The teacher asked questions such as: "See if there is a pattern on my wrapping paper" "[Child A] could you tell me if you can see a pattern? What is the pattern?" Child A child volunteered that he saw another pattern. The teacher asked him to describe it. The teacher then asked other children "What pattern do you see?", "[Child B] can you see any pattern there? Is there a repeating pattern?" A number of patterns were identified and described. The teacher emphasized the nature of the patterns, that is, the repetition.

After the introduction which took about 15 minutes, the children sat in a circle on the floor. One child was given a container of square, plastic tiles of various colours. The teacher asked, "[Child C] see if you can make a repeating pattern and we'll see if we can work out what it is." Once the child had made a pattern the teacher asked her: "[Child C] could you tell us your pattern please?" The teacher then addressed the class: "If that is a pattern, what would be the next colour you put in?" She praised the child for her pattern: "That was a very hard pattern [Child C], you were very good at that".
The teacher then made a pattern – Green, red, green, red, and asked: "[Child D] what's my pattern? What would go next?" The teacher then informed the class: "I am going to give you some containers. I would like you all to make your own pattern. Try and make it a repeating pattern. When you finish we will look at your pattern". The children made a variety of patterns (e.g., Figure 4).

Rather than commenting herself on the children's patterns, the teacher gave this role to the class by saying things such as "[Child E] tell us your pattern. Is that a repeating pattern? Let's all say [Child E's] pattern. How many colours make up [Child E's] pattern?" She did the same for another child: "[Child F] tell us your pattern. Is that a repeating pattern?"

Figure 4. Making patterns

She then asked another child to comment on the pattern made: "[Child G] why is it not a repeating pattern? Where did it stop being the same? How can we make it a repeating pattern?" For another she said "How could he make that a repeating pattern" Anybody got any suggestions?"

Another 15 minutes had passed. The lesson then continued with the same mathematical focus but with different materials. Within the context of preparing for a party that was to occur the following day, the teacher introduced the idea of decorating a clear plastic cup. After demonstrating how to stick coloured stars on the cup she stated: "I'd like you to use your imagination and try and make a repeating pattern around the top of the cup". The children were free to choose the number of colours in their pattern, thus there was some openness in the task. While each child decorated a cup, the teacher assessed whether the children were able to make a repeating pattern; she teacher roved and recorded this on a piece of paper.

Following this task the children moved to the floor and the class discussed some of the patterns. The teacher read them out and the children were asked to tell her whether each was a repeating pattern. She asked questions such as "Is that a repeating pattern? Does it do the same thing over and over again?" Some children were observed to self-correct, such as one child who stated: "I forgot a red".

Reflecting upon this lesson

This lesson contained many elements that suggest the teacher possessed and implemented a range of characteristics of effective teachers of numeracy as identified from the ENRP case studies. Some of these are discussed below.

One element that stands out is a continuing *focus on the important mathematical idea* of pattern. The idea of repetition within pattern was reinforced through the use of *a range of materials/representations/contexts for the same concept*. The teacher also believed that the use of the book and the bag of shopping allowed children to "relate patterns to elsewhere, patterns in the environment" and thus the teacher *made connections to mathematical ideas from children's experiences*.

The mathematical focus was made clear to the children, for example, through questioning the children about whether they and others had made repeating patterns, and by stating the idea of a "repeating pattern" a number of times, as well as by giving the children the opportunity to examine and make a range of varying patterns.

The teacher also *structured purposeful tasks that enabled different possibilities, strategies and products to emerge*, that is, the tasks had some openness. A range of patterns was possible when using both the counters and the stars. Each pattern might have had anything from two to about five or six repeating elements. There were also a number of tasks and changes of activity within the lesson. The teacher explained: "I have to. Preps lose interest if you dwell on one aspect too long. You have to have short sharp activities with Prep". Nonetheless, each task was focused on the mathematical idea and through the use of a range of materials allowed the children to draw out the common element, that is, the mathematics.

Many characteristics within the *Organisational style(s), teaching approaches*, and *Learning community and classroom interaction* themes (Figure 3) were evident in this lesson. For example, the children were encouraged to *explain their mathematical thinking/ideas* and to e*valuate others' mathematical thinking/ideas, and help with methods and understanding.* The teacher was not the only person in the class responsible for reflecting upon whether the children had met the requirements. In the interview the teacher stated:

I don't think you can just say that's wrong. … I prefer the other children to say no, that's not a repeating pattern than for me to say it. I don't like to be the bad guy perhaps, but I think it's good when the other children can recognise it's not. And I can say, well how can we make it one. We all pitch in their ideas to make it.

She also stated, "If I say, how did you work that out, it just reinforces the strategy that they've used". However, she also believed there were times when she had to actively intervene:

I like them to be involved with what we're doing. And as much as possible I like them to try and work it out or see it before I actually tell them. But sometimes you just have to tell them. Sometimes you do. Some children just don't pick it and if you tell them, they think, Oh yeah! Perhaps that will give them that little start and then they'll start to think things. I do like to get them involved and I do try to let them point things out as much as I can before I'll show them … as if they've discovered it and they…. I'll look, Oh yes, well done, well let's have a look at that, if they're leading where we're going perhaps.

Ms Prep also seemed to *listen attentively to individual children* when interacting with them. These discussions had the characteristics of an individual conversation, even when student initiated. The summary report following the first three visits included the following comment:

[Ms Prep] was very positive towards the children. She praised them for their thinking. She showed caring and respect for individual children, giving them her attention even within a group situation. She answered individual children's questions and followed their train of thinking, unless it was totally off the topic".

Another feature of this lesson was that Ms Prep *drew out key mathematical ideas during and/or towards the end of the lesson.* While there was a share time and drawing together towards the end of the lesson, this reflection also occurred through the lesson.

Another of the 25 characteristics evident within this lesson was the *collection of data by observation and/or listening to children, taking notes as appropriate*. Being clear on the mathematical focus of the lesson, Ms Prep was able to do this meaningfully.

The intention of presenting the detail of Ms Prep's lesson was not to outline a perfect lesson! However, it is a lesson that illustrates many of the 25 characteristics of effective teachers identified within the ENRP individual case studies. Even for this teacher these characteristics became evident in different ways in other lessons observed. No two lessons looked the same just as the teaching of no two of the highly effective teachers studied within the case studies looked the same. However, it is reiterated that the 25 themes were each based on evidence from within observed lessons taught by at least four of the six case study teachers.

Differences

As mentioned earlier, the research also identified differences between these teachers. These included the timing and sequencing of mathematics topics, with for example, Ms Prep not usually teaching more than two consecutive lessons on a topic, but Ms NESB, who was also a Prep teacher, focusing mainly on Number for four days of the week. Organisation of children in the main part of the lesson was another difference with, for example, Ms NESB usually having two groups of children based on like needs, Ms Grade 1/2 taking a separate teaching group, and Ms Grade 2 having all children do the same broad activity and roving around the classroom to question, supervise, check, and have quieter conversations with some children, challenging them as she saw appropriate.

These differences reiterate the point made earlier that quality of teaching is more important than class organisation.

Insights into highly effective teachers

One feature of these teachers was their ability to integrate a number of the 25 features within any one lesson and make it look effortless. However, there was evidence that these teachers reflected upon what they knew of their children's mathematical understandings, upon the mathematics that the children were moving towards, and upon the effectiveness of their teaching strategies. In turn they planned their mathematics classes with care. Having undertaken this reflection and planning they could make effective fine grain decisions as they were teaching their numeracy lessons. As identified by Boaler (2003), it is not only the features apparent in teaching that are important, but it is how they are enacted that is important. It is clear that teachers should consider not only what they do but *how* they do it.

The ENRP case studies also revealed as common to these highly effective teachers, personal attributes related to beliefs, confidence and pride in their children. This illustrates that the study was not only about *teaching* if we take the definition provided by Hiebert and Grouws (2007), that is, that teaching is classroom interactions between teachers and students around mathematical content, but that the study was more broadly about *teachers* as it included the characteristics of the teachers themselves.

Conclusion

Teaching is a complex craft involving many skills. It can also involve the artistry and enjoyment of combining these skills into cohesive and effective lessons that are productive learning opportunities for children, and can be influenced by teacher beliefs and attitudes. As indicated by Hargreaves (1994) in the following quote, teachers each play a key role in the learning experiences of the children they teach.

Teachers don't merely deliver the curriculum. They develop, define it and reinterpret it too. It is what teachers think, what teachers believe, and what teachers do at the level of the classroom that ultimately shapes the kind of learning that young people get. (Hargreaves, 1994, p. ix)

The purpose of this paper was to consider characteristics of highly effective teachers of mathematics. The listing of themes resulting from the ENRP and the identification of features from other research provide the opportunity for the reader to reflect upon, and perhaps extend or refine, his or her list of perceived characteristics of effective teachers of mathematics. The findings from the ENRP case studies have challenged me to examine what I value in mathematics teaching. Since having the privilege of observing many lessons taught by ENRP teachers, and most particularly 15 lessons within the classrooms of three of the ENRP highly effective teachers of numeracy, and analysing and reflecting upon what makes the six teachers highly effective, I have found myself implementing some of these characteristics, sometimes consciously and sometimes unconsciously, in both primary and tertiary settings. It is my belief that the characteristics of highly effective teachers of numeracy, as outlined in this paper, can apply to teaching in the early years and beyond. As suggested earlier in this paper, there is potential also for teacher education courses to be informed by the findings from this research on highly effective teachers of mathematics.

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Heading in the Right Directions? Logo as Part of the Pre-Service Mathematical Education of Primary Teachers

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This paper addresses the use of Logo with student-teachers. The work reported is done in a short unit on turtle graphics within the Mathematics course taken by all Bachelor in Education students attending Froebel College. After outlining relevant research, the paper describes the Logo unit; it then focuses on recent students' reactions to learning Logo, recorded in written reflections. Aspects identified include adjustment – with varying degrees of enjoyment and success – to constructivist approaches to learning mathematics, and enhanced understanding of mathematical concepts, notably with regard to angles. Consideration is given to the form and value of the unit.

Introduction

The programming language Logo attained cult status in the 1980s, being viewed as an ideal vehicle for developing students' problem-solving skills as well as a way of introducing students to mathematical concepts and thinking. To some extent simple Logo then went out of fashion, with more enhanced products such as Lego Logo and Lego Mindstorms providing the transformative power and excitement that had characterised early experiments with the original Logo. However, the author has continued to use simple Logo in the Mathematics course that she teaches to students reading for a Bachelor in Education degree at Froebel College of Education. This paper provides an account of the rationale for and evolution of the Logo unit; it then reports research into the students' reactions to and reflections on their experience. It aims both to provide information on the unit and to allow for reflection on the extent to which it fulfils its intended roles, in particular with regard to providing the students with opportunities to construct their mathematical knowledge in a suitable ICT environment.

The first section of the paper briefly describes the philosophy and development of Logo, and (also briefly) notes work done with it in Ireland. The second section explains the origins, aims and content of the three-year Mathematics course of which the Logo unit forms a small part. The unit itself is described in the third section. The methodology and outcomes of the research, using the students' written reflections, are presented in the fourth section. Discussion and conclusions are presented in the final section.

Logo: philosophy and development

Papert's philosophy of constructivist learning as encapsulated in Logo became widely known from his book *Logo: Children, Computers and Powerful Ideas* (Papert, 1980). For the 'turtle graphics' aspect of Logo, the turtle – a robot the movements of which are controlled by the computer, or an on-screen symbol that likewise can be made to move around the screen by being given appropriate commands – is an 'object-to-think-with' (Papert, 1980, p. 11). The turtle is equipped with an actual or virtual 'pen' that enables it to leave a trail as it moves, and hence to draw pictures. The computer environment provides a 'microworld' (Papert, 1980, p. 117) in which certain kinds of learning – for example, problem analysis and solution using trial and improvement methods – may be facilitated. Mathematical topics such as geometry (through the 'body syntonic' approach {Papert, 1980, p. 63} associated with 'playing Turtle' {Papert, 1980, p. 58}) and algebra (through the use of variables) can be met in ideal surroundings: 'learning mathematics in Mathland' (Papert, 1980, p. 6). The fact that work with Logo can be started by using a small set of primitive commands with simple names (such as

FORWARD [a specified distance] and RIGHT [a specified number of degrees]) mean that it is accessible to young children. Its power comes from a number of features. These include the fact that it is extensible (new commands can be defined by means of *procedures*, and then used as if they were primitives); that the procedures facilitate top down analysis and structured programming (breaking a problem into 'mind-sized bites' {Papert, 1980, p. 103}); that it supports recursion; and that it has full list-processing capabilities.[1]

Initial optimism about the role of programming in general, and Logo in particular, in developing problem-solving skills was tempered by research such as that by Pea and Kurland (for example, Pea & Kurland, 1984). They asserted that 'findings to date of transfer from learning to program have not been *encouraging*' (Pea & Kurland 1984, p. 162) and that Logo in particular did not live up to the claims made for it. A few years later, in a major review of IT in mathematics education, Kaput referred to the 'inconclusive impacts of Logo programming' and suggested that 'interest in Logo as a medium for learning mathematics was waning' (Kaput 1992, p. 521). However, he also reported a number of studies from the late 1980s with more positive outcomes in the context of learning geometry. Moreover, Noss and Hoyles (1996) claimed that the evidence cited by Pea and Kurland does not support their main conclusions. Noss and Hoyles continue to address the role of programming in learning mathematics. (For a somewhat fuller discussion relevant to this paper, see Oldham & Butler, 2006; for a general summary with relevance to teaching, see Grabe & Grabe, 2001, pp. 80-104 {especially pp. 92-99}).

In a move from constructivism to *constructionism* (involving the design of an artefact), the ideas underlying Logo were developed further during the 1990s; Lego Logo and Lego Mindstorms reflect this approach (Papert, 1993, 1996; Kafai & Resnick, 1995; Resnick, 1998). It is exciting, but has less obvious application to the learning of mathematics, and discussion is outside the scope of this paper.

The study of Logo and its developments in Ireland owes much to work done in St. Patrick's College by Fred Klotz, Sean Close, and Deirdre Butler: in particular, for mathematically gifted and talented students (Butler & Close, 1989), with regard to mental models of recursion (Dicheva & Close, 1996) and through the Empowering Minds project (Butler, 2002). At school level, notable work was done by a group of teachers in East Cork. They produced a book, *Logo ar Scoil* (Grupa Múinteoirí Ríomhaireachta in Oirthear Corcaí, 1992), that not only provided a scheme of work but also emphasised the philosophy of Logo: 'LOGO is a theory about thinking and problem solving…. Using LOGO is a unique way for children to learn by doing…' (Ó Floinn,1992, pp. 3, 4). However, uptake nationally was not large. A study by O'Leary (2002) of a cohort of primary teachers who used ICTs in their classrooms found little use of Logo. Nonetheless, in what may come to be regarded as a turning point for use of simple Logo in the classroom, Logo activities are recommended in the revised

[1] Examples of Logo commands and programs together with their outputs (drawings in the case of turtle graphics), and further discussion of the educational rationale for using Logo, can be accessed readily on the Internet. A good overview is provided on the commercial website http://www.terrapinlogo.com/why-use-logo.php [retrieved 23 August 2007].

Primary Curriculum for mathematics (DES/NCCA, 1999a, p. 100; DES/NCCA, 1999b, pp. 62-63).

The Mathematics course in the Froebel College Bachelor in Education programme
In order to describe the aims and scope – and limitations – of the Logo unit in Froebel College, it is necessary first to outline some features of the Bachelor in Education programme of which it forms a very small part.

The B. Ed. programme offered at Froebel College is that run by Trinity College Dublin in association with three colleges of education – Coláiste Mhuire (Marino) and the Church of Ireland College of Education (CICE) as well as Froebel College. A feature of the 'associated colleges' model for the B. Ed. is that all students are required to follow three-year courses in what might be considered the 'core' disciplines of Irish, English and Mathematics, in addition to studying the teaching and learning of these subjects in primary schools. In the original discussions with regard to the design of the programme, emphasis was put on the idea that these courses would be offered *at the students' own level*: they would contribute to the students' academic and personal development – a role played in the other Colleges of Education (St. Patrick's, Mary Immaculate and – originally – Carysfort) by the study of 'academic' subjects unrelated to education. Thus, the courses were not intended just to revise or re-teach the content of the primary curriculum in Irish, English and Mathematics, though it was envisaged that, by enhancing students' overall knowledge and understanding in these areas, the courses would in fact contribute to improvement of their teaching.[1]

The programme was introduced in 1975, and was adjusted and reviewed within its first five years of operation. It is interesting to note the original and/or projected content of the Mathematics courses discussed at early meetings.[2] The present author, who initially taught the students from CICE (until that college appointed a mathematics and science specialist) and also those from Froebel College, outlined what she would now call a 'liberal arts' course in mathematics; it focused on thinking and talking about the nature of the subject, as well as addressing various topics at the students' own level with particular relevance to illuminating aspects of the school curriculum. For the review in 1979, the lecturer at Marino (Con Carroll) presented a 'first arts' course of university type; it contained topics such as calculus and vectors. The outline tabled by Leo Frost of CICE contained elements of each approach. After discussion and sharing of insights, the three lecturers agreed on proposed aims, objectives and assessment procedures; they also agreed not to specify content, leaving it to each lecturer to select topics by means of which he or she could best implement the aims with students in the relevant college. Of interest for the present paper is that the resulting document mentioned, *inter alia*, the issues of intuition versus algorithmics and of process versus product: aspects relevant to computer use in general and to Logo in particular.

The lecturers dealing with Mathematics Methods also produced a projected course outline. The two groups ultimately met together; another matter of interest – in view of a general debate with regard to mathematics *at the students' own level* or *from the*

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[1] This account reflects the author's personal experience, supported by a personal communication (summer 2007) from the historian of education Susan Parkes, who was deeply involved in the process of course development.

[2] The account in this and the following paragraph is drawn from the author's notes and records.

primary curriculum – is an indication that the Mathematics Methods group raised the issue of focusing on primary content. A joint document was produced, incorporating abbreviated forms of the aims and objectives put forward by the two groups, and outlining a syllabus that was intentionally rather vague about the division between Mathematics and Mathematics Methods and the way in which the former would support or complement the latter.[1] The Mathematics lecturers at the three colleges continued to implement the spirit of the agreement: they chose content through which they could implement the aims as they felt best able. This flexibility was useful also in dealing with a situation in which students entered the course with widely differing levels of attainment in mathematics (from good grades in the Higher course Leaving Certificate, to 'C' or 'D' grades in the Ordinary course achieved only with difficulty). With a typical intake to each college of only about thirty students per year, at least some mixed-ability teaching approaches were relatively easy to implement.

A major review of the B. Ed. programme was initiated in the late 1990s. In tune with the times, more detailed course outlines were required; moreover, with increasing disquiet about the mathematical attainment levels of school leavers (see NCCA (2005) for details and further references), greater priority was given to coverage of material in the primary Mathematics curriculum (DES/NCCA, 1999a). In the B. Ed. Mathematics course, therefore, 'core' content is designed around curricular topics. However, 'options' can be selected from themes such as 'Mathematical Enrichment and Awareness' and 'ICT and Mathematics'. This, together with the fact that each college sets its own examination papers in Mathematics and Mathematics Methods, means that there is still room for variety in the material covered, and indeed also in the way in which it is divided between the Mathematics and Mathematics Methods lectures. A point worth noting here is that student intake to the programme has increased, making it harder to accommodate and challenge all students appropriately, especially in trying to fulfil the original objective of allowing the students to develop at their own level. The revised Mathematics course was introduced into Froebel College from 2002.[2]

The fact that the original B. Ed. programme was devised and reviewed in the 1970s explains, if it does not altogether excuse, the lack of reference to ICTs – or, in the language of the time, to 'computers in education'. Naturally, the omission was corrected in the revised programme. In the interim, the three colleges addressed the issue in a variety of ways. A general description is outside the scope of this paper. The mathematics lecturers had shown their interest in the area even prior to 1980, for example through the reference to algorithmics noted above, and the various Mathematics courses reflected that interest over the ensuing years. It is a task for others to record the work done at CICE and Marino; as indicated earlier, this paper concentrates on the work done in Mathematics with Froebel College students.

Logo as part of the Mathematics course for Froebel College students

The 'liberal arts' course initially devised by the current author introduced some elementary computer work: the second-year course included a unit on flowcharting and

[1] The document formed a section of the B. Ed. course handbook in operation for some twenty years from 1980.

[2] The revised B. Ed. programme is available from all the colleges involved in teaching it.

on programming in the low-level (pseudo-assembler) language CSSP.[1] In 1979-80 – by which time the School of Education owned an Apple II computer that could be brought to the lecture room – the unit was replaced by one on BASIC, which was a resident language in most microcomputers of the day. The units aimed to provide

- appropriate algorithmic work at the students' own level
- content that could also be offered as extension material to primary school students
- experience with 'computers in education' consonant with contemporary thinking and availability.

Once Logo became available, its greater suitability for primary school students made it an obvious choice as a replacement for BASIC. It was introduced into the Froebel College second-year course in 1983-84, and has figured there ever since. The remainder of this section outlines the duration, structure and resourcing of the unit; the aims and content; the teaching/learning approaches; and the modes of assessment. It documents the evolution of the unit over the years as resources increased and as the B. Ed. programme was revised.

Duration, structure and resourcing

Issues of relevance here are those of the time given to the Logo unit, the size of the year group to be taught, and the number of computers available. A contributory factor is the extent to which it is possible to divide the class for lectures and/or hands-on sessions.

A fundamental issue is the *time* allocated the unit. For many years from 1983-84, it was typically from eight to ten one-hour lectures (in a second-year Mathematics course of around twenty hours' duration). When the revised B. Ed. programme was phased in from 2002, giving increased emphasis to primary school mathematical topics, the time available for Logo was reduced; only five or six lecture sessions could be accommodated. One question considered in this paper is whether the short time is sufficient to achieve the aims of the unit.

The early *computer facilities* were minimal (a few BBC computers, some with Acornsoft and some with Logotron Logo). Various machines were added, with a turning point coming when a suite of networked IBM PCs was installed in the mid-1990s. From that time on, the unit has been taught using a version of Berkeley Logo for Windows. Its advantages include its being in the public domain and available on old-fashioned floppy disks, hence being easily transferable to almost all computers. The present suites of computers in the college, and many of the students' own machines, could accommodate more extensive versions; however, continued presence in many primary classrooms of old machines, not connected to the Internet, has contributed to retention (to date) of the limited version as giving greater scope for implementation in schools. In any case, the short length of the unit means that advanced features of stronger versions of Logo would probably not be addressable during the course.

As indicated above, student *numbers* have grown during the period. In the mid-1980s, the intake was around thirty students per year. Numbers started to rise in the 1990s, with the cohort in Froebel College rising to around forty for a while and then growing to more than eighty. Resource matters (timetabling constraints for the students and the

[1] The first cohort of students was also introduced in the third year of their course to programming in BASIC.

lecturer, access to the computer facilities, and so forth) meant that until the last increase, the class was generally taught Mathematics as a whole group – a situation that could not be described as appropriately Froebelian – although special arrangements were eventually put in place for formally dividing the class at least for some of the Logo sessions. Now, the Mathematics course is taught in two parallel sessions, with a new staff member (Ruth Forrest, a former student of the college) sharing the lecturing. For the Logo unit, managing groups of forty or so is still a challenge. In the past year (2006-07) the author taught both groups in turn, while her colleague taught a different topic to the remaining students. In the coming year, it is planned that each lecturer will take her own 'half' of the class for Logo – at different times of the year, so as to distribute the pressure on the computer facilities. The input from a different lecturer will provide opportunities for revising both the content and the teaching of the unit.

Aims and content

The aims and content are best described together. Since the unit is part of a Mathematics course, the main emphasis has always been on turtle graphics. From the beginning, the unit was intended to be a vehicle for

- problem solving, with emphasis on
- top down analysis, breaking problems into 'mind-sized bites'
- use of procedures to implement the analysis
- trial-and-improvement learning and de-bugging (Papert, 1980, pp. 113-115)
- addressing specific mathematical topics
- geometry (via turtle graphics)
- basic ideas of algebra (via variables)
- encouraging creativity, consonant with Froebelian philosophy.
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The label 'constructivist' was not used initially, but was implicit in the approach. In the first few years, some work was done on list processing using the FIRST, LAST, BUTFIRST and BUTLAST primitives, and brief references have been made from time to the facilities for calculating, but these have been peripheral issues.

In the revised Mathematics course, Logo is a natural way to address the optional themes of 'Mathematical Enrichment and Awareness' and 'ICT and Mathematics' as well as geometric topics from the primary curriculum. With fewer hours available for the unit, it has still been possible to address most of the aims and areas outlined above, for example because students start with better IT skills. However, formal introduction of variables (and hence work leading to algebra) has had to be dropped from the taught course. For a few years, some students were able to augment their experience by taking a module on Lego Mindstorms; they could feed their extra knowledge and skills into the Mathematics course, so that all students could benefit. Unfortunately this module (taught by the college's IT specialist, Richard Butler) has been discontinued for the present because of pressure of time and other resources.

Teaching and learning

Throughout the period, the approach to teaching and learning has involved scheduled whole-class lecture-and-discussion sessions and hands-on tutorials. Students are also expected to work in their own time. The whole-class sessions, held in a lecture room with a computer and projection facilities, are intended

- to outline the philosophy of Logo and to introduce primitives and techniques (via expository teaching and demonstration)
- to engage students in whole-class problem-solving (with ideas tried out on the machine) or in small-group work (using physical activity and/or pencil and paper)
- after hands-on sessions, to reflect on and internalise what has been learnt.

For the hands-on sessions, students move to a computer room. Detailed notes containing instructional material, worksheet-type exercises and more open-ended challenges are available, allowing students (at their own speed, and working individually or in groups as they choose)

- to try out the features that have been introduced
- to address appropriate exercises or challenges
- to progress to more advanced work if and when they are ready.

The notes still contain material on variables, so students can investigate them if they wish.

An alternative approach would be to hold all lecture sessions in the computer room, with the lecturer leading the students through the various activities. However, in dealing with any computer application, the author has a personal preference for the model described here, on a number of grounds. The initial whole-group sessions are intended to set the context and provide advance organisers (Ausubel, 1968), rather than allowing students to engage with details before seeing the purpose of the exercise. The overall layout of the screen and the types of image that appear on it should be familiar before the students start their own explorations. Naturally, students may miss some of the finer points until they engage with the machines – but that should serve to emphasise the importance of them eventually *constructing* their own knowledge and indeed recognising some of the limitations of expository teaching *not* supported by activity. The plenary sessions that follow hands-on work, as indicated above, are intended to promote reflection: 'minds on' as well as hands-on (Lamon, 2003), with ultimate benefits in being able to solve problems mentally rather than (or in addition to) by physical trial and improvement. The time that has to be invested in developing suitable notes for the hands-on sessions – so that students can work independently at their own pace – is considerable, but the author finds it very worth while. (By contrast, when lecturers lead students through hands-on sessions, students cannot easily set the pace that suits them; a student who gets lost may remain lost until the end of the session.) The model, taken in conjunction with the approach to assessment described below, is compatible with the teaching advice summarised by Grabe & Grabe (2001): for example, it allows for highlighting of required skills and 'thinking aloud' in the whole-class sessions, for independent student work and one-to-one discussions in the hands-on sessions, and for group work and analytical discussions in the context of the assessment process.

In fact the current resources (time and machines) do not allow for a situation in which all teaching would take place in a computer room, so the model is used of necessity as well as by choice. Some reconsideration would be appropriate if the students had laptops with them, but the author's present inclination would be to ask for 'lids down' at certain periods in order to try to gain the benefits of the whole-class interaction, reflection and discussion. Of course, different learners may have different preferences; this is another issue to be noted below.

Assessment

Assessment takes place as part of the end-of-year Mathematics examination. In the early years there were two aspects, both occurring in the written examination:

- questions testing a range of techniques
- an essay question or questions, typically asking about the philosophy of Logo and its application to learning aspects of mathematics or general problem solving

In the 1990s, a practical element was added. The form has varied somewhat over the years, but the main features are as follows.

- Students are required to design and implement – in their own time – a set of procedures to draw a picture:
- the picture must be drawn by typing in a single word (the name of the main procedure)
- typical requirements are that the main procedure calls at least two sub-procedures (to encourage good problem analysis and structured programming) and should contain colour (to enliven the presentation)
- in recent years, the explicit inclusion of some mathematical aspects (such as a polygon and parallel or perpendicular lines) has been required;

The students come individually to the computer room for a short practical test in which they demonstrate that they can

- get into Logo, load their file, and type the procedure name that causes the picture to be drawn
- list procedures and explain their workings to the examiner
- edit procedures
- print the picture and save the file.

For several years now, the students have been asked to work in groups of two or three in preparing their pictures. The practical exercise is intended to achieve a number of goals:

- to provide an opportunity (not so easy to organise for other parts of the Mathematics course) for involving elements of creativity and problem solving that can be geared to the students' own (diverse) levels of expertise
- to allow students to experience co-operative work and to share expertise – now particularly important in view of the restricted time available for the unit.

The fact that students present for the practical test individually is meant to ensure that – whereas they may make different levels of contribution to the *process* – at least they all understand the *product*; also, that they have acquired the basic skills of working in the turtle graphics environment. The practical test lasts around ten minutes for each student (two days for the examiner {the author}). To gain credit for the Mathematics course, students have to pass the written examination and show suitable proficiency at the practical test.

The model again reflects a personal preference of the author with regard to teaching programming: a formal introduction through expository teaching and hands-on practice, followed by learning on a 'need to know' basis in the context of working on a project of the students' choice. The best possible timing of the move between the more structured and less structured environment is likely to vary from student to student (and of course some students might benefit most from an entirely structured or an entirely free approach); however, the timing is meant to allow for 'the greatest good of the greatest number' within the constraints of the situation. The approach is another issue discussed below.

The students' reactions

Attention is now given to the research carried out over the last three years in order to investigate the students' reactions to the Logo unit. The background and methodology, and analysis of the students' responses, are presented in turn.

Background and methodology

The idea of collecting students' reactions to and reflections on their experience of learning Logo stemmed in particular from remarks made by students in the essay questions in the end-of-year examinations. As indicated above, the questions usually addressed issues related to the philosophy of Logo and its application to learning aspects of mathematics or general problem solving. However, the question in 2003 was based on the quotation from *Logo ar Scoil* given earlier (Ó Floinn, pp. 3, 4), with students additionally being asked to reflect on their own experience of 'thinking … problem solving … learn[ing] by doing' as they learnt Logo. Comments typified by 'I never understood angles until I did Logo' gave insights to deep-rooted student problems that had not been apparent in the standard teaching and examining situation, and suggested that further insights would be gained by asking the students to write more about their learning. Such writing would also be advantageous in encouraging students to reflect. Hence, the examination papers for the last three years have asked students to provide comments, as follows:

Summer 2005: Comment briefly on your own experience of learning Logo (and Lego Mindstorms, where relevant). What do you feel that you learnt, both about mathematics and about your own learning style? To what extent did the concept of 'hard fun' apply, and why?[1]
Summer 2006: Briefly relate your experience of solving problems: in Logo [and Lego Mindstorms, where relevant], and in other settings.
Summer 2007: Comment on your own experience of learning Logo, with regard both to your mathematical learning and to the insights that the experience has given you about your own learning style.

In general, the students' responses were read initially in the context of marking the papers, but were then studied further for the purposes of research. Multiple readings were undertaken in order to identify common themes and specific comments of interest. Using the examination answers together with other data, a number of research questions have been addressed over the period. The most relevant one for this paper was whether the short time devoted to Logo at Froebel College yields sufficient benefits to make the exercise worthwhile, especially in view of the increasing pressure to concentrate explicitly on topics and skills in the Primary Curriculum (Oldham & Butler, 2006). Data of the type collected cannot provide a full answer to the question of whether it

[1] For Papert's concept of 'hard fun' – involving genuine, meaningful challenge that is (therefore) enjoyable – see Papert (1996).

would be ‘better’ to focus more on the basic skills; however, by indicating some of the benefits or difficulties that the students encounter and the attitudes that they develop through studying Logo, the data can point to the value or otherwise of the Logo experience. Other research questions were more concerned with the Froebelian aspects of Logo and Lego Mindstorms (Butler & Oldham, 2006), and in particular with whether the digital manipulatives constructed with the Lego Mindstorms kit could play a role equivalent to that of Froebel’s ‘gifts’ in developing students’ learning (Butler & Oldham, 2007). Major issues addressed in this paper are those regarding the overall worth of the Logo unit in view of the benefits, challenges and frustrations encountered during the experience, as reported by the students in their examination answers.

There are caveats to be considered in handling such data. Despite the best efforts of the author in trying to establish that ‘honesty is the best policy’ and that it was the quality of reflection that was important, students may have written what they thought she would wish to read – or, at least, may have refrained from writing what they thought she would not wish to read – or may have tried, entirely understandably, to portray themselves and their learning in a good light. However, the fact that it is socially acceptable to dislike mathematics and find it difficult, while otherwise regrettable, is helpful in this context.

Analysis of the students’ responses

Themes emerging from the students’ responses are obviously (and appropriately) determined by the exact form of the questions asked. The main themes emerging from analysis of the students’ answers in 2005 (Oldham & Butler, 2006) were ‘hard fun’, ‘mathematical concepts and processes’, ‘active learning’, ‘benefits of group work’ and ‘classroom potential’. For the 2006 data (Butler & Oldham, 2006), in addition to the problem-solving aspect elicited by the question, similar themes emerged though with different emphasis; also, the ‘active learning’ theme can be described more broadly as reflecting ‘engaged learning’. Summaries are drawn from the published papers.

> ‘Hard fun’ includes both cognitive and affective aspects. About one student in five began his or her answer with a positive affective response such as ‘*I enjoyed my logo experience*,’ ‘*I loved it,*’ or even ‘*My experience of Logo was an eye-opener*.’ About the same number emphasised their initial difficulty, duly overcome: for example, ‘*At the beginning I found Logo difficult…. I improved and could enjoy Logo,*’ and ‘*To begin with I thought it was a very difficult and complex program…. I began to enjoy the concepts of exploration and design*’….
>
> Students’ initial difficulties with Logo were sometimes identified as stemming from weaknesses in their own mathematics. This leads to the second theme, ‘mathematical concepts and processes’: improving or revising both skills and understanding. In this context, some two-thirds of the class highlighted specific topics, skills or strands from the elementary mathematics curriculum. Many of the references involved the strand ‘Shape and space,’ with the most frequently mentioned topic being angles, for example: ‘*It really developed my knowledge of angles*’ and ‘*I have finally managed to get to grips with angles*’ (Oldham & Butler, 2006, p. 3806).

> [With regard to active or engaged learning] it was apparent that [the students] had taken the opportunity both to construct their own knowledge and to reflect on their learning. One frequently mentioned aspect was that of hands-on and/or physically active learning. Another was use of trial and error, for example enabling them to correct mistakes – or rather, improve attempts that were less than fully correct – without automatic recourse to the lecturer. Some students’ comments indicated that mathematics previously considered irrelevant had acquired meaning for them when used for drawing pictures in which they had a considerable emotional involvement. One person commented that having to solve a ‘real’ problem (making a robot move), rather than trying to solve a hypothetical problem, put the

> mathematical concepts into context. All these aspects can be seen as consonant with constructivist approaches. It can be noted here that that the same can be said of some of the references to *group work* (Butler & Oldham, 2006, pp. 35-36).

Further analysis here focuses on the 2007 responses, as they have not previously been reported. It should be noted that the available lecture time, only five hours in 2006-2007, was further decreased by a number of interruptions outside the control of the lecturer (the author) and/or the class. Despite this, the students in general performed very creditably in both the written examination and the practical test. The overall impression is captured by the author's note written after she marked the scripts: 'They EXPERIENCED [a] CONSTRUCTIVIST APPROACH'. More detailed examination of scripts and comments of interest led to the identification of themes similar to those occurring previously. The best descriptions on this occasion are 'initial confusion, eventual enjoyment', 'different learning', 'understanding mathematics', 'classroom potential' and 'benefits of group/project work.' They are discussed in turn.

'Initial confusion, eventual enjoyment' was perhaps the most prominent theme (especially if the responses of students who did not report any confusion, and of students who did not report any enjoyment, are included). Archetypal comments were '*initially I was of the opinion that it was very confusing but I found myself getting better…. I think it is a fantastic mathematical tool now* ', '*I thoroughly enjoyed my experience with Logo, at the start I was very confused as to what it was all about*' and '*At first, I found logo to be impossible and very difficult…. Gradually…, I realised I enjoyed doing it and trying to draw different pictures ... and felt a great sense of achievement when I eventually got it [right]*'. The initial confusion stemmed from two distinct causes: difficulty in grasping the concepts, especially until the students worked at the computers ('*once we went to the computer, practiced [sic] and saw the result of each command I found logo much easier to understand*'), and problems with understanding the rationale for the exercise or seeing its relevance to the primary classroom ('*At first ...[I] could not see any use for it at all*', '*It seemed somewhat pointless especially in the contexts of a primary school classroom. However, once I tried it out for myself I realised the opposite is true'*).

The 'different learning' that students experienced in Logo was a factor here. One aspect of this was active learning, as reported in previous years: hands-on work in the computer room ('*when I actually sat down and typed in some commands everything was so much clearer*') and what might be called feet-on or body syntonic work in playing the role of the turtle ('*… felt the need to pretend I was the turtle and turn to face certain directions to help myself* ', '*I actually stood up at first and walked as the turtle would'*). A second aspect, notable this year, was provided by a set of comments typified by '*you are using your brain in a different way*'. Thirdly, some students identified their familiarity with, or preference for, working with rules and formulae – something that they could not do in the Logo environment – and the necessity, or difficulty, or benefit, of taking a different approach. Notable comments were: '*I am very used [to] working out a sum using a standard formula but with logo it is really about trial and error and I found that difficult*', '*… difficult to go from a system that I had used for 20 years where I had always been told what to do and how to approach each problem, to a system where I was the director and teacher*' and '*In maths I prefer to learn a rule and apply that rule to the questions…. I can see however how [Logo] is useful in the classroom and how it develops logical thinking and mathematical understanding*').

Evidence that the different learning paid dividends can be seen in the responses on 'understanding mathematics'. As usual, references to enhanced understanding of angles were particularly notable (typically, '*it helped especially in my concept of angles*'). A more general comment, from someone who reported that she struggled with mathematics in general and Logo in particular, was '*If I had learned logo at school my ability to deal with things like trigonometry and area in secondary school probably would have been better*'. The fact that the project provided a context for the mathematical concepts was a source of enhanced comprehension or pleasure for some ('*these [geometrical] concepts are no longer just rules, they apply to real situations*').

While discussing their own reactions to Logo, a number of students commented – in most cases very positively – on its 'classroom potential': for example, '*I felt that it is a great way to teach about angels [sic] to a class as it is different and interesting*'. Despite the fact that pressures of time meant that the students had been pointed to much less background reading than usual, some noted the potential for cross-curricular integration ('*I see how it can be very beneficial for mathematical skills and also in visual arts and in other subjects such as English').* One student, a self-confessed Logo addict, went so far as to note: '*I found myself in art lectures muttering commands to myself!*'.

For the final theme, 'benefits of group/project work', some references were implicit rather than explicit. However, comments included '*I feel that I learnt a lot by working in a group on our logo project. It was through discussion as a group that we managed to figure out the different procedures* ' and '*the people in my group helped me along, it was nice to work in a group for this reason*'.

Owing to the constraints of space, the quotations given in this paper do not do full justice to the many insightful remarks in the scripts. One further comment can illustrate the extent to which students experienced and grasped the philosophy of Logo: a student said that she '*learned to enter a "mathland"... I found that you can in fact "programme ideas"'.*

Discussion and conclusion

This paper set out to reflect on the work done in using Logo with student-teachers at Froebel College. It described the scope and aims of the Logo unit: limited compared with some of the work done elsewhere, notably at St. Patrick's College, but involving all B. Ed. students as an element of their compulsory course in Mathematics. The Logo unit is given as part of the second-year course, and nowadays occupies some five or six hours of lecture time.

Students' written accounts of their reactions to the unit over the past three years, as recorded in their end-of-year examinations, have been collected and examined. Even with due allowance being made for some students presenting their reactions in a rosy light, the evidence suggests that overall the Logo unit achieves its aims of giving students an experience of constructivist learning at their own pace and at their own level. More than other areas of the Mathematics course, it provides an antidote to the rule-bound learning style that many of them have developed in the course of their school mathematics programmes. The project (allowing for group work, problem solving and creativity) and the practical test (checking individual skills and providing for one-to-one discussion) are particularly important in this respect. For some though certainly not all students, the project allows them to 'fall in love' with Logo (Papert, 1980, 1993) and so have a very positive affective experience of mathematics. Also, many students report enhanced understanding, in particular of geometry. In judging the worth of the exercise, however, it should be noted that students' retention of their insights, and the comparative effects of direct instruction, have not been investigated.

With regard to the models used for teaching and assessment, the assessment element seems to work particularly well. However, some difficulties remain with regard to teaching, particularly in maximising the benefits of the whole-group sessions. The extent of initial confusion reported, and the inability of some students to follow quite simple and standard demonstrations mentally, are causes for concern. It is the author's impression – somewhat confirmed by examining students' performance on old examination papers dating from a time at which much less hands-on experience was available – that the proportion of students who find the mental aspects difficult has increased. Likely reasons must be considered. One possibility is that the author is teaching less well than she did in the past; however, she would prefer to find alternative explanations! The size of the groups – around forty students – is certainly a disadvantage; when half of the number are sent the computer room, discussion with the remainder is easier to manage, but the hands-on group are unsupervised and so may suffer at times from lack of guidance. This is an organisational issue that could be addressed if further resources were available. In academic terms, however, the problems are consistent with recent reports of poor mathematical attainment as mentioned earlier (NCCA, 2005) and in particular with the heavy reliance on rule-bound learning noted above.

Presentations that do not lead to cognitive conflict, or at least to cognitive challenge, are unlikely to facilitate deep learning. Some initial confusion (rather than a false sense of simplicity) is entirely compatible with constructivist approaches, and developing the confidence to work through it is an important skill. However, many of the students seem to have suffered unduly on their route to greater understanding. Perhaps, in some cases, their difficulties stem from very poor spatial and visualisation skills (one student wrote '*I am not very capable of imagining how things should look …*').[1] Many would doubtless prefer the whole unit to be done in hands-on mode, in the computer room or in the privacy of their own homes, where they could continue to use trial and improvement, albeit perhaps without developing their skills of reflection and internalisation. For some students, such an approach might indeed be most appropriate. The increase in IT skills – a positive development in recent years – is beneficial here. However, on the theoretical grounds discussed earlier, the author remains unwilling (as well as currently unable) to dispense entirely with classroom sessions. At the risk of annoying the more able students, it is intended to start in the coming year with a 'walkabout' in which students can, if necessary, establish which is their right hand and which their left; practise rotating through full, half and quarter turns; make attempts at estimating angles measured in degrees; and program each other to describe figures such as squares and circles. This could be followed by inputs from a teacher who is using

[1] Another contributory factor in 2006-07, in particular, may have been the disruptions of the normal lecture sessions as mentioned above. Students coming from visits outside the college were arriving in lectures late, tired and hungry – or, in some cases, were not arriving in lectures (an aspect that did not figure in their comments!).

Logo and/or Lego Mindstorms in the classroom: making the point that it may actually be easier for children than for student-teachers. Such an approach may overcome at least some confusion, with regard both to the spatial concepts involved and to the purpose of the exercise and its role for the primary curriculum (Butler & Oldham, 2007). To allow for this, a minimum of six lecture/practice sessions may be needed for the Logo unit. More would be very welcome, but the pressures to engage with other parts of the course render such a development unlikely.

In conclusion, it is appropriate to reiterate key outcomes from the study. After their short Logo experience, students reported benefits with regard to knowledge and understanding of geometry, notably of angles; personal recognition of the value of active learning in mathematics; and adjustment, albeit with varying degrees of enjoyment and success, to unfamiliar approaches to mathematics and problem solving. They entered Papert's 'Mathland'. The author and her colleagues look forward to working in Mathland with further generations of students in the years to come.

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Teaching Mixed Ability Mathematics Classes

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Teachers confront substantial challenges in teaching mathematics to mixed groups of student. On one hand, there is often a wide diversity in the mathematical competence of the students. On the other hand, there is variety in their interest and level of engagement. The following examines some guidelines, developed after extensive classroom trialling, that teachers can use to accommodate diversity in readiness to learn mathematics. The guidelines are based on the choice of classroom tasks with a range of entry and exit levels, a commitment to building a coherent classroom community, and the use of specific prompts to address the needs of students experiencing difficulty, and those who complete the set work readily.

Introduction

Whatever structures schools and teachers use to group students for mathematics, teachers still confront classes of diverse interests and readiness to learn. Schools in Australia, for example, are confronting serious challenges from disengaged students (e.g., Russell, Mackay, & Jane, 2003), with the implication that teachers should choose more interesting and relevant everyday type tasks (see Klein, Beishuizen, & Treffers, 1998). At the same time, there is a serious decline in the number of students entering university level mathematics courses, threatening international competitiveness and innovation, fostering calls for more mathematical rigour.

This article reports an aspect of a project that sought to investigate ways of addressing the mix of abilities and interests that are found in most mathematics classes. The research involved the gradual development, trailing, evaluation, and adjustment of a model for planning and teaching mathematics (see Sullivan, Mousley, & Zevenbergen, 2006, for a full discussion of details of the project).

Initially, the research identified and described aspects of classroom teaching that may act as barriers to mathematics learning for some students, and drew on responses from focus groups of teachers and academics to suggest strategies for overcoming such barriers (see Sullivan, Zevenbergen, & Mousley, 2002). Next, the project analysed some partially scripted experiences taught by participating teachers (see Sullivan, Mousley, & Zevenbergen, 2004). This analysis allowed reconsideration of the emphasis and priority of respective teaching elements.

The overall study can be described as design research that “attempts to support arguments constructed around the results of active innovation and intervention in classrooms” (Kelly, 2003, p. 3). Cobb, Confrey, DiSessa, Lehrer, and Schaubble (2001) argue that design experiments are appropriate for classroom explorations where the research team and teachers collaborate on determining the research focus. As a result the data are the products of the investigation, rather than particular measures on specific aspects. It also means that the reports of the research, as you can see below, are reports and analysis of excepts from trials.

There were four key elements of the model, developed through this research, that are particularly relevant for considering approaches to teaching mixed ability classes. The key elements are: the type of task; the building of mathematical community; enabling prompts that can be used to support students experiencing difficulty in learning; and extending prompts that can be used to challenge students who have completed the set work. These are discussed in the following sections, with examples drawn from the research to illustrate what such actions might look like.

The Type Of Task

Many commentators have argued that the choice of tasks is the key element of any planning model. Christiansen and Walther (1986), argued that non-routine tasks, provide optimal conditions for cognitive development. Brousseau (1997) proposed that "the teacher must imagine and present to the students situations within which they can live and within which the knowledge will appear as the optimal and discoverable solution to the problems posed" (p. 22). Hiebert and Wearne (1997) proposed that "instructional tasks and classroom discourse moderate the relationship between teaching and learning" (p. 420). Stein and Lane (1995), argued that optimal tasks engage students in "doing mathematics or using procedures with connection to meaning" (p. 50).

One of the more powerful notions that can guide teachers' thinking in choosing tasks is Vygotsky's (1978) zone of proximal development (ZPD), which he described as the "distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined by problem solving under adult guidance or in collaboration with more capable peers" (p. 86). Mason, Drury, and Bills (2007) argued that teachers need to go beyond merely promoting proximal behaviour to fostering proximal generality, proximal relevance and proximal awareness. In particular, ZPD describes the work of the class as going beyond tasks or problems that students can solve independently by working on challenges that they can achieve with some support (Lerman, 1998). In other words, one aspect of the teacher's task is to pose to the class problems that most students are not able to do. This creates a tension, especially for classes with a diversity of mathematical readiness.

While the teaching and planning model can be applied to a range of task types and levels of cognitive demand (e.g. Stein & Lane, 1996), the research focussed on open-ended tasks that can support mathematics learning by fostering operations such as investigating, creating, problematising, communicating, generalising, and coming to understand—as distinct from merely recalling—procedures.

There is a substantial support for this assumption. Examples of researchers who have found that tasks or problems that have many possible solutions contribute to such learning include those working on investigations (e.g., Wiliam, 1998), those using problem fields (e.g., Pehkonen, 1997), those exploring problem posing by students (e.g., Leung, 1992), and the open approach (e.g., Nohda & Emori, 1997). It has been shown that opening up tasks can engage students in productive exploration (Christiansen & Walther, 1986), enhance motivation through increasing the students' sense of control (Middleton, 1995), and encourage pupils to investigate, make decisions, generalise, seek patterns and connections, communicate, discuss, and identify alternatives (Sullivan, 1999). The nature of content specific open-ended tasks can best be illustrated by an example:

> After 5 games, the mean number of points that a basketballer had shot was 6, and the median number of points was 4. What scores might the basketballer have shot in each game?

The first observation is that this task is not readily solved by the application of a formula, and students must consider the meaning of the concepts of mean and median.

Second, assuming that students have met mean and median, it has an easy entry in that students can choose possible scores with which they are familiar. Third, there are obvious and ready extensions possible for students who find a few responses quickly. Fourth, such tasks are content-specific in that they address the type of mathematical operations that form the basis of textbooks and the conventional mathematics curriculum. Teachers can include these as part of their teaching without jeopardising students' performance on subsequent internal or external mathematics assessments.

Further examples are presented later in the article. (For a wide range of such tasks at the primary level, see Sullivan & Lilburn, 2002)

Such tasks are also open-ended in that there is a variety of possible operations and a variety of ways of communicating responses. In each case, it is assumed that the teacher will imagine the actions in which the students will engage and pose the task but, other than clarifying the expectations and language, will provide limited direction for the students. This does not release the teacher from teaching, as it is the teacher who then facilitates discussion between students during and after their explorations and who asks challenging questions to encourage thinking.

In each open-ended task there is considerable choice in relation to operations: different strategies and solution types are possible. Some students might use trial and error to seek a variety of arithmetically derived solutions, and others may apply or develop a generalised algebraic approach using a formula and graphs, while others may satisfy themselves by exploring further combinations and perhaps discovering and employing patterns. Class discussion about the range of approaches used and range of solutions found can lead to an appreciation of their variety and relative efficiencies, and the power of some mathematical methods as well as the thinking that underpins these. When all students can contribute to such discussions in their own ways, there is potential for thoughtful questioning by the teacher to draw students into new levels of engagement and learning.

The research on content-specific open-ended questions demonstrated that the combination of such tasks with strategic questioning by teachers can lead to extension of mathematical thinking, since students can explore a range of options as well as considering forms of generalised response.

Learning Community

In this teaching and planning model, a deliberate intention is that all students progress through learning experiences in ways that allow them to feel part of a class community and contribute to it, including being able to participate in reviews and summative class discussions about the work. To this end, it is assumed that all students will benefit from participation in at least some core activities that can form the basis of common discussions and shared experience, both social and mathematical, as well as a common basis for any following lessons and assessment items on the same topic. This is both contentious and challenging.

As an aside, we did not consider ability grouping, either within or across classes, to be a solution to diversity because of its potential to exacerbate disadvantage due to self-fulfilling prophesy effects and poor modelling and mentoring for students who really need it (Brophy, 1983).

A key challenge in the common discussions is the way that teachers manage the contributions of the students. Of course, the teacher must observe what students are doing, so that they can choose contributors to the discussion in an appropriate sequence, perhaps from least sophisticated to more spohisticated. Another key aspect is that, given that there will be a range of types of responses, the teacher must emphasise the dimensions of variation, and prompt for generality (Mason et al., 2007).

For example, in the basketball task above, a student might choose to list various solutions such as

4, 4, 4, 4, 14

4, 4, 4, 5, 13

4, 4, 4, 6, 12 etc

Whereas another student might suggest that, "if the first 3 scores are 4, then the latter two scores can be any two numbers that add to 18, so long as they are both at least 4". This latter response is moving toward formulating a generalisation.

In the research, the use of tasks and classroom processes that support the participation of all students has resulted in classroom interactions that build a sense of learning community (Brown & Renshaw, 2006), with wide-ranging participation in leaning activities as well as group and whole-class discussions.

Enabling Prompts

Very much connected to the building of community, along with the commitment to posing challenging tasks, is a need to provide students experiencing difficulty with the necessary support. Open-ended tasks have been shown to be generally more accessible than closed examples, in that students who experience difficulty with traditional closed and abstracted questions can approach such tasks in their own ways (see Sullivan, 1999). Nevertheless there will be students who require hints or prompts, to proceed with the work. These are called enabling prompts.

Enabling prompts can involve slightly lowering an aspect of the task demand, such as the form of representation, the size of the numbers, or the number of steps, so that a student experiencing difficulty can proceed at that new level, and then if successful can proceed with the original task. This approach can be contrasted with the more common requirement that such students (a) listen to additional explanations; or (b) pursue goals substantially different from the rest of the class. In the project, the use of enabling prompts has generally resulted in students experiencing difficulties being able to start (or restart) work at their own level of understanding and enabled them to overcome barriers met at specific stages of the lessons.

This notion of adapting tasks has been a consistent theme in advice to teachers for many years. For example, the Association of Teachers of Mathematics produced a handbook (ATM, 1988) that described 34 different strategies that teachers might use when intervening while students are working. The strategies were grouped under headings that described major decisions such as: whether or not to intervene; how to initiate an intervention; and whether to withdraw or proceed with the intervention. There were 14 specific suggestions related to interventions to support students experiencing difficulty, about half of which relate to specific prompts for students experiencing difficulty. Similarly, Christiansen and Walther (1986), in describing the nature of student engagement in their learning, argued that:

> Through various means, actions are envisaged, discussed and developed in a co-operation between the teacher and the students. One of the many aims of the teacher is here to differentiate according to the different needs for support but to ensure that all learners recognise that these processes of actions are created deliberately and with specific purposes. (p. 261)

Likewise, Griffin and Case (1997) described teaching as involving knowing what individual learners understand, being aware what knowledge is within the ZPD, providing carefully constructed tasks to engage students in learning, helping learners as they construct their knowledge, and "constantly shifting or changing the 'bridge' to accommodate the learner's growing knowledge" (p. 4). Teachers should make it clear to students that the differentiation is to enable them to pursue the same learning goals as other students, thereby giving them confidence in the guidance as well as communicating the expectation that all can succeed. In other words, the teacher formulates enabling prompts that support students' developmental movement from current knowledge to a learning goal, setting up a further ZPD.

Some examples

The following are examples of enabling prompts observed in particular lessons. In each case, an observer wrote a naturalistic record of classroom events, as well as noting specific entries against each aspect of the teaching and planning model.

This following lesson was from an observation during the phase of the research where teachers were given lesson suggestions proposed by the project team. The teacher's planning notes were examined, and generally the teachers were interviewed before and after the observed lessons.

In this lesson, as an introductory task, the students were asked to use isometric paper to draw shapes that could be made using 3 cubes, each with at least one face touching another cube. In the teachers' notes there were two suggestions of differentiated tasks that were ready to be offered to students experiencing difficulty. The first was to "Have some isometric paper with the 2 cubes already drawn", and an observer recorded the successful use of this suggested prompt as follows:

> … there were students who had not appreciated the way the isometric paper could be used. … They were aware, however, that their drawing was not working as intended. This created anxiety for a number of reasons not the least of which was that they did not like having messy work. Once they became aware that their drawing was not what was intended, they sought to erase their work. Some additional isometric sheets with two cubes already drawn had been prepared. The teacher merely replaced the sheets of the students experiencing difficulty with this new sheet. In all cases, this action was all that was needed.

The second recommended differentiated task was to "Have some cubes. Ask the students to model what the 3 (cubes) might look like":

> (This prompt was) suggested to students who had drawn one configuration appropriately but seemed to be experiencing difficulty in visualizing what another would look like. The teacher used three large cubes to model the less obvious configuration on the teacher's desk: this was drawn to the attention of these students. No further prompt was necessary, and all were able to proceed with exploring the mathematical task.

In the case of the first of these, it seems that the combination of the fresh start, and the hint provided by the cubes already drawn allowed all students to complete this aspect of the task. The difficulty for students who needed the second prompt may have been that the task was abstract or that visualizing an alternate configuration was difficult, and so the concrete example assisted in overcoming this barrier. The lesson was successful in that all students progressed to the ultimate goal task, and were able to participate productively in the class discussions. Both prompts supported students experiencing difficulties on a one-to-one basis, without resorting to extended explanations or publicly emphasising the difficulties that some students had. The prompts allowed all students to experience success (for a fuller description of this lesson, see Sullivan et al., 2006).

The following are further examples of enabling prompts that were observed. All of the following examples were taken from lessons that the teachers themselves had developed.
One teacher had posed the basketball task (above) to the class:

> After 5 games, the mean number of points that a basketballer had shot was 6, and the median number of points was 4. What scores might the basketballer have shot in each game?

The record from the observer noted:

> For one student, the teacher said that he could forget about the median for now, and that there are 5 games and the mean points was 6.
>
> After 5 games, the mean number of points that a basketballer had shot was 5. What scores might the basketballer have shot in each game?

The effect of this enabling prompt is to reduce the variables from two to one, while preserving the open-ended nature of the exploration.

In another observed lesson the posed task was:

> Draw some closed shapes that have 6 right angles

The observer commented that the teacher asked one student:

> For a start, draw some closed shapes that have 4 right angles

This has the effect of reducing the complexity of the task and 4 right angles is a more familiar shape.

In another lesson, the posed task was:

> What might be the numbers in the boxes? $\square$ % of $\square$ = 200

The observer noted that, for some students, the teacher changed the target from 200 to 20. This reduces the size of the numbers to be manipulated, presumably making it easier for the students who may be experiencing difficulty with the original task.

There was also an observed lesson in which a task was:

> For the equation 3a + 2b = 70, what might be the values of a and b?

The observer noted that the teacher wrote the following on the board

$$3 \times \square + 2 \times \square = 70$$

and asked some students to find the missing numbers. This reduces the abstractness and simplifies the forms of representation by avoiding the algebra.

Each of these possibilities could have the effect of changing the task so that students' experiencing difficulty can engage with the adapted task, and hopefully attempt the original task within the scope of the lesson. This was the effect in many of the lessons observed.

In terms of guidance for teachers, it is proposed that aspects that might contribute to students' difficulties include the nature of the communication used in both the framing the problem and responding, language and reading or listening skills, difficulty in connecting the words of a problem to a practical situation, or unfamiliarity with contextual knowledge. These factors suggests guidance in the form of enabling prompts that could support students to deal with comprehension of the task, its semantic features, word elements; abstractness; or the need to identify aspects of the problem that are relevant compared with those that are just elaboration. Alternatively, the teacher could make inferences about the cognitive demand of the task—such as the number of steps required to formulate a solution, the number of unknowns sought, the degree of operational complexity, or the level of abstraction or visualisation required. Analysis of these elements informs teacher's formulation of a differentiated task such as reducing the number of steps or variables, varying the problem to allow the elements to be manipulated independently, or making the problem more concrete through a physical representation.

Extending Prompts

A further challenge in mixed ability classes is that some students finish the set tasks quickly, and so need to be set work that extends their thinking on the task that they have worked on.

Well-designed open-ended tasks also create opportunities for extension of mathematical operations and dimensions of thinking, since students can explore a range of options as well as considering forms of generalised response.

Teachers can pose prompts that extend the thinking of students who complete tasks readily in ways that do not make them feel that they are merely getting more of the same. Students who complete the planned tasks quickly are posed supplementary tasks or questions that extend their thinking and activity. Extending prompts have proved effective in keeping higher-achieving students profitably engaged and supporting their development of higher-level, generalisable understandings that we associate with learning.
To give an example, for the basketball task, some extending prompts might be:

> How many different responses are there?
>
> If you know that the mode is 3, how many possible responses are there?

The intent of the first of these prompts would be to encourage students to search for patterns, and orderly ways of recording responses. The purpose of the second prompt was to pose an additional complexity, adding a further dimension of variation.

As it happens, not many instances were observed in which extending prompts were used. Our sense is that some students "paced themselves" so that they did not finish quickly. It is an aspect of our model that requires additional investigation.

A Specific Example

The following is the report of a particular lesson that was taught as part of the research. It is presented here to give a more holistic view of the way the planning and teaching model might be implemented.

The teacher of this lesson, pseudonomically Mr Smith, was not a teacher at the particular school, but was experienced, and was also familiar with the project and the planning and teaching model. There was a trained observer present, as was the usual mathematics teacher. The lessons had been planned beforehand, and Mr Smith and the observer met with the class teacher before and after the lesson.

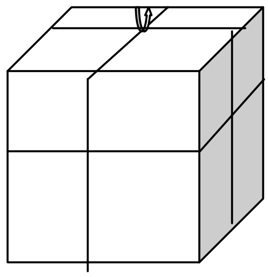
The report is an amalgamation of the lesson plans, Mr Smith's recollections captured after the lesson, and the observer's notes written at the time.

T

he teacher intended that the students work on the following problem (goal task):

You have a box that needs 1 m of string to tie it up like this. What might be the dimensions of the box?

Assume that 30cm is needed to make the bow.



The class teacher had earlier suggested both the lesson focus and this context of wrapping presents, and so it was assumed that it was familiar to the students, and it proved to be so in the lesson.

The task is illustrative of the type that formed the basis of the project, in that:

it addresses challenging and useful mathematics, and focuses on concepts that may have applicability in other contexts;

there are many ways of solving the task, and different possible interpretations of the task demand, and so it could be assumed that most students would be willing to make an attempt, given that they have some choice over their approach, and most would be able to do so because of the different entry levels;

the students would have the opportunity both to find a solution and to describe their solution to the class, allowing both opportunities for problem solving and for considering challenges in communicating the solution, as well as experiencing the advantage of being part of a community working on the same learning task;

the range of possible solutions, when viewed together, could allow students to see the potential variability in the box within the constraints of the tasks, and that the key dimensions (L, W, H) can vary;

there would be limited need to listen to explanations by the teacher at the start, not only allowing students opportunity to solve the task for themselves, but also avoiding the potentially disengaging effect of teacher explanations;

because of the openness, students who finish quickly could be posed extension exercises readily.

As a first step in the lesson, Mr Smith gave each pair of students a box wrapped and tied with string as in the photo above, and asked them to calculate the length of the string, without untying it. Mr Smith later explained that he considered this to be a way of introducing the students to the key concepts without requiring extensive explanations, and the demand of the task would be readily communicated to the students.

Mr Smith first explained that the bow used 30 cm of string. The observer recorded the students' response as follows:

> The students refused to believe that the bow was actually 30 cm. (As in the photo, it did not look like 30 cm). The teacher asked the students to assume that it was. The students still did not start on the task, so the teacher untied the bow and asked one of the students to measure the loose ends. Each end of the string was 15 cm, and so the students then agreed that the bow was 30 cm.

Clarifying aspects of the problem, including specific language, is universally important. Note that this is different from telling the students solve to do the problem. In this case, the consideration of the bow from then on in the lesson was not an issue.

As enabling prompts, Mr Smith had ready some boxes and loose string for students who might need to tie up a box, but no students seem to need this prompt. Mr Smith also had a box covered in a streamer that could be cut into sections, and this was not used either.

After the students had worked on the problem, Mr Smith invited individual students, on behalf of their pair, to explain their methods of solution to the group. The observer recorded this as follows:

> One student explained that they had measured separately each section of string, written down the lengths and added them up. There were 4 sections on the top (from each edge to the centre where the string crosses), 4 on the bottom and 4 on the sides.
>
> Another student explained that they had measured each of the lengths and then coloured in the string to show they had done it, adding the lengths as they went.
>
> Another student said all the pieces of string on the sides were the same. There was some debate about this (there were slight differences depending on whether the wrapping on the box was neat and regular), and (Mr Smith) explained that for this purpose those sections were the same. The student then said that the top and the bottom was the same, and that this is how they worked it out.

This initial task was a key part of the lesson, because the subsequent task required the assumption of regularity of the box. This is an example of the teacher considering the trajectory of the class learning.

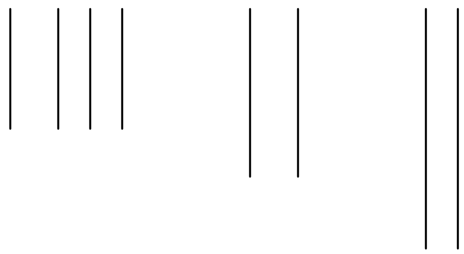
The next task, as presented above, was then posed, the terms and assumptions clarified, and the students set to work. Note that all of the boxes in the previous task required more than 2 m of string, and so did not detract from the visualisation needed to address the problem. All pairs retained their box and handled it throughout the activity.
The observer noted:

> Again the students worked in pairs. The students required constant monitoring in that their interactions with each other were confrontational but they engaged with the task, and knew what to do.

Mr Smith had prepared some enabling prompts for students who may have experienced difficulty in engaging with this task, including a box covered in white paper on which they could draw, and a box that used approximately 1 m of string. Again neither of these prompts was used.

Mr Smith later commented that he had asked students experiencing difficulty to try out a particular dimension, such as assuming that the height was 5 cm. This had the effect of reducing the task demand for some students, and so is an example of an enabling prompt.
The observer recorded the subsequent class discussion of the task as follows:

> One student reported that the group had assumed that it was a cube, and that the lengths would add to 70 cm, and that there would be 8 equal lengths but did not progress beyond that. The teacher did not follow this up.
>
> Another student reported that his pair had said the 4 sides were the same, the 2 lengths the same, and the 2 widths. He represented this as follows on the whiteboard:



> He said that the total had to be 70, and that they were working out what numbers to use to make the total 70. They had first tried unrealistic numbers, but using trail and error the lengths turned out to be 5, 10 and 15 cm respectively, which made 70 together. (Mr Smith) asked the other students to ask any questions. No one did.
>
> Another student drew a table of values, with columns headed length, width and height, and made guesses of the dimensions, but then made errors in the calculation.

In summary, this lesson experience is illustrative of the planning and teaching model. There was a potentially engaging task, whose openness indicated the possibility of student control and choice, a hypothetical trajectory to lead students to key aspects of the goal task, an intent that the class work together as a community and review their work as part of the learning experience, both pre-prepared and impromptu enabling prompts for students experiencing difficulty, and the potential for ready extension, if needed.

Conclusion
It is proposed that teachers can address the diverse needs of students in most classes but choosing tasks that have potential for student constructive explorations and which are accessible at a variety of levels. Connected to this is a commitment to building a classroom community, and posing enabling prompts and extending prompts for students who might benefit from them.

In all, it is clear from our work with primary and secondary teachers that these ideas can be implemented in everyday classrooms. It offers an important alternative to the obvious disadvantages arising from grouping students by like ability.

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Development in the Mathematics Teaching of Beginning Elementary School Teachers: An Approach Based on Focused Reflections

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This paper discusses some early results from a longitudinal study investigating how development in the mathematics teaching of beginning elementary school teachers might be promoted. Two questions are addressed. The first concerns how such development might be evidenced and the second whether it is possible to attribute development to participation in the study. A framework for observation and discussion of mathematics teaching – the Knowledge Quartet, was used by teachers in the study to facilitate reflection on their teaching over a two year period. Initial findings would suggest that it is possible to show that development has taken place and that this development is at least in part attributable to the focused reflections of participants

.

Introduction

Several research reports would suggest there is reason for concern about the mathematics knowledge held by elementary school teachers, in England, Ireland and elsewhere, (e.g. Ball, 1990; Cooney and Jones, 1990; Corcoran 2006; OFSTED, 2000). It has also been argued that this weakness in content knowledge has a negative impact on teaching effectiveness (Ball, 1988; Rowland, Martyn, Barber and Heal, 2000). Mathematics educators have identified the need to improve the mathematics content knowledge - both subject matter knowledge per se and pedagogical content knowledge of elementary teachers during their initial training (Ball, 1990; Goulding, Rowland and Barber, 2002) However developments made during training do not appear to be sustained once teachers are in post (Brown, McNamara, Jones and Hanley, 1999; Mcnamara, Jaworski, Rowland, Hodgen and Prestage, 2002).

It would seem that a longer-term approach is needed to improve teacher knowledge and consequently the teaching of elementary mathematics. Reflection *in* and *on* practice has often been suggested as a means by which teaching might be improved (Schon, 1983). Burton and Povey (1996) found such reflection to be an important factor in increasing competence in mathematics teaching. My research aims to explore the effect that reflection, with a focus on mathematics content knowledge, may have on teacher development in the early years of teaching. I used the *Knowledge Quartet framework* (Rowland and Turner, 2006) as a tool to facilitate this focused reflection.

The *Knowledge Quartet* was developed as a framework for discussion of mathematics teaching between trainees and their mentors, providing shared understandings for dialogue and reflection on mathematics teaching. The framework consists of four dimensions termed *foundation, transformation, connection* and *contingency*. The *foundation* dimension includes the theoretical knowledge of both SMK and PCK as well beliefs about mathematics and mathematics teaching. This dimension is seen as underpinning the other three. *Transformation* encompasses the ways in which the teacher's own knowledge is transformed to make it accessible to the learner. *Connection* includes issues of sequencing and connectivity as well as complexity and conceptual appropriateness. Finally, *contingency* covers the way in which teachers respond to unplanned instances in a lesson.

Methods

The Knowledge Quartet framework was introduced to the 2004-5 cohort of 160 trainee teachers during a lecture in the second term of their one year post graduate initial teacher training. I outlined the longitudinal study I intended to carry out using the framework and asked for volunteers to participate in the study. Over 30 people volunteered and twelve participants were selected to cover a range of confidence levels in mathematics as well as to give a balance of those intending to teach in foundation (3-5years), Key Stage 1 (5-7 years) and Key Stage 2 (7-11 years) classes. Their intention to stay in the local area was also a major consideration.

It was explained to the selected volunteers that the Knowledge Quartet framework was to be used as a basis for the analysis of their teaching and to provide a framework for discussion. They were given a document, which had been produced for teacher-mentors, setting out a number of example questions to ask of observed teaching under each of the four dimensions of the Knowledge Quartet. The trainees were encouraged to use the framework to focus reflections on their mathematics teaching. Participants continued to work with the Knowledge Quartet framework to discuss their lessons, and to aid reflections, in the first and second years of their teaching. Over these two years the number of participants reduced in the first year to nine and in the second year to six and then five.

All participants were observed teaching during the final placement of their training year, twice during their first year of teaching and three times during their second year of teaching. These lessons were all video-taped. In the training year the video-tapes were used for stimulated recall discussions using the Knowledge Quartet to focus on the mathematical content of the lesson. During the first year of teaching feedback using the Knowledge Quartet was given following the two observed lessons. The discussion that took place during these feedback sessions was audio-taped and transcribed. Participants were sent a DVD of their lesson on which to write their own reflections. In the second year of their teaching only minimal feedback was given following the observed lessons as I wanted to see how the teachers would independently make use of the Knowledge Quartet. They were sent DVDs of their three lessons and wrote reflections on each of these.

In the first year of their teaching participants were also asked to write reflections on their mathematics teaching over the course of the year. In the second year this requirement was made more specific and participants completed reflections each half term (every 6-7 weeks). We met as a group at the end of the training year, at the end of the first year of teaching and at the end of each of the three terms in the second year of teaching. The discussions from each of these meetings was recorded on audio-tape and transcribed. For further details of methods used in the study see Turner (2007).

Findings

Two key questions became apparent towards the end of the third year of working with teachers on this project:

- How can development in mathematics teaching be identified and evidenced?
- Can development be demonstrated to be attributable to reflection using the Knowledge Quartet framework?

In this paper I begin to try to answer these questions with reference to three types of data from my research. These data came from observations of teaching, participant group meetings and participants' written reflections on teaching. My longitudinal observation of the teaching of individual participants presents some evidence that their teaching had become more effective, and that this development was likely, at least partially to result from the teachers' focused reflections. There is still a great deal of data analysis to be completed, however I have been able to identify some examples of developments in teaching through longitudinal observation.

Evidence of development from observations of teaching

In her first year of teaching Linda was observed teaching a method for subtraction to 4 and 5 year old children. She modelled a very procedural method, holding up fingers to represent the subtrahend, folding down the number of the minuend and then counting how many were left. Though this was a reasonable method, there were no opportunities for children to explore or develop their own strategies. When reflecting on this lesson, during the feedback session, we discussed giving children opportunities to develop and explain their own methods. In reflections of her teaching in the first term of the following year Linda wrote:

> I have tried this term to let the children take the lead more in their learning. This started when I assessed each child individually on their counting skills. One aspect I assessed was how they count objects in a structured line and how they count a random pile of objects. I asked the children "How many objects are there?" and let the children work out the problem in their own way. I got a range of ways that demonstrated different stages of counting e.g. some looked at the objects and told me how many with the smaller amounts, some counted and pointed to each object. I made notes on the different methods used which gave me a good insight in to where to start with my teaching.

This development toward allowing children to make use of their own methods was also apparent in observations of her teaching. In a lesson observed in November 2006, Linda asked the children to demonstrate how they would keep track when counting pictures of objects displayed on a screen, counting physical objects and when counting actions. She allowed them to explain and demonstrate their strategies and made suggestions which developed the children's ideas.

Another teacher, Kate, was observed in her training year teaching a lesson about doubling. The representation she chose to use for recording the process of doubling was problematic. In an attempt to engage the children she used a picture of a witch's cauldron with bubbles coming from it. The rhyme 'hubble bubble what's my double' was used to initiate completion of the calculations. The double was written as an addition calculation in the cauldron e.g. '3 + 3' and the answer written in a bubble. For two digit doubling Kate introduced a second bubble and wrote the answers in the bubbles as the number of units and the number of tens e.g. $14 + 14 \rightarrow 2 + 8 \rightarrow 28$. During the stimulated recall discussion of the lesson, I suggested that this was problematic since such recording allows only for an additive model of doubling and does not allow for a multiplicative model. We also discussed the problems associated with her recording of doubling the tens digit. I suggested this modelled a column method of addition which did not hold the value of the numbers. Later, in her reflections made at the beginning of her first year of teaching, Kate demonstrated that she was thinking more deeply about recording and representing numbers and calculations.

> I have tried to avoid talking in terms of 'tens and units' which I learned as I thought this was no longer good practice, but I have realised that my colleagues use this terminology and in working with place value set out numbers under these headings, so we used this method of recording this week. I think I had previously been disinclined to use this as I

> associate it with column methods which we are not teaching in year 1/2. As visual aids this week we have used place value cards, cubes, 100squares and real objects.

In a subtraction lesson on difference that I observed in the second year of Kate's teaching, she made more considered use of representations to record calculations. She began by building towers of cubes to represent the two numbers between which they wanted to find the difference. Kate then asked the children to think of their own ways of finding the difference between two small numbers and show these on their individual white boards. After discussion of the varied methods illustrated by the children, Kate focused on using an empty number line. She demonstrated putting the two numbers on the line and counting on from the smaller to the larger.

A third participant, Amy was observed teaching lessons on counting in her training year and again in her first year of teaching. This allowed more direct comparisons to be made about the teaching and clearly suggested that development had taken place. In the lesson observed during Amy's training she showed some useful pedagogical content knowledge in relation to the pre-requisites for counting (Gelman and Gellistel, 1978). When asked if she remembered what children need to know in order to be able to count, Amy responded that "It's number names and order" and went on to say "It's not anything to do with cardinality or anything yet, it's just the rote, the reciting of numbers". When asked explicitly whether she had thought about this in her planning she replied "I think I had, 'cus on my plan I wrote down the objective for that, so I think I had". A year later my observation of another lesson on counting clearly demonstrated that Amy was now more confidently using her knowledge of the pre-requisites for counting in her teaching. Throughout the lesson she made these explicit to the children and reminded them of related strategies to use in order to be able to count - put objects in a line, point to or touch each one only once. Her planning of the lesson would seem to have been based around this knowledge and this was confirmed during discussion.

> Yes I planned it from that framework (the foundation dimension of the Knowledge Quartet) really, of that progression of saying the numbers in order, being able to do one to one correspondence and then the cardinal principle.

Amy seemed to recognise that a common error children make when counting is not knowing when to stop. She reminded children that they should stop counting when she stopped hitting the chime and that the last number is the answer to 'How many?' In the post-lesson discussion, Amy demonstrated that she was aware of this difficulty:

> Well I picked Katy 'cus she often, well when she started she would count things and then she would get to the last thing and then you would say 'How many?' and she would start counting again. She didn't have like the cardinal principle … that's why I asked 'How many are there?' instead of saying 'yes there's seven'.

These three examples suggest that it is possible to show development in the teaching practices of beginning teachers through observations of their teaching. There would also seem to be evidence that this development has been facilitated by the teachers' reflections on the mathematical content of their teaching and that this focus on content was a result of using the Knowledge Quartet framework.

Evidence of development from group meetings

Participants' perception of their own development as evidenced in transcripts of group meetings and in written reflections presents some further evidence of development in their teaching. I will begin by discussing some evidence from the group discussions. I pursued similar themes in all the group meetings, one of which was concerned with how the participants perceived their mathematics teaching to have changed or progressed over the last year. At the end of the first year of teaching seven of the nine participants attended the meeting and some responses suggested that they were focusing on content knowledge. Four participants suggested that they made use of knowledge they had gained during their training course and two admitted to being worried about their content knowledge. A developing belief that the mathematical understandings of their pupils might be different from their own was also apparent. Specific aspects of subject knowledge detailed in the Knowledge Quartet framework were also mentioned. One participant said that they now thought more about their use of examples and another that they made more links in their teaching. Development in the participants' beliefs about mathematics and mathematics teaching was also apparent in their responses. Three of the participants suggested they were more focused on understanding and developing children's own strategies and three suggested that they had come to recognise the advantages of a problem solving approach but thought it probably wouldn't work with their children. Two teachers said that they had successfully made use of such an approach.

A growing confidence in their own teaching was a feature of the discussion at the end of the first term of their second year of teaching. Out of the six teachers still in the project, four participants attended the meeting and all said they felt more confident. Two participants said they were more enthusiastic and a teacher who had previously thought a problem solving approach unworkable for pragmatic reasons had now adopted this approach. Such growing confidence may also be related to the greater understanding participants had of the needs of their pupils. Three participants claimed to be more aware of progression in children's learning while another said they had a better idea of what children find difficult. Some participants seemed to be thinking more deeply about the content knowledge needed for teaching mathematics. Three teachers suggested that they now had a greater understanding of the complex nature of mathematics and others seemed to have applied mathematical content knowledge to their evaluation of practice. Three said they had become more critical of resources used in their schools for teaching mathematics and more critical of the mathematics planning and teaching of their colleagues.

The theme of critical evaluation was again apparent in the discussions of teachers at the end of term two in their second year of teaching. Again four out of the six participants were able to attend the meeting. One participant said they were more critical of their own teaching and another that they had learned from their previous teaching. Two of the group mentioned being critical of some aspects of curriculum directives from the government and a further participant said they were now less reliant on curriculum directives and available resources. One teacher said she now felt able to be more flexible in the organisation of mathematics teaching and learning in her classroom. This seemed to suggest a growing independence in some participants however one participant valued discussion with colleagues particularly recognising the learning opportunities presented by joint planning. Two comments focused on the children, with one teacher saying that she had higher expectations of them and another suggesting that they were responding better to her teaching. Only one comment referred specifically to confidence in teaching mathematics and this suggested that the teacher had become more confident in teaching less familiar mathematical topics.

Evidence that development might be attributable to using the Knowledge Quartet as a framework for reflection

The evidence from participant group discussions above, suggests that some development in the group's understanding of effective teaching has taken place. However, further evidence is needed in order to determine whether this development is the result of reflections on teaching through the lens of the Knowledge Quartet. In order to answer this question, another theme that I pursued at all of the group meetings was that of how useful participants had found using the framework. At the end of their first year of teaching, five participants out of the seven present made comments suggesting that using the quartet had made them think more. More specifically three said that using it had alerted them to things they had missed out in their teaching and two said they liked the way it had helped them find different ways of doing things. One participant said that it had improved their knowledge, another that it had improved their teaching and another that using the quartet encouraged dialogue with colleagues.

Participants also considered the impact of using the Knowledge Quartet at the end of the first term in the second year of teaching. There seemed to be more consistency in the responses they gave. Of the five teachers present four suggested that they thought about the Knowledge Quartet when planning and two when in the act of teaching. Three people made comments suggesting it had improved their content knowledge and three suggested it helped them recognise when their teaching had been unhelpful to their pupils. One participant thought that using the Knowledge Quartet helped her understand children's thinking.

At the end of the second term of their second year of teaching four of the six participants were able to attend the meeting. Three of these suggested that using the Knowledge Quartet made them more thoughtful about the match between activities or resources and their learning objectives. Three teachers also thought that it helped them to act contingently in response to children's ideas and interests. Two teachers specifically mentioned that the Knowledge Quartet helped them to think about connections and one said it helped them think about their use of examples. One teacher suggested that the Knowledge Quartet made her think about how children make their own mathematical meanings. This was a different participant to the one who had said that the Knowledge Quartet helped her understand children's thinking at the previous meeting. Data from these group discussions would suggest that participants' use of the Knowledge Quartet, to focus reflection on content knowledge, has been influential in the development of their mathematics teaching.

The discussion above has tried to answer two questions by looking for themes in longitudinal data from a number of beginning teachers. Examination of this cross-case data suggests that it is possible to identify development in the mathematics teaching of beginning teachers and that this development may be the result of reflections which focus on subject knowledge through the lens of the Knowledge Quartet. To further support these contentions I now focus on the development of an individual participant and consider longitudinal data from her written reflections. This data comes from written reflections on her observed lessons and half-termly reflections of mathematics teaching generally. The participant is Kate who was also discussed above in relation to observations of her teaching.

Evidence of development that might be attributable to using the Knowledge Quartet framework from the written reflections of one participant: Kate

Kate's reflections seem to suggest four emerging themes in her development which may be attributed, at least in part, to working with the Knowledge Quartet. The first of these themes is the development of her mathematics teaching through an increased understanding of children's learning. This is a theme that seems to be particularly prevalent at the beginning of Kate's second year of teaching. In her reflection on the lesson I observed in the autumn term Kate commented on how she drew on her previous teaching experience when planning the lesson.

> I had given a lot of thought to the teaching points that I wished to make drawing on my experience of last year and the lesson I taught in my PGCE placement.

In her reflections at the end of the first half term of her second year of teaching, she demonstrated a growing awareness of progression in children's learning.

> I was much more aware this year than last year of the stages children have to go through at the early stages of learning numbers.
>
> My understanding of the various stages of knowledge that have to be acquired before we can understand place value is much better than it was last year.

In reflections at the end of the autumn term Kate showed that she drew on her previous experience of teaching a topic to make assessments of children's understanding

> From last time we covered place value, I realised that the majority of my Year Ones were not clear on this concept
> …because I know that my last year's class found these really hard to get to grips with

It may be argued that as a teacher gains experience of teaching mathematics they should be able to judge the needs of their pupils more appropriately even without the aid of focused reflections. However the reflections discussed would seem to be concerned with understanding the conceptual appropriateness of the subject matter. This is a key aspect of the connection dimension of the Knowledge Quartet and it could be argued that working with the framework may have helped Kate to make better use of her experience than if she did not have this focus.

A second theme emerging from Kate's reflections was that of an increased confidence in her own pedagogical content knowledge. This confidence was apparent in her willingness to debate with colleagues and to make her own decisions about how she wanted to teach. In her reflection of the lesson I observed at the beginning of her second year of teaching Kate expressed her reservations about using resources produced by her colleague.

> My colleague had produced 10cm long strips and I was not convinced that these were as useful as they could be because she had not marked the number of centimetres on the lines but between two lines.

In her reflections on the first half term in the second year of teaching, Kate showed that she had become confident enough to make amendments to what she saw as the 'flawed' ideas of her colleagues.

> So I unilaterally decided to change the investigation when I did it with my class and we investigated what I thought was a slightly less flawed question.
>
> … so I can politely say something straight away if I am uncomfortable with the mathematical ideas behind our planning in any other case!

In her reflections at the end of the third half term, Kate referred on a number of occasions to differences of opinion about pedagogical practices between herself and colleagues.

> My colleague doesn't like these methods … I am not sure though because … so I think it is worth practicing with small numbers too.
>
> … my colleague believes we should only be teaching 'counting on' along the ENL for subtraction… following my chat with Fay…these seem to me to lend themselves to different methods on an ENL.
>
> Also, we have been debating whether to teach adding and subtracting together or separately (Kate decided to teach them together and her colleague separately)

By the middle of her second year of teaching Kate seemed to feel sufficiently confident in her own mathematical pedagogy to question the practices of her more established colleagues and to make amendments that were consistent with her own understanding. It is possible that this confidence has grown out of her reflections on, and discussions of, pedagogical content knowledge facilitated by the framework. Such reflections and discussions might also be seen as catalytic in the third theme emerging from Kate's reflections. By the middle of her second year of teaching Kate's reflections demonstrated some quite profound thinking about the content of the mathematics she was teaching.

> Lots of them are becoming dependant on the hundred squares and looking above or below but can't explain why this strategy works and don't seem to have generalised what happens to the digits when we add/take away 10.
>
> I got into a mess with number lines … I did not teach this very well because I had looked at the planning and thought the areas were connected, so did not draw out sufficiently for the children the different skills which were needed.
>
> In the first lesson we did several activities which involved putting numbers into order … but I think this exercise had more to do with place value then ordering numbers as they had to work out how many ten marks to count along and then think about the units.

The final theme that emerged from analysis of Kate's reflections was that concerning a developing coherence in her beliefs about effective mathematics teaching. This was a key feature of her reflections at the end of the second term in her second year of teaching.

> I started this week with a clear idea of the maths I wanted to teach …which is a little different from the approach we have often taken before ...
>
> We took an investigative approach to data handling which was fantastic! This worked really well because they had to think for themselves …
>
> Lots of boring subtraction number sentences this week to see how they tackle them. It was a really good exercise as they had to use their initiative and think about the investigation and how to record. It was very 'real life' and motivating.
>
> It was interesting to discuss different types of triangle. We got very interested in discussing how triangles can have different numbers of lines of symmetry … I had never thought of that before!
>
> I was really pleased – my upper group have finally started to work through a problem systematically of their own initiative.

Towards the end of her second year of teaching, Kate seemed to be developing an increasingly learner-focused view of teaching (Kuhs and Ball, 1986). In terms of Ernest's (1989) models of teachers' conceptions of mathematics teaching, she was somewhere between a model where conceptual understanding along with problem solving form the basis of teaching and one in which investigations and problem solving form the basis of teaching. Reflecting on her mathematics teaching through the lens of the Knowledge Quartet would seem to have helped Kate develop a view of what she considered effective learning and teaching against which to assess her own practice.

Conclusion and next steps

The four themes emerging from Kate's reflections suggest that it is possible to identify aspects of development in her mathematics teaching. It would also seem that at least some of this development can be attributed to the reflections on her teaching that have been focused on mathematical content knowledge through the lens of the Knowledge Quartet framework. Earlier I attempted to show that development in mathematics teaching could be identified through longitudinal observations and analysis of teaching. I also demonstrated that themes in development could be identified through analysis of participants' contributions to discussions. It was suggested that the developments observed in teaching and the themes that emerged from group discussions might be, at least in part, attributable to participants' reflections through the lens of the Knowledge Quartet framework. In the year remaining of this project I hope to further investigate the themes that have arisen from analyses of observed teaching, group meetings and written reflections. I shall make use of triangulation of the different types of data in order to build a more conclusive case for the suggested developments in teaching I have discussed above. It is also likely that further themes will emerge as more structured qualitative analysis using NVivo software is completed. During the final year I will conduct interviews with the participants, in which themes and ideas emerging from the data collected over the past two years will be pursued and their validity questioned or strengthened.

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